Particle motion in Hamiltonian Formalism I

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Introduction to Accelerator Physics

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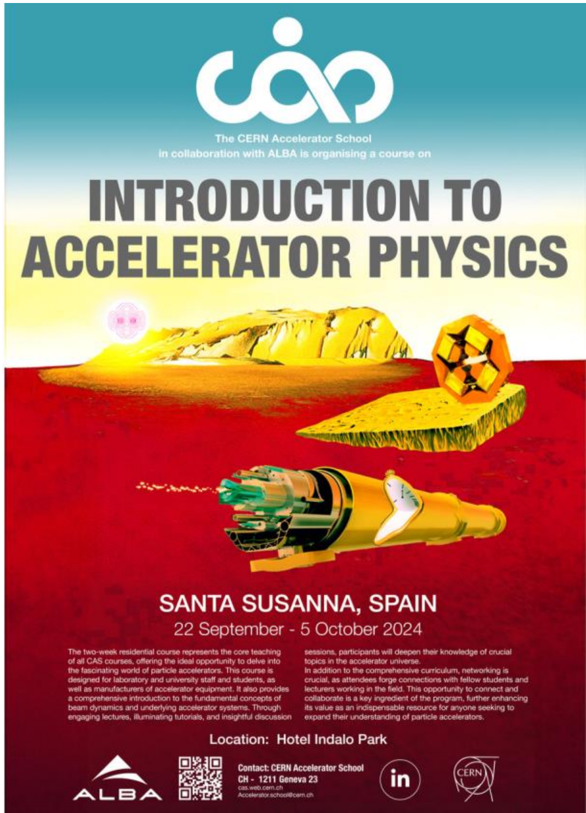

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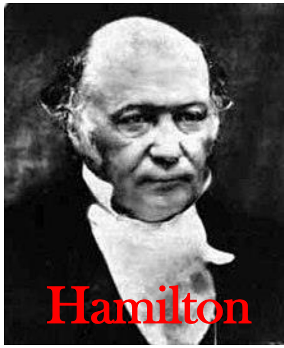
Purpose

- The objective is finding methods to **derive** and **solve** (**integrate**) **equations** of **motion**, in order to describe the **evolution** (dependence with "**time**") of a **system** ("**particle**")
- Introduce **formalism** of **theoretical (classical) mechanics** for analysing motion in general (linear or non-linear) **dynamical systems**, including **particle accelerators**
- Connect this **formalism** with **concepts** already studied in the introductory CAS (matrices for transverse motion, synchrotron motion, invariants,…)
- Prepare the **ground** for approaches followed for **studying non-linear particle motion** in accelerators (in the advanced CAS)

Principles of classical mechanics

- Some **key concepts** of **classical (analytical) mechanics** reviewed in this lecture, including
  - **Integrals** of **motion**
  - **Integration** by **quadrature**
  - **Period** and **Frequency**
  - **Hamilton's principle**
  - **Lagrangian, Euler-Lagrange equations**
  - **Hamiltonian, Hamilton's equations**
  - **Canonical variables, Symplecticity**
  - **Poisson brackets**
  - **Canonical transformations**



Equations of motion

Reminder: Newton's law

- The motion of a "classical" particle in a force field is described by **Newton's law**:

$$m\frac{d^2u(t)}{dt^2} = \frac{dp_u(t)}{dt} = F(u) = -\frac{\partial V(u)}{\partial u}$$

with u the position

p_u the momentum

$F(u)$ the force

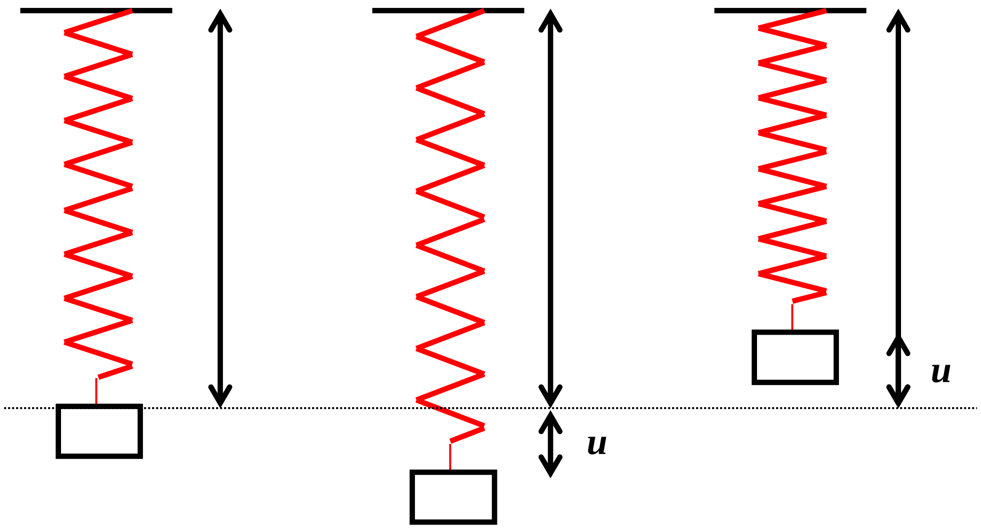
$V(u)$ the corresponding potential

- It is essential to solve (**integrate**) the differential equation for understanding the evolution of the physical (dynamical) system

Reminder: Harmonic oscillator

- A linear restoring force (**Harmonic oscillator**) is described by

$$\frac{d^2u(t)}{dt^2} + \omega_0^2 u(t) = 0 \quad \text{with} \quad \omega_0 = \sqrt{\frac{k}{m}}$$



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  through the **characteristic polynomial**

$\lambda^2 + \omega_0^2 = 0 \Rightarrow \lambda_\pm = \pm i\omega_0,$ which yields the **general solution**

$$u(t) = ce^{i\omega_0 t} + c^* e^{-i\omega_0 t} = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) = A \sin(\omega_0 t + \phi)$$

  with the "**velocity**"

$$\frac{du(t)}{dt} = -C_1\omega_0 \sin(\omega_0 t) + C_2\omega_0 \cos(\omega_0 t) = A\omega_0 \cos(\omega_0 t + \phi)$$

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- Note that a **negative sign** in the differential equation provides a solution described by **hyperbolic sine/cosine** functions
- Note also that for **no restoring force** $\omega_0 = 0$, the motion is **unbounded**

Matrix solution

- The **amplitude** and **phase** depend on the **initial conditions**

$$u(0) = u_0 = C_1 \ , \ \frac{du(0)}{dt} = u_0' = C_2\omega_0 \ , \ A = \frac{\left(u_0'^{\,2} + \omega_0^2 u_0^2\right)^{1/2}}{\omega_0} \ , \ \tan(\phi) = \frac{u_0'}{\omega_0 u_0}$$

- The solutions can be re-written thus as

$$u(t) = u_0 \cos(\omega_0 t) + \frac{u_0'}{\omega_0}\sin(\omega_0 t)$$
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or in **matrix form**

$$\begin{pmatrix} u(t) \\ u'(t) \end{pmatrix} = \begin{pmatrix} \cos(\omega_0 t) & \frac{1}{\omega_0}\sin(\omega_0 t) \\ -\omega_0 \sin(\omega_0 t) & \cos(\omega_0 t) \end{pmatrix} \begin{pmatrix} u_0 \\ u_0' \end{pmatrix}$$

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- By replacing $\omega_0 \rightarrow \sqrt{k_0}$ and $t \longrightarrow s$, this becomes the solution of a **quadrupole** (see **Transverse Linear Beam Dynamics** lectures)

Matrix formalism

- General **transfer matrix** from s_0 to s

$$\begin{pmatrix} u \\ u' \end{pmatrix}_s = \mathcal{M}(s|s_0) \begin{pmatrix} u \\ u' \end{pmatrix}_{s_0} = \begin{pmatrix} C(s|s_0) & S(s|s_0) \\ C'(s|s_0) & S'(s|s_0) \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}_{s_0}$$

- Note that $\det(\mathcal{M}(s|s_0)) = C(s|s_0)S'(s|s_0) - S(s|s_0)C'(s|s_0) = 1$ which is always true for **conservative systems ("energy" is constant)**
- Note also that $\mathcal{M}(s_0|s_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathcal{I}$

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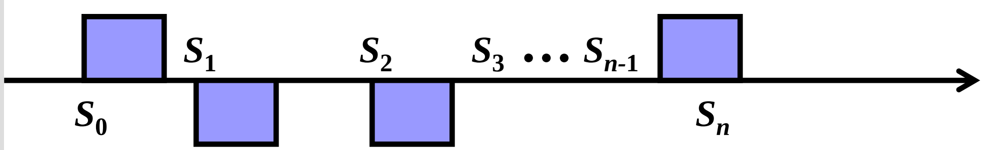
- Note also that $\mathcal{M}(s_0|s_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathcal{I}$
- The **general solution** can be built by a series of matrix multiplications

$$\mathcal{M}(s_n|s_0) = \mathcal{M}(s_n|s_{n-1}) \ldots \mathcal{M}(s_3|s_2) \cdot \mathcal{M}(s_2|s_1) \cdot \mathcal{M}(s_1|s_0)$$

from s_0 to s_1



from s_0 to s_2

from s_0 to s_3

from s_0 to s_n

(see **Transverse Linear Beam Dynamics lectures**)

Integral of motion

- Rewrite the differential equation of the harmonic oscillator as a **pair of coupled 1st order equations**

$$\begin{aligned} \frac{du(t)}{dt} &= p_u(t) \\ \frac{dp_u(t)}{dt} &= -\omega_0^2 u(t) \end{aligned}$$

Integral of motion

- Rewrite the differential equation of the harmonic oscillator as a **pair of coupled 1st order equations.**

$$\frac{du(t)}{dt} = p_u(t)$$
$$\frac{dp_u(t)}{dt} = -\omega_0^2 u(t)$$

By dividing the two sides of the equations, they can be combined to provide

$$\frac{dp_u}{dt}p_u + \omega_0^2 u\frac{du}{dt} = \frac{1}{2}\frac{d}{dt}\left(p_u^2 + \omega_0^2 u^2\right) = 0$$

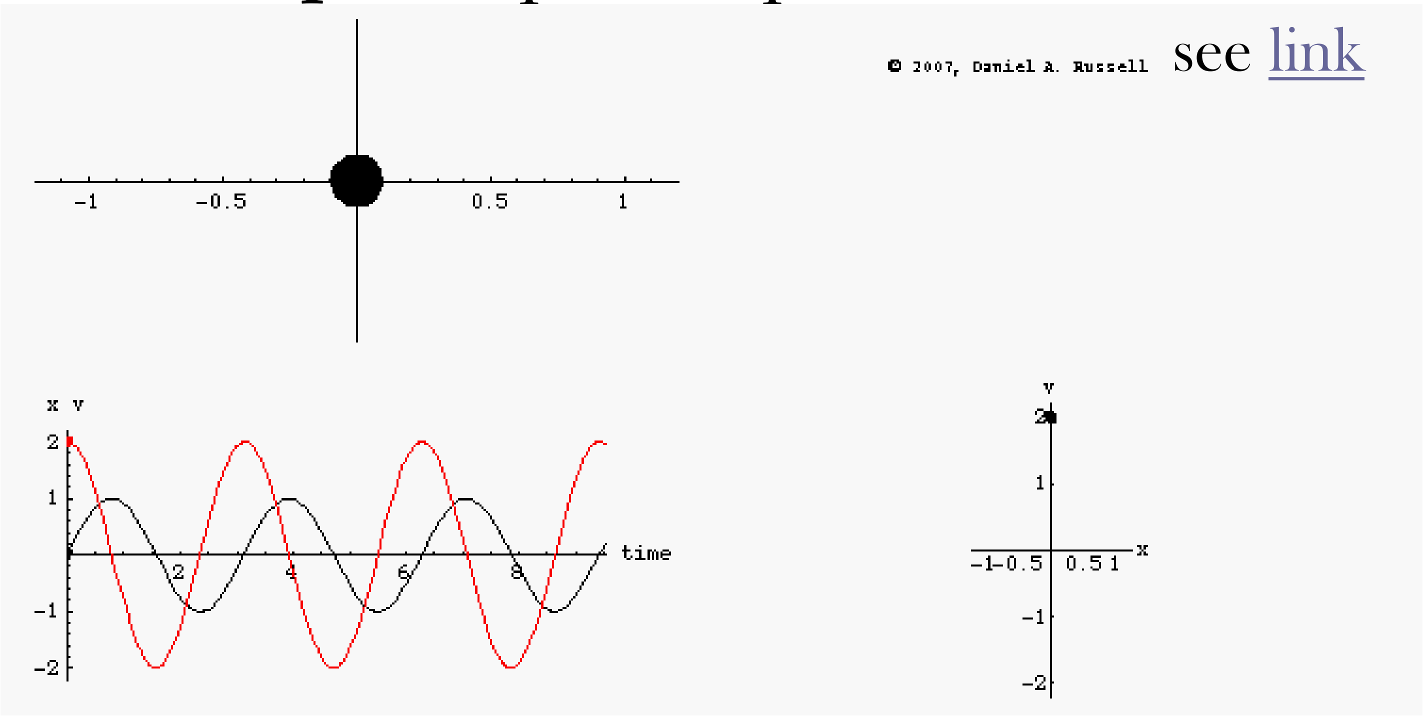
or

$\frac{1}{2}\left(p_u^2 + \omega_0^2 u^2\right) = I_1$ with I_1 an **integral of motion**

identified as the **mechanical energy** of the system

Integral of motion

- The equation $\frac{1}{2}\left(p_u^2 + \omega_0^2 u^2\right) = I_1$ describes in general an **ellipse** in phase space



see link

- Solving the previous equation for p_u, the system can be reduced to a first order equation

$$\frac{du}{dt} = \sqrt{2I_1 - \omega_0^2 u^2}$$

Integration by quadrature

- The last equation can be solved as an explicit integral or **"quadrature"**

$$\int dt = \int \frac{du}{\sqrt{2I_1 - \omega_0^2 u^2}}\,, \text{ yielding } \quad t + I_2 = \frac{1}{\omega_0} \arcsin\left(\frac{u\omega_0}{\sqrt{2I_1}}\right)$$

or the well-known solution $u(t) = \dfrac{\sqrt{2I_1}}{\omega_0} \sin(\omega_0 t + \omega_0 I_2)$

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- **Note**: Although the previous route may seem complicated, it becomes more natural when **non-linear** terms appear, where an **ansatz** of the type $u(t) = e^{\lambda t}$ is **not applicable**
- The ability to integrate a differential equation is not just a nice mathematical feature, but deeply characterizes the **dynamical behavior** of the system described by the equation

Frequency of motion

- The **period** of the harmonic oscillator is calculated through the previous integral after integration between two extrema (when the velocity $\frac{du}{dt} = \sqrt{2I_1 - \omega_0^2 u^2}$ vanishes), i.e. $u_{\rm ext} = \pm \frac{\sqrt{2I_1}}{\omega_0}$:

$$T = 2\int_{-\frac{\sqrt{2I_1}}{\omega_0}}^{\frac{\sqrt{2I_1}}{\omega_0}} \frac{du}{\sqrt{2I_1 - \omega_0^2 u^2}} = \frac{2\pi}{\omega_0}$$

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- The integral of motion is $I_1 = \frac{1}{2}p_u^2 + \frac{1}{4}k\;u^4$ and the

integration yields
$$T = 2\int_{-(4I_1/k)^{1/4}}^{(4I_1/k)^{1/4}} \frac{du}{\sqrt{2I_1 - \frac{1}{2}k\;u^4}} = \sqrt{\frac{1}{2\pi}}\Gamma^2(\frac{1}{4})\,(I_1\;k)^{-1/4}$$

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- This means that the **period** (frequency) **depends** on the **integral of motion** (energy) i.e. the maximum "**amplitude**"

The pendulum

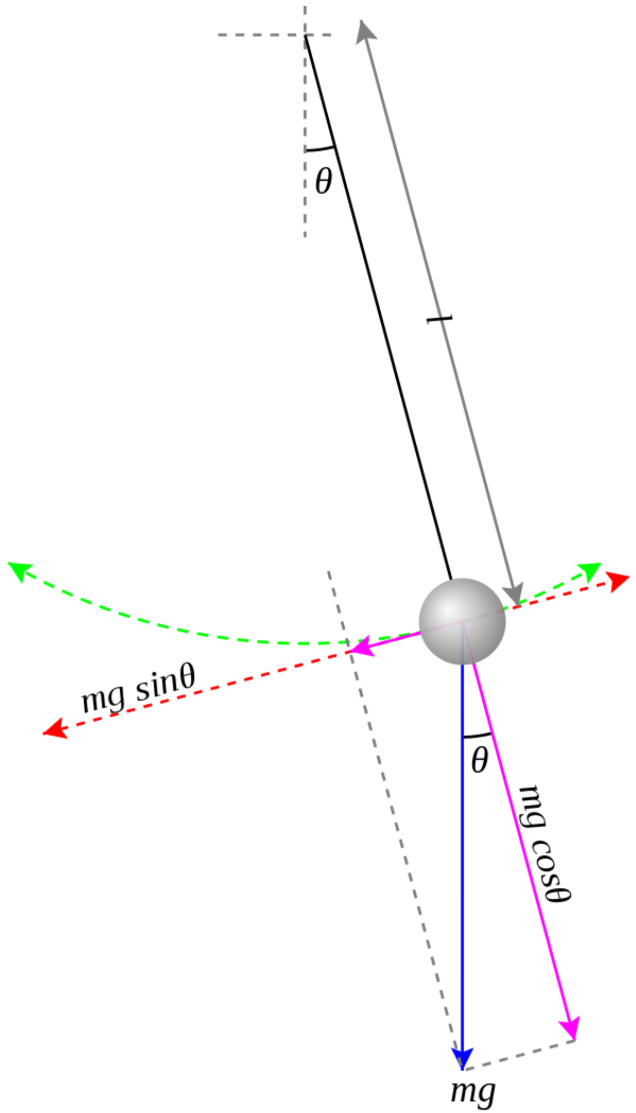
- An important non-linear equation which can be integrated is the one of the **pendulum**, for a string of length L and gravitational constant g

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

- For small displacements it reduces to a **harmonic oscillator** with frequency

$$\omega_0 = \sqrt{\frac{g}{L}}$$

- By appropriate substitutions, this becomes the equation of **synchrotron motion (see Longitudinal beam dynamics lectures)**



Solution for the pendulum

- The **integral of motion** (scaled energy) is

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - \frac{g}{L}\cos\theta = I_1 = E'$$

and the quadrature is written as $t = \int \frac{d\theta}{\sqrt{2(I_1 + \frac{g}{L}\cos\theta)}}$

assuming that for $t = 0$, $\theta_0 = \theta(0) = 0$

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- Using the substitutions $\cos\theta = 1 - 2k^2\sin^2\phi$ with $k = \sqrt{1/2(1 + I_1 L/g)}$, the integral is

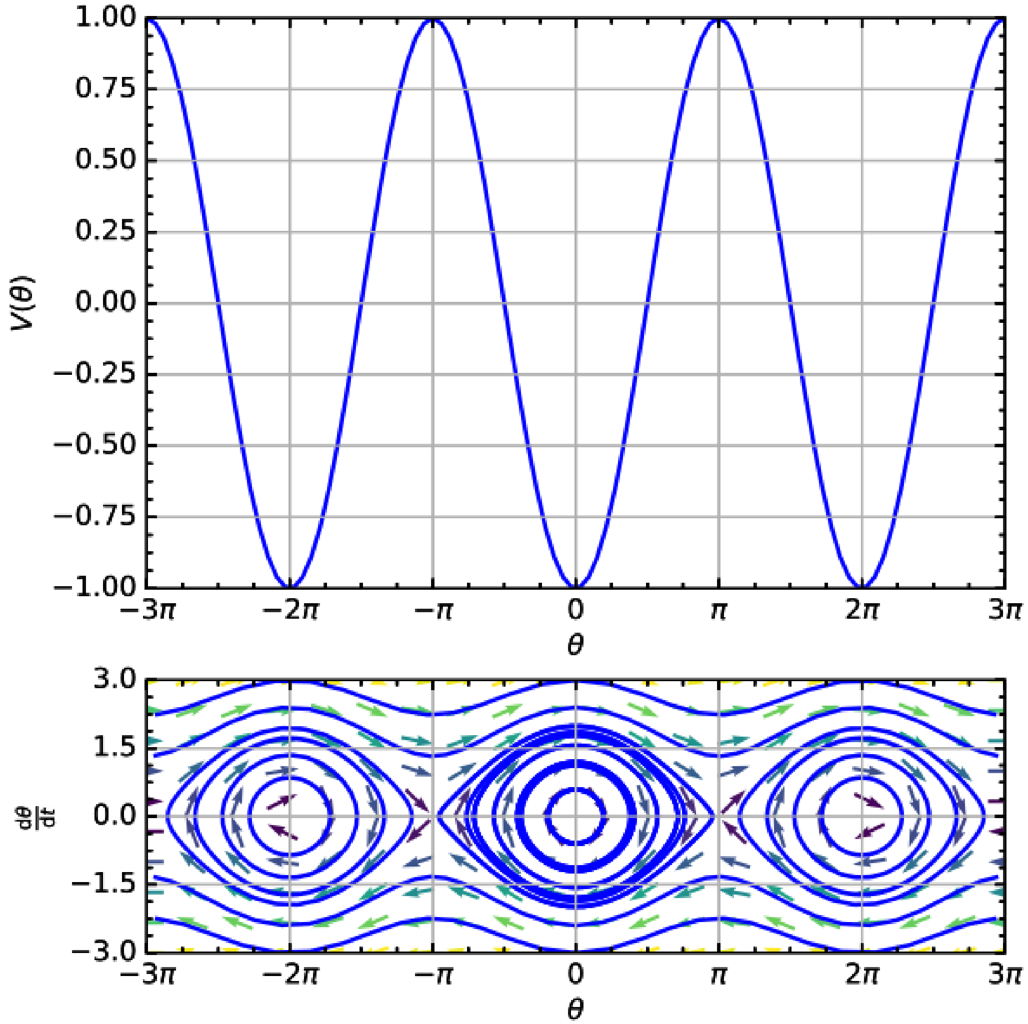
$$t = \sqrt{\frac{L}{g}}\int_0^\theta \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}$$

and can be solved using

Jacobi elliptic functions: $\theta(t) = 2\arcsin\left[k\ \mathrm{sn}\left(t\sqrt{\frac{g}{L}}, k\right)\right]$

with "sn " representing the **Jacobi elliptic sine**

Solution for the pendulum



- **Minima** and **maxima** of the potential correspond to stable and unstable fixed points

Period of the pendulum

- For recovering the **period**, the integration is performed between the two extrema, i.e. $\theta = 0$ and $\theta = \arccos(-I_1 L/g)$, corresponding to $\phi = 0$ and $\phi = \pi/2$

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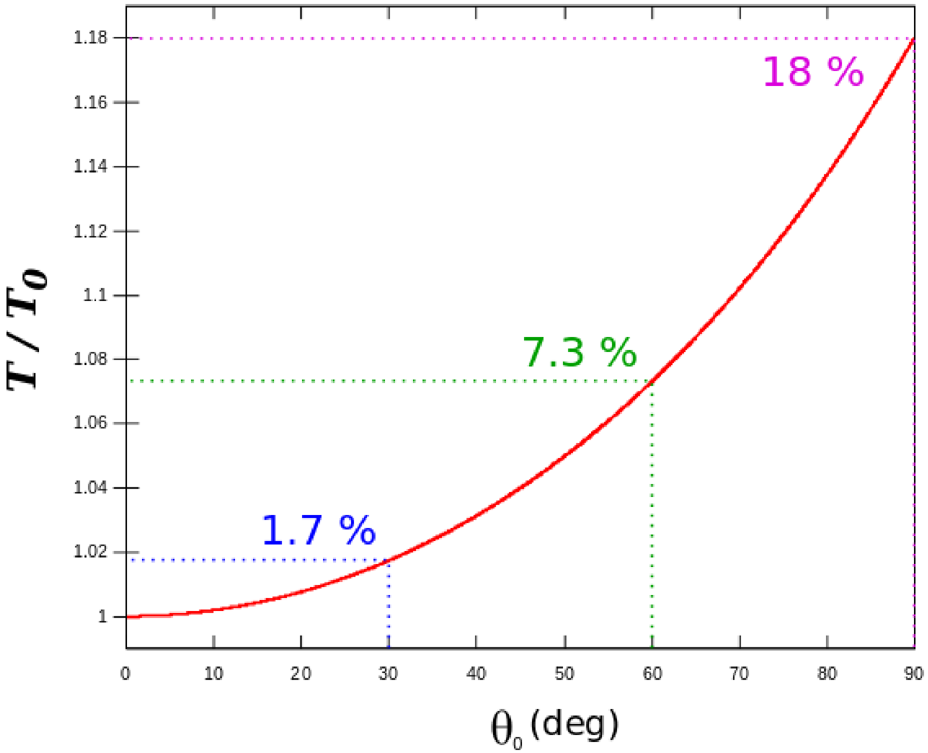
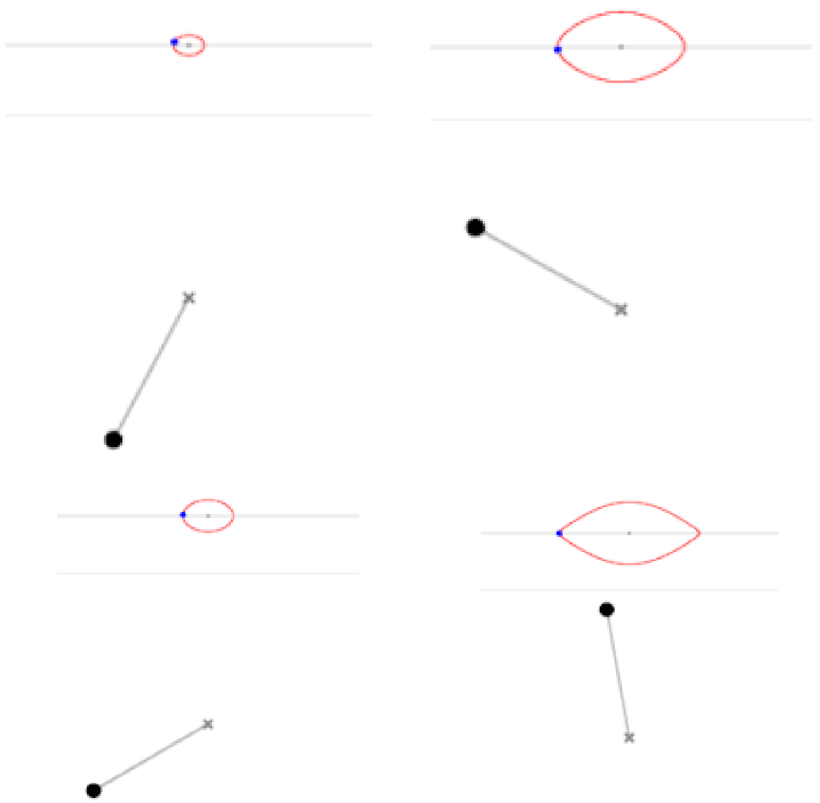
- By expanding $\mathcal{K}(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left(\frac{(2n)!}{2^{2n}(n!)^2} \right)^2 k^{2n} = \frac{\pi}{2} \left(1 + \frac{1}{4} k^2 + \cdots \right)$ with $k = \sqrt{1/2(1 + I_1 L/g)}$, the **"amplitude" dependence** of the frequency becomes apparent

Period of the pendulum

- The deviation from the linear approximation becomes important at **large amplitudes**
- The dependence of frequency with amplitude (**spread**) is useful for **(Landau) damping (“beam”) instabilities**





Langrangian and Hamiltonian

Lagrangian formalism

- Describe motion of particles in q_n coordinates (***n* degrees of freedom)** from time t_1 to time t_2
- It can be achieved by the **Lagrangian function** $L(q_1,\ldots,q_n,\dot{q}_1,\ldots,\dot{q}_n,t)$ with $(q_1,\ldots,q_n)$ the **generalized coordinates** and $(\dot{q}_1,\ldots,\dot{q}_n)$ the **generalized velocities**

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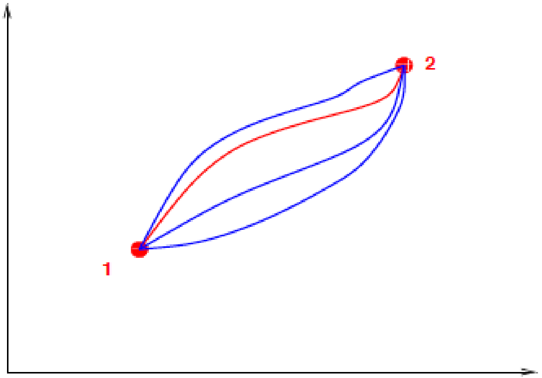
- The Lagrangian is defined as $L = T - V$, i.e. difference between **kinetic** and **potential** energy
- The integral $S = \int L(q_i, \dot{q}_i, t)dt$ defines the **action**
- **Hamilton's principle**: system evolves so as the action becomes extremum (principle of **stationary action**)



Euler- Lagrange equations

- By using **Hamilton's principle,** i.e. $\delta S = 0$, over some time interval t_1 and t_2 for two stationary points $\delta q(t_1) = \delta q(t_2) = 0$ (see appendix), the following differential equations for each degree of freedom are obtained, the **Euler-Lagrange equations**

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

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- In other words, by knowing the form of the Lagrangian, the **equations of motion** can be **derived**

Lagrangian mechanics

- For a simple **force law** contained in a potential function, governing motion among interacting particles, the (classical) **Lagrangian** is (or as Landau-Lifshitz put it "experience has shown that…")

$$L = T - V = \sum_{i=1}^{n} \frac{1}{2} m_i \dot{q}_i^2 - V(q_1, \ldots, q_n)$$

- For velocity independent potentials, Lagrange equations become

$$m_i \ddot{q}_i = -\frac{\partial V}{\partial q_i}$$

i.e. **Newton's equations**.

From Lagrangian to Hamiltonian

- Some **disadvantages** of the Lagrangian formalism:
  - **No uniqueness**: different Lagrangians can lead to same equations
  - **Physical significance** not straightforward (even its basic form given more by “experience” and the fact that it actually works that way!)
- **Note:** The (relativistic) Lagrangian is very useful in **particle physics** (invariant under Lorentz transformations)

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- **Note:** The (relativistic) Lagrangian is very useful in **particle physics** (invariant under Lorentz transformations)
- Lagrangian function provides in general n **second order differential equations** (**coordinate space**)
- Already observed advantage to move to system of $2n$ **first order differential equations**, which are more straightforward to solve (**phase space**)

Hamiltonian formalism

- The **Hamiltonian** of the system is defined as the **Legendre transformation** of the Lagrangian

$$H(\mathbf{q},\mathbf{p},t) = \sum_i \dot{q}_i p_i - L(\mathbf{q},\dot{\mathbf{q}},t)$$

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- The **generalised velocities** can be expressed as a function of the **generalised momenta** if the previous equation is invertible, and thereby define the Hamiltonian of the system
- **Example**: consider $L(\mathbf{q},\dot{\mathbf{q}}) = \dfrac{1}{2}\sum_i m_i \dot{q}_i^2 - V(q_1,\ldots,q_n)$
- From this, the momentum can be determined as $p_i = \dfrac{\partial L}{\partial \dot{q}_i} = m\dot{q}_i$

which can be trivially inverted to provide the Hamiltonian

$$H(\mathbf{q},\mathbf{p}) = \sum_i \frac{p_i^2}{2m_i} + V(q_1,\ldots,q_n)$$

Hamilton's equations

- The **equations of motion** can be derived from the Hamiltonian following the same variational principle as for the Lagrangian ("stationary" action) but also by simply taking the differential of the Hamiltonian (see appendix)

$$\dot{q}_i = \frac{\partial H}{\partial p_i}\ , \quad \dot{p}_i = -\frac{\partial H}{\partial q}\ , \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

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- These are indeed $2n+2$ equations describing the motion in the **"extended" phase space** $(q_1, \ldots, q_n, p_1, \ldots, p_n, t, -H)$

Properties of Hamiltonian flow

- The variables $(q_1, \ldots, q_n, p_1, \ldots, p_n, t, -H)$ are called **canonically conjugate** (or canonical) and define the evolution of the system in **phase space**
- These variables have the special property that they preserve volume in phase space, i.e. satisfy the well-known **Liouville's theorem**
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- The variables used in the **Lagrangian do not** necessarily **have** this **property**
- Hamilton's equations can be written in **vector form** $\dot{\mathbf{z}} = \mathbf{J} \cdot \nabla H(\mathbf{z})$ with $\mathbf{z} = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ and $\nabla = (\partial q_1, \ldots, \partial q_n, \partial p_1, \ldots, \partial p_n)$
- The $2n \times 2n$ matrix $\mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}$ is called the **symplectic** matrix

Poisson brackets

- Crucial step in study of Hamiltonian systems is identification of **integrals of motion**
- Consider a **time dependent function** of phase space. Its time evolution is given by

$$\frac{d}{dt}f(\mathbf{p},\mathbf{q},t) = \sum_{i=1}^{n}\left(\frac{dq_i}{dt}\frac{\partial f}{\partial q_i} + \frac{dp_i}{dt}\frac{\partial f}{\partial p_i}\right) + \frac{\partial f}{\partial t}$$

$$= \sum_{i=1}^{n}\left(\frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i}\right) + \frac{\partial f}{\partial t} = [H, f] + \frac{\partial f}{\partial t}$$

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- If a quantity is explicitly **time-independent** and its Poisson bracket with the Hamiltonian vanishes (i.e. **commutes** with H), it is a **constant** (or **integral**) of motion (as an **autonomous** Hamiltonian itself)

Canonical transformations

Canonical Transformations

- Find a **function** for transforming the Hamiltonian from variable $(\mathbf{q},\mathbf{p})$ to $(\mathbf{Q},\mathbf{P})$, so system becomes **simpler** to study
- Transformation should be **canonical** (or **symplectic**), so that **Hamiltonian** properties (**phase-space volume**) are preserved

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- These "mixed variable" **generating** functions are derived by

$$F_1(\mathbf{q},\mathbf{Q}) : p_i = \frac{\partial F_1}{\partial q_i}, \quad P_i = -\frac{\partial F_1}{\partial Q_i} \qquad F_3(\mathbf{Q},\mathbf{p}) : q_i = -\frac{\partial F_3}{\partial p_i}, \quad P_i = -\frac{\partial F_3}{\partial Q_i}$$

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- A general **non-autonomous Hamiltonian** is transformed to

$$H(\mathbf{Q},\mathbf{P},t) = H(\mathbf{q},\mathbf{p},t) + \frac{\partial F_j}{\partial t}\ , \quad j = 1,2,3,4$$

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- One generating function can be constructed by the other through **Legendre transformations**, e.g.

$$F_2(\mathbf{q},\mathbf{P}) = F_1(\mathbf{q},\mathbf{Q}) - \mathbf{Q}\cdot\mathbf{P}\ ,\quad F_3(\mathbf{Q},\mathbf{p}) = F_1(\mathbf{q},\mathbf{Q}) - \mathbf{q}\cdot\mathbf{p}\ ,\quad \ldots$$

with the inner product defined as $\mathbf{q}\cdot\mathbf{p} = \sum_i q_i p_i$

Preservation of Phase Volume

- A fundamental property of canonical transformations is the **preservation** of **phase space volume**
- This **volume** preservation in phase space can be represented in the **old** and **new variables** as

$$\int \prod_{i=1}^{n} dp_i dq_i = \int \prod_{i=1}^{n} dP_i dQ_i$$

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$$\prod_{i=1}^{n} dp_i dq_i = \frac{\partial(P_1, \ldots, P_n, Q_1, \ldots, Q_n)}{\partial(p_1, \ldots, p_n, q_1, \ldots, q_n)} \prod_{i=1}^{n} dP_i dQ_i$$

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- These two relationships imply that the **Jacobian** of a **canonical transformation** should have **determinant** equal to **1**

$$\left|\frac{\partial(P_1,\ldots,P_n,Q_1,\ldots,Q_n)}{\partial(p_1,\ldots,p_n,q_1,\ldots,q_n)}\right| = \left|\frac{\partial(p_1,\ldots,p_n,q_1,\ldots,q_n)}{\partial(P_1,\ldots,P_n,Q_1,\ldots,Q_n)}\right| = 1$$

Examples of transformations

- The transformation $Q = -p\,,\quad P = q$, which **interchanges conjugate variables** is area preserving, as the Jacobian is

$$\frac{\partial(P,Q)}{\partial(p,q)} = \begin{vmatrix} \frac{\partial P}{\partial p} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial Q}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1$$

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- On the other hand, the transformation from **Cartesian to polar** coordinates $q = P\cos Q\ , \quad p = P\sin Q$ is not, since

$$\frac{\partial(q,p)}{\partial(Q,P)} = \begin{vmatrix} -P\sin Q & P\cos Q \\ \cos Q & \sin Q \end{vmatrix} = -P$$

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- There are actually **"polar" coordinates** that are **canonical**, given by $q = -\sqrt{2P}\cos Q\ , \quad p = \sqrt{2P}\sin Q$ for which

$$\frac{\partial(q,p)}{\partial(Q,P)} = \begin{vmatrix} \sqrt{2P}\sin Q & \sqrt{2P}\cos Q \\ -\frac{\cos Q}{\sqrt{2P}} & \frac{\sin Q}{\sqrt{2P}} \end{vmatrix} = 1$$

Summary of Lecture I

- 2nd order dif. equations of motion from Newton's law (**configuration space**) can be solved by **transforming** them to pairs of 1st order ones (in **phase space**)
- Natural appearance of **invariant** of motion ("**energy**")
- Non-linear oscillators have **frequencies** which **depend** on the **invariant** (or "**amplitude**")
- Connected invariant of motion to system's **Hamiltonian** (derived through **Lagrangian**)
- Shown that through the **Hamiltonian** , the **equations** of **motions** can be **derived**
- **Poisson bracket** operators are helpful for discovering integrals of motion
- **Canonical** (or **symplectic**) transformations are necessary for preserving the phase space-volume

Appendix

Derivation of Lagrange equations

- The variation of the action can be written as

$$\delta W = \int_{t_1}^{t_2} \left(L(q+\delta q, \dot{q}+\delta\dot{q}, t) - L(q,\dot{q},t)\right) dt = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q}\right) dt$$

- Taking into account that $\delta\dot{q} = \frac{d\delta q}{dt}$, the 2nd part of the integral can be integrated by parts giving

$$\delta W = \left|\frac{\partial L}{\partial \dot{q}}\delta q\right|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right) \delta q dt = 0$$

- The first term is zero because $\delta q(t_1) = \delta q(t_2) = 0$ so the second integrant should also vanish, providing the following differential equations for each degree of freedom, the **Lagrange equations**

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

Derivation of Hamilton's equations

- The **equations of motion** can be derived from the Hamiltonian following the same variational principle as for the Lagrangian ("least" action) but also by simply taking the differential of the Hamiltonian

$$dH = \sum_i p_i d\dot{q}_i + \dot{q}_i dp_i - \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{p_i} d\dot{q}_i - \underbrace{\frac{\partial L}{\partial q_i}}_{\dot{p}_i} dq_i - \frac{\partial L}{\partial t} dt$$

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or

$$dH(q,p,t) = \sum_i \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial L}{\partial t} dt = \sum_i \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial t} dt$$

- By equating terms, **Hamilton's equations** are derived

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q}, \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

- These are indeed $2n+2$ equations describing the motion in the **"extended" phase space** $(q_i, \ldots, q_n, p_1, \ldots, p_n, t, -H)$

Poisson brackets' properties

- The Poisson brackets between two functions of a set of canonical variables can be defined by the differential **operator**

$$[f, g] = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i} \frac{\partial f}{\partial q_i} \right)$$

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- From this definition, and for any three given functions, the following **properties** can be shown

$$[af+bg,h] = a[f,h]+b[g,h]\ ,\ a,b\in\mathbb{R}$$ **bilinearity**

$$[f,g] = -[g,f]$$ **anticommutativity**

$$[f,[g,h]]+[g,[h,f]]+[h,[f,g]] = 0$$ **Jacobi's identity**

$$[f,gh] = [f,g]h+g[f,h]$$ **Leibniz's rule**

- Poisson brackets operation satisfies a **Lie algebra**

Preservation of Phase Volume

- A fundamental property of Hamiltonian systems is the **preservation** of **phase space volume** as they evolve
- Let's have a system evolving from $(p_i q_i) \rightarrow (p'_i q'_i)$ after time δt. By Taylor-expanding and using Hamilton's equations we have:

$$q'_i = q_i(t+\delta t) = q_i(t) + \frac{dq_i}{dt}\delta t + O(\delta t^2) = q_i - \frac{\partial H}{\partial p_i}\delta t + O(\delta t^2)$$

$$p'_i = p_i(t+\delta t) = p_i(t) + \frac{dp_i}{dt}\delta t + O(\delta t^2) = p_i + \frac{\partial H}{\partial q_i}\delta t + O(\delta t^2)$$

- Differentiating, we have

$$dq'_i = dq_i - \frac{\partial}{\partial q_i}\left(\frac{\partial H}{\partial p_i}\right) dq_i \delta t + O(\delta t^2)$$

$$dp'_i = dp_i + \frac{\partial}{\partial p_i}\left(\frac{\partial H}{\partial q_i}\right) dp_i \delta t + O(\delta t^2)$$

- Multiplying the two equations

$$dq'_i dp'_i = dq_i dp_i \left[1 - \frac{\partial}{\partial q_i}\left(\frac{\partial H}{\partial p_i}\right) + \frac{\partial}{\partial p_i}\left(\frac{\partial H}{\partial q_i}\right)\right]\delta t + O(\delta t^2) \approx dq_i dp_i$$