Accelerator & Technology Sector
Beams Department
Accelerator Beam Physics Group

Particle Accelerators and Beam Dynamics

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Eurolabs 2024

Disclaimer

Based on:

- Y. Papaphilippou : “Introduction to Accelerators”
- Summer student lectures:
    - B. Holzer, V. Kain, and M. Schaumann
- CERN accelerator school (CAS):
    - F. Tecker: *“Longitudinal beam dynamics”*
- Joint Universities Accelerator School (JUAS):
    - F. Antoniou, H. Bartosik and Y. Papaphilippou: *“Linear imperfections”* and *“nonlinear dynamics”*
- Books:
    - K. Wille: *“The Physics of Particle Accelerators”*
    - S.Y. Lee: *“Accelerator Physics”*
    - A. Wolski: *“Beam Dynamics in High Energy Particle Accelerators”*
    - H. Wiedemann: “Particle Accelerator Physics”

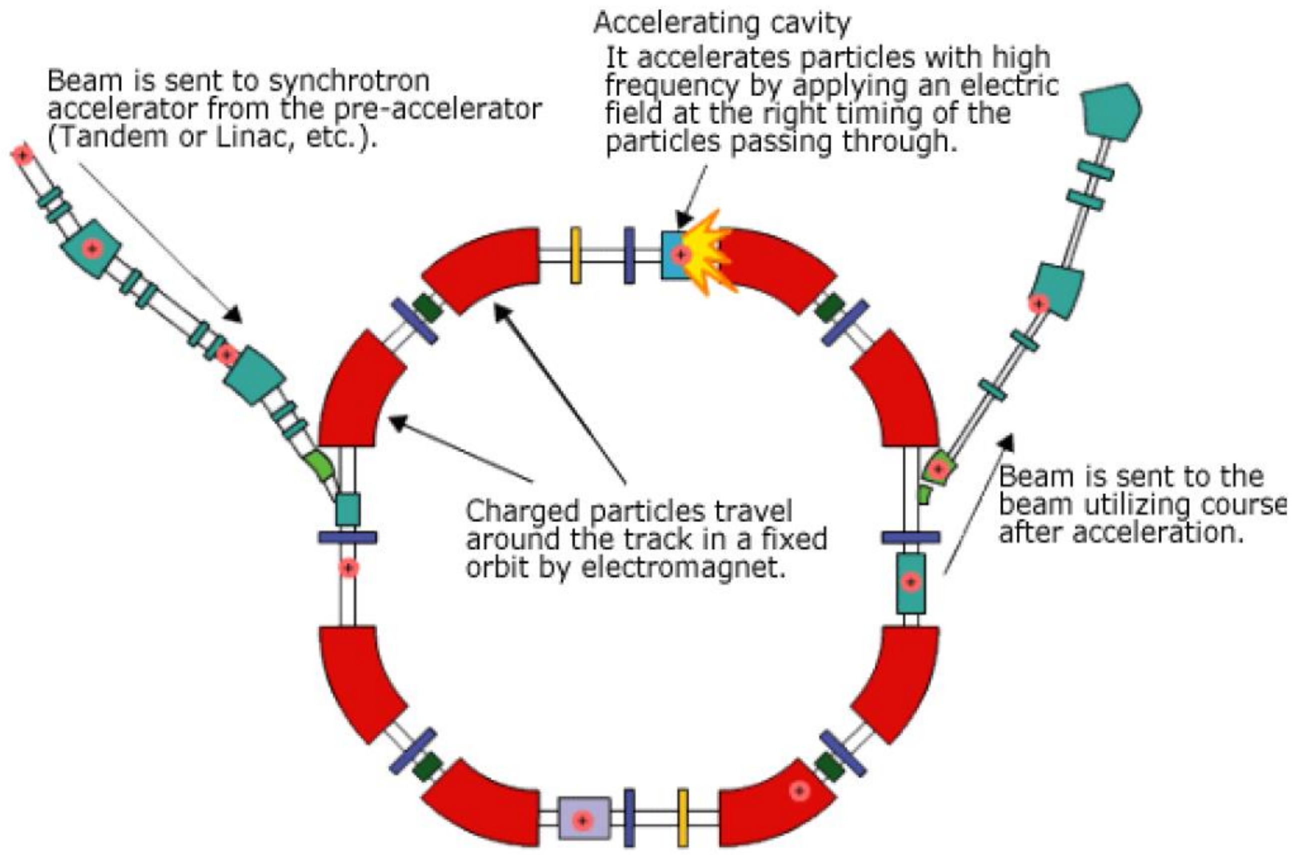
Images: cds.cern.ch

Overview

I. Introduction to Accelerators

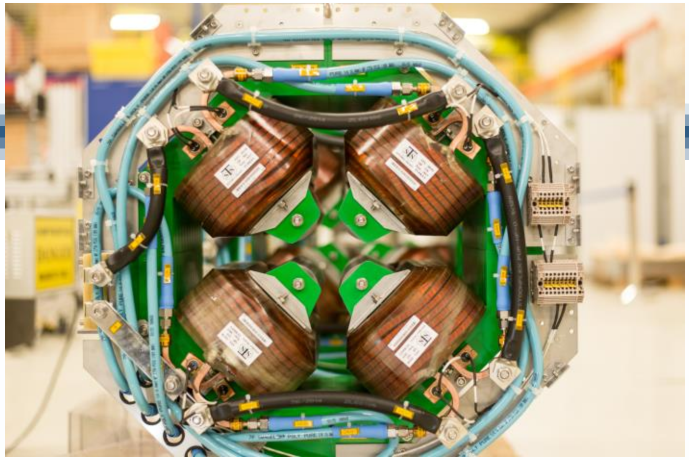
II. Accelerator beam dynamics
- Transverse beam dynamics
  - Optics functions
  - Tune and resonances
- Longitudinal beam dynamics
  - Acceleration
  - Synchrotron motion

Reminder – Synchrotron



The most common accelerator

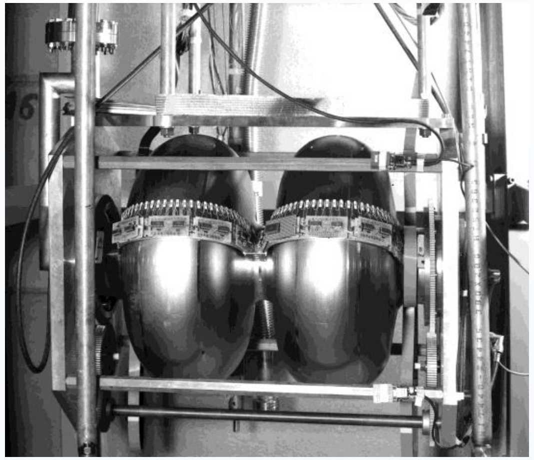
- Fixed beam trajectory | magnetic field changes synchronous to the energy
- Magnets around the beam path to control the motion | **bending** (dipoles) & **focusing** (quadrupoles)
- Electric fields used to **accelerate** (RF cavity) the beam



How do particles move under the influence of these elements?

→ Transverse & Longitudinal Beam Dynamics

Charges in electromagnetic fields

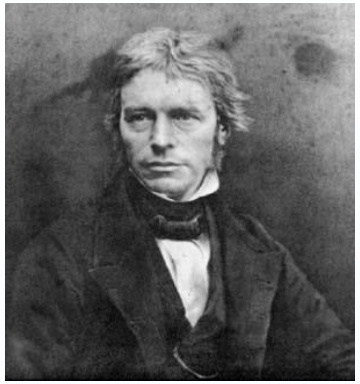


Maxwell's equations for electromagnetism

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}}$$

Gauss law for electricity

electric field diverges from electric charges



$$\boxed{\nabla \cdot \mathbf{B} = 0}$$

Gauss law for magnetism

no isolated magnetic poles

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}}$$

Faraday's law of induction

changing magnetic fields produce electric fields

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}}$$

Ampere-Maxwell law

changing electric fields and currents produce circulating magnetic fields

Lorentz force

- Force acting on **charged particles** moving under the influence of **electromagnetic fields**

$$\boxed{\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$$

- Kinetic energy (T) change is caused by the **electric field** – ***acceleration***

$$\boxed{\frac{dT}{dt} = \mathbf{v} \cdot \mathbf{F} = q\mathbf{v} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}) = q\mathbf{v} \cdot \mathbf{E}}$$

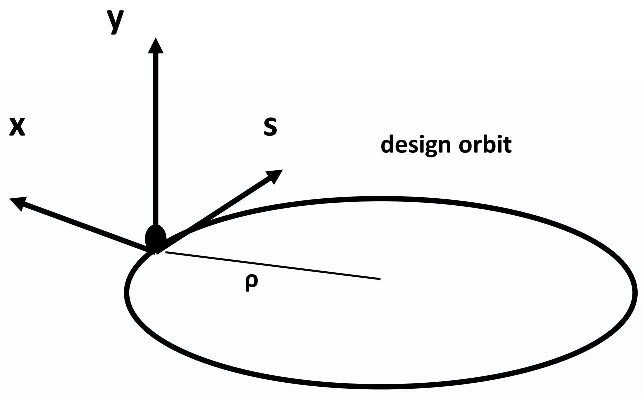
- **Horizontal component** of the Lorentz force (particle moving on the longitudinal plane

$$\boxed{\mathbf{F_x} = q(E_x - v_z B_y)}$$

- For high energy (relativistic limit): $\upsilon_z \approx c$ & $\upsilon_z B_y >> E_x$ (1 T corresponding to 300 MV/m)

  → **Magnetic fields** much more efficient for ***steering***

Transverse motion – Field expansion



- In a synchrotron we want to study particles on the design orbit
- Magnetic fields are present all along s
- The magnetic field at the vicinity of the particle can be expanded as:

$$\frac{e}{p}B_y(x) = \frac{e}{p}B_{y0} + \frac{e}{p}\frac{dB_y}{dx}x + \frac{1}{2!}\frac{e}{p}\frac{d^2B_y}{dx^2}x^2 + \frac{1}{3!}\frac{e}{p}\frac{d^3B_y}{dx^3}x^3 + \cdots = \frac{1}{r} + kx + \frac{1}{2!}mx^2 + \frac{1}{3!}ox^3 + \cdots$$

Linear terms

Dipole ***Quadrupole***

Sufficient terms for a synchrotron

Higher order terms

Sextupole ***Octupole*** …

Transverse motion – Dipoles

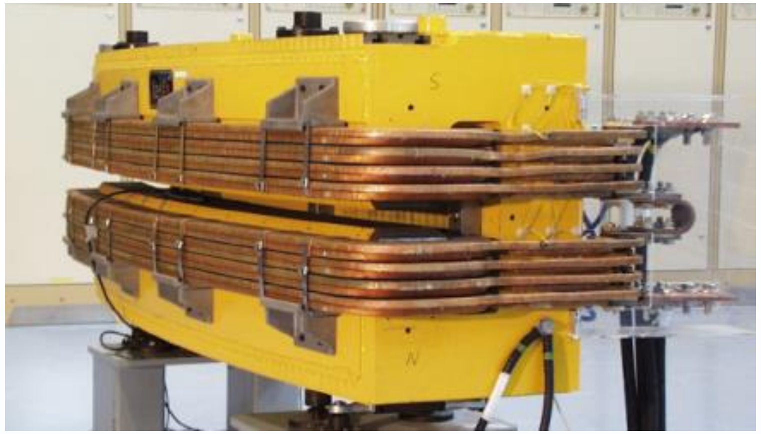
In a circular accelerator of energy ***E***, with ***N*** dipoles, each of length ***L***

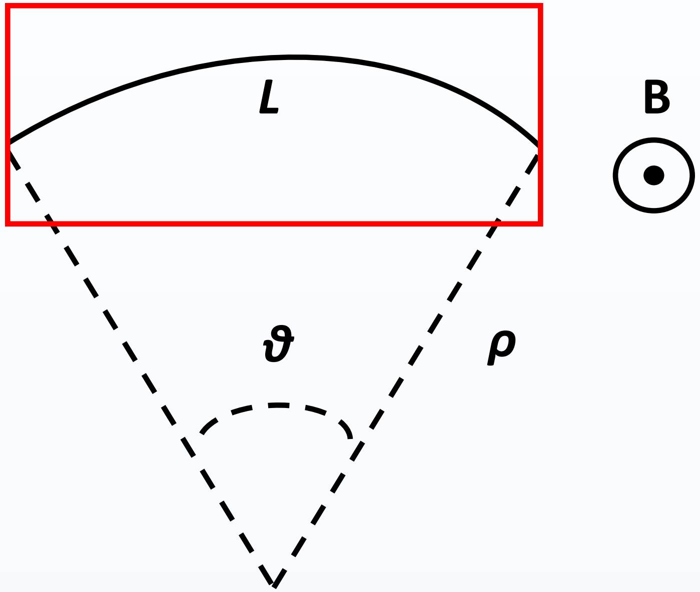
- Bending angle: $$\theta = \frac{2\pi}{N}$$
- Bending radius: $$\rho = \frac{L}{\theta}$$
- Dipole field: $$B = 2\rho p / (qNl)$$

→ Choosing a dipole magnetic field: the length is determined (and vice versa)

→ For higher fields, smaller and fewer dipoles can be used

→ Ring circumference (cost) depends on field selection





Example LHC:

7000 GeV Proton storage ring
dipole magnets N = 1232
l = 15 m
q = +1 e

$$\int B\,dl \approx N\,l\,B = 2\pi\;p/e$$

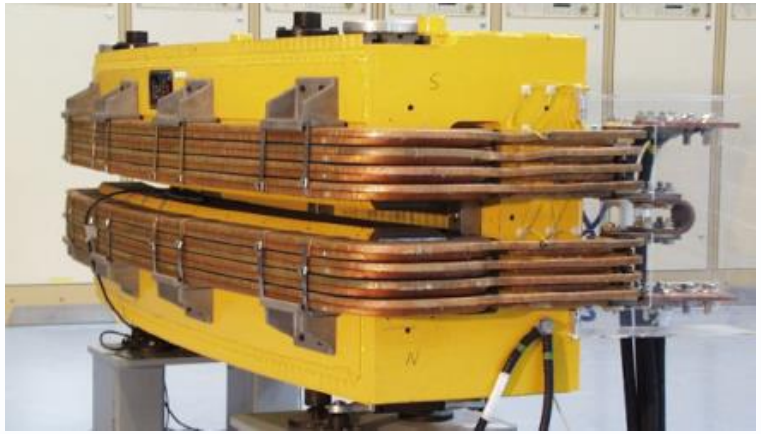
$$B \approx \frac{2\pi\;\;7000\;\;10^{9}\,eV}{1232\;\;15\,m\;\;3\;10^{8}\,\frac{m}{s}\;\;e} = 8.3\;Tesla$$

Transverse motion – Dispersion

Reminder:

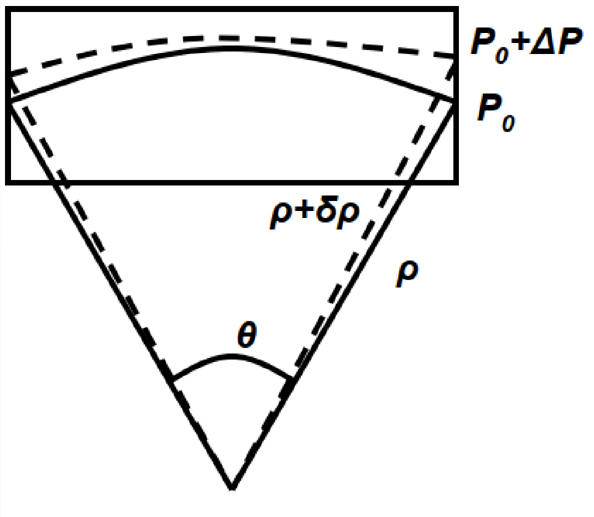
- From the RF cavities | **bunches formation**:
  - *The particles forming a bunch have a spread of momenta around the reference particle*

  → *Off-momentum particles (Δp/p,* with respect to the reference*)*



- From the **beam rigidity** (& dipole field):
  - *The synchrotron has a constant radius if the field follows the momentum*

  → *Off-momentum particles:* $B(\rho + \Delta\rho) = \dfrac{P_0 + \Delta P}{q} \Rightarrow \dfrac{\Delta\rho}{\rho} = \dfrac{\Delta P}{P_0}$



- ***The off-momentum particles follow a different orbit than the reference!***

→ The different orbit when *Δp/p = 1* is called: ***Dispersion***

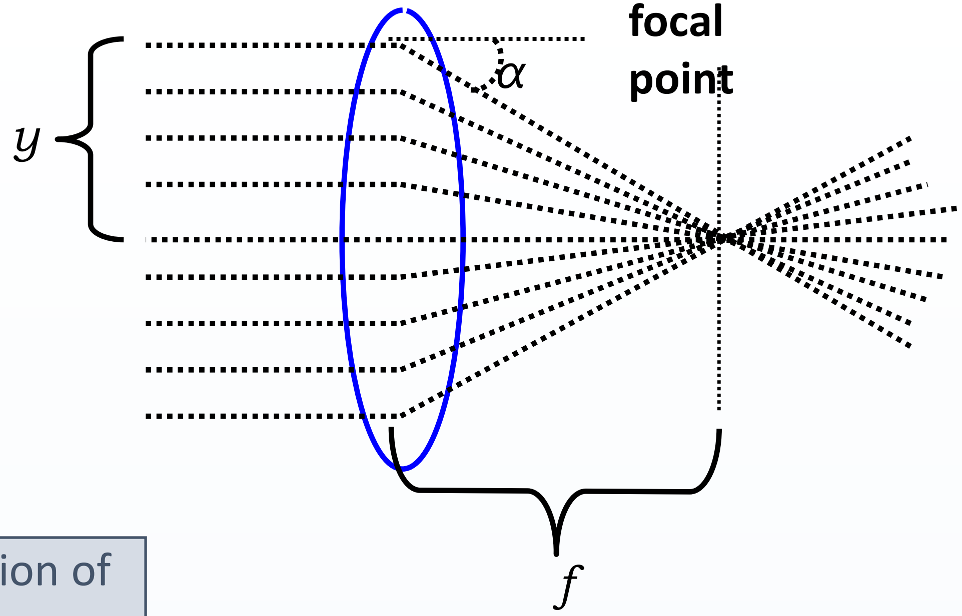
Transverse motion – Quadrupoles

- Quadrupoles can have a focusing effect similar to lenses, where: $\alpha = -\frac{y}{f}$

Quadrupole with field: $(B_x, B_y) = G \cdot (y, x)$

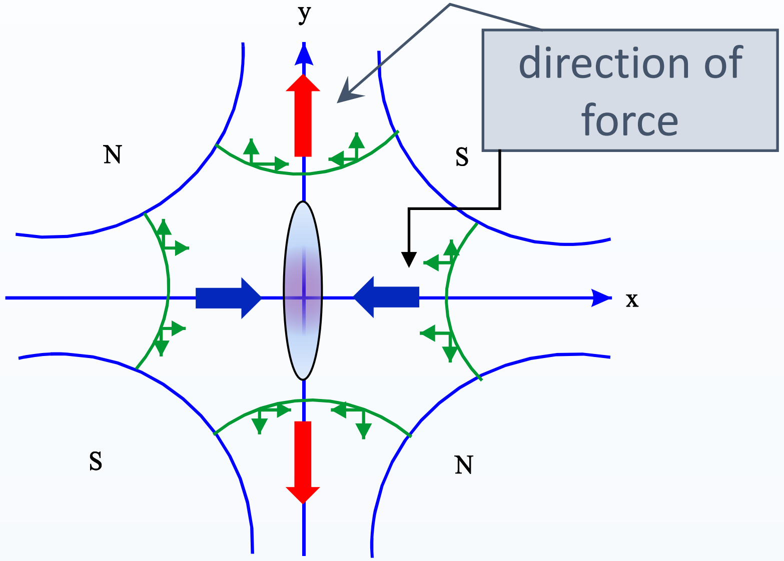
and force: $(F_x, F_y) = k \cdot (-x, y)$, $k = \frac{G}{B\rho}$

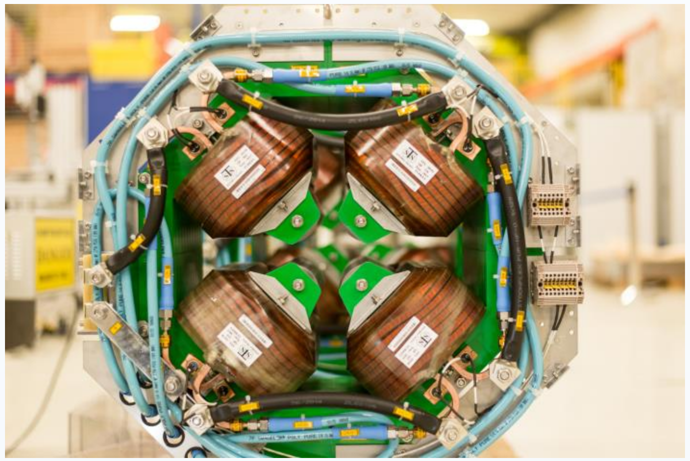
- Acts as a lens with focal length: $f = \frac{1}{k \cdot l_Q}$



Reminder:

- ***Quadrupoles with a focusing effect in one plane have a defocusing in the other***





Transverse motion – FODO

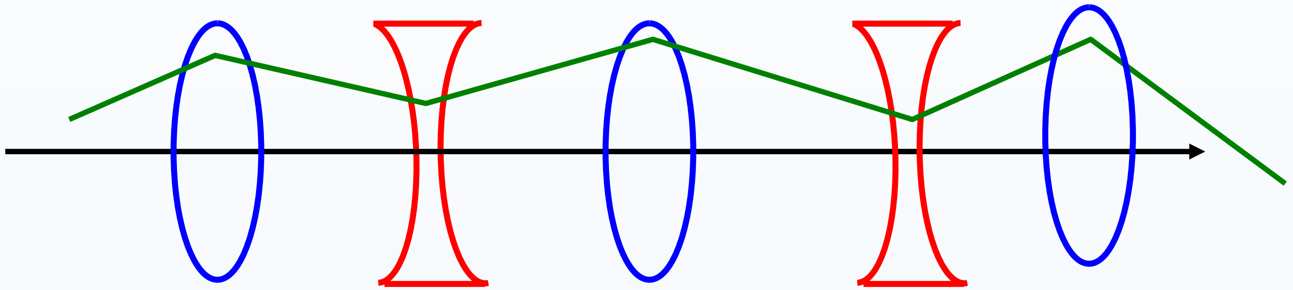
Alternating gradient focusing:

- ***Alternating focusing and defocusing lenses can have an overall focusing effect***
- Combination of lenses with focal lengths, f_1 and f_2 in a distance *d* gives a focal length:

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1\ f_2}$$

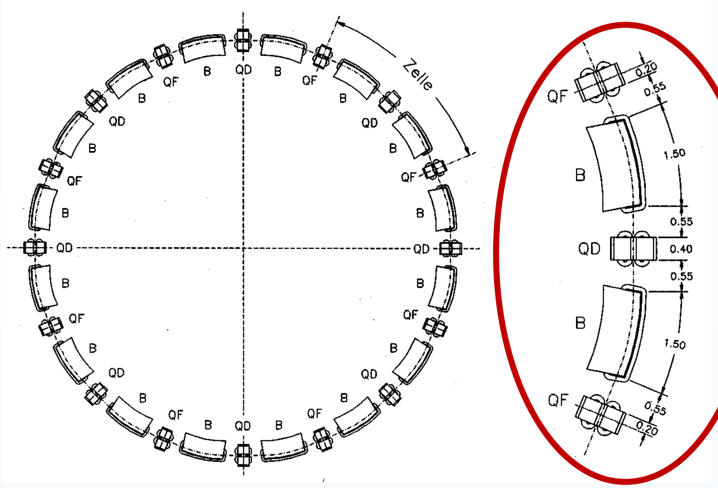
- if $f_1 = -f_2$, we get an overall focusing effect:

$$\frac{1}{f} = \left|\frac{d}{f_1\ f_2}\right|$$

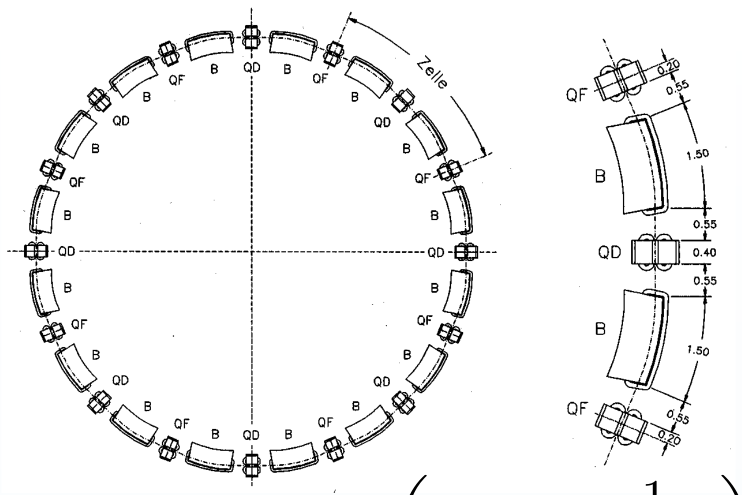


FODO structure

- *"**Cell**" of alternating focusing and defocusing elements (along with drifts, dipoles etc)*
- Structure repeats itself giving a strong periodicity in the ring



Transverse motion – FODO



The equations of motion moving along the FODO structure:

$$x'' + \left(k(s) + \frac{1}{\rho(s)^2}\right) x = \frac{1}{\rho(s)} \frac{\Delta p}{p}$$

$$y'' - k(s)\, y = 0$$

Dispersive contribution

Quadrupole contribution (strong focusing)

Dipole contribution (weak focusing)

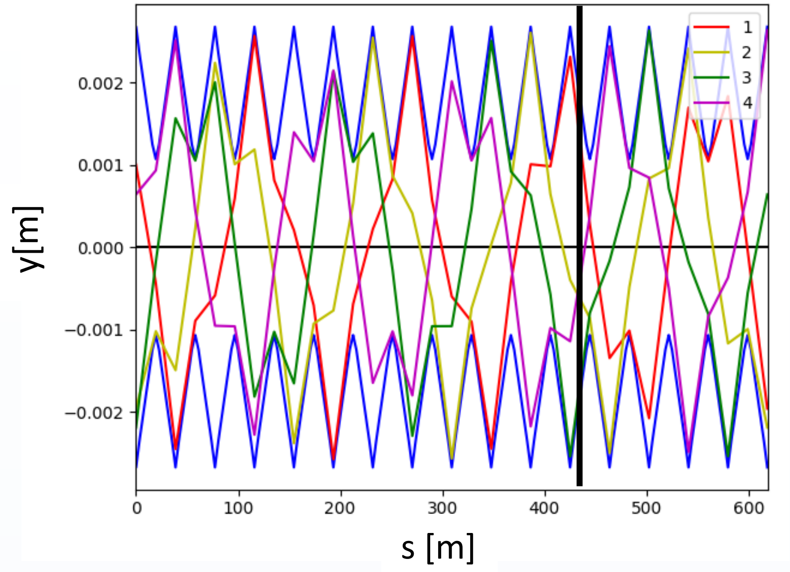
Setting $K_x(s) = \left(k(s) + \frac{1}{\rho(s)^2}\right)$ and $K_y(s) = -k(s)$ we obtain *Hill's equations, with* $K_{x,y}(s)$ depending on the position *s and* $K_{x,y}(s+L) = K_{x,y}(s)$ periodic functions, where *L* is the periodicity

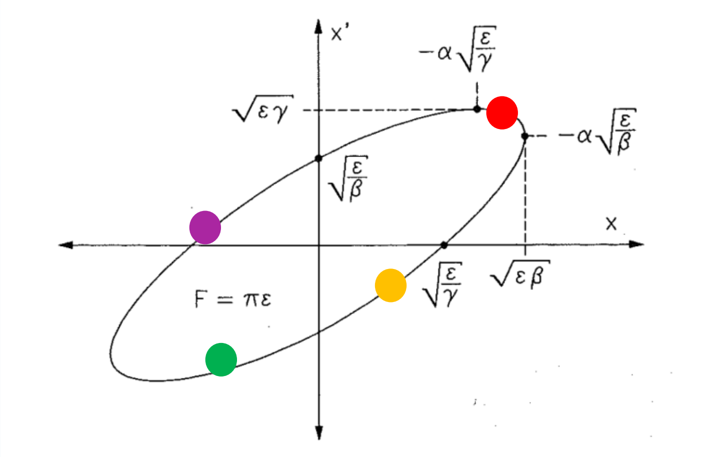
- Solutions describe a quasi harmonic oscillation, where **amplitude**, **phase** (and **dispersion**) depend on the position s in the ring

$$\boldsymbol{y(s) = \sqrt{\varepsilon_y \beta_y(s)}\, cos\left(\varphi_y(s)\right)}$$

$$\boldsymbol{x(s) = \sqrt{\varepsilon_x \beta_x(s)}\, cos\left(\varphi_x(s)\right) + D(s)\frac{\Delta p}{p}}$$

Transverse motion – betatron oscillations





- Particles perform **oscillations (betatron)** around the **design orbit**
- The motion is bound from the **envelope** ($\sqrt{\boldsymbol{\varepsilon_y \beta_y(s)}}$),
  - $\boldsymbol{\beta_y(s)}$: **beta function** characteristic of the ring
  - $\boldsymbol{\varepsilon_y}$: **emittance** is a **constant** of the motion (Liouville's theorem: the area is preserved)
    - It defines an **ellipse in the phase space** (Courant-Snyder invariant)

      $\varepsilon_y = \gamma_y(s)y^2(s) + 2\alpha_y(s)y'(s)y(s) + 2\beta_y(s)y'^2(s)$,

      α,β,γ: optics functions
    - It cannot be changed by the optics functions
  - The envelope gives the **beam size** of a particle ensemble
- # of oscillations per turn, **tune:** $Q = \frac{1}{2\pi}\int \frac{ds}{\beta(s)}$

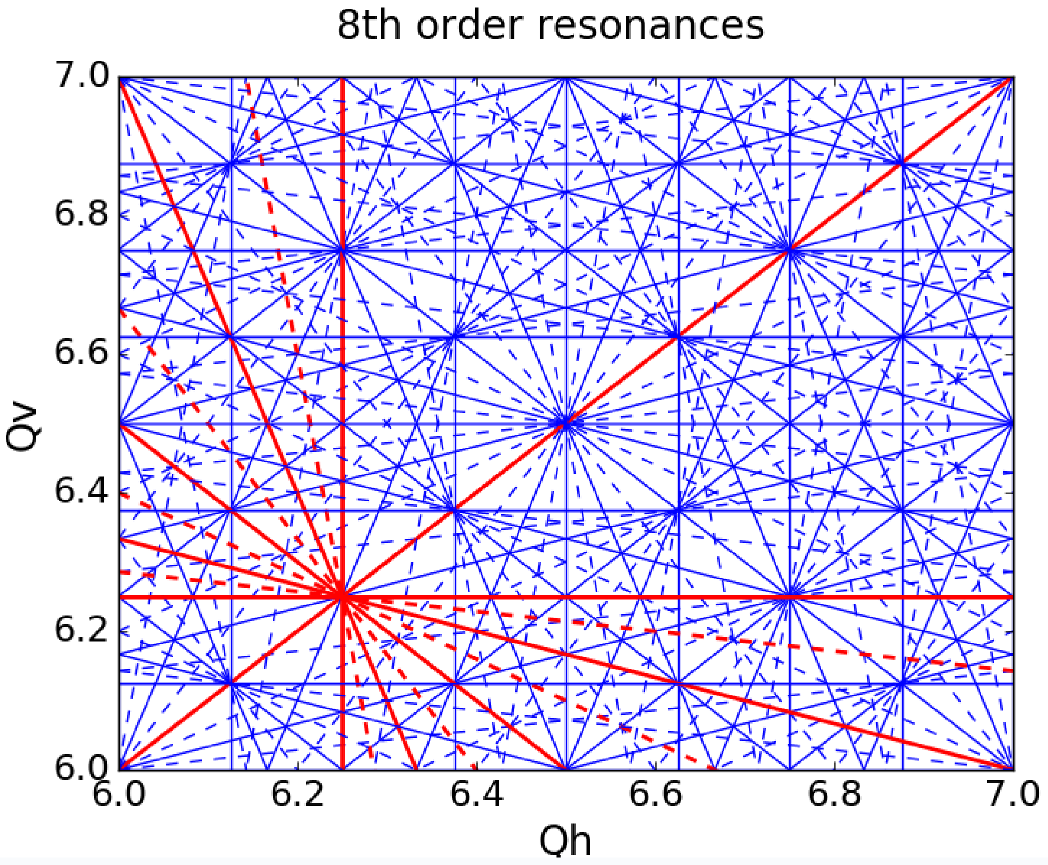
Resonances

The tunes in the respective planes: (Q_x, Q_y)

Define resonance conditions described by: $mQ_x + nQ_y = l,$

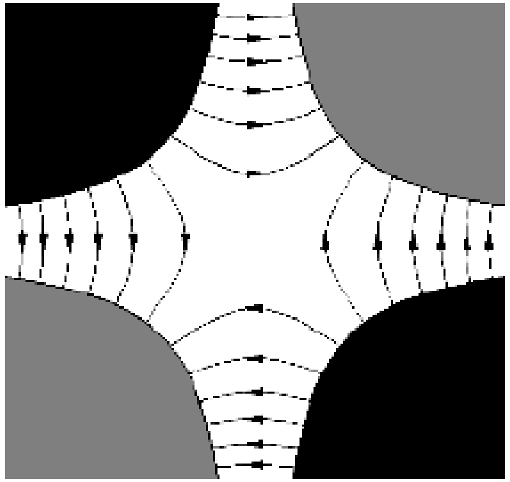
- where m,n,l integers
- $m|+|n|$ the resonance order
- If the above condition is satisfied:
  - ***Particle losses***
  - ***Emittance increase***

Resonances	Machine Periodicity	
Magnetic Field Component		
	Systematic	Non Systematic
Skew		
Normal		



Magnetic Field Component

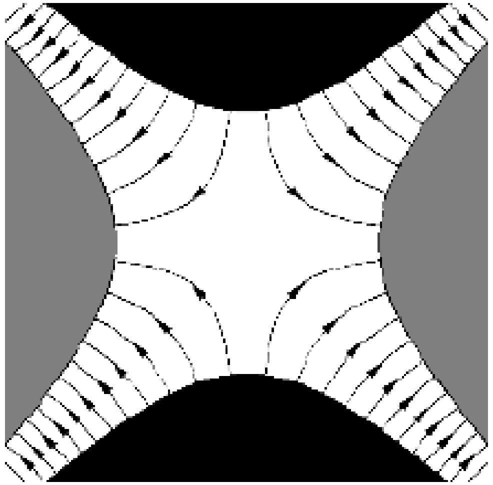
Normal components become skew when rotated by half the rotation symmetry

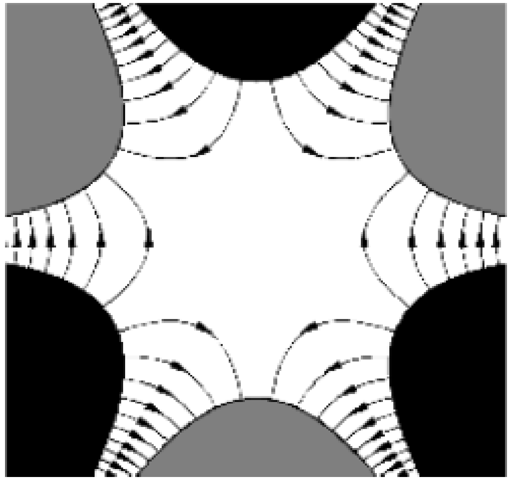


Normal Quadrupole

45° Rotation

Skew Quadrupole

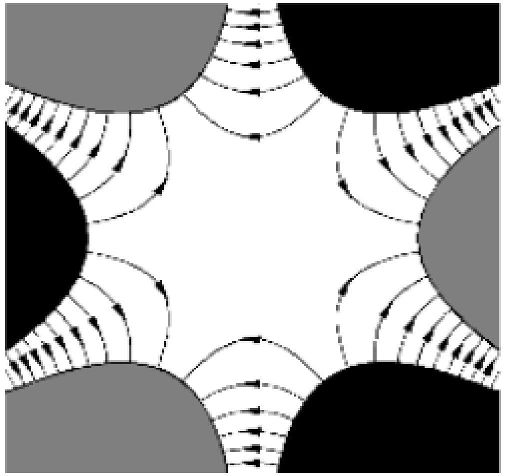




Normal Sextupole

30° Rotation

Skew Sextupole

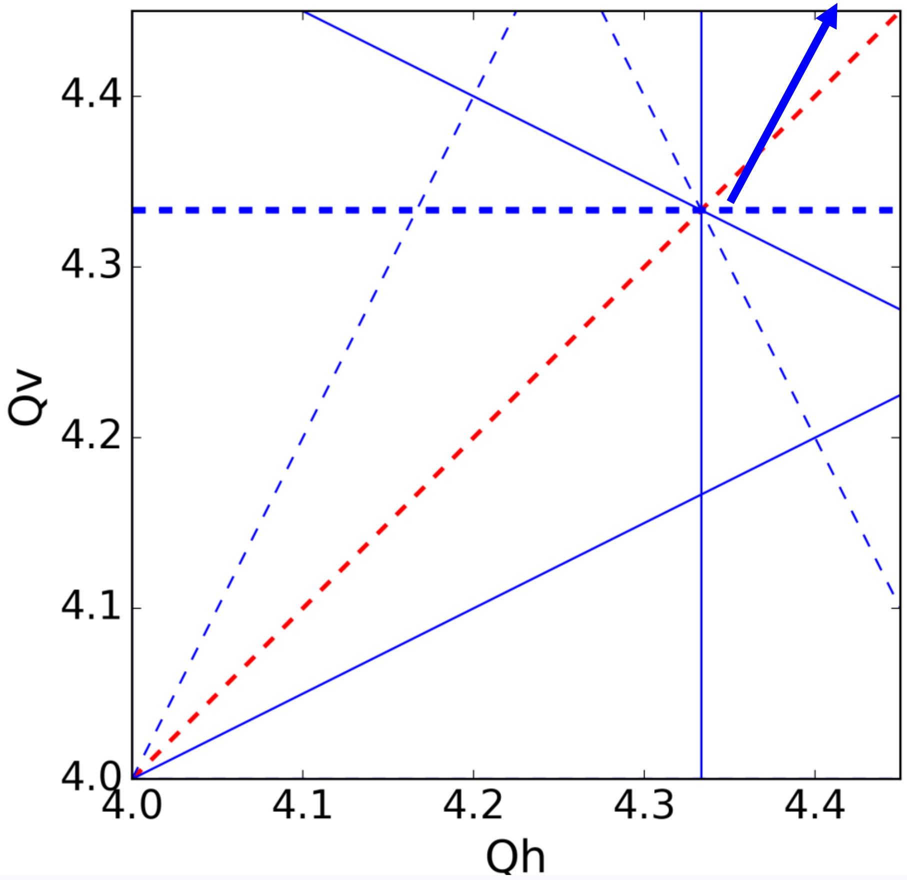


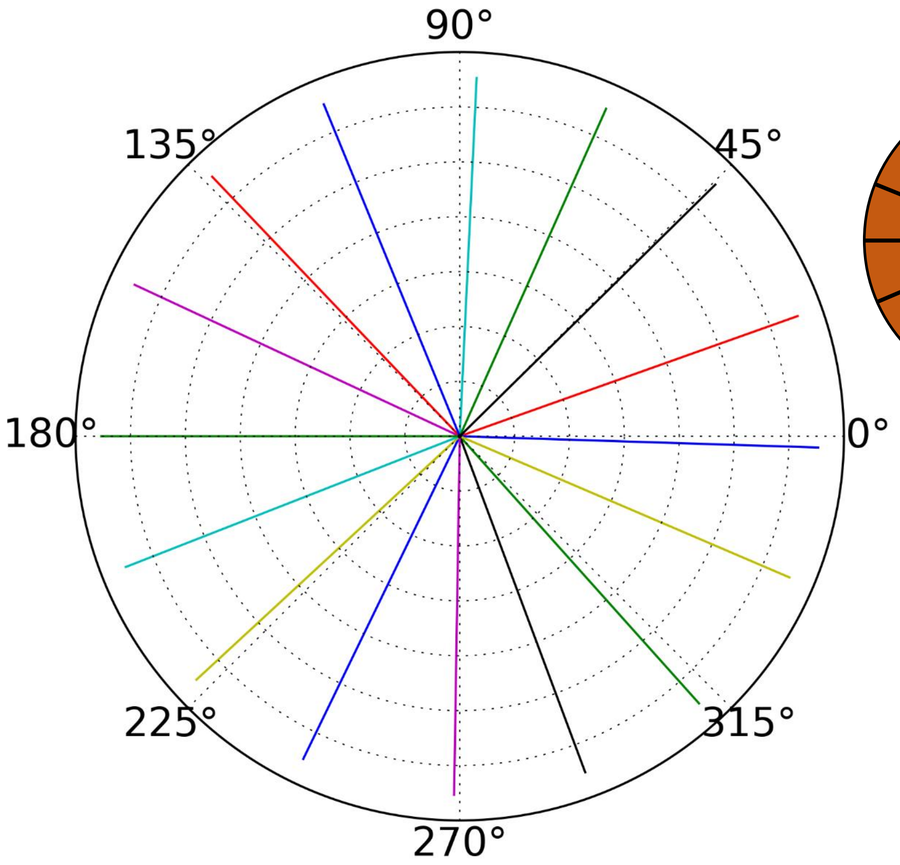
Plots from:
T. Satogata *et al.*, "*Magnets and Magnet Technology*", USPAS2013

Machine Periodicity

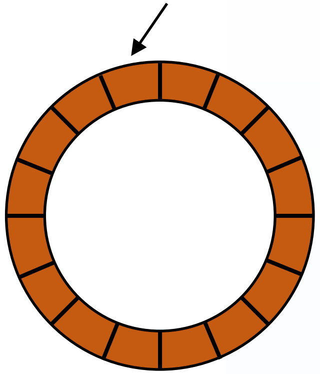
Non-Systematic Resonance

$3Q_v = \mathbf{13}$





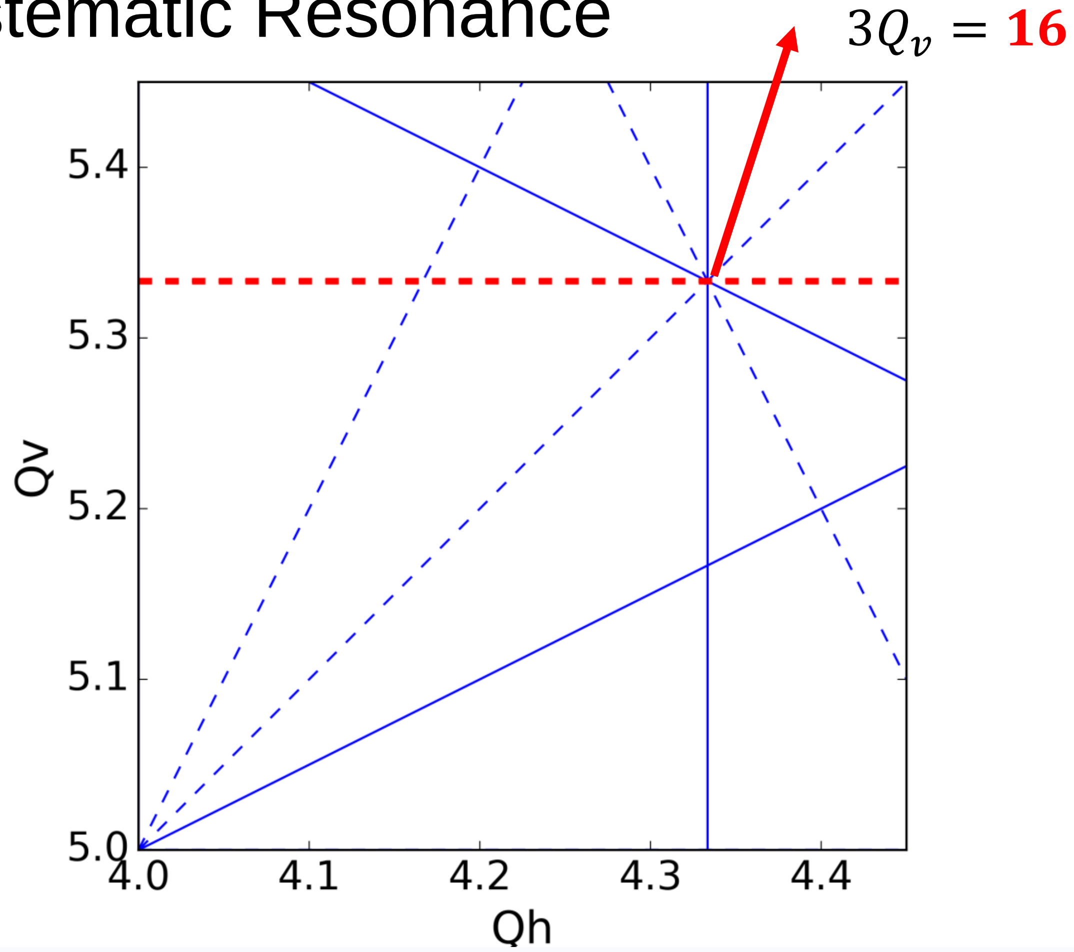
16 cells

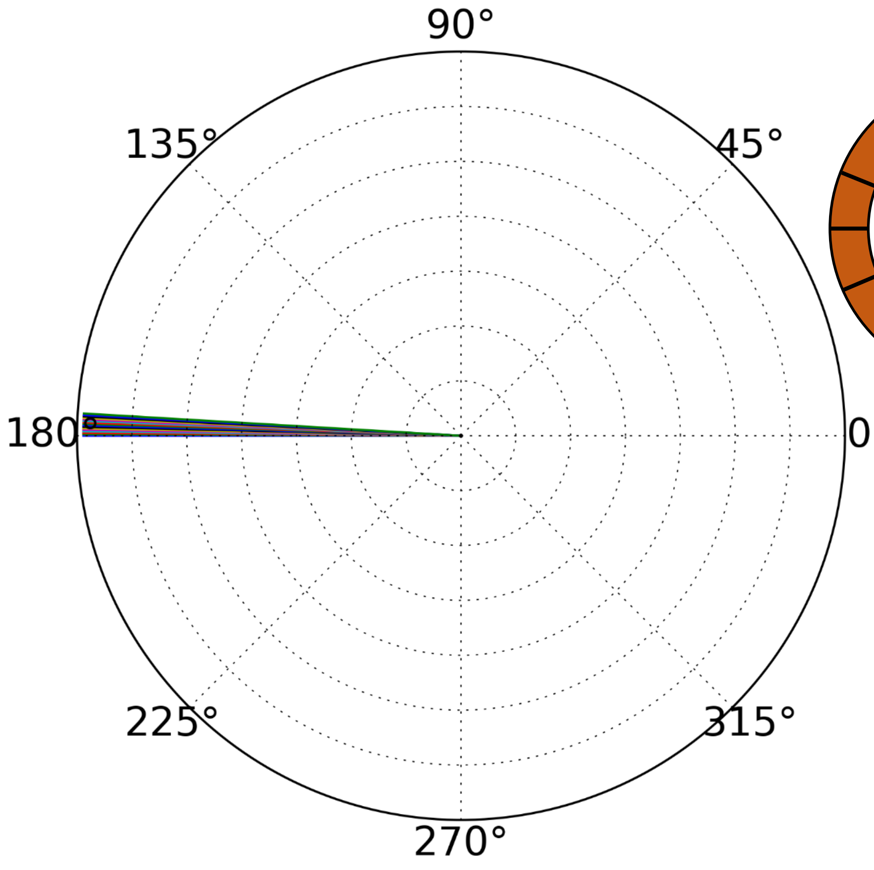


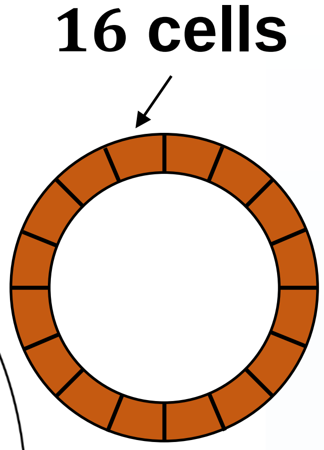
The errors **cancel out due to the periodicity**

Machine Periodicity

Systematic Resonance





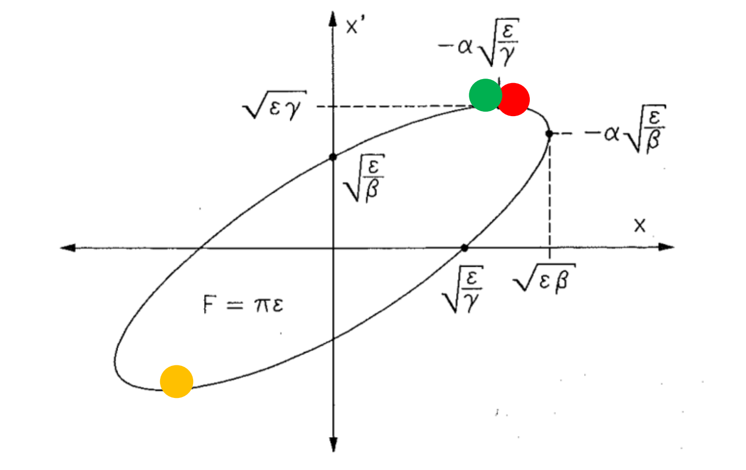


The errors **sum up** – strong resonance

Transverse motion – betatron resonances

Under normal conditions the emittance is preserved turn after turn:

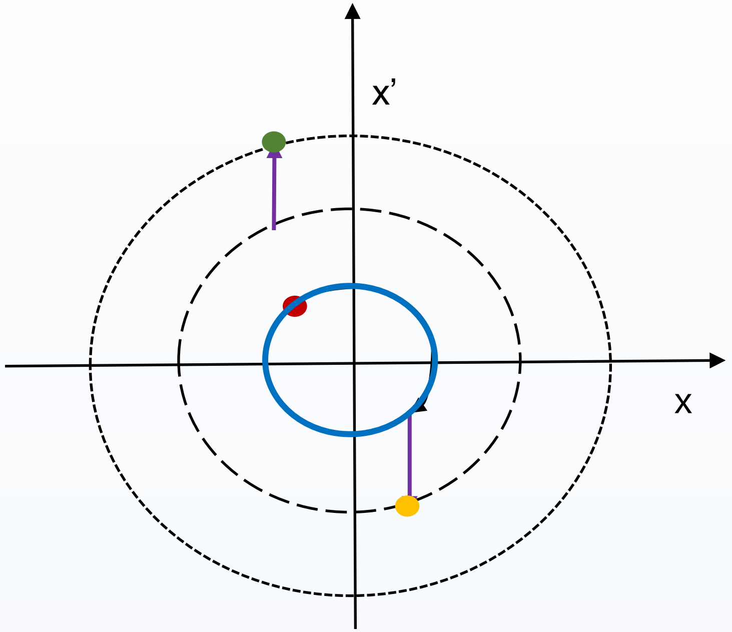
- Observing the phase space turn-by-turn we get the emittance ellipse



In the presence of a strong **resonance:**

- Emittance of a particle on the **1st turn**
- Emittance increases on the **2nd turn**
- Emittance increases further on the **3rd turn**

Emittance (and amplitude) will keep increasing until the particle is lost



Mitigation measures:

1. **Careful tune choice – avoid resonance condition**
2. **Higher order elements – corrections to cancel the effect of the resonance**

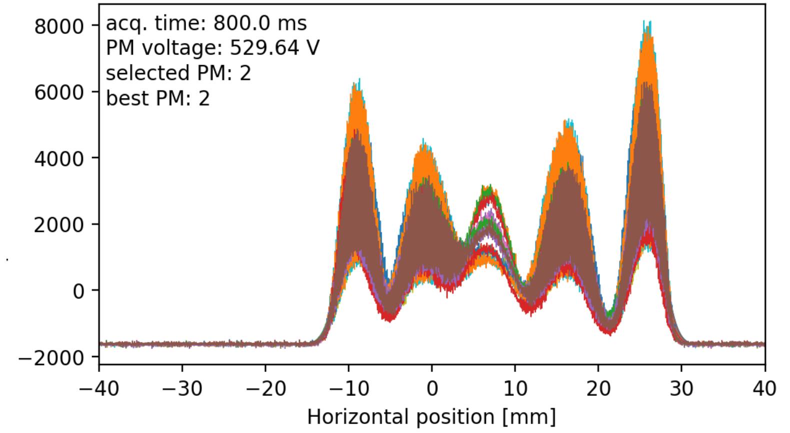
Transverse motion – phase space

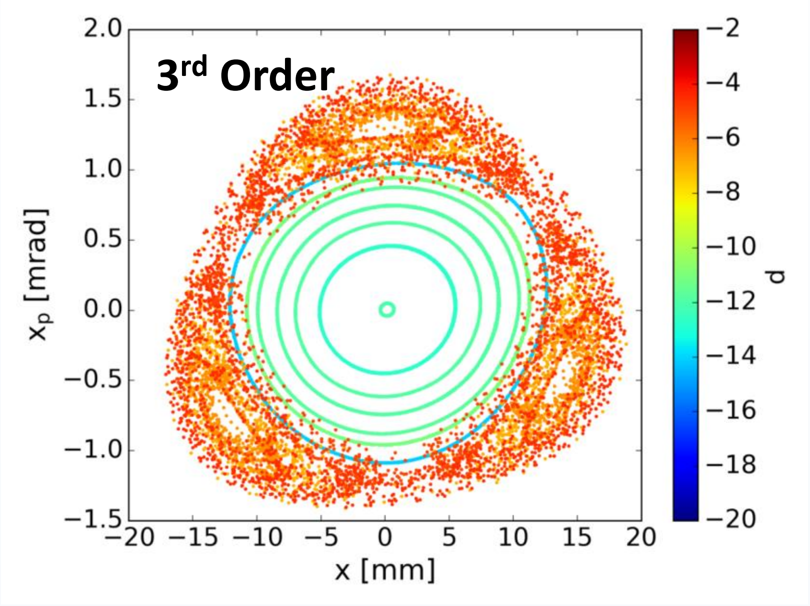
The phase space gets distorted in the vicinity of driven reso

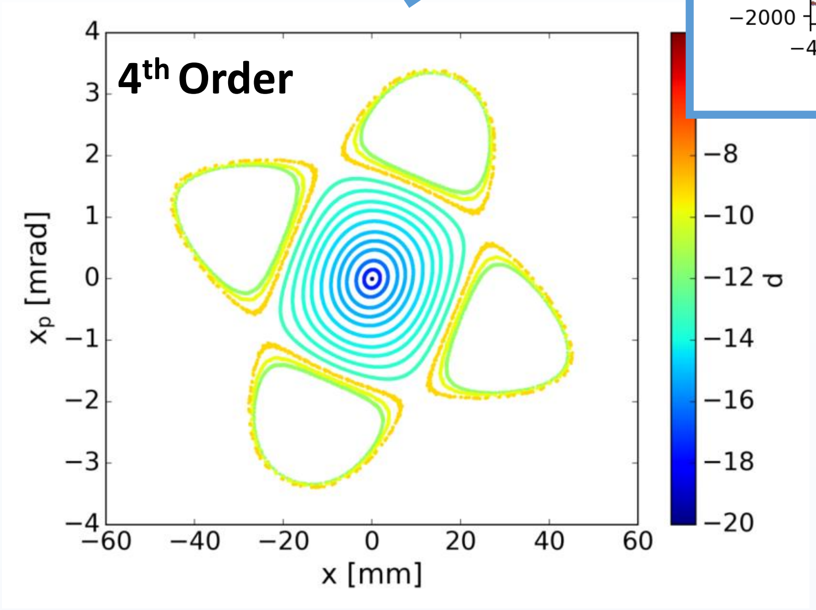
- *Changing our tune to approach a resonance:*
  - Starting with well defined ellipses
  - Approaching the resonances – ***forming islands depe***

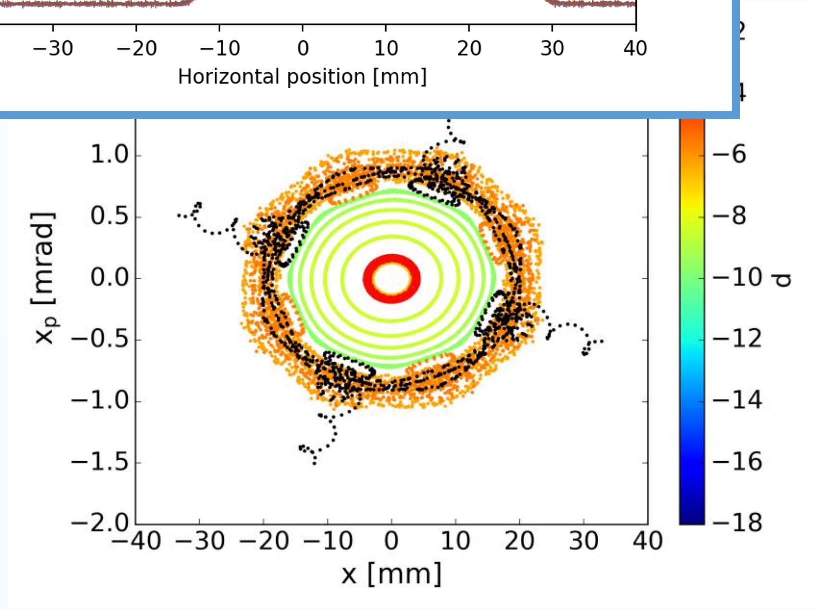
Used in PS Multi-Turn-Extraction

Profile (projection in x) measurements









Longitudinal motion - Acceleration

Reminder:

- Acceleration in a synchrotron is achieved in the **RF cavities**, using a voltage V
- During operations, we have a **synchronous RF phase** for which the **energy gain** fits the **increase of the magnetic field** at each turn. → *condition for constant radius*

- Energy gain per turn:

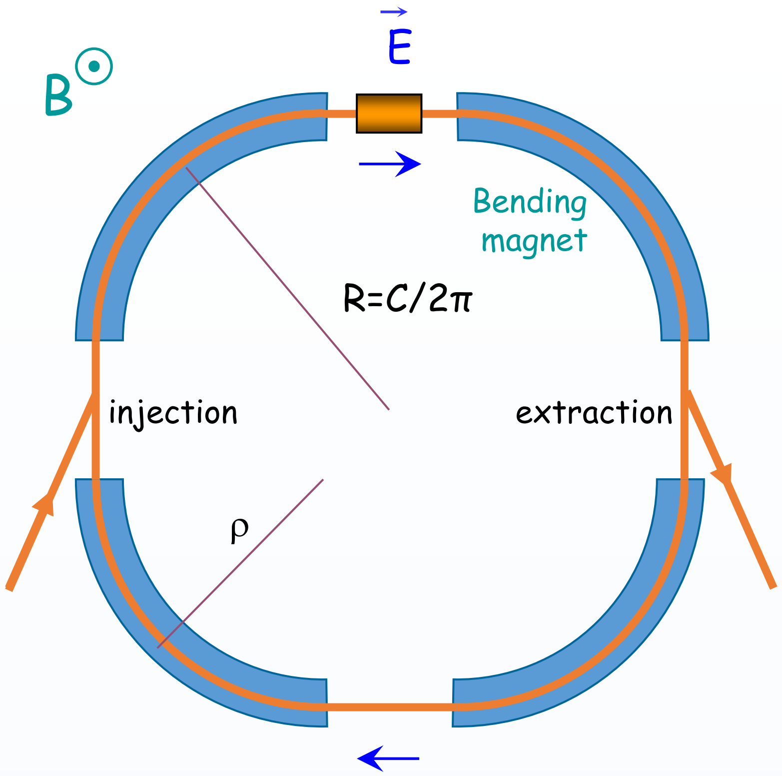
$$qV\sin\phi = qV\sin\omega_{RF}t$$

- synchronous phase:

$$\phi = \phi_s = const$$

- RF synchronism - frequency must be on the revolution frequency (1 turn around the ring):

$$\omega_{RF} = h\omega_{rev}$$



h (integer): **harmonic number**

- number of RF cycles per revolution
- ➢ ***Defines the maximum number of bunches in the synchrotron*** *(available RF buckets)*

Longitudinal motion – f_{RF} and ϕ_s change

During acceleration "***ramping***" energy & the magnetic field are changing:

- The revolution frequency changes: $\omega(B, R_s)$
- From the synchronism condition RF frequency needs to follow (using $p(t) = eB(t)\rho,\ E^2 = (m_0c^2)^2 + p^2c^2$):

Can be omitted at the relativistic limit where B >> $m_0c^2/(ec\rho)$

$$\frac{f_{RF}(t)}{h} = \frac{v(t)}{2\pi R_s} = \frac{1}{2\pi}\frac{ec^2}{E_s(t)}\frac{\rho}{R_s}B(t) = \frac{c}{2\pi R_s}\left\{\frac{B(t)^2}{\cancel{(m_0c^2/ec\rho)^2} + B(t)^2}\right\}^{1/2}$$

- Similarly, the phase, $\boldsymbol{\varphi_s}$ needs to follow
- From Bρ:

$$(\Delta p)_{turn} = e\rho \dot{B} T_r = \frac{2\pi e\rho R\dot{B}}{v} \Rightarrow (\Delta E)_{turn} = (\Delta W)_s = 2\pi e\rho R\dot{B} = e\hat{V}\sin\phi_s$$

$$\phi_s = \arcsin\left(2\pi\rho R\frac{\dot{B}}{\hat{V}_{RF}}\right)$$

Longitudinal motion – Dispersion effects

Reminder:

Off-momentum particles follow a different orbit than the design: ***Dispersion***

- The ***orbit length*** is different – the ***momentum compaction factor*** shows the variation of the orbit length with respect to the variation of the momentum:

$$\alpha_c = \frac{dR/R}{dp/p}$$

- From different momentum, different velocity & different path: different time (& revolution frequency) to arrive to the RF cavity – ***slip factor***, variation of the revolution frequency with respect to the variation of the momentum:

$$h = \frac{\mathrm{d}f_r/f_r}{\mathrm{d}p/p}$$

Longitudinal motion – Dispersion effects

- The revolution frequency change depends both on the **orbit** and **velocity** change:

$$\frac{df_r}{f_r} = \frac{d\beta}{\beta} - \frac{dR}{R} = \frac{d\beta}{\beta} - \alpha_c \frac{dp}{p}$$

- For $\frac{d\beta}{\beta}$: $p = mv = bg\frac{E_0}{c}$ Þ $\frac{dp}{p} = \frac{db}{b} + \frac{d\left(1-b^2\right)^{-½}}{\left(1-b^2\right)^{-½}} = \underbrace{\left(1-b^2\right)^{-1}}_{g^2}\frac{db}{b}$

- Finally, we get the relation between momentum compaction and slip factor: $\eta = \frac{1}{\gamma^2} - \alpha_c$

- The energy in which $\eta = 0$, is called ***transition energy***: $\boldsymbol{\gamma_t = \frac{1}{\sqrt{\alpha_c}}}$

➢ **Below γ_t (η>0)** the arrival at the cavity depends on the **velocity**

➢ **At γ_t (η=0)** the velocity change and the path length change **compensate each other**

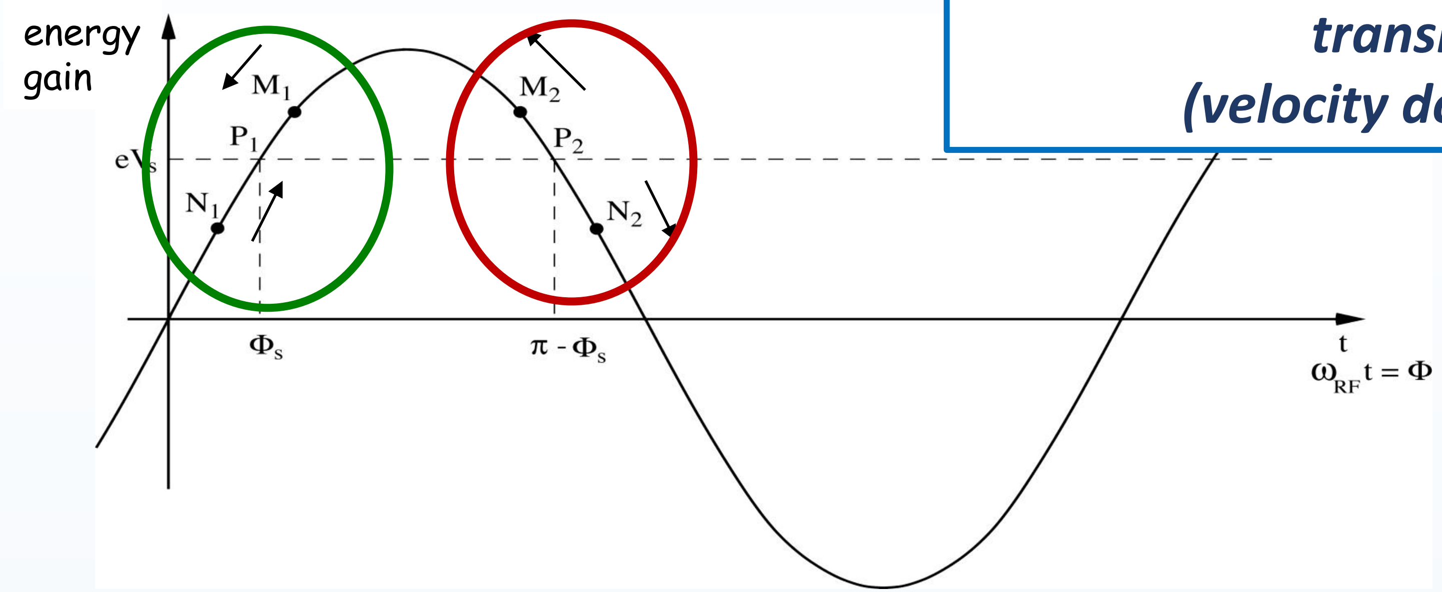
➢ **Above γ_t (η<0)** the arrival at the cavity depends ***only*** on the **path length**

Longitudinal motion – Phase stability

Reminder:

- Phase focusing: bunches are formed as particles arriving at the cavity before or after the synchronous particle are “brought closer” to it
- This stands for $\varphi_s < \pi/2$ as:
  - M_1 & N_1 will move towards P_1 => **stable**
  - M_2 & N_2 will go away from P_2 => **unstable**

This behaviour is true below transition (velocity dominates)

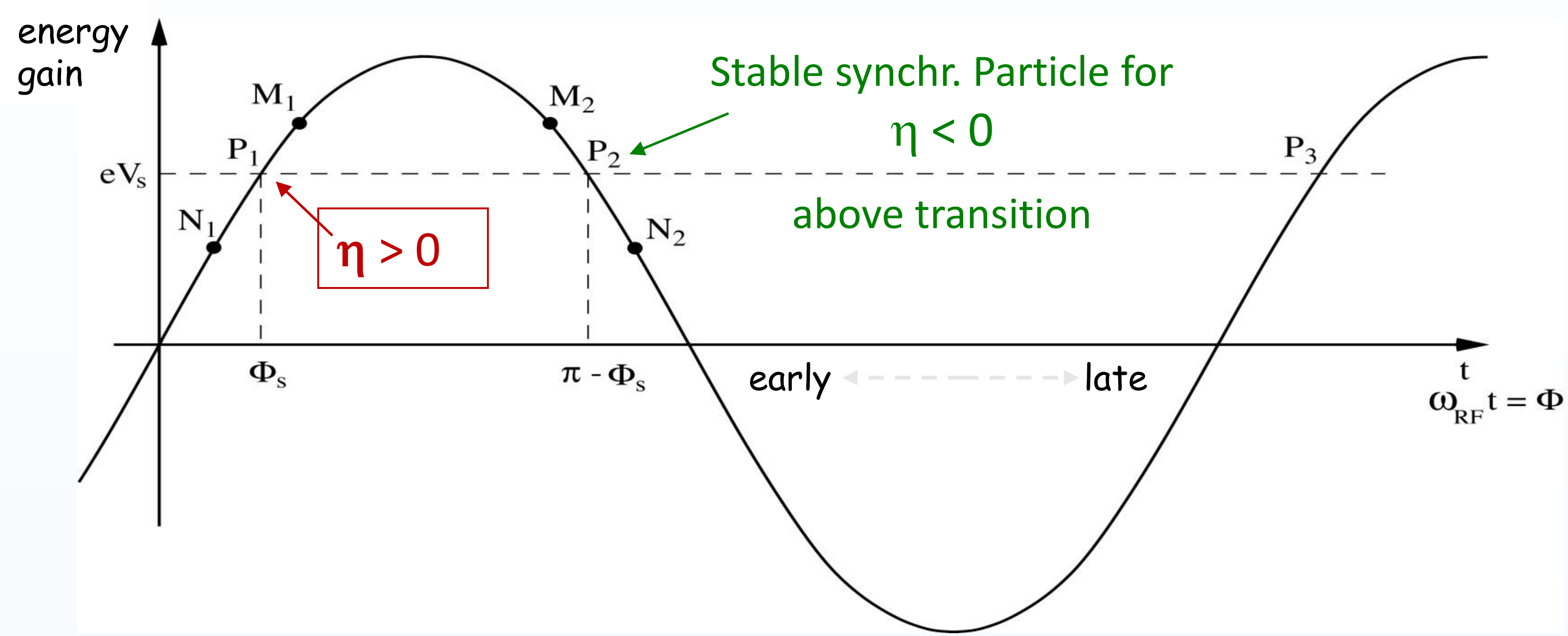


Longitudinal motion – Phase stability

Since:

- **Below γ_t (η>0)** the arrival at the cavity depends on the **velocity**
- **Above γ_t (η<0)** the arrival at the cavity depends ***only*** on the **path length**

➢ **The behaviour for the phase stability is reversed around transition crossing**



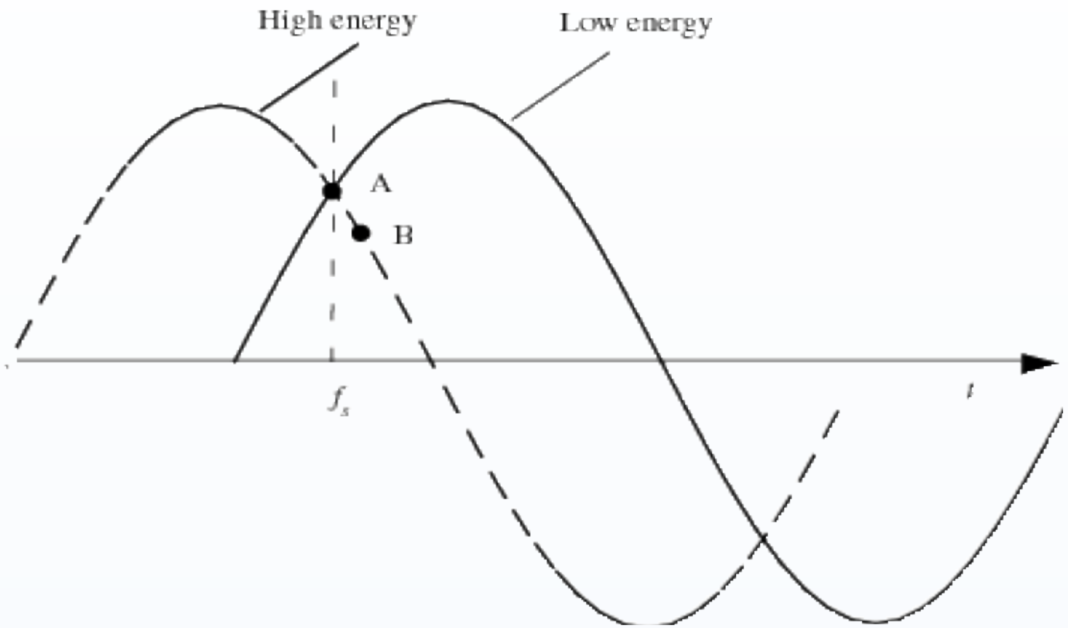
Longitudinal motion – Transition crossing

- Change of stable phase implies:

- ***Crossing transition during acceleration makes the previous stable synchronous phase unstable.***

➢ The RF system needs to make a rapid change of the RF phase, a "phase jump".

- ***Such a manipulation is needed at the CERN PS***

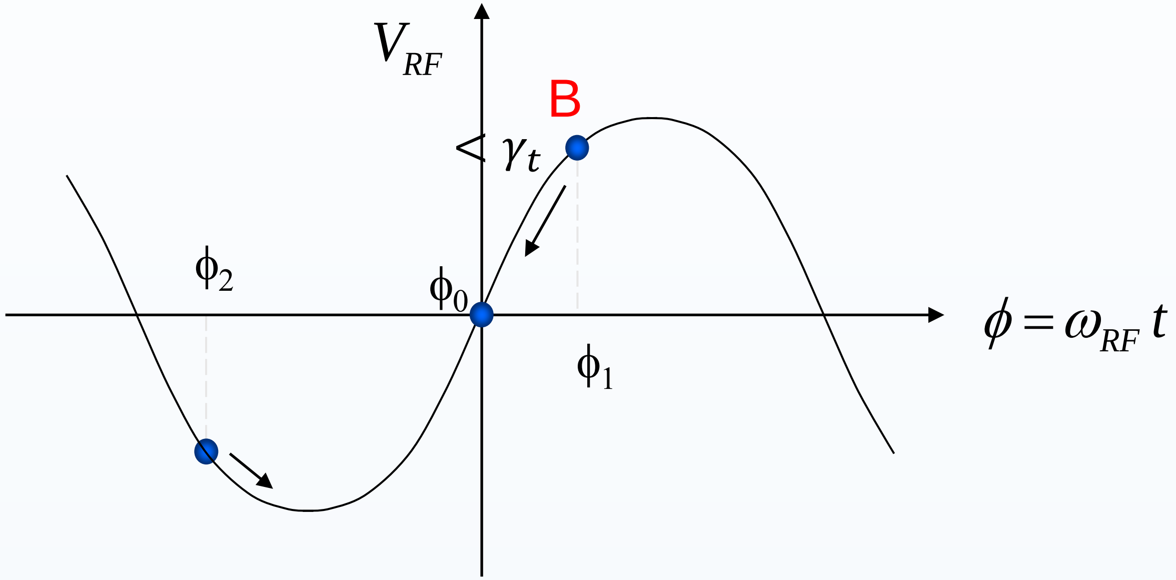


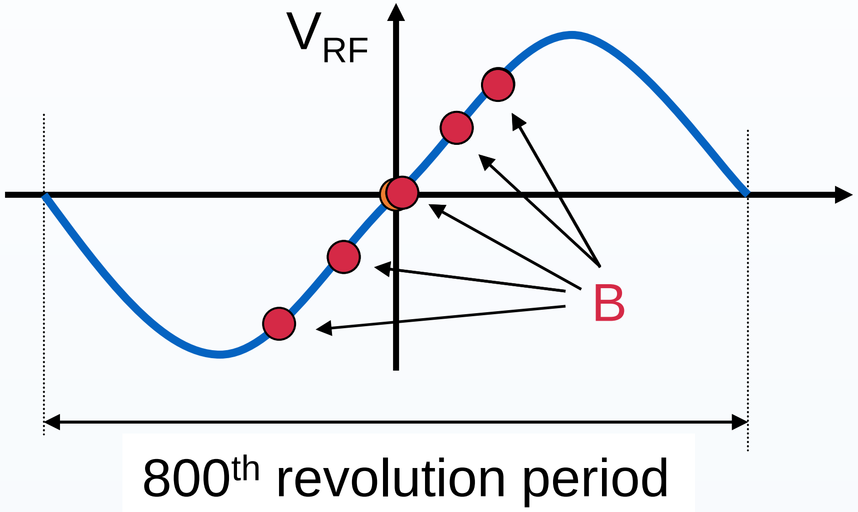
Longitudinal motion – Synchrotron oscillations

Operating below transition & at constant energy (and B)

- Synchronous phase ϕ_0=0
- Particle with a $\phi>\phi_0$: particle gets accelerated and moves towards ϕ_0
- Particle with a $\phi<\phi_0$: particle gets decelerated and moves towards ϕ_0

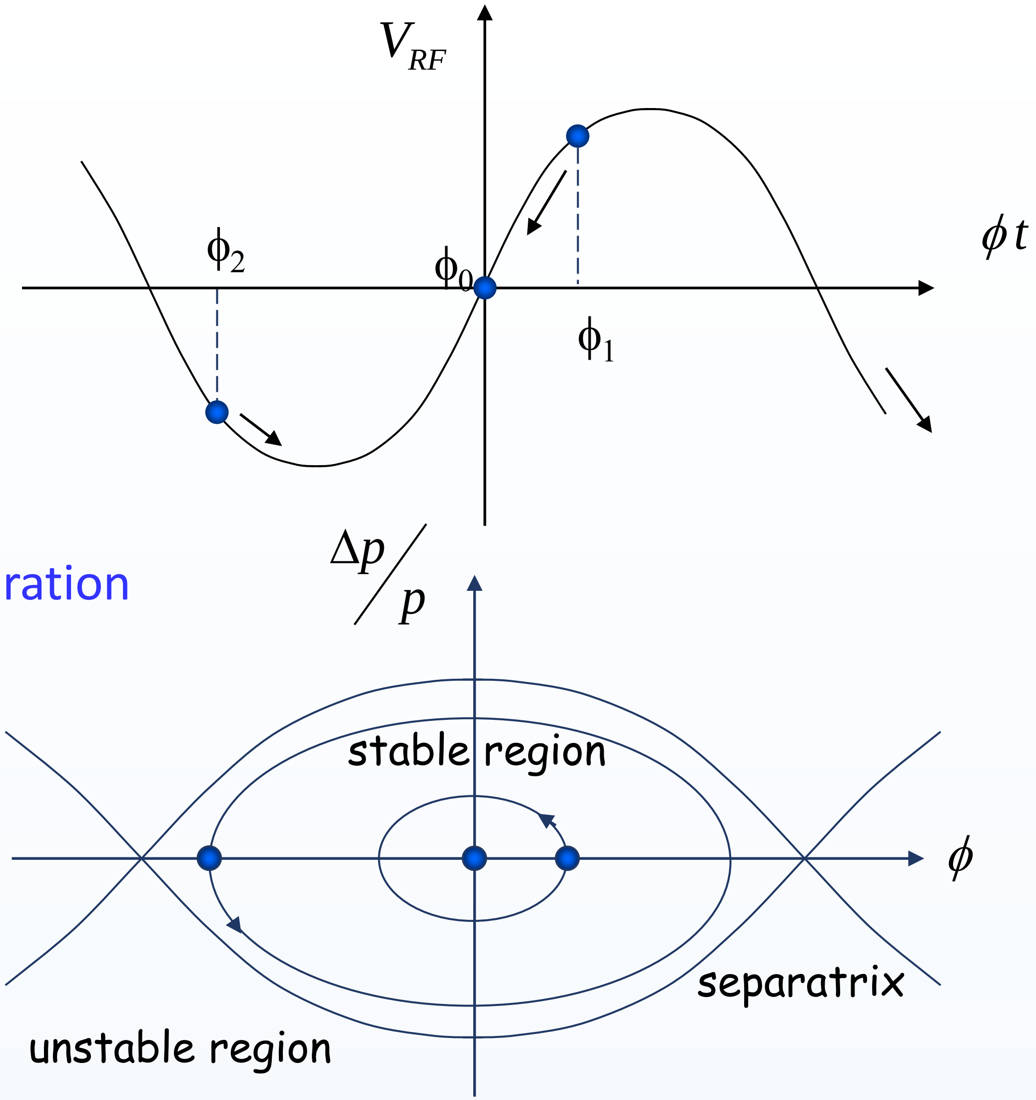
➢ ***Particles will start performing oscillations around the synchronous particle***



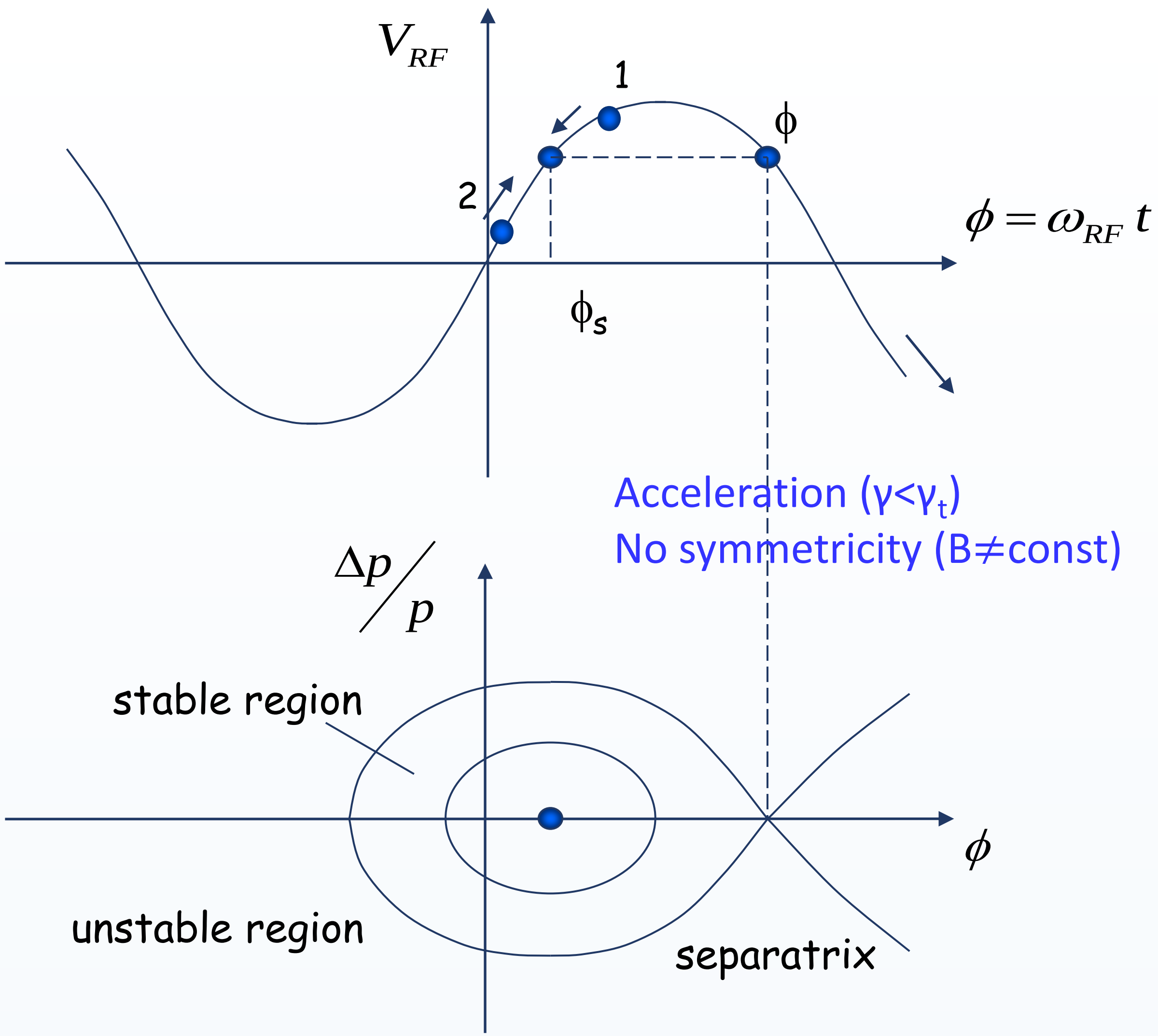


Longitudinal motion – Phase space

No Acceleration



Acceleration ($\gamma<\gamma_t$)
No symmetricity ($B\neq$const)



Takeaways

Transverse motion

- The beam moves in ***FODO structure***
- Particles perform oscillations around the design orbit called ***betatron***
- Turn-by-turn the ellipse formed in the phase space is called ***emittance***
- The number of betatron oscillations in 1 turn is called **tune**
- Emittance remains **constant** for "normal" conditions
- In the presence of ***resonances*** in the tune space, the emittance increases
- The ***beam size*** is defined as $\sqrt{\varepsilon_y \beta_y(s)}$

Longitudinal motion

- ***Synchronism:*** RF frequency needs to be locked to revolution frequency
- During acceleration the ***phase and frequency*** need to adjust to the ***energy & B increase***
- Path length changes with momentum – ***momentum compaction factor***
- Frequency changes with momentum – ***slip factor***
- Phase stability depends on ***transition energy***
- ***Phase jump*** to cross transition
- Particles perform oscillations around ϕ_s called ***synchrotron***