Unfolding in the context of the muon $g-2$

Laurent Lellouch

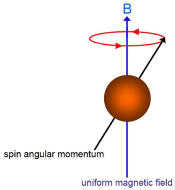
CPT Marseille
CNRS & Aix-Marseille U.

for the BMW and DHMZ collaborations

PRD 109(7) (2024) 076019 $\rightarrow$ BMWc-DHMZ'23
DHMZ, EPJC 80(3) (2020) 241 $\rightarrow$ DHMZ'19
BMW, Nature 593 (2021) 51 $\rightarrow$ BMW'20
Aoyama et al., PR 887 (2020) 1 $\rightarrow$ WP'20
PRL 131 (2023) $\rightarrow$ Fermilab'23

Introduction and motivation

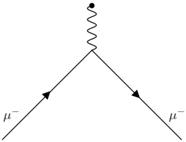
- Muon behaves like a tiny magnet with a magnetic dipole moment



$$\vec{\mu}_\mu = -g_\mu \frac{e}{2m_\mu} \vec{S}$$

Leading order SM :

$$g_\mu = 2$$

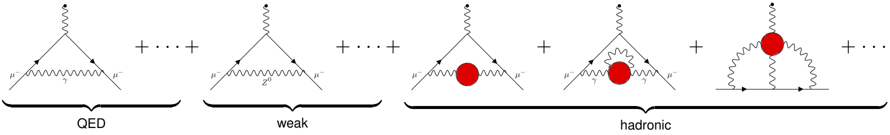


- Quantity of interest is the *anomalous* contribution

$$a_\mu = \frac{g_\mu - 2}{2}$$

$\rightarrow$ given by quantum corrections (loops)

- a_μ can be measured very precisely
- a_μ can be computed "equally" precisely in the SM

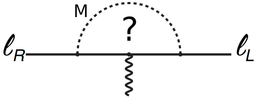


Introduction and motivation

Big question:

$$a_\mu^{\text{exp}} = a_\mu^{\text{SM}} ?$$

- YES $\rightarrow$ another success for the SM *(at given level of precision)*
- NO $\rightarrow$ new fundamental physics must be contributing to a_μ^{exp}, e.g.



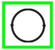
- Complementarity w/ *direct* searches (e.g. LHC): may be sensitive to dofs that are too massive or too weakly coupled to be produced or measured directly
- Complementarity w/ other *indirect* searches (FCNCs (e.g. in s and b decays), EDMs, ...)
  - $\rightarrow$ a_μ is flavor & CP conserving and chirality flipping ($L \leftrightarrow R$)
  - $\Rightarrow$ probes mass generating mechanism of the theory

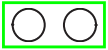
Summary of contributions to $a_\mu^{\text{LO-HVP}}$ [BMW'23]

Isospin symmetric

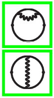

connected light
633.7(2.1)(4.2)

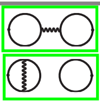

connected strange
53.393(89)(68)


connected charm
14.6(0)(1)


disconnected
-13.36(1.18)(1.36)

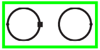
QED isospin-breaking: valence


connected -1.23(40)(31)

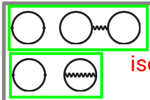

disconnected -0.55(15)(10)

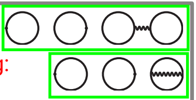
Strong isospin-breaking


connected
6.60(63)(53)


disconnected
-4.67(54)(69)

QED isospin-breaking: sea


connected 0.37(21)(24)

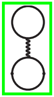

disconnected -0.040(33)(21)

Etc.

bottom; higher order; perturbative

0.11(4)

QED isospin-breaking: mixed


connected -0.0093(86)(95)


disconnected 0.011(24)(14)

Finite-size effects

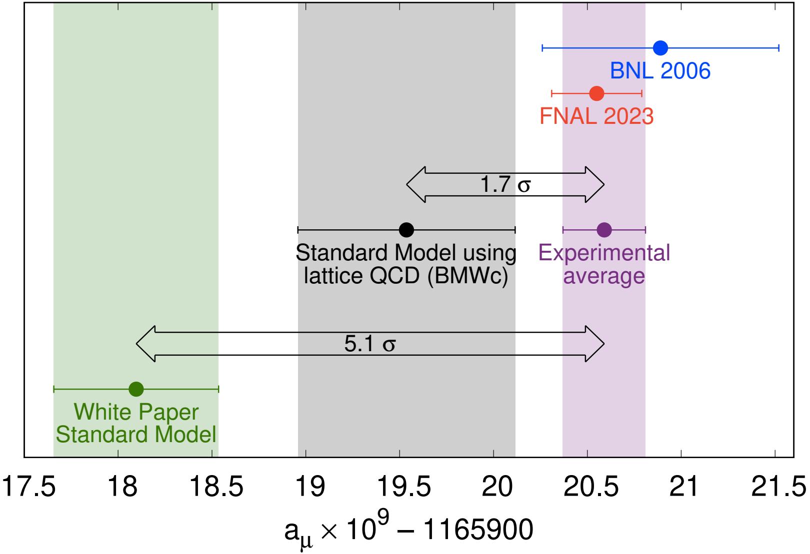
isospin-symmetric
18.7(2.5)

isospin-breaking
0.0(0.1)

$$10^{10}\times a_\mu^{\text{LO-HVP}} = 707.5(2.3)_{\text{stat}}(5.0)_{\text{sys}}[5.5]_{\text{tot}}$$

Corresponds to a 0.78% total uncertainty

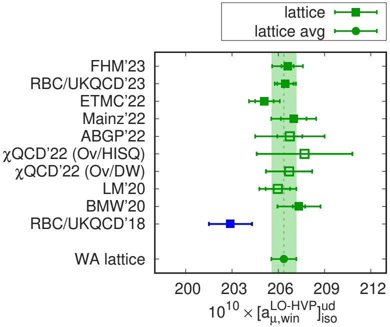
Experiment vs BMWc and WP

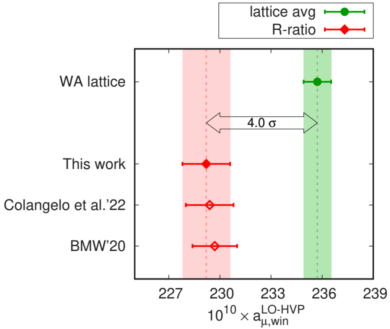


Has new fundamental physics really been uncovered?

Significant tensions between lattice and data-driven (DD) results for HVP

- $[\Delta a_\mu^{\text{LO-HVP}}]_{\text{lat-DD}} \sim 2.1\sigma$ [BMW'20, WP'21]
- Simpler $[\Delta a_{\mu,\text{win}}^{\text{LO-HVP}}]_{\text{lat-DD}} \gtrsim 4\sigma$ [Observable proposed in RBC/UKQCD'18]





→ origin of tensions?

→ comparison not trivial

Primary observables

- Lattice: compute with QCD simulations (spacelike)

$$C(t) = \frac{a^3}{3e^2}\sum_{i=1}^{3}\sum_{\vec{x}}\langle J_i(\vec{x},t)J_i(0)\rangle$$

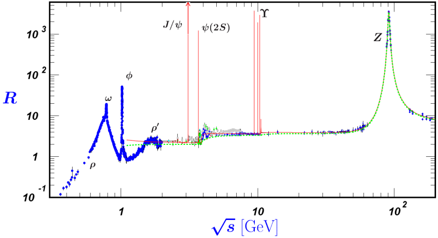
w/ $\frac{J_\mu}{e} = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c - \frac{1}{3}\bar{b}\gamma_\mu b + \frac{2}{3}\bar{t}\gamma_\mu t$

$a_{\mu(,\text{win})}^{\text{LO-HVP}}$, $\hat{\Pi}(Q^2)$, ... are weigthed sums of $C(t)$ over imaginary time t

- Data-driven: measure (timelike)

$$R(s) \equiv \frac{\sigma(e^+e^-(s) \to \text{hadrons}(+\gamma))}{4\pi\alpha^2(s)/(3s)}$$

$a_{\mu(,\text{win})}^{\text{LO-HVP}}$, $\hat{\Pi}(Q^2)$, ... are weigthed integrals of $R(s)$ over s



[PDG compilation]

Lattice $\leftrightarrow$ R-ratio

$$C(t) = \frac{1}{24\pi^2}\int_0^\infty ds\sqrt{s}\,R(s)\,e^{-|t|\sqrt{s}}$$

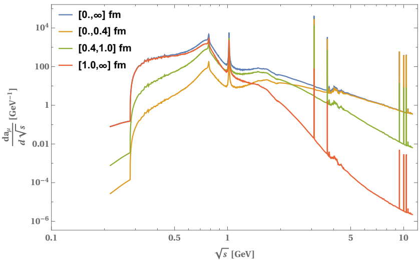
[Bernecker et al '11]

- R-ratio $\longrightarrow$ lattice: "straightforward"

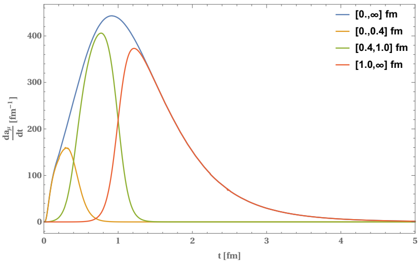
  $\rightarrow$ integrate R-ratio

- Lattice $\longrightarrow$ R-ratio: inverse Laplace transform

  $\rightarrow$ ill-posed unfolding problem



$\longleftrightarrow$



Requirements for comparison methodology

1. Very few HVP quantities computed on lattice w/:
   - all contributions to $C(t)$: flavors, quark Wick contractions, QED and SIB corrections
   - all limits taken: $a \to 0$, $L \to \infty$, $M_\pi \to M_\pi^\phi$, …
2. None w/ correlations among lattice observables (quantitative comparisons)
3. None w/ uncertainties on these correlations (checking stability of inverse problem)

→ Want approach that:

- provides useful information w/ limited lattice input
- can be systematically improved w/ more lattice input
- can incorporate theoretical constraints (e.g. Colangelo et al '20)
- includes measure of agreement of lattice & data-driven results w/ comparison hypothesis
- accounts for all correlations in lattice and data-driven observables …
- … including uncertainties on these

4. Here use BMW'20: $a_\mu^{\text{LO-HVP}}$, $a_{\mu,\text{win}}^{\text{LO-HVP}}$ & $\delta(\Delta_{\text{had}}^{(5)}\alpha) \equiv \Delta_{\text{had}}^{(5)}\alpha(-1 \to -10\,\text{GeV}^2)$ (preliminary)

Lattice covariances: method

- Uncertainties and correlations critical for quantitative comparisons

- Use extension of BMW error method with stat resampling and syst histogramming w/ flat and AIC weights [BMW '08, '15, '20, see also Neil et al '23, Pinto et al '23]

  $\rightarrow$ for $N_{\mathcal{O}}$ observables $\{a_j\} = \{a_\mu^{\text{LO-HVP}}, a_{\mu,\text{win}}^{\text{LO-HVP}}, \delta(\Delta_{\text{had}}^{(5)}\alpha), \cdots\}$

$$H(\{a_j\}) = \sum_{\psi^{\text{corr}},\{\psi_j^{\text{aic}},\psi_j^{\text{flat}}\}} \mathcal{N}_{N_{\mathcal{O}}}[\{a_j\}, \{\overline{a}_j(\psi^{\text{corr}}, \psi_j^{\text{aic}}, \psi_j^{\text{flat}})\}, C^{\text{stat}}(\psi^{\text{corr}}, \{\psi_j^{\text{flat}}, \psi_j^{\text{aic}}\})] \times \Pi_j \omega_j(\psi^{\text{corr}}, \psi_j^{\text{aic}}, \psi_j^{\text{flat}})$$

$$\omega_j(\psi^{\text{corr}}, \psi_j^{\text{aic}}, \psi_j^{\text{flat}}) = \frac{\text{aic}(\psi^{\text{corr}}, \psi_j^{\text{aic}}, \psi_j^{\text{flat}})}{\sum_{\psi_j^{\text{aic}},\psi_j^{\text{flat}}} \text{aic}(\psi^{\text{corr}}, \psi_j^{\text{aic}}, \psi_j^{\text{flat}})}$$

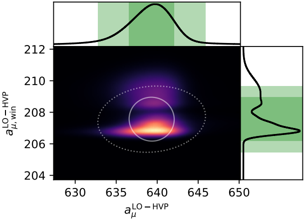
- Build matrix from 1D distributions for $\{a_\mu^{\text{LO-HVP}}, a_{\mu,\text{win}}^{\text{LO-HVP}}, a_\mu^{\text{LO-HVP}} + a_{\mu,\text{win}}^{\text{LO-HVP}}\}$

- Separate stat & syst by solving ($\lambda = 2$)

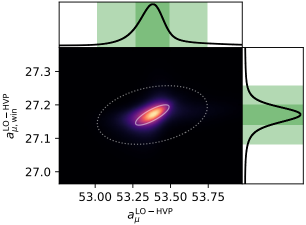
$$\begin{aligned} C &= C^{\text{stat}} + C^{\text{syst}} \\ C_\lambda &= \lambda\, C^{\text{stat}} + C^{\text{syst}} \end{aligned}$$

Lattice covariances: results

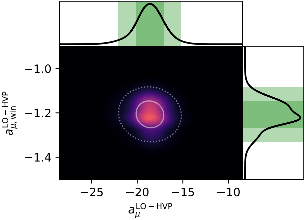
- $\delta(\Delta^{(5)}_{\text{had}}\alpha)$ largely uncorrelated w/ other two observables
- Uncertainties and correlations of $a_\mu^{\text{LO-HVP}}$ & $a_{\mu,\text{win}}^{\text{LO-HVP}}$ contributions (units of 10^{-10})



(connected *ud*)



(connected *s*)



(disconnected)

- Double peak $\rightarrow$ consider 1σ & 2σ intervals

Uncertainties on lattice covariances

- Uncertainties on covariance matrix can compromise the inverse problem
- Stat error estimated from bootstrap on only 48 reconstructed samples (sufficient for this study)
- Syst from:
  - For: *ud*, *s*, QED, SIB connected, and disconnected
    $\rightarrow$ get uncertainties from 1 or 2σ quantiles
    $\rightarrow$ 0 or 100% correlations in $a \rightarrow 0$ uncertainties of $T = a_\mu^{\text{LO-HVP}}$ and $W = a_{\mu,\text{win}}^{\text{LO-HVP}}$, w/ $C = T - W$

$$C_{TW} = C_{TW}^{\text{other}} + \begin{bmatrix} (dW)^2 + (dC)^2 & \{0,1\} \times (dW)^2 \\ \{0,1\} \times (dW)^2 & (dW)^2 \end{bmatrix}_{\text{cont}}$$

  - Similarly for *c*

$\Rightarrow$ in units of 10^{-20}:

$$C_{\text{lat}}^{1\sigma,0\%} = \begin{bmatrix} 30.13(4.88) & -0.05(0.03) \\ -0.05(0.03) & 1.95(0.47) \end{bmatrix} \qquad C_{\text{lat}}^{2\sigma,0\%} = \begin{bmatrix} 34.04(16.80) & 0.32(0.05) \\ 0.32(0.05) & 1.12(0.07) \end{bmatrix}$$

$$C_{\text{lat}}^{1\sigma,100\%} = \begin{bmatrix} 30.13(4.88) & 1.56(0.03) \\ 1.56(0.03) & 1.95(0.47) \end{bmatrix} \qquad C_{\text{lat}}^{2\sigma,100\%} = \begin{bmatrix} 34.04(16.80) & 1.94(0.05) \\ 1.94(0.05) & 1.12(0.07) \end{bmatrix}$$

Testing lattice

- 1-by-1 comparisons

Observable	lattice [BMW '20]	data-driven	diff.	% diff.	σ	p-value [%]
$a_\mu^{\text{LO-HVP}} \times 10^{10}$	707.5(5.5)	694.0(4.0)	13.5(6.8)	1.9(1.0)	2.0	4.7
$a_{\mu,\text{win}}^{\text{LO-HVP}} \times 10^{10}$	236.7(1.4)	229.2(1.4)	7.5(2.0)	3.2(0.8)	3.8	0.01
$\delta(\Delta_{\text{had}}^{(5)}\alpha) \times 10^4$	48.67(0.32)	48.02(0.32)	0.65(0.45)	1.3(0.9)	1.4	15

$\Rightarrow$ excess in $C = a_\mu^{\text{LO-HVP}} - a_{\mu,\text{win}}^{\text{LO-HVP}}$: $[\Delta C]_{\text{lat-DD}} \sim 6.0(7.9) \times 10^{-10}$

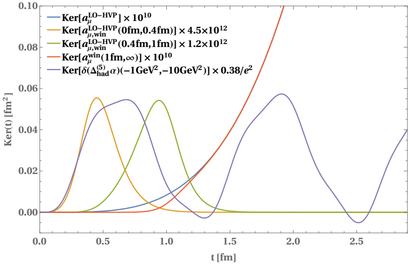
- Simultaneous comparisons w/ correlations

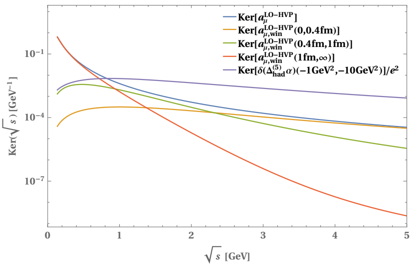
$$\chi^2(a_j) = \sum_{j,k} \left[a_j^{\text{lat}} - a_j\right] \left[C_{\text{lat}}^{-1}\right]_{jk} \left[a_k^{\text{lat}} - a_k\right] + \sum_{j,k} \left[a_j^{\text{R}} - a_j\right] \left[C_{\text{R}}^{-1}\right]_{jk} \left[a_k^{\text{R}} - a_k\right]$$

# observ.	χ^2/dof	p-value [%]
2	$14.4/2 - 18.8/2$	$0.008 - 0.07$
3	$14.4/3 - 18.8/3$	$0.03 - 0.23$

- Some dilution compared to $a_{\mu,\text{win}}^{\text{LO-HVP}}$ alone, but still significant tensions

Consequences for lattice $C(t)$





$\Rightarrow$ SD:ID:LD windows: [using KNT'18 $e^+e^- \to$ hadrons compilation]

- 10%:33%:57% for $a_\mu^{\text{LO-HVP}}$
- 70%:29%:1% for $\delta(\Delta_{\text{had}}^{(5)}\alpha)$

$+$ tensions and agreements above

$\Rightarrow$ excess in $C(t)$ for $t \sim [0.4, 1.5]\,\text{fm}$

$\Rightarrow$ probably for $t \gtrsim 1.5\,\text{fm}$

$\Rightarrow$ possible suppression for $t \lesssim 0.4\,\text{fm}$ (mainly based on preliminary $\delta(\Delta_{\text{had}}^{(5)}\alpha)$)

Testing R-ratio: methodology

- Chop a_j^{R} into contributions a_{jb}^{R} from same $\sqrt{s}$-intervals I_b for all j

$$a_j^{\mathrm{R}} = \sum_b a_{jb}^{\mathrm{R}}$$

- To accommodate lattice results a_j^{lat}, allow common rescaling of a_{jb}^{R}, for all j, in certain I_b

$$a_j^{\mathrm{lat}} = \sum_b \gamma_b a_{jb}^{\mathrm{R}} = \sum_b (1+\delta_b) a_{jb}^{\mathrm{R}} \qquad (1)$$

$\rightarrow$ can take some $\gamma_b = 1$

$\rightarrow$ simplest interpretation: R-ratio rescaled by γ_b in I_b

$\rightarrow$ however, constrains shape of R-ratio modification in limited way

$\rightarrow$ Φ deformation may be allowed

- If $N_j \geq N_\gamma$, system (over-)constrained

- Here single γ_1 in I_1 w/ $a_1 = a_\mu^{\text{LO-HVP}}$ & $a_2 = a_{\mu,\text{win}}^{\text{LO-HVP}}$ (2 observables) or w/ additional $a_3 = \delta(\Delta_{\text{had}}^{(5)}\alpha)$ (3 observables)

Testing R-ratio: methodology

- Solve 2 or 3 eqs in (1) for γ_1

$$\tilde{\gamma}_j \equiv \frac{a_j^{\rm lat} - a_{j1}^{\rm R}}{a_{j1}^{\rm R}}$$

  w/ $j = 1, 2(, 3)$

- γ_1 weighted average from minimization of

$$\chi^2(\gamma_1) = \sum_{j,k} [\gamma_1 - \tilde{\gamma}_j] \left[\left(C_{\rm lat}^{\tilde{\gamma}} + C_{\rm R}^{\tilde{\gamma}} \right)^{-1} \right]_{jk} [\gamma_1 - \tilde{\gamma}_k]$$

- Minimization w/ diagonal covariance to avoid possible biases

- $\delta\gamma_1$ and $\chi^2(\gamma_1)_{\rm min}$ w/ full covariance

- Alternative approach via

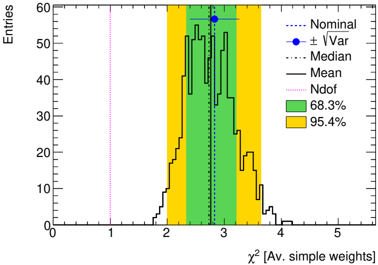
$$\chi^2(a_{jb}, \gamma_b) = \sum_{j,k} \left[a_j^{\rm lat} - \sum_b \gamma_b a_{jb} \right] \left[C_{\rm lat}^{-1} \right]_{jk} \left[a_k^{\rm lat} - \sum_c \gamma_c a_{kc} \right] + \sum_{(jb)(kc)} \left[a_{jb}^{\rm R} - a_{jb} \right] \left[C_{\rm R}^{-1} \right]_{(jb)(kc)} \left[a_{kc}^{\rm R} - a_{kc} \right]$$

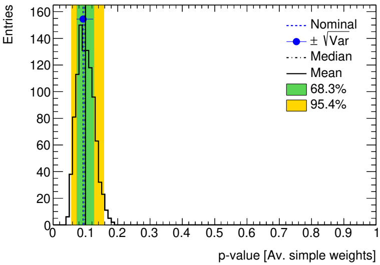
$\rightarrow$ compatible results

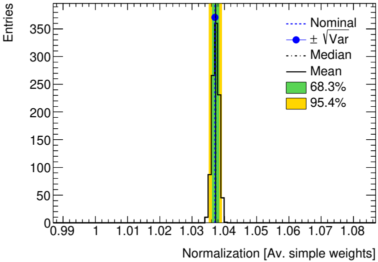
- Solve for 1000 stat bootstrap samples and 4 syst variations of $C_{\rm lat}$

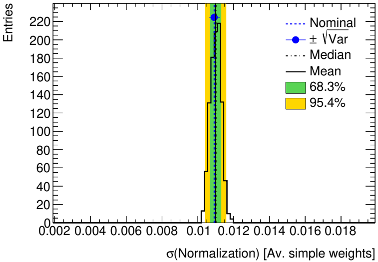
Statistical distributions for rescaling in $[\sqrt{s_{th}}, 0.96\,\mathrm{GeV}]$

From 1000 stat bootstrap variations of lattice C_0 w/ $a_\mu^{\text{LO-HVP}}$ and $a_{\mu,\text{win}}^{\text{LO-HVP}}$ constraints



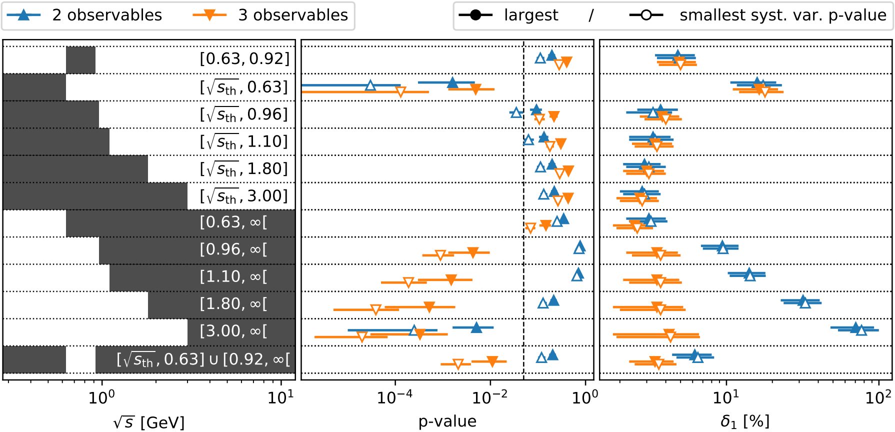






Testing R-ratio: results

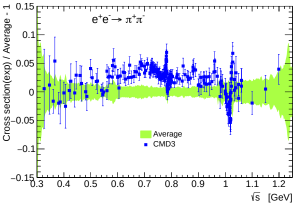
Consider $a_1 = a_\mu^{\text{LO-HVP}}$, $a_2 = a_{\mu,\text{win}}^{\text{LO-HVP}}$ (2 obs.) w(/out) $a_3 = \delta(\Delta_{\text{had}}^{(5)}\alpha)$ (3 obs.)



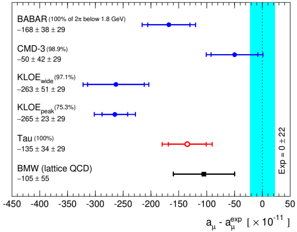
- Stat and syst uncertainties on lattice covariance matrices do not change overall picture

Situation evolving fast

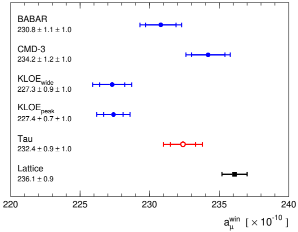
- February 2023: new measurement of $\sigma(e^+e^- \to \pi^+\pi^-)$ [CMD-3, 2302.08834] gives data-driven $a_\mu^{\text{LO-HVP}}$:
  - $\sim 3\sigma$ larger than WA data-driven $a_\mu^{\text{LO-HVP}}$!
  - compatible w/ BMWc'20 lattice $a_\mu^{\text{LO-HVP}}$!
  - many questions were asked, but no significant problem found
- December 2023: taking data-driven approach apart [Davier et al, 2312.02053]



CMD-3 R-ratio is $\sim 5\%$ than previous WA around ρ-peak !



W/out KLOE, data-driven $a_\mu^{\text{LO-HVP}}$ agrees with lattice & tension of former w/ Fermilab a_μ measurement much reduced !



W/out KLOE, tension between data-driven and lattice $a_{\mu,\text{win}}^{\text{LO-HVP}}$ reduced to $\sim 1-3\sigma$!

Problems w/ NLO QED effects in KLOE (& BES III) not covered by systematic uncertainties ? [BaBar '23]

Conclusions

- Presented flexible method for comparing lattice QCD and data-driven HVP results
- Find that discrepancies/agreements between lattice and data-driven results for $a_\mu^{\text{LO-HVP}}$, $a_{\mu,\text{win}}^{\text{LO-HVP}}$ and $\delta(\Delta_{\text{had}}^{(5)}\alpha)$:

  On lattice side, result, from:

  - a $C(t)$ that is enhanced in $t \sim [0.4, 1.5]\,\text{fm}$
  - also probably for $t \gtrsim 1.5\,\text{fm}$
  - w/ possible suppression for $t \lesssim 0.4\,\text{fm}$ (mainly based on preliminary $\delta(\Delta_{\text{had}}^{(5)}\alpha)$)

  On data-driven side, could be explained by:

  - enhancing measured R-ratio around ρ-peak
  - or in any larger interval including ρ-peak
- Lattice and measured R-ratio correlations critical for drawing such conclusions
- Conclusions limited by uncertainties and correlations on lattice and data-driven results

Conclusions

- Important to check that uncertainties on uncertainties and correlations do not spoil picture, especially for inverse problem

  $\rightarrow$ checked here for lattice stat and syst uncertainties

  $\rightarrow$ must do so for measured R-ratio uncertainties

- Also important not to share results between 2 approaches before they are final (mutual blinding)

- W/ more HVP observables, many generalizations possible, also including Φ constraints (e.g. Colangelo et al '20)

- However, limit on independent HVP observables in data-driven and lattice approaches (not shown)

- Same methods can be used to combine determinations of lattice and data-driven results for HVP observables, once differences are understood

- No problems w/ EWPO fits in case of 3-observable comparisons (not shown)

Some references to related work on HVP

- Windows proprosed in RBC/UKQCD arXiv:1801.07224 . . .
- . . . discussed in context of future detailed comparisons in Colangelo et al arXiv:2205.12963
- Consequences of rescaling of measured R-ratio studied in Crivellin et al arXiv:2003.04886, Keshavarzi et al arXiv:2006.12666, de Rafael arXiv:2006.13880, Malaescu et al arXiv:2008.08107
- Consequences of lattice $a_\mu^{\text{LO-HVP}}$ on $\pi^+\pi^-$ contributions to R-ratio w/ Φ constraints in Colangelo et al arXiv:2010.07943
- Use of Backus-Gilbert method for reconstructing smeared R-ratio from lattice $C(t)$ in Hansen et al arXiv:1903.06476, Alexandrou et al arXiv:2212.08467
- Proposal for comparing measured R-ratio and lattice $C(t)$ via spectral-width sumrules in Boito et al arXiv:2210.13677
- . . . (many other references for reconstructing spectral functions from lattice correlators)