RECENT DEVELOPMENTS
IN FORMAL THEORY

Towards all-loop scattering amplitudes

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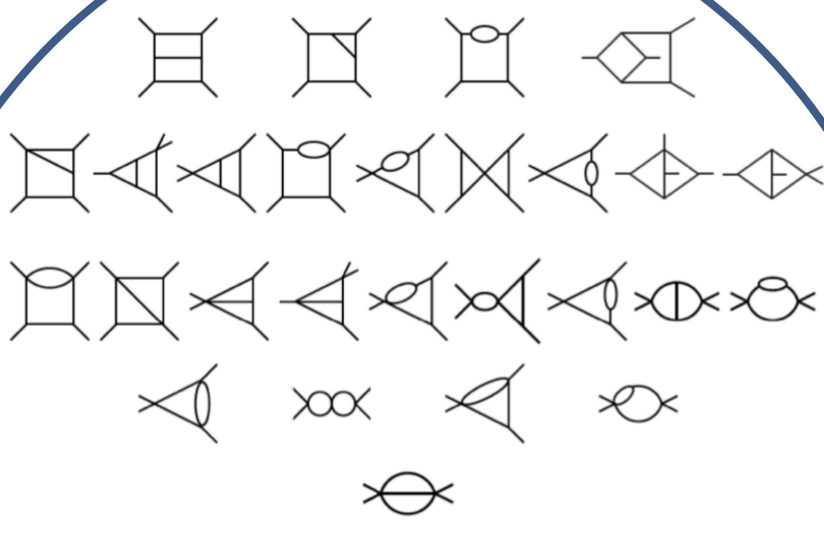
DPF-Pheno 2024, Pittsburgh, May 15, 2024

PROGRESS IN AMPLITUDES

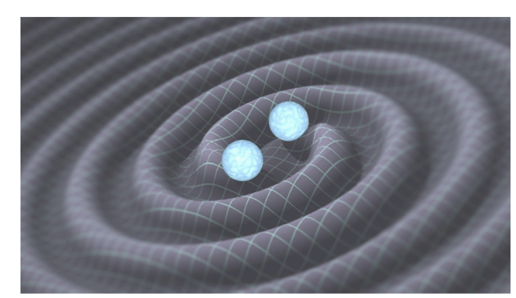
Wide field which spreads between formal theory, collider physics and newly also gravitational physics



precision physics, calculation of higher-loop Feynman integrals relevant for colliders

exploring new theoretical methods in "toy models": with potential general applications

applications to gravitational wave physics and cosmology



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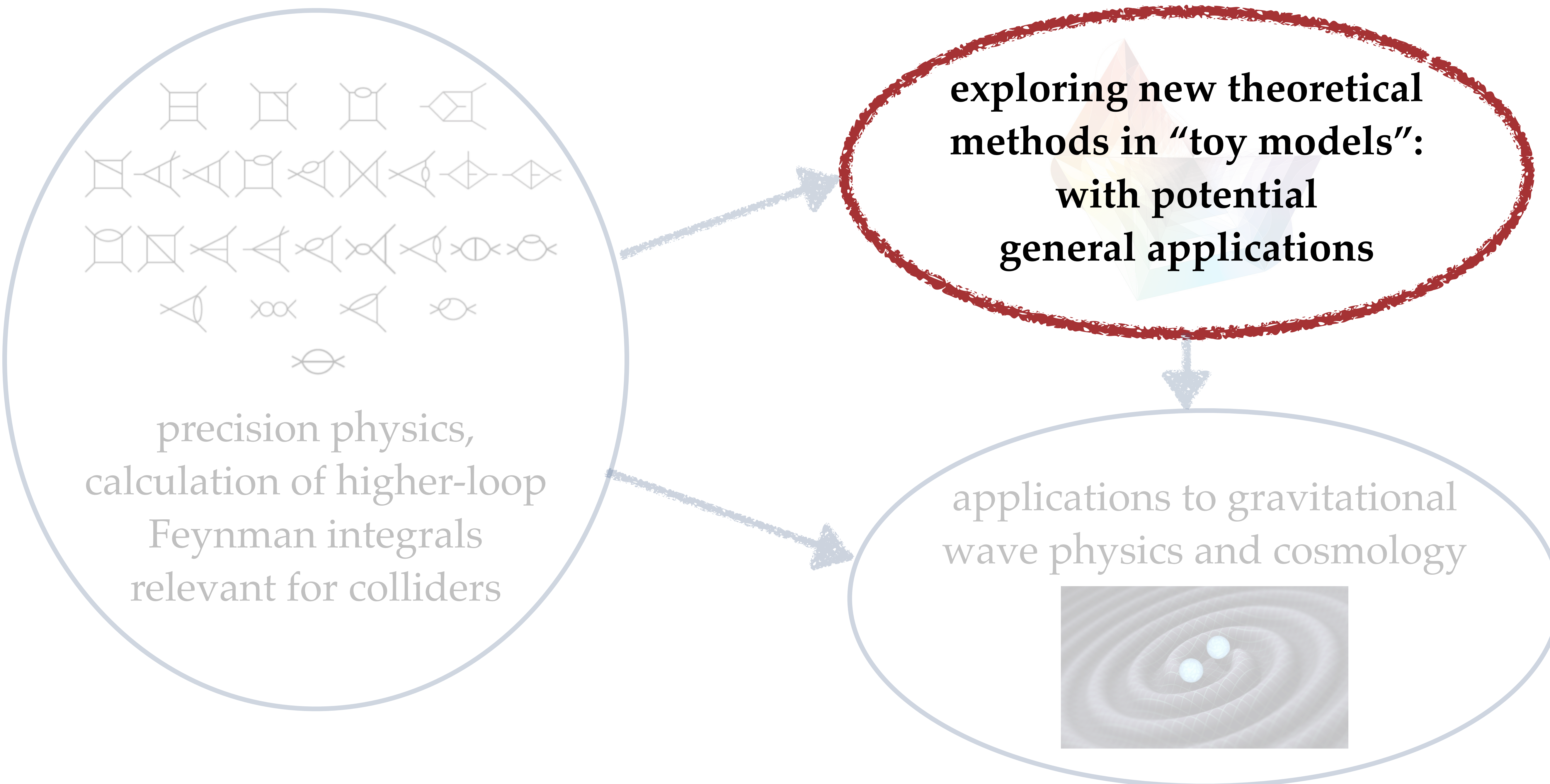
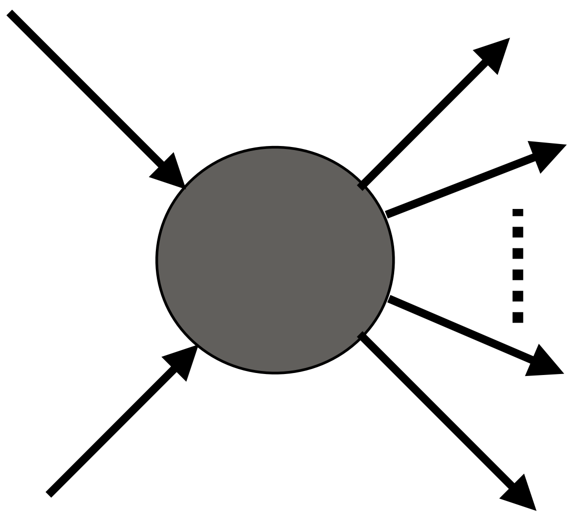


SCATTERING PROCESS

Scattering of elementary particles: predictions of outcomes



many possible outcomes

we can only talk about probabilities

$$\sigma = \int \mathrm{d}\Omega \, |A|^2$$

cross section

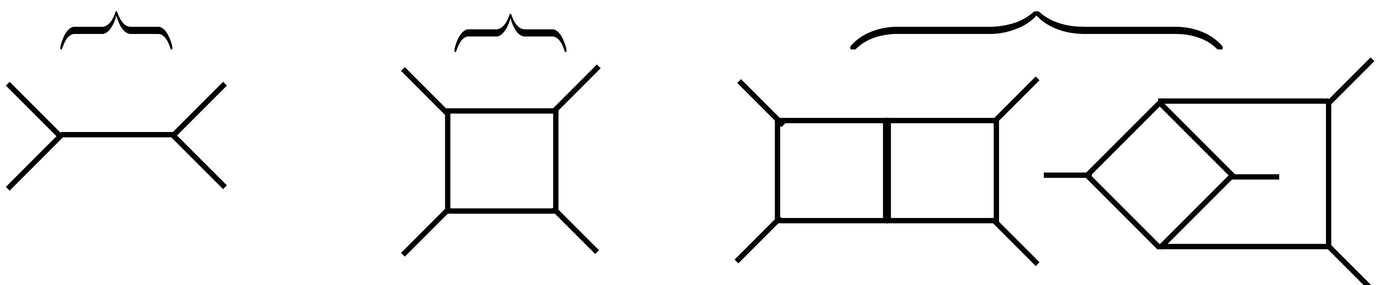
amplitude

Calculating amplitudes: at weak coupling in principle solved

PERTURBATION THEORY

Perturbation theory: series expansion in g

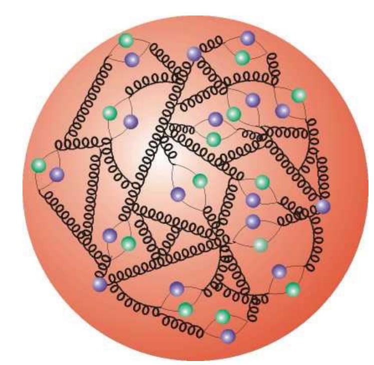
$$A_n \sim A_n^{\text{tree}} + g^2 A_n^{1-\text{loop}} + g^4 A_n^{2-\text{loop}} + g^6 A_n^{3-\text{loop}} + \ldots$$



asymptotic series

Feynman diagrams: diagrammatic organization

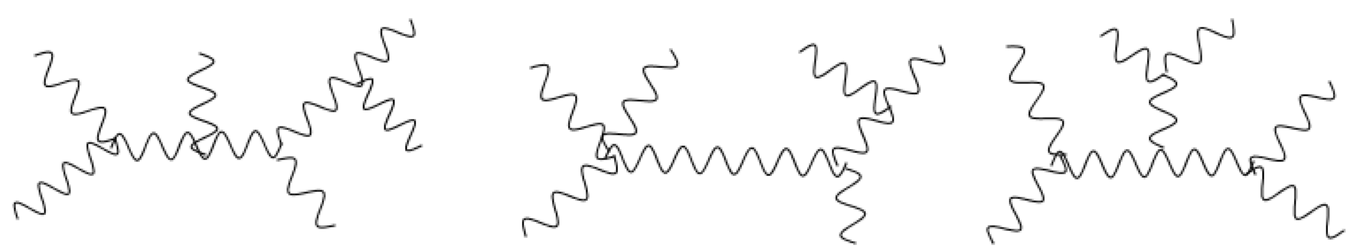
Main object of interest: amplitudes in QCD



needed for Standard model background

most complex part: **gluon amplitudes**

standard procedure:
Feynman diagrams



PRE-HISTORY

Calculation of gluon amplitudes: status in late 1970s

Most complicated process:
2->3 process = 5pt amplitude
at tree-level

Brute force calculation:
24 pages of result

$(k_1 \cdot k_4)(\epsilon_2 \cdot k_1)(\epsilon_1 \cdot \epsilon_3)(\epsilon_4 \cdot \epsilon_5)$

SSC approved in 1983: higher-point calculations needed

Energy 40 TeV: many gluons!
Next on the list: **6pt amplitude**
$gg \to gggg$

PARKE-TAYLOR FORMULA

Calculation completed in 1984

220 Feynman diagrams, 100 pages of algebra

(Parke, Taylor)

GLUONIC TWO GOES TO FOUR

Stephen J. Parke and T.R. Taylor
Fermi National Accelerator Laboratory
P.O. Box 500, Batavia, IL 60510
U.S.A.

ABSTRACT

The cross section for two gluon to four gluon scattering is given in a form suitable for fast numerical calculations.

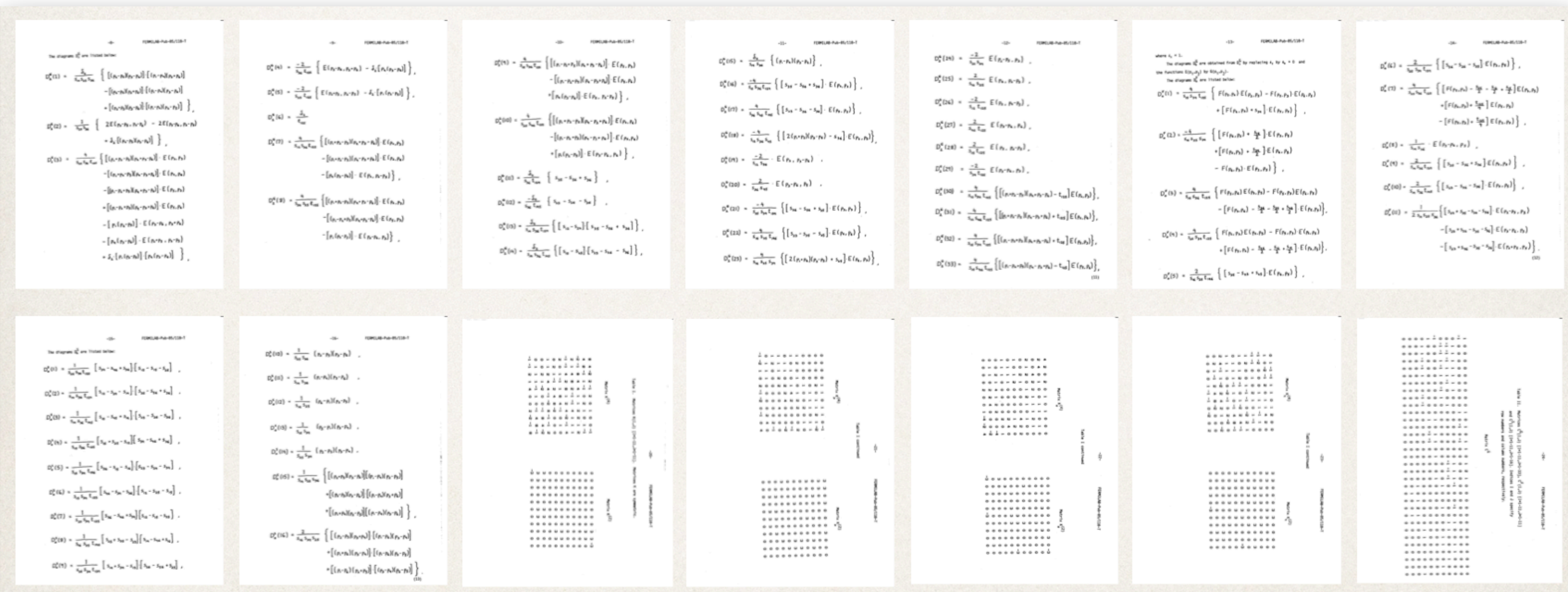
result compressed
to 14-page paper

PARKE-TAYLOR FORMULA

Calculation completed in 1984

220 Feynman diagrams, 100 pages of algebra

(Parke, Taylor)



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(Parke, Taylor)

Conclusion of the paper:

Our result has succesfully passed both these numerical checks.

Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.

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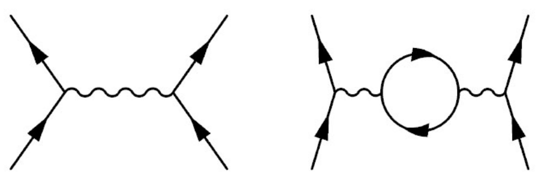
Within a year of the publication they found an extraordinary simplification:

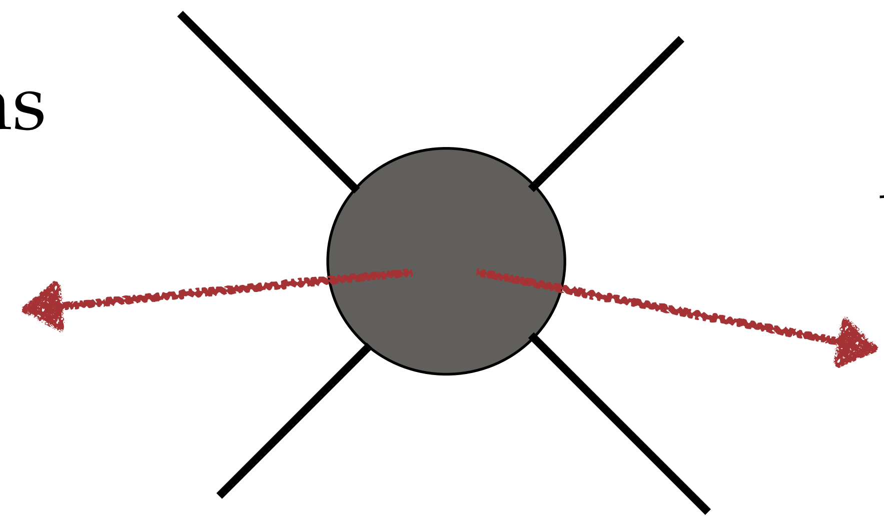
$$|A_6|^2 \sim \frac{(p_1 \cdot p_2)^3}{(p_2 \cdot p_3)(p_3 \cdot p_4)(p_4 \cdot p_5)(p_5 \cdot p_6)(p_6 \cdot p_1)}$$

MODERN METHODS

What is the scattering amplitude?

Feynman diagrams





Analytic S-matrix:
not successful

Modern methods use both:

- Expose simplicity of amplitudes
- Use perturbation theory

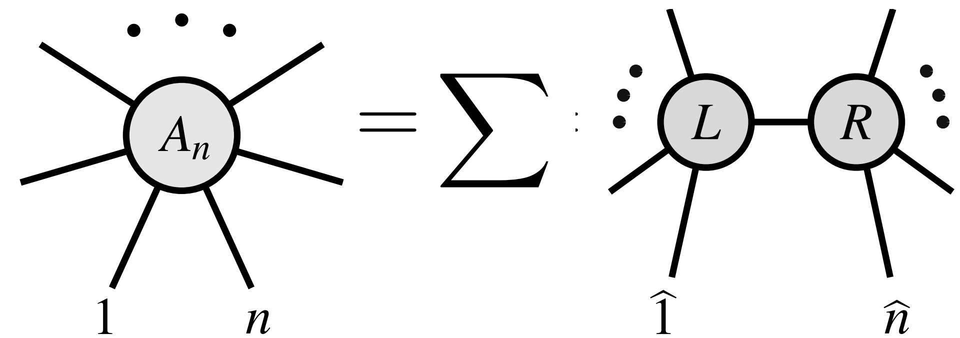
Many new methods: generalized unitarity, recursion relations, string-based methods,…

(Bern, Dixon, Kosower)

TREE-LEVEL AMPLITUDES

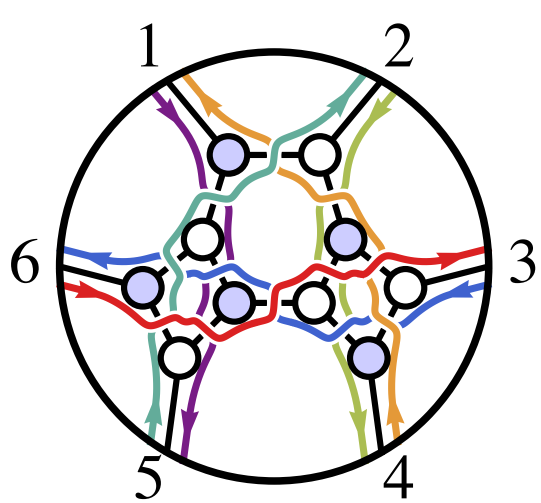
Very efficient recursion relations

	$gg \to 4g$	$gg \to 5g$	$gg \to 6g$
Feynman diagrams	220	2485	34300
Terms in recursion	3	6	20



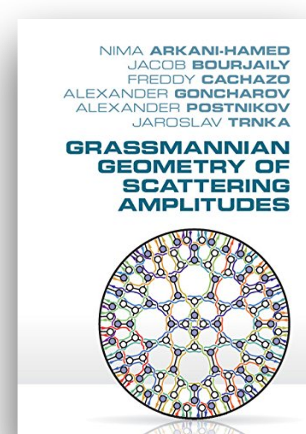
(Britto, Cachazo, Feng, Witten 2005)

For gluon amplitudes: very efficient implementation

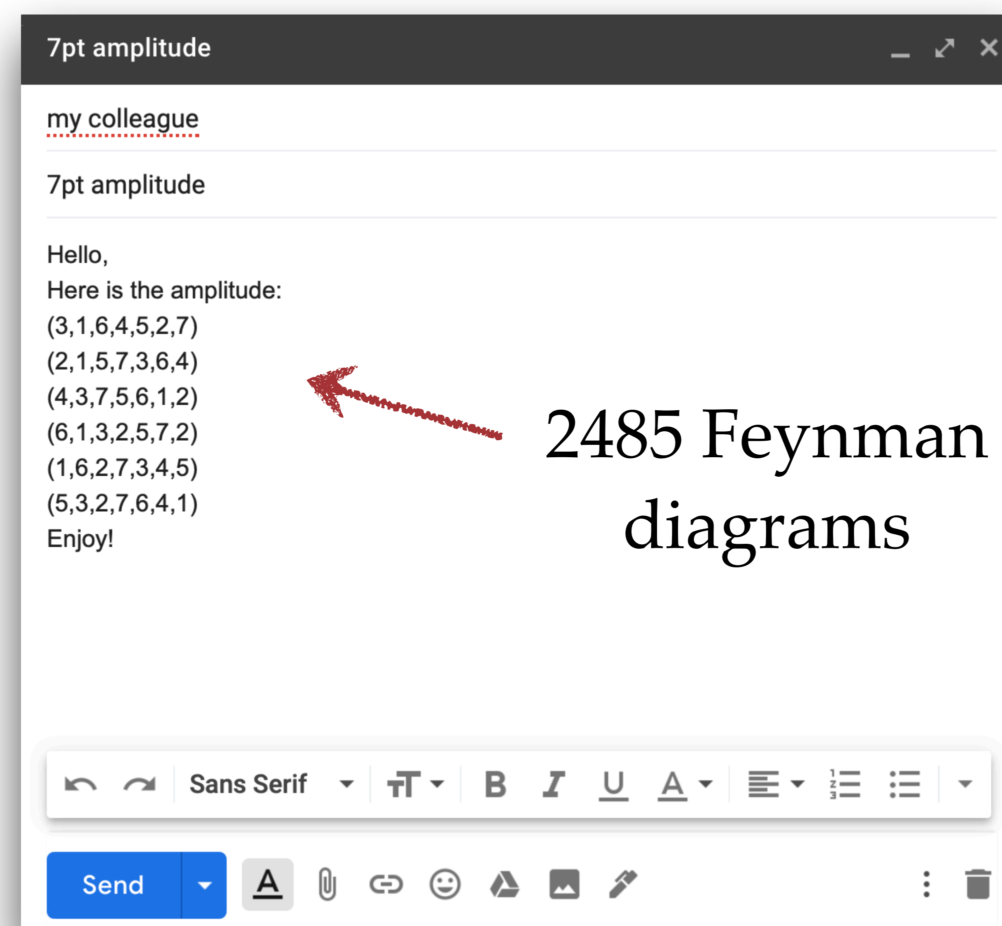


on-shell diagrams

positive Grassmannian

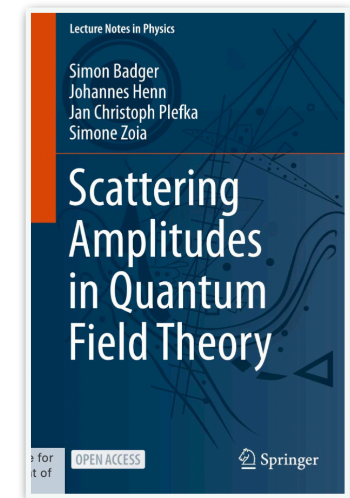


(Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, JT)



LOOP AMPLITUDES

Loop amplitude: complicated function of kinematics



$$A_n^{\ell-\text{loop}} = \int d^{4\ell}L \; \mathcal{I}_n^{\ell-\text{loop}}$$

transcendental functions

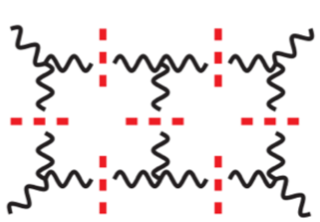
$\log, \mathrm{Li}_2, G_{a_1,a_2,\ldots a_m}, \ldots$

loop integration

rational function: "loop integrand"

Unitarity methods: re-organizaton of the expansion

$$\mathcal{I}_n^{\ell-loop} = \sum_k c_k B_k$$



calculate coefficients from cuts

basis integrals to calculate: critical to choose good basis

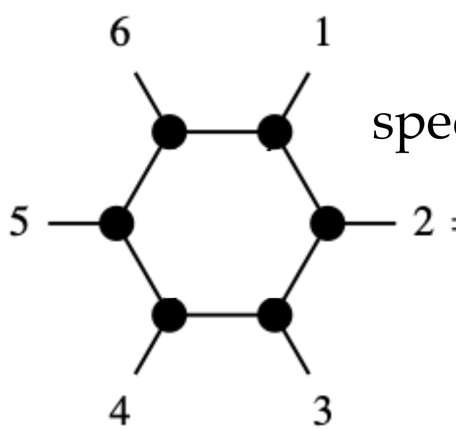
IBP methods, differential equations,…

LOOP INTEGRATION

Lesson learnt: convenient choice of integrals dramatically simplifies calculation, sometimes you get result almost for free

make analytic properties, UV and IR manifest

special numerator



$$= \mathrm{Li}_2(1-u_1) + \mathrm{Li}_2(1-u_2) + \mathrm{Li}_2(1-u_3) + \log(u_3)\log(u_1) - \frac{\pi^2}{3}$$

state of the art: 2-loop 5pt, 3-loop 4pt

Mathematical properties: transcendentality, bootstrap methods, cluster algebras, differential equations

Time-proven method: develop the methods in toy models, provides a useful organizational principle

SIMPLEST QFT

Planar N=4 super Yang-Mills theory: our favorite toy model

- maximal supersymmetry in D=4, superpartners to gluons, cancelations
- limit of an infinite number of colors, only planar diagrams contribute
- AdS/CFT correspondence: dual to supergravity

What do we loose?

- UV finite theory, no confinement

Why is it a good toy model?

- tree-level amplitudes of gluons: same as in QCD
- loop amplitudes simpler, convergent perturbative series
- past experience: new computational methods developed first here

HIGHER LOOP CALCULATIONS

Huge simplifications in planar N=4 SYM amplitudes

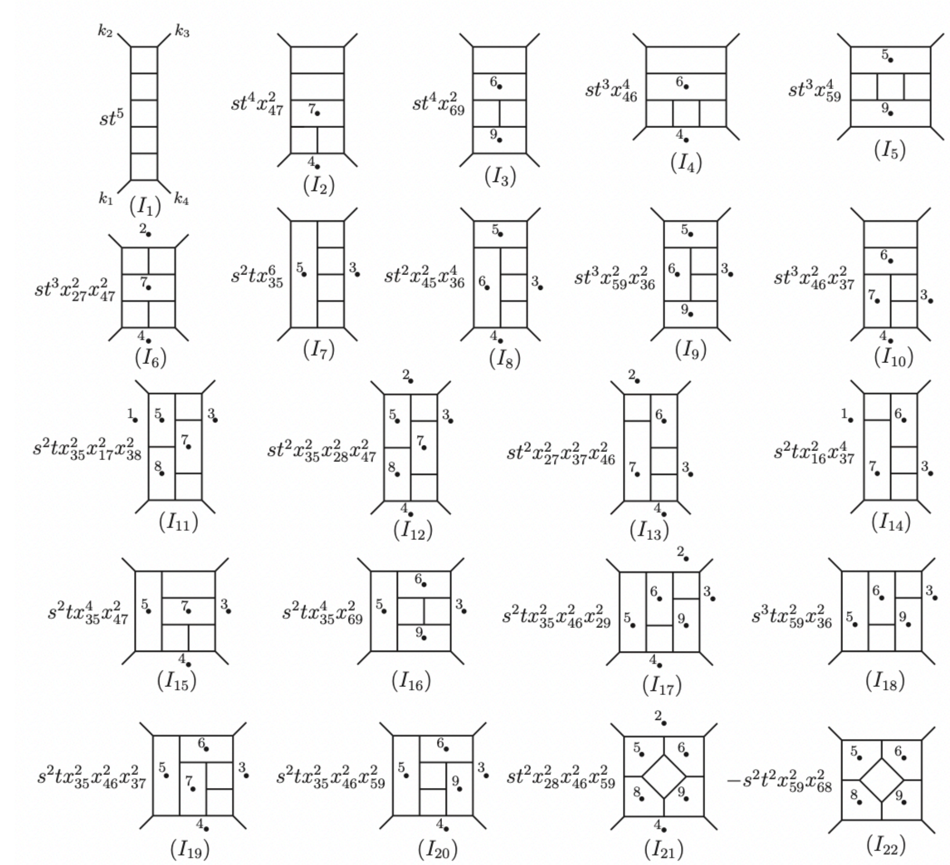
in some cases due to symmetries of the theory, otherwise unexplained

Integrands of 2->2 scattering amplitudes calculated up to 11-loops

(Bern, Dixon, Kosower 2005,…. Bourjaily, Heslop, Tran 2016)

number of Feynman diagrams grows very very fast, but even in the most compressed form the result is complicated

ℓ	number of planar DCI integrands
1	1
2	1
3	2
4	8
5	34
6	284
7	3,239
8	52,033
9	1,025,970
10	24,081,425
11	651,278,237



progress on integration using new techniques (dlog forms, differential equations)

SYMBOLS

Skip the integrand step, using the knowledge of the function space to construct the amplitude directly

Reduce information in the function to a **symbol**

(Goncharov, Spradlin, Vergu, Volovich 2009)

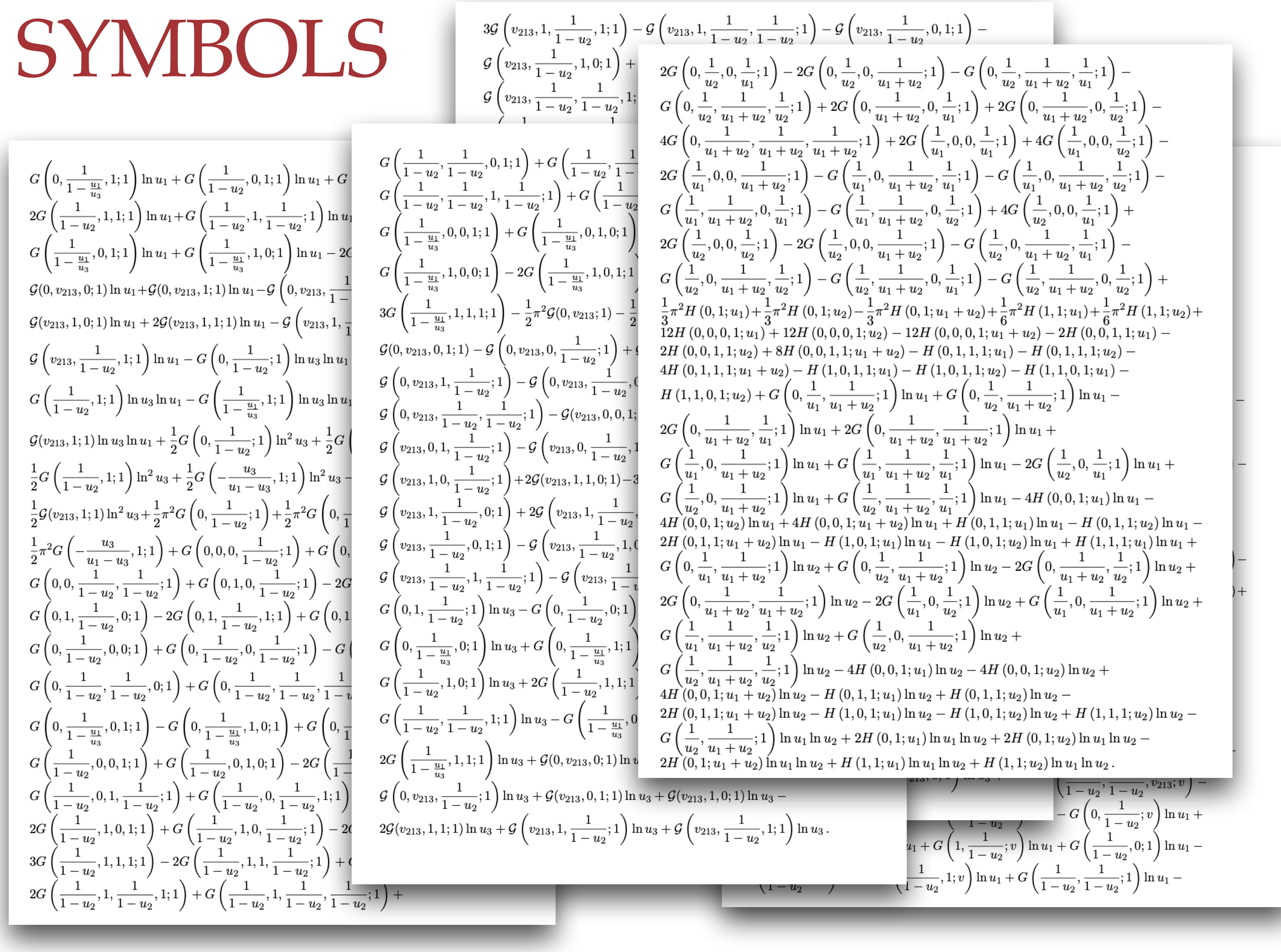
$$\mathcal{S}(f^{(k)}) = \sum_{\vec{\alpha}} \underbrace{\phi_{\alpha_1} \otimes \ldots \otimes \phi_{\alpha_k}}$$ ← encodes branch cuts

$$\{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$$ ← symbol letters

This originated from another surprising simplification

In 2009 Del Duca, Duhr and Smirnov calculated a certain 2-loop 6pt amplitude in the planar N=4 SYM theory, 30 pages of result

SYMBOLS



SYMBOLS

The 30 pages result simplifies into a single line formula

$$R_6^{(2)}(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}.$$

This can be seen when writing down the symbol

$$\begin{aligned}&\mathcal{S}(R_6^{(2)})\\ &= -\frac{1}{8}\Big\{\Big[u\otimes(1-u)\otimes\frac{u}{(1-u)^2} + 2\,(u\otimes v + v\otimes u)\otimes\frac{w}{1-v} + 2\,v\otimes\frac{w}{1-v}\otimes u\Big]\otimes\frac{u}{1-u}\\ &\quad + \Big[u\otimes(1-u)\otimes y_u y_v y_w - 2\,u\otimes v\otimes y_w\Big]\otimes y_u y_v y_w\Big\} + \text{ permutations}\,,\end{aligned}$$

Write the symbol for 30 pages, almost everything cancels, and then promote the symbol to a function

Hexagon bootstrap: write down consistent symbol

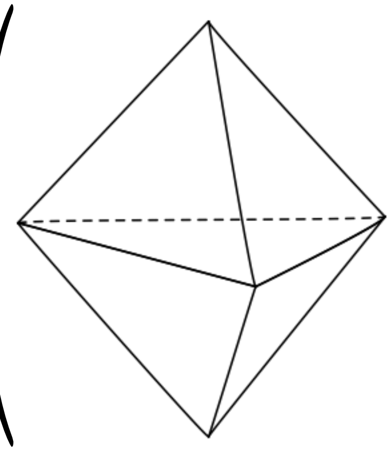
(Dixon and collaborators) results up to 8-loops

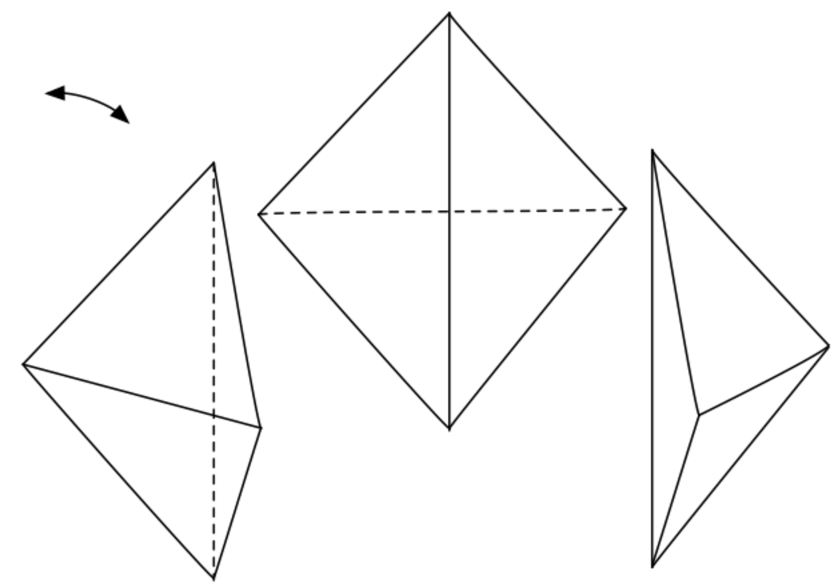
AMPLITUHEDRON

Geometric picture for tree-level amplitudes and loop integrands

(Arkani-Hamed, JT 2013)

$$\mathcal{I}_n^{\ell-\text{loop}} = \Omega\left(\quad \right)$$





"**Volume**": differential form with logarithmic singularities on the boundaries of the space

Amplituhedron defined by a set of inequalities in the kinematical space

Different ways to express amplitudes (Feynman diagrams, unitarity methods, recursion relations,…) correspond to different triangulations

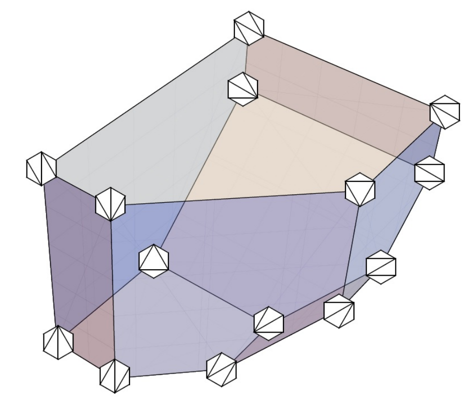
Dynamics of particle scattering -> **static geometry**

POSITIVE GEOMETRY

Amplituhedron is one example of positive geometries

Geometric picture for ϕ^3 amplitudes - **associahedron**

(Arkani-Hamed, Bai, He, Yan 2017)



Later also generalized to loops - **surfacehedron**

(Arkani-Hamed, Cao, Dong, He, Figueiredo 2023)

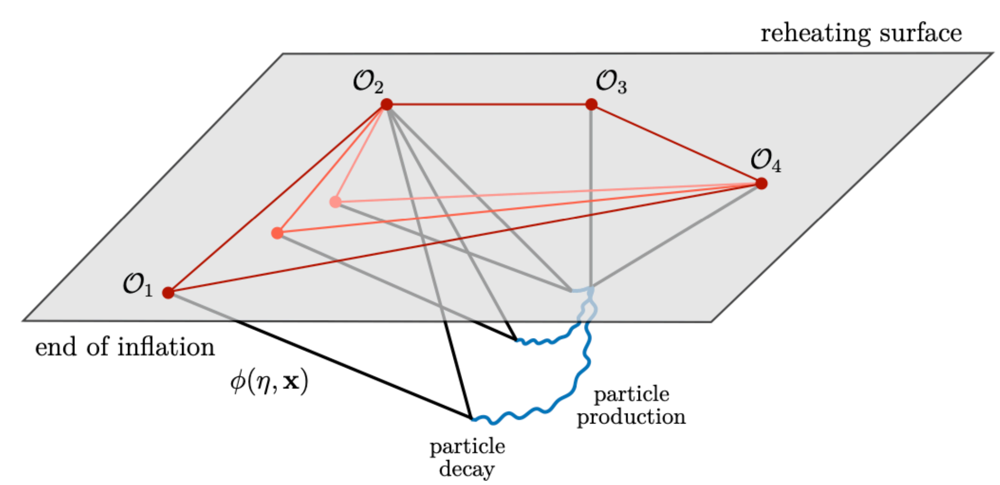
hidden zeroes, also for NLSM and other theories

(see talk by Tonnis ter Veldhuis on Tuesday)

More examples of positive geometries: **cosmological correlators**

$$\lim_{E_{\rm tot}\to 0} \mathcal{C}_n = A_n$$

reproduce amplitudes on the energy poles



Positive geometries seem to be a more general language for amplitudes

TOWARDS ALL-LOOPS IN PLANAR N=4 SYM THEORY

COLLABORATORS

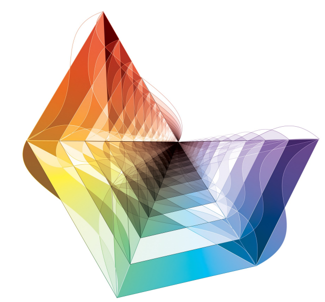
I will review the progress more broadly but my work is done with

Nima Arkani-Hamed

Johannes Henn

JHEP 03 (2022) 108

based on an earlier work on the Amplituhedron



Taro Brown

Umut Oktem

Shruti Paranjape

2312.17736

+

Lance Dixon

in-progress

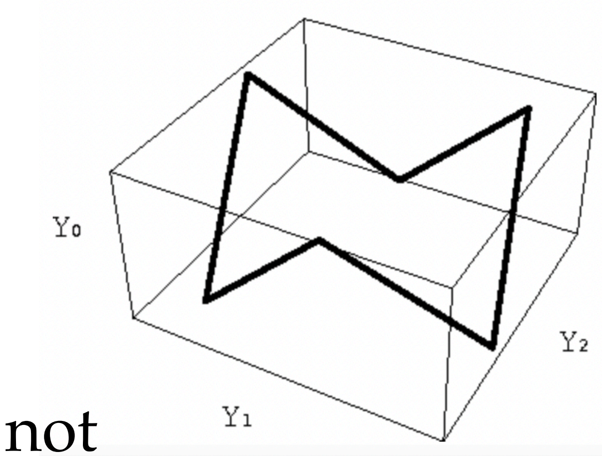
MAIN GOAL

Both the integrands and symbols get eventually more complicated at higher loop order -> new ideas needed

We want to calculate amplitudes to all loops: **full non-perturbative result** (no non-perturbative effects in this theory)

We have indirect evidence some simplicity must be there

- calculation of the amplitude at strong coupling from AdS/CFT
- integrability predicts the **gamma cusp**



↓

exponentiation of IR divergencies

$$\ln \mathcal{M}_n = \frac{\gamma_{\rm cusp}}{\epsilon^2} + \ldots$$

conjectured to all loops but can not be derived from amplitudes

(in dim-reg)

AMPLITUDE LOGARITHM

Instead of the amplitude, we consider a logarithm for 4pt amplitude

$$\ln \mathcal{M}_4 = \frac{\gamma_{\text{cusp}}}{\epsilon^2} + \ldots$$ ← very mildly divergent

$$\gamma_{\text{cusp}} = \underbrace{4g^2 - 8\zeta_2 g^4 + \cdots}_{\text{weak coupling}} = \underbrace{2g - \frac{3\log 2}{2\pi} + \ldots}_{\text{strong coupling}}$$

$$\ln \mathcal{M}_4 = \sum_{\ell=0}^{\infty} g^{2\ell}\, \widetilde{M}_4^{\ell-\text{loop}}$$ integrand for the logarithm $$\widetilde{M}_4^{\ell-\text{loop}} = \int d^{4\ell}L\, \widetilde{\mathcal{I}}_4^{\ell-\text{loop}}$$

Integrate over all loops except one: **IR finite function**

$$\mathcal{F}_\ell(z) = \int d^{4(\ell-1)}L\, \widetilde{\mathcal{I}}_4^{\ell-\text{loop}} \longrightarrow \mathcal{F}(g,z) = \sum_{\ell=0}^{\infty} g^{2\ell} F_\ell(z)$$

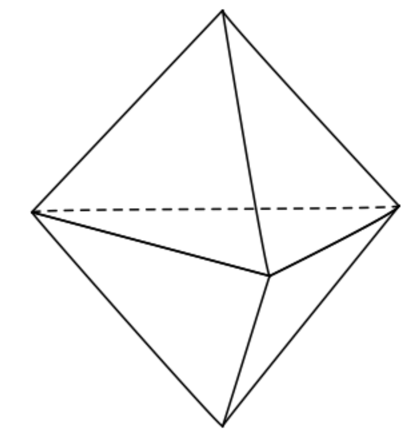
amplitude-like function of one kinematical variable

contains the last unintegrated loop

AMPLITUDE LOGARITHM

Fortunately, there is an Amplituhedron also for the logarithm

geometric definition for the integrand $\widetilde{\mathcal{I}}_4^{\ell-\text{loop}}$



Triangulate the geometry and find $\widetilde{\mathcal{I}}_4^{\ell-\text{loop}}$ **too difficult**

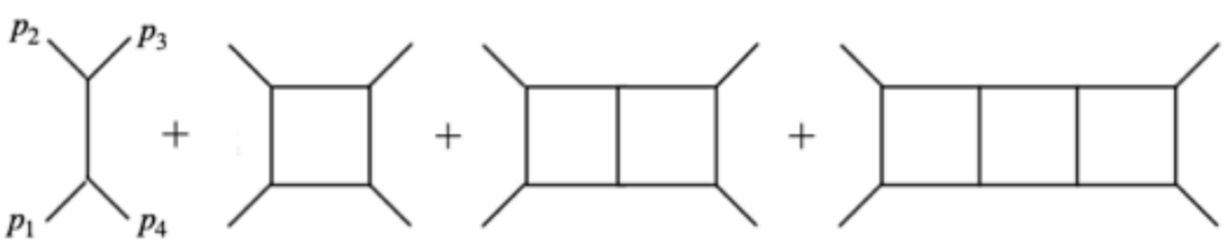
Integrate the object to get $\mathcal{F}_\ell(z)$ **too difficult**

Are we completely stuck here?

Can we keep only "simple" terms in the triangulation we can calculate?

Tried before with Feynman diagrams: **ladder resummation**

(Broadhurst, Davydychev 2010)



$+ \ldots \sim e^{-g}$ for $g \gg 1$

exponentially suppressed vs linear growth

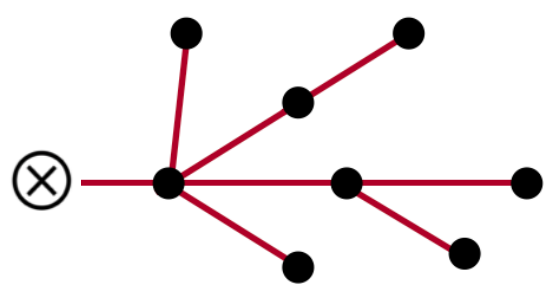
very bad approximation

APPROXIMATION

(Arkani-Hamed, Henn, JT 2021)

We use a specific triangulation in terms of **negative geometries**

only keep simplest pieces



"tree graphs" in the loop space

satisfy a differential equation

$$(\partial_z^2 + g^2)\mathcal{F}_{\text{tree}}(g,z) = 0$$

which we can solve

$$\mathcal{F}_{\text{tree}}(g,z) = \frac{A^2}{g^2}\frac{z^A}{(z^A+1)^2} \quad \text{where} \quad \frac{A}{2g\cos\frac{\pi A}{2}} = 1$$

Easy to expand at strong coupling:

$$\mathcal{F}_{\text{tree}}(g,z) = -\frac{z}{(1+z)^2} + \mathcal{O}\left(\frac{1}{g}\right)$$

→ misses the leading term

→ has 1/g expansion

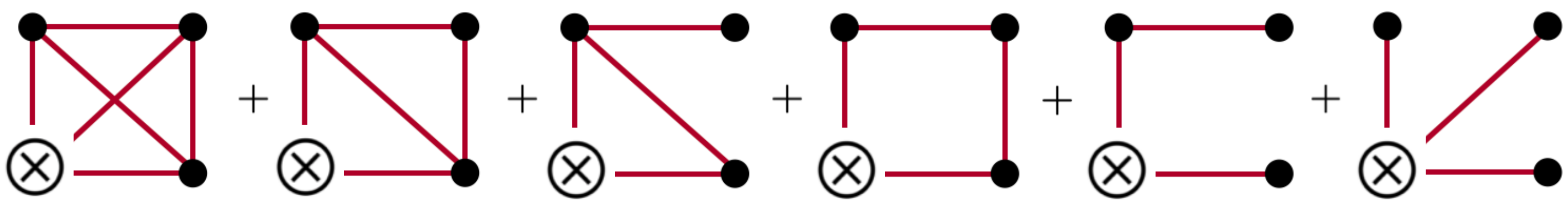
$$\gamma_{\text{cusp}} \to \begin{cases} 2g - \frac{3\log 2}{2\pi} + \cdots & \to \text{exact} \\ \frac{8}{\pi}g - 1 + \ldots & \to \text{our approximation} \end{cases}$$

qualitatively correct behavior at strong coupling

LOOP OF LOOPS EXPANSION

(Brown, Oktem, Paranjape JT 2023)

Systematic expansion in terms of "negative geometries"



each vertex represents a loop

more internal cycles
= more complicated

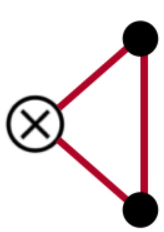
Solved for integrands for all one-cycle geometries

still needs to be integrated / find the differential equation

(Brown, Dixon, Oktem, Paranjape, JT in progress)

$$= -\frac{1}{12}\left[\pi^2 + \log^2 z\right] \times \left[5\pi^2 + \log^2 z\right]$$

} simple "tree" geometry



$$\begin{aligned} = &-\frac{1}{6}\log^4 z + \log^2 z\left[-\frac{2}{3}\mathrm{Li}_2\left(\frac{1}{z+1}\right) - \frac{2}{3}\mathrm{Li}_2\left(\frac{z}{z+1}\right) + \frac{\pi^2}{9}\right] \\ &+ \log z\left[4\mathrm{Li}_3\left(\frac{z}{z+1}\right) - 4\mathrm{Li}_3\left(\frac{1}{z+1}\right)\right] - \frac{2}{3}\left[\mathrm{Li}_2\left(\frac{1}{z+1}\right) + \mathrm{Li}_2\left(\frac{z}{z+1}\right) - \frac{\pi^2}{6}\right]^2 \\ &- \frac{8}{3}\pi^2\left[\mathrm{Li}_2\left(\frac{1}{z+1}\right) + \mathrm{Li}_2\left(\frac{z}{z+1}\right) - \frac{\pi^2}{6}\right] - 8\mathrm{Li}_4\left(\frac{1}{z+1}\right) - 8\mathrm{Li}_4\left(\frac{z}{z+1}\right) - \frac{\pi^4}{18} \end{aligned}$$

} complicated "one-loop" geometry

SUMMARY

Huge progress on all fronts of amplitudes field: progress in the loop integrations, applications to gravitational waves and also planar N=4 SYM amplitudes and positive geometries

All-loop resummation: defined an IR finite function containing 4pt scattering amplitudes, used negative geometries to approximate it to all loops -> surprisingly good strong coupling behavior

Future: more orders in loops of loops expansion, geometry at strong coupling? Can we use the same set of ideas for amplitudes in other theories?


Thank you!