An Area Law for Entanglement Entropy in Particle Scattering

Based on 2405.08056 and 240X.XXXXX with Ian Low

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Introduction

- Entanglement is a key phenomenon in quantum theories; characterized by entropy
- Area laws exist for entropy; may be related to holography
- We consider entanglement entropy in $2 \to 2$ scattering

Wave packets

Ingredient 1: Distinguishable particles

Consider the elastic scattering of AB $\to$ AB. The initial state:

$$|\text{in}\rangle = |\psi_\text{A}\rangle \otimes |\psi_\text{B}\rangle$$

The density matrix: $\rho^\text{i} = |\text{in}\rangle\langle\text{in}|$: We need $\text{Tr}\,\rho^\text{i} = 1$

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$$|\psi\rangle = \int_p \psi(p)|p\rangle, \ \langle\psi|\psi\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3}\, |\psi(p)|^2 = 1$$

with $\langle p|q\rangle = (2\pi)^3\, 2E_p\, \delta^3(\vec{p} - \vec{q})$.

The final state

Ingredient 2: Unitary S-matrix

- $|\text{out}\rangle = S|\text{in}\rangle$, $S = 1 + iT$. $S^\dagger S = 1$, $2\,\text{Im}\,T = T^\dagger T$.

$$\langle\{k_\text{f}\}|T|\{k_\text{i}\}\rangle = (2\pi)^4\delta^4\left(\sum k_\text{f} - \sum k_\text{i}\right) M(\{k_\text{i}\};\{k_\text{f}\})$$

- The "complete" final state density matrix is $|\text{out}\rangle\langle\text{out}|$

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Probabilities:

$$\mathcal{P}_\text{tot} = \langle\text{in}|T^\dagger T|\text{in}\rangle,\ \mathcal{P}_\text{el} = \langle\text{in}|T^\dagger P_\text{AB} T|\text{in}\rangle,\ \mathcal{P}_\text{inel} = \langle\text{in}|T^\dagger(1 - P_\text{AB})T|\text{in}\rangle$$

The plane wave limit

Ingredient 3: The plane wave limit

We require $\psi_{\mathsf{A/B}}(p)$ to

- Be centered around $\vec{k}_{\mathsf{A/B}}$. In the CoM frame, $\vec{k}_{\mathsf{A}} = -\vec{k}_{\mathsf{B}} \equiv \vec{k}$
- Have a width of δ_{p} in momentum space
- Take the plane wave limit: $\delta_{\mathsf{p}} \ll |\vec{k}|$, and expand around small $\delta_{\mathsf{p}}/|\vec{k}|$

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Probabilities ("$\doteq$" means keeping the leading piece in $\delta_{\mathsf{p}}/|\vec{k}|$):

$$\mathcal{P}_{\mathsf{tot}} \doteq I_0(|\vec{k}|)\sigma_{\mathsf{tot}},\ \mathcal{P}_{\mathsf{el}} \doteq I_0(|\vec{k}|)\sigma_{\mathsf{el}},\ \mathcal{P}_{\mathsf{inel}} \doteq I_0(|\vec{k}|)\sigma_{\mathsf{inel}}$$

where

$$I_0(|\vec{k}|) = 4|\vec{k}|\sqrt{s}\int_{p_1,p_2,q_1,q_2} \psi_{\mathsf{A}}(p_1)\psi_{\mathsf{B}}(p_2)\psi^*_{\mathsf{A}}(q_1)\psi^*_{\mathsf{B}}(q_2) \times (2\pi)^4\delta^4(q_1+q_2-p_1-p_2)$$

The uniform wave packet

$\mathcal{P} \doteq I_0(|\vec{k}|)\sigma$ is small in the plane wave limit

- $I_0(|\vec{k}|) \sim {\delta_\mathsf{p}}^2$ has the dimension of area^{-1}
- The size of $I_0(|\vec{k}|)$ depends on the overlap of the wave functions in the transverse directions in position space

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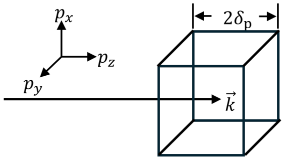

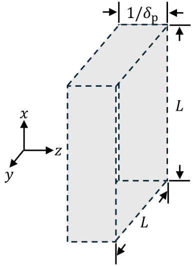

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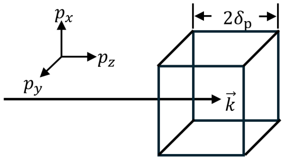

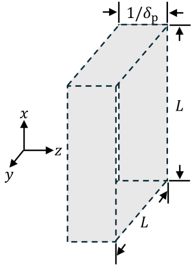

- Roughly uniform in the transverse direction: $L \gg 1/\delta_\mathsf{p}$, with $\mathcal{O}(1/(\delta_\mathsf{p} L))$ corrections
- For head-on collisions, $I_0(|\vec{k}|) \doteq 1/L^2$, $\mathcal{P} \doteq \sigma/L^2$

Final state entropy

For the elastic scattering AB $\to$ AB, $|\text{out}\rangle_{\text{el}} = P_{\text{AB}}|\text{out}\rangle$

- $\text{Tr}(|\text{out}\rangle\langle\text{out}|) = \langle\text{in}|S^\dagger S|\text{in}\rangle = 1$
- Properly normalized ρ^{f}:

$$\rho^{\text{f}} = \frac{1}{1-\mathcal{P}_{\text{inel}}}|\text{out}\rangle_{\text{el}}\,{}_{\text{el}}\langle\text{out}|$$

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The key result

For the linear entropy $\mathcal{E}_2(\rho) = 1 - \text{Tr}\,\rho^2$, the final entanglement entropy is

$$\mathcal{E}_2^{\text{f}} \doteq 2I_0(|\vec{k}|)\sigma_{\text{el}} \doteq 2\mathcal{P}_{\text{el}}$$

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For uniform wave packets,

$$\mathcal{E}_2^{\text{f}} \doteq 2\frac{\sigma_{\text{el}}}{L^2}$$

For Tsallis and Rényi entropies

$$\mathcal{E}_n^{\text{f}} \doteq \frac{n}{n-1}I_0(|\vec{k}|)\sigma_{\text{el}} \doteq \frac{n}{n-1}\mathcal{P}_{\text{el}}$$

Generalization

Consider 3 type of quantum numbers:

- What distinguishes A and B, which we will call "charge"
- Kinematics, described by wave packets
- Other discrete quantum numbers $f_{\mathsf{A/B}}$ ("flavors"), labelled by i and $\bar{i}$

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The initial state:

$$|\mathrm{in}\rangle = |\psi_{\mathsf{A}}\rangle \otimes |i\rangle \otimes |\psi_{\mathsf{B}}\rangle \otimes |\bar{i}\rangle,$$

with $\langle i|j\rangle = \delta^{ij}$, $\langle \bar{i}|\bar{j}\rangle = \delta^{\bar{i}\bar{j}}$

Cutting the cake

	A	B
p	p_A	p_B
f	f_A	f_B

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i, $\bar{i}$ to any j, $\bar{j}$

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Discussions

Ingredients

1. Distinguishable particles
2. Unitary S-matrix
3. The plane wave limit

- Entanglement entropy as cross section/probability
- General and non-perturbative (in coupling)
- An area law for a 2-body system: What is the boundary? Holography?
- Other entropies: Specific momenta? Other kinds of final states?
- Beyond the plane wave limit: Constraints from entropy?