Two different types of series expansions valid at strong coupling

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DPF-Pheno Meeting 2024, May 13-17, Pittsburgh, PA

Perturbative expansion and asymptotic series

- The path integral in quantum mechanics and quantum field theory is typically expanded perturbatively in powers of of the coupling. This is well-known to yield an asymptotic series.
- An asymptotic series can still be useful at weak coupling but fails completely at strong coupling.
- We study two different types of series expansions: the first is the usual one in powers of the coupling but the second is a series expansion of the quadratic part (the interaction is left alone).
- The first is an asymptotic series but the second is an absolutely convergent series that is valid at strong coupling.
- We revisit the first series, identify why it diverges and fix the problem to obtain an absolutely convergent series.

The prototypical example: a one-dimensional integral

In non-perturbative studies, the prototypical example used to illustrate how perturbative expansions yield an asymptotic series is the following one-dimensional integral:

$$I = \int_{-\infty}^{\infty} e^{-a\,x^2 - \lambda\,x^4}\,dx$$

where a and λ are positive real constants. The above integral has an exact analytical expression given by

$$I = \frac{1}{2}\,e^{\frac{a^2}{8\lambda}}\,\sqrt{\frac{a}{\lambda}}\,\text{BesselK}\left[\frac{1}{4}, \frac{a^2}{8\lambda}\right]$$

where BesselK$[n, z]$ is the modified Bessel function of the second kind.

First series
Expansion of quartic term in powers of coupling λ

A series expansion in powers of λ of the quartic term to order n is given by

$$F_1(n) = \int_{-\infty}^{\infty} dx\, e^{-a\,x^2} \sum_{j=0}^{n} \frac{(-\lambda\, x^4)^j}{j!} = \sum_{j=0}^{n} \frac{(-\lambda)^j}{j!} \int_{-\infty}^{\infty} dx\, e^{-a\,x^2} x^{4\,j}$$

$$= \sum_{j=0}^{n} \frac{(-1)^j}{j!} \left(\frac{\lambda}{a^2}\right)^j a^{-1/2}\, \Gamma[1/2 + 2j]\,.$$

This is an asymptotic series since $\lim_{n\to\infty} \left(\frac{\lambda}{a^2}\right)^n \Gamma[1/2 + 2n]/n! \to \infty$.

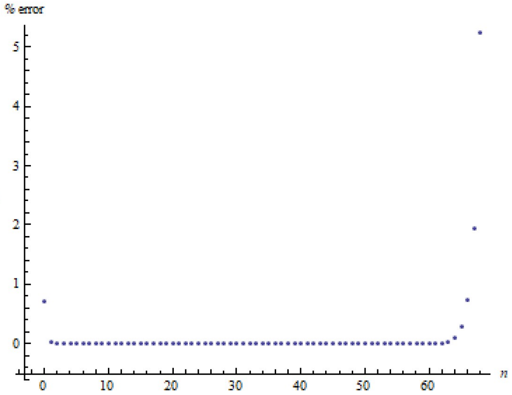
We plot $F_1(n)$ for three values of λ: 0.01, 0.1 and 1.0 (setting $a = 1$).

For each λ, we present a table comparing $F_1(n)$ to the exact analytical value. All values are quoted to eight digit accuracy.

Case $\lambda = 0.01$: weak coupling

Plateaus to the correct value before diverging $\Rightarrow$ reliable perturbative expansion at weak coupling

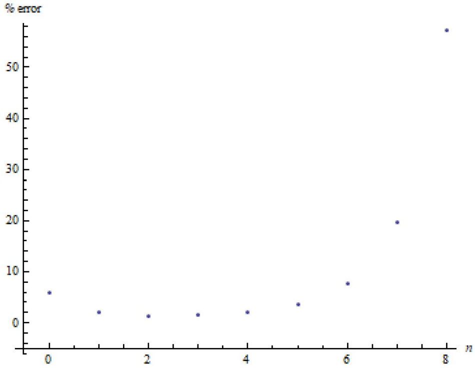
λ=0.01 Exact value is **I= 1.7596991**		
n	**F_1(n)**	**% error**
0	1.7724539	0.73
1	1.7591604	0.031
2	1.7597420	2.5×10^{-3}
3	1.7596941	2.9×10^{-4}
4	1.7596999	4.0×10^{-5}
5	1.7596990	8.2×10^{-6}
6-51	**1.7596991**	**0**
60	1.7597507	2.9×10^{-3}
67	1.7254544	1.95
70	2.4570073	40
80	39560.681	> 100
90	$\sim 10^{10}$	> 100
200	$\sim 10^{92}$	> 100



Case $\lambda = 0.1$: intermediate value

No plateau region but dips close to correct value early on before diverging $\Rightarrow$ less reliable

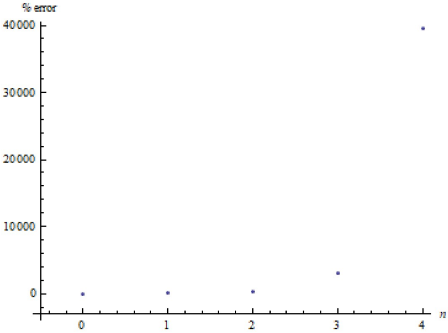
λ=0.1 Exact value is **I = 1.6740859**		
n	**$F_1(n)$**	**% error**
0	1.7724539	5.88
1	1.6395198	2.06
2	1.6976785	1.41
3	1.6496976	1.46
4	1.7081743	2.04
5	1.6137344	3.61
6	1.8037946	7.75
7	1.3456137	19.6
8	2.6328157	57.3
9	-1.4969574	189
10	13.401199	700
11	-46.293005	>1000
12	216.73458	>1000
13	-1047.3153	>1000



Case $\lambda = 1$: strong coupling

Diverges early on (never close to correct value) $\Rightarrow$ completely unreliable

λ=1.0 Exact value is **I = 1.3684269.**		
n	**$F_1(n)$**	**% error**
0	1.7724539	29.5
1	0.44311346	67.6
2	6.2589777	357
3	-41.721902	2.95×10^3
4	543.04507	3.96×10^4
5	-8900.9415	6.50×10^5
6	181159.29	1.32×10^7
7	-4.4006498×10^6	3.22×10^8
8	1.2431955×10^8	9.08×10^9
9	-4.0054535×10^9	2.92×10^{11}



Second series
Expansion of the quadratic term

We perform a series expansion of the quadratic term in the original integral to order n. This yields

$$F_2(n) = \int_{-\infty}^{\infty} dx\, e^{-\lambda\, x^4} \sum_{j=0}^{n} \frac{(-a\, x^2)^j}{j!} = \sum_{j=0}^{n} \frac{(-a)^j}{j!} \int_{-\infty}^{\infty} dx\, e^{-\lambda\, x^4}\, x^{2\,j}$$

$$= \sum_{j=0}^{n} \frac{(-1)^j}{j!} \left(\frac{a^2}{\lambda}\right)^{j/2} \frac{1}{2\,\lambda^{1/4}}\, \Gamma[1/4 + j/2\,]\,.$$

Note that this a series expansion in powers of the inverse coupling λ.

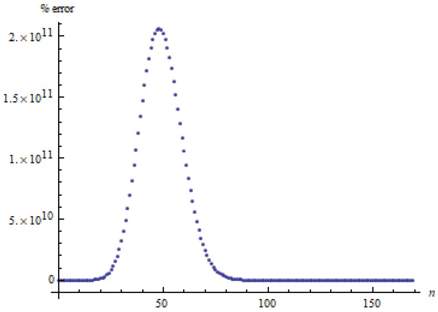
The series is absolutely convergent (ratio test): $\lim_{n\to\infty} \frac{\Gamma[1/4+(n+1)/2]}{(n+1)\,\Gamma[1/4+n/2]} \to 0$.

Converges faster at strong coupling!

Case: $\lambda = 0.01$: weak coupling

Converges at weak coupling to the exact value but very slowly (at order $n = 159!$) $\Rightarrow$ not *convenient* to use at weak coupling

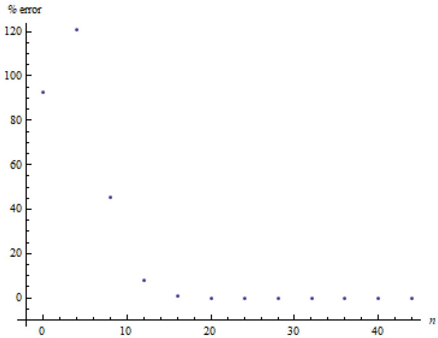
λ=0.01 Exact value is **I= 1.7596991**		
n	**$F_2(n)$**	**% error**
0	5.7325926	225
20	2.5103285×10^{7}	1.43×10^{9}
40	2.5963201×10^{9}	1.48×10^{11}
60	1.8581749×10^{9}	1.06×10^{11}
80	4.9648270×10^{7}	2.82×10^{9}
100	112774.14	6.41×10^{6}
120	37.346246	2.022×10^{3}
140	1.7618630	0.123
159	**1.7596991**	**0**
180	1.7596991	0
200	1.7596991	0
220	1.7596991	0



Case: $\lambda = 0.1$: intermediate value

Converges relatively quickly to the exact value (below 1% error at $n = 16$).

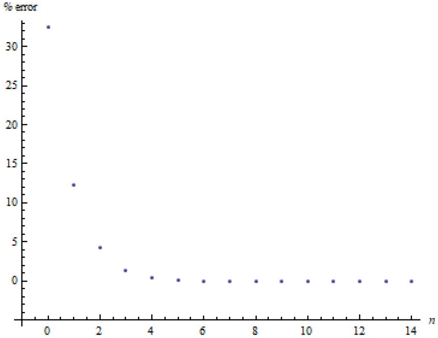
λ=0.1 Exact value is **I=** **1.6740859**		
n	**$F_2(n)$**	**% error**
0	3.2236737	92.5
4	3.6983551	121
8	2.4374890	45.6
12	1.8032981	7.72
16	1.6864361	0.738
20	1.6748432	4.52×10^{-2}
24	1.6741182	1.93×10^{-3}
28	1.6740869	5.79×10^{-4}
32	1.6740859	0
36	1.6740859	0
40	1.6740859	0
44	1.6740859	0



Case: $\lambda = 1.0$: strong coupling

Converges very quickly at strong coupling to the exact value (at $n = 4$ the error is already less than 1%) $\Rightarrow$ very useful series to use at strong coupling

λ=1.0 Exact value is **I= 1.3684269**		
n	**F_2(n)**	**% error**
0	1.8128050	32.5
1	1.2000966	12.3
2	1.4266972	4.26
3	1.3501087	1.34
4	1.3737129	0.386
5	1.3670114	0.103
6	1.3687817	2.59×10^{-2}
7	1.3683429	6.14×10^{-3}
8	1.3684457	1.37×10^{-3}
9	1.3684228	3.00×10^{-4}
10	1.3684277	5.16×10^{-5}
11	1.3684267	2.19×10^{-5}
12	**1.3684269**	**0**
13	1.3684269	0
14	1.3684269	0



Revisiting the first series $F_1(n)$

Why does the first series $F_1(n)$, obtained by expanding the quartic term, diverge when the original integral is finite?

The reason is that the integrand $e^{-a x^2 - \lambda x^4}$ in the limit as $x \to \infty$ is dominated by the quartic part λx^4 but the power series expansion of $e^{-\lambda x^4}$ up to any finite order n diverges in the limit as $x \to \infty$.

To capture the asymptotics of the quartic part properly, one must integrate x to a finite value β instead of infinity and then sum the series.

In particular, $\lim_{n\to\infty} \int_{-\beta}^{\beta} e^{-a x^2} (\lambda x^4)^n / n!$ tends to zero instead of infinity for any finite β.

One obtains the resulting series $S(n, \beta)$ in powers of λ which converges absolultely for any arbitrarily large value of β.

The series $S(n, \beta)$ and the incomplete Gamma function

Expanding the quartic term of the original integral I but integrating to finite β yields the following series in powers of the coupling λ:

$$S(n,\beta) = \int_{-\beta}^{\beta} dx\, e^{-a\,x^2} \sum_{j=0}^{n} \frac{(-\lambda\, x^4)^j}{j!} = \sum_{j=0}^{n} \frac{(-\lambda)^j}{j!} \int_{-\beta}^{\beta} dx\, e^{-a\,x^2}\, x^{4\,j}$$

$$= \sum_{j=0}^{n} \frac{(-\lambda)^j}{j!}\, a^{-2j-\frac{1}{2}}\, \gamma(2j+\tfrac{1}{2},\; a\,\beta^2)$$

where the incomplete gamma function $\gamma(z, \alpha)$ is defined as

$$\gamma(z,\alpha) = \int_0^{\alpha} e^{-t}\, t^{z-1}\, dt\,. \tag{1}$$

The series $S(n, \beta)$ is an absolutely convergent series for any finite β and valid at weak and strong coupling λ.

Table of values of $S(n, \beta)$ for different λ

$\boldsymbol{\lambda = 0.01}$ (exact value=1.7596991)				
n	$S(n,\beta=1)$	$S(n,\beta=2)$	$S(n,\beta=3)$	$S(n,\beta=4)$
1	1.4916429	1.7529462	1.7591604	1.7591605
2	1.4916478	1.7532172	1.7597216	1.7597419
3	1.4916478	1.7532097	1.7596811	1.7596941
4	1.4916478	1.7532099	1.7596847	1.7596999
5	1.4916478	1.7532099	1.7596844	1.7596990
6	1.4916478	1.7532099	1.7596844	**1.7596991**
7	1.4916478	1.7532099	1.7596844	**1.7596991**
8	1.4916478	1.7532099	1.7596844	**1.7596991**

$\boldsymbol{\lambda = 0.1}$ (exact value=1.6740859)				
n	$S(n,\beta=1)$	$S(n,\beta=2)$	$S(n,\beta=3)$	$S(n,\beta=4)$
10	1.4740801	1.6731653	1.6781192,	3.2144919
20	1.4740801	1.6731653	1.6740878	59.452736
30	1.4740801	1.6731653	**1.6740859**	31.420652
40	1.4740801	1.6731653	**1.6740859**	2.3645137
50	1.4740801	1.6731653	**1.6740859**	1.6755770
60	1.4740801	1.6731653	**1.6740859**	1.6740863
70	1.4740801	1.6731653	**1.6740859**	**1.6740859**
80	1.4740801	1.6731653	**1.6740859**	**1.6740859**
90	1.4740801	1.6731653	**1.6740859**	**1.6740859**

$\boldsymbol{\lambda = 1}$ (exact value=1.3684269)				
n	$S(n,\beta=1)$	$S(n,\beta=2)$	$S(n,\beta=3)$	$S(n,\beta=4)$
20	1.3336109	212.23528	5.5923449×10^{14}	9.9289902×10^{21}
40	1.3336109	1.3686641	9.1530933×10^{22}	1.5667467×10^{40}
60	1.3336109	**1.3684269**	7.3181151×10^{26}	1.3017096×10^{54}
80	1.3336109	**1.3684269**	8.1171151×10^{27}	1.5057863×10^{65}
100	1.3336109	**1.3684269**	6.4811079×10^{26}	1.2481108×10^{74}
200	1.3336109	**1.3684269**	3.0781473	3.6929318×10^{97}
300	1.3336109	**1.3684269**	**1.3684269**	3.4250715×10^{98}
400	1.3336109	**1.3684269**	**1.3684269**	6.8956011×10^{84}
500	1.3336109	**1.3684269**	**1.3684269**	1.6689893×10^{60}
600	1.3336109	**1.3684269**	**1.3684269**	7.8776498×10^{26}
700	1.3336109	**1.3684269**	**1.3684269**	**1.3684269**
800	1.3336109	**1.3684269**	**1.3684269**	**1.3684269**
1000	1.3336109	**1.3684269**	**1.3684269**	**1.3684269**

Properties of $S(n, \beta)$

- The series converged to the correct value (to eight digit accuracy) for all three values of λ: at weak coupling $\lambda = 0.01$, at intermediate coupling $\lambda = 0.1$ and at strong coupling $\lambda = 1$.

- The series $S(n, \beta)$ has a remarkable property: it is an expansion in powers of λ but it is an absolutely convergent series valid at both strong and weak coupling λ.

- The value of the integral limit β required for convergence was very low. With $\beta \leq 4$, convergence up to eight digit accuracy was reached for all three values of λ.

  $\Rightarrow$ For practical calculations, small β suffices. The limit $\beta \to \infty$ is not required.

Thank you

Quantum Mechanical Path Integral: quartic anharmonic oscillator

The Euclidean path integral for the quartic anharmonic oscillator with source term J is given by

$$
\begin{aligned}
K_E &= \int \mathcal{D}x(\tau)\, e^{-S_E/\hbar} \\
&= \int \mathcal{D}x(\tau) \exp\left[\frac{-1}{\hbar}\int_{\tau_a}^{\tau_b}\left(\frac{1}{2}\, m\dot{x}^2 + \frac{1}{2}\, m\,\omega^2\, x^2 + \lambda\, x^4 - J(\tau)\, x\right) d\tau\right].
\end{aligned}
$$

Analog of second series: expanding quadratic term

Expanding the quadratic term we obtain

$$k_E = \int \mathcal{D}x(\tau)\, e^{\frac{1}{\hbar}\int_{\tau_a}^{\tau_b} \left(-\lambda\, x^4 + J(\tau)\, x\right)\, d\tau}$$

$$\Bigg(1 - \frac{1}{\hbar}\int_{\tau_a}^{\tau_b} (\frac{1}{2}\, m\dot{x}^2 + \frac{1}{2}\, m\,\omega^2\, x^2)\, d\tau$$

$$+ \frac{1}{\hbar^2}\,\frac{1}{2!}\left(\int_{\tau_a}^{\tau_b}\left(\frac{1}{2}\, m\dot{x}^2 + \frac{1}{2}\, m\,\omega^2\, x^2\right)\, d\tau\right)^2 + \ldots\Bigg)\,.$$

Discretized path integral

We divide the time interval $\tau_b - \tau_a$ into N segments. This yields a discretized path integral over $N-1$ variables $x_i = x(\tau_i)$:

$$K_E = \Big(\frac{m}{2\,\pi\,\epsilon\,\hbar}\Big)^{N/2} \int_{-\infty}^{\infty} dx_1\, dx_2 ... dx_{N-1}\, \exp\Big(- \frac{\lambda\,\epsilon}{\hbar} \sum_{i=1}^{N-1} x_i^4 + \vec{J}\cdot\vec{x}\Big)$$

$$\Bigg(1 - \frac{m}{2\,\epsilon\,\hbar}\Big((2+\omega^2\,\epsilon^2)\sum_{i=1}^{N-1} x_i^2 - 2\sum_{i=2}^{N-1} x_i\, x_{i-1}\Big)$$

$$+ \Big[- \frac{m}{2\,\epsilon\,\hbar}\Big((2+\omega^2\,\epsilon^2)\sum_{i=1}^{N-1} x_i^2 - 2\sum_{i=2}^{N-1} x_i\, x_{i-1}\Big)\Big]^2 \frac{1}{2!} + ...\Bigg)\,.$$

where $\epsilon = (\tau_b - \tau_a)/N$.

Generating functional

We define the generating functional

$$Z[\vec{J}] = \int_{-\infty}^{\infty} \exp\Big(-\frac{\lambda\,\epsilon}{\hbar}\sum_{i=1}^{N-1} x_i^4 + \vec{J}\cdot\vec{x}\Big)\, dx_1\, dx_2 ... dx_{N-1}$$

$$= \int_{-\infty}^{\infty} dx_1 \,\exp\Big(-\frac{\lambda\,\epsilon}{\hbar}\, x_1^4 + J_1\, x_1\Big) \int_{-\infty}^{\infty} dx_2 \,\exp\Big(-\frac{\lambda\,\epsilon}{\hbar}\, x_2^4 + J_2\, x_2\Big)$$

$$... \int_{-\infty}^{\infty} dx_{N-1} \,\exp\Big(-\frac{\lambda\,\epsilon}{\hbar}\, x_{N-1}^4 + J_{N-1}\, x_{N-1}\Big)$$

$$= \prod_{i=1}^{N-1} I[J_i]$$

where $I[J_i]$ is a one-dimensional integral which can be expressed in terms of generalized hypergeometric functions.

Generalized hypergeometric functions instead of Gaussians

$$I[J_i] = \int_{-\infty}^{\infty} dx_i \, \exp\Big(-\frac{\lambda\,\epsilon}{\hbar}\,x_i^4 + J_i\,x_i\Big) = 2\,\Gamma\big(\frac{5}{4}\big)\Big(\frac{\hbar}{\epsilon\,\lambda}\Big)^{1/4} {}_0F_2\left(;\frac{1}{2},\frac{3}{4};\frac{J_i^4\,\hbar}{256\,\epsilon\,\lambda}\right)$$
$$+\frac{1}{4}\,\Gamma\big(\frac{3}{4}\big)\Big(\frac{\hbar}{\epsilon\,\lambda}\Big)^{3/4} {}_0F_2\left(;\frac{5}{4},\frac{3}{2};\frac{J_i^4\,\hbar}{256\,\epsilon\,\lambda}\right) J_i^2\,.$$

where ${}_0F_2\left(;\frac{1}{2},\frac{3}{4};\frac{J_i^4\,\hbar}{256\,\epsilon\,\lambda}\right)$ and ${}_0F_2\left(;\frac{5}{4},\frac{3}{2};\frac{J_i^4\,\hbar}{256\,\epsilon\,\lambda}\right)$ are generalized hypergoemetric functions ${}_pF_q(a;b;z)$.

Since ${}_pF_q(a;b;0) = 1$, it follows that

$$I[0] = 2\,\Gamma\big(\frac{5}{4}\big)\,\Big(\frac{\hbar}{\epsilon\,\lambda}\Big)^{1/4}$$
$$Z[0] = I[0]^{N-1} = \left[2\,\Gamma\big(\frac{5}{4}\big)\,\Big(\frac{\hbar}{\epsilon\,\lambda}\Big)^{1/4}\right]^{N-1}.$$

Note the inverse powers of λ.

Series via functional derivatives of generating functional

A functional derivative with respect to J_i of $Z[\vec{J}]$ brings down a factor of x_i. We can therefore express the series as

$$K_E = C \sum_{n=0}^{\infty} \frac{1}{n!}\, \hat{Q}^n\, Z[\vec{J}]\,\Big|_{\vec{J}=0}$$

$$= C\Big[Z[0] + \hat{Q}\, Z[\vec{J}]\,\Big|_{\vec{J}=0} + \frac{1}{2!}\, \hat{Q}^2\, Z[\vec{J}]\,\Big|_{\vec{J}=0} + \ldots \Big]$$

where the operator $\hat{Q}$ is given by

$$\hat{Q} = -\frac{m}{2\,\epsilon\,\hbar} \left((2 + \omega^2\, \epsilon^2) \sum_{i=1}^{N-1} \Big(\frac{\delta}{\delta J_i}\Big)^2 - 2 \sum_{i=2}^{N-1} \frac{\delta}{\delta J_i}\, \frac{\delta}{\delta J_{i-1}} \right)$$

and the prefactor C is

$$C = \Big(\frac{m}{2\,\pi\,\epsilon\,\hbar} \Big)^{N/2} .$$

Evaluating the functional derivatives

$Z[\vec{J}]$ is a product of the $I[J_i]$s. We therefore need to determine the functional derivatives of $I[J_i]$. After J_i is set to zero, only even derivatives survive and this is given by the simple expression

$$\left(\frac{\delta}{\delta J_i}\right)^{2n} I[J_i]\Big|_{J_i=0} = \int_{-\infty}^{\infty} dx_i \exp\left(-\frac{\lambda\,\epsilon}{\hbar}\,x_i^4\right) x_i^{2n}$$
$$= \frac{1}{2}\,\Gamma\left(\frac{2n+1}{4}\right)\left(\frac{\hbar}{\epsilon\,\lambda}\right)^{\frac{2n+1}{4}}$$

where n is any non-negative integer. When $n = 0$ we recover the expression for $I[0]$. The above result is central to evaluating the series for K_E.

First order contribution

The first order ($n = 1$) contribution to the series is given by

$$\hat{Q}\, Z[\vec{J}]\Big|_{\vec{J}=0} = -\frac{m}{2\,\epsilon\,\hbar}\Big((2+\omega^2\,\epsilon^2)\sum_{i=1}^{N-1}\big(\frac{\delta}{\delta J_i}\big)^2 - 2\sum_{i=2}^{N-1}\frac{\delta}{\delta J_i}\,\frac{\delta}{\delta J_{i-1}}\Big)\prod_{i=1}^{N-1} I\,[J_i]\Big|_{\vec{J}=0}$$

$$= -\frac{m}{2\,\epsilon\,\hbar}\,(2+\omega^2\,\epsilon^2)\,\sum_{i=1}^{N-1}\big(\frac{\delta}{\delta J_i}\big)^2\,\prod_{i=1}^{N-1}\,I\,[J_i]\Big|_{\vec{J}=0}$$

$$= -\frac{m}{2\,\epsilon\,\hbar}\,(2+\omega^2\,\epsilon^2)\sum_{i=1}^{N-1} I[J_1]\,I[J_2]...\big(\frac{\delta}{\delta J_i}\big)^2 I\,[J_i\,]...I[J_{N-1}]\Big|_{\vec{J}=0}$$

$$= -\frac{m}{2\,\epsilon\,\hbar}\,(2+\omega^2\,\epsilon^2)(N-1)\,I\,[0\,]^{N-2}\,\frac{1}{2}\,\Gamma\big(\frac{3}{4}\big)\,\Big(\frac{\hbar}{\epsilon\,\lambda}\Big)^{\frac{3}{4}}$$

$$= -Z\,[0\,]\,(N-1)\,\frac{1}{\lambda^{1/2}}\,\frac{\Gamma\big(\frac{3}{4}\big)}{\Gamma\big(\frac{5}{4}\big)}\,(2+\omega^2\,\epsilon^2)\,\frac{m}{8\,\hbar^{1/2}\,\epsilon^{3/2}}\,.$$

Series up to first order

The series up to first order (subscript (1)) is given by the analytical formula

$$K_{E_{(1)}} = C\,Z[0]\left(1 - \frac{1}{\lambda^{1/2}}\,(N-1)\,\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}\,(2+\omega^2\,\epsilon^2)\,\frac{m}{8\,\hbar^{1/2}\,\epsilon^{3/2}}\right).$$

The expression is a function of N, the coupling constant λ and the parameters ω and m as well as the constant $\hbar$. It depends also on the time interval $\mathcal{T}$ via $\epsilon = \mathcal{T}/N$. Having an expression as a function of N is very useful since numerically, N is the number of integrations required in the original path integral and this can become computationally intensive in the continuum limit where N is large and formally infinite.

First order analytical formula matches numerical integration

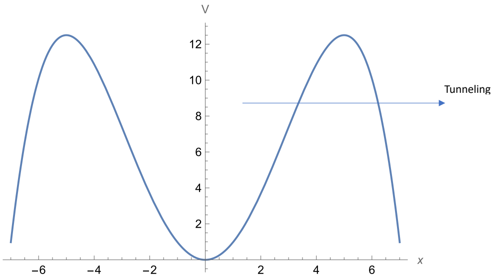
As a simple check on the first order analytical formula, we performed a first order numerical integration of the series for the case $N = 4$ which involves $N - 1 = 3$ integrals. We used the following numerical values for the parameters: $m = \hbar = \omega = \mathcal{T} = 1$. Hence $\epsilon = \mathcal{T}/N = 1/4$. The numerical value of λ was not specified. The analytical formula and first order numerical integration *matched* and gave the following result:

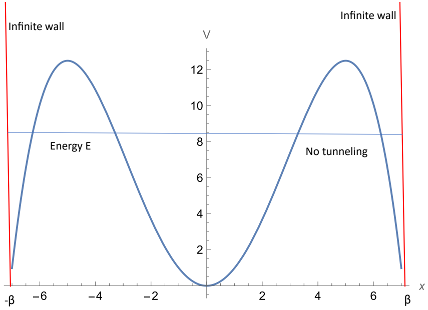
$$K_{E_{(1)}} = \frac{64\sqrt{2}\,\Gamma\left(\frac{5}{4}\right)^3}{\pi^2\,\lambda^{3/4}} - \frac{99\,\Gamma\left(\frac{1}{4}\right)}{2\,\pi\,\lambda^{5/4}} \qquad \text{for } N = 4 \text{ and } m = \hbar = \omega = \mathcal{T} = 1.$$

The inverse powers of λ above illustrates again that this series is outside the usual perturbative regime and is well suited to the strong coupling non-perturbative regime.

Circumventing Dyson's argument on asymptotic series

- Dyson would argue that a perturbative series expansion about $\lambda = 0$ in powers of λ should yield an asymptotic series. If it were absolutely convergent then the series would also be convergent for negative λ assuming its absolute value is sufficiently small.
- The original integral diverges for negative λ which implies the series $F_1(n)$ with positive λ must be an asymptotic series.
- In quantum mechanics (QM), Dyson's argument would be that the potential $V(x) = \lambda\, x^4 + a\, x^2$ with negative λ exhibits tunneling and hence an instability so the series must diverge.
- How did we circumvent Dyson's argument with our series $S(n, \beta)$? The answer is that x ranges between the finite values of $-\beta$ and β so that our original integral with those limits is finite when λ is negative.
- In the QM case, the particle is confined between $-\beta$ and β and this requires placing infinite walls at $x = \pm\beta$ in the potential $V(x)$. The walls prevent tunneling from occuring.


V
12
10
8
6
4
2
−6
−4
−2
2
4
6
x
Tunneling


Infinite wall
Infinite wall
V
12
10
8
6
4
2
Energy E
No tunneling
-β
−6
−4
−2
2
4
6
β
x