A Geometrical Formalism of Functional Matching

Xu-Xiang Li 黎旭翔 (University of Utah)

May 13, 2024

University of Pittsburgh

DPF – PHENO 2024

Based on work in preparation collaborating with Xiaochuan Lu and Zhengkang (Kevin) Zhang

Field Space Geometry in Higgs Sector

- **SM Higgs doublet under different parametrization**

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} -\phi^2 - i\phi^1 \\ \phi^4 + i\phi^3 \end{pmatrix}$$

$$\Delta\mathcal{L} = \frac{1}{2}\delta_{ij}(\partial_\mu \phi^i)(\partial^\mu \phi^i) - V(|\phi|^2)$$

$$\Delta\mathcal{L} = (\partial_\mu H)^\dagger(\partial^\mu H) - V(H^\dagger H)$$

$$H = U(x)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+h \end{pmatrix}$$

$$U(x) = \exp\left(-\frac{i\sigma^a\pi^a}{v}\right)$$

$$\Delta\mathcal{L} = \frac{1}{2}(\partial_\mu h)^2 + \frac{1}{2}\left(1+\frac{h}{v}\right)^2 g_{ab}(\pi)(\partial_\mu \pi^a)(\partial^\mu \pi^b) - V(h)$$

Field Space Geometry in Higgs Sector

- SM Higgs doublet under different parametrization

Cartesian Coodinates

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} -\phi^2 - i\phi^1 \\ \phi^4 + i\phi^3 \end{pmatrix}$$

$$\Delta\mathcal{L} = \frac{1}{2}\delta_{ij}(\partial_\mu \phi^i)(\partial^\mu \phi^i) - V(|\phi|^2)$$

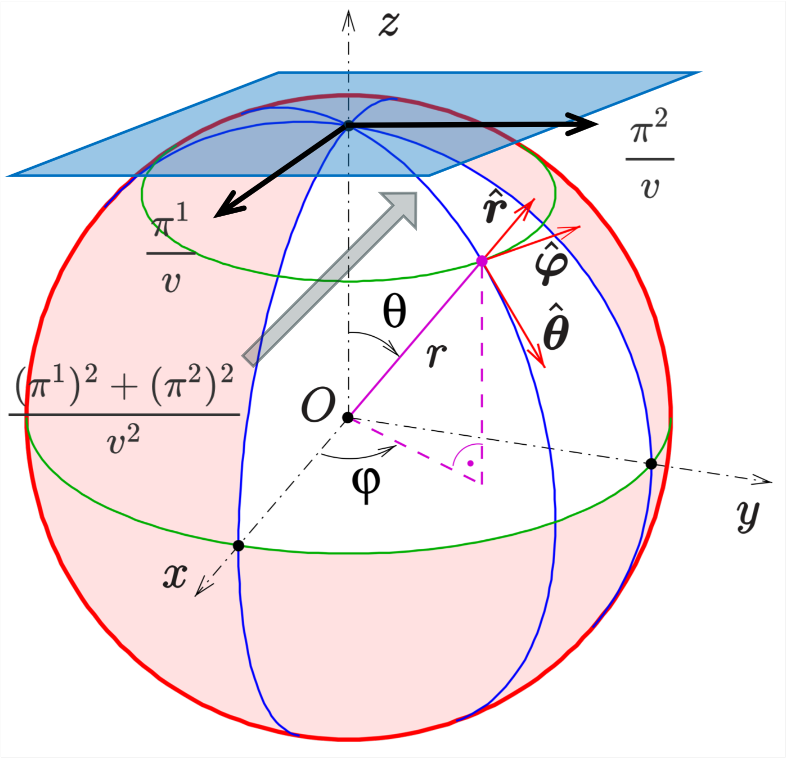
$$\Delta\mathcal{L} = (\partial_\mu H)^\dagger(\partial^\mu H) - V(H^\dagger H)$$

$$H = U(x)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+h \end{pmatrix}$$

$$U(x) = \exp\left(-\frac{i\sigma^a\pi^a}{v}\right)$$

$$\Delta\mathcal{L} = \frac{1}{2}(\partial_\mu h)^2 + \frac{1}{2}\left(1+\frac{h}{v}\right)^2 g_{ab}(\pi)(\partial_\mu \pi^a)(\partial^\mu \pi^b) - V(h)$$

Spherical Coodinates



Field Space Geometry in Higgs Sector

- SM Higgs doublet under different parametrization

Cartesian Coodinates

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} -\phi^2 - i\phi^1 \\ \phi^4 + i\phi^3 \end{pmatrix}$$

$$\Delta\mathcal{L} = \frac{1}{2}\delta_{ij}(\partial_\mu\phi^i)(\partial^\mu\phi^i) - V(|\phi|^2)$$

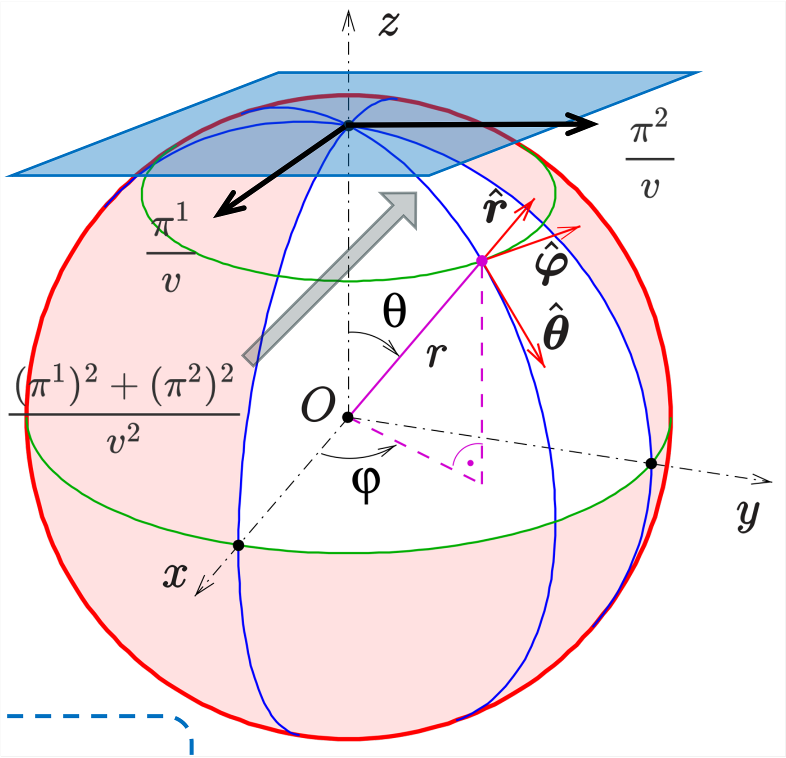
$$\Delta\mathcal{L} = (\partial_\mu H)^\dagger(\partial^\mu H) - V(H^\dagger H)$$

$$H = U(x)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+h \end{pmatrix}$$

$$U(x) = \exp\left(-\frac{i\sigma^a\pi^a}{v}\right)$$

$$\Delta\mathcal{L} = \frac{1}{2}(\partial_\mu h)^2 + \frac{1}{2}\left(1+\frac{h}{v}\right)^2 g_{ab}(\pi)(\partial_\mu\pi^a)(\partial^\mu\pi^b) - V(h)$$

Spherical Coodinates



Field Space Geometry in General Scalar Sector

- **A general scalar theory with two derivatives**

$$\mathcal{L} = \frac{1}{2} g_{ij}(\phi)(\partial_\mu \phi^i)(\partial^\mu \phi^j) - V(\phi)$$

- **Under non-derivative field redefinition**

$$\phi^i \to \phi'^i = \phi'^i(\phi)$$

$$(\partial_\mu \phi^i) \to (\partial_\mu \phi'^i) = \frac{\partial \phi'^i}{\partial \phi^j}(\partial_\mu \phi^j)$$

$$g_{ij} \to g'_{ij} = \frac{\partial \phi^k}{\partial \phi'^i}\frac{\partial \phi^l}{\partial \phi'^j} g_{kl}$$

Field Space Geometry in General Scalar Sector

- A general scalar theory with two derivatives

$$\mathcal{L} = \frac{1}{2} g_{ij}(\phi)(\partial_\mu \phi^i)(\partial^\mu \phi^j) - V(\phi)$$

- Under non-derivative field redefinition

$$\phi^i \to \phi'^i = \phi'^i(\phi)$$

$$(\partial_\mu \phi^i) \to (\partial_\mu \phi'^i) = \frac{\partial \phi'^i}{\partial \phi^j}(\partial_\mu \phi^j)$$

$$g_{ij} \to g'_{ij} = \frac{\partial \phi^k}{\partial \phi'^i}\frac{\partial \phi^l}{\partial \phi'^j} g_{kl}$$

⟺

Coodinate transformation

Vector

(0, 2)-tensor ⟹ metric

Field Space Geometry

Alonso, Jenkins, Manohar [arXiv: 1511.00724, 1605.03602]

- SM Higgs under Unitary representation:

$$g_{ij} = \begin{pmatrix} \left(1+\frac{h}{v}\right)^2 g_{ab}(\pi) & 0 \\ 0 & 1 \end{pmatrix} \implies R_{ijkl} = 0 \quad \text{Flat space} \implies \text{Renormalizable}$$

Partition Function & Effective Action

- **Partition function**

$$Z[J] = \int \mathcal{D}\phi \sqrt{\mathrm{Det}\, g(\phi)} e^{i(S[\phi]+J\phi)}$$

  - **Tree level: Stationary point**

$$\left.\frac{\delta S}{\delta \phi}\right|_{\phi=\bar{\phi}} + J = 0$$

  - **One-loop level: Gaussian approximation**

$$\phi = \bar{\phi} + \phi'$$

$$S[\phi] + J\phi = S[\bar{\phi}] + J\bar{\phi} + \frac{1}{2}\phi' \left.\frac{\delta^2 S}{\delta \phi^2}\right|_{\phi=\bar{\phi}} \phi' + O(\phi'^3)$$

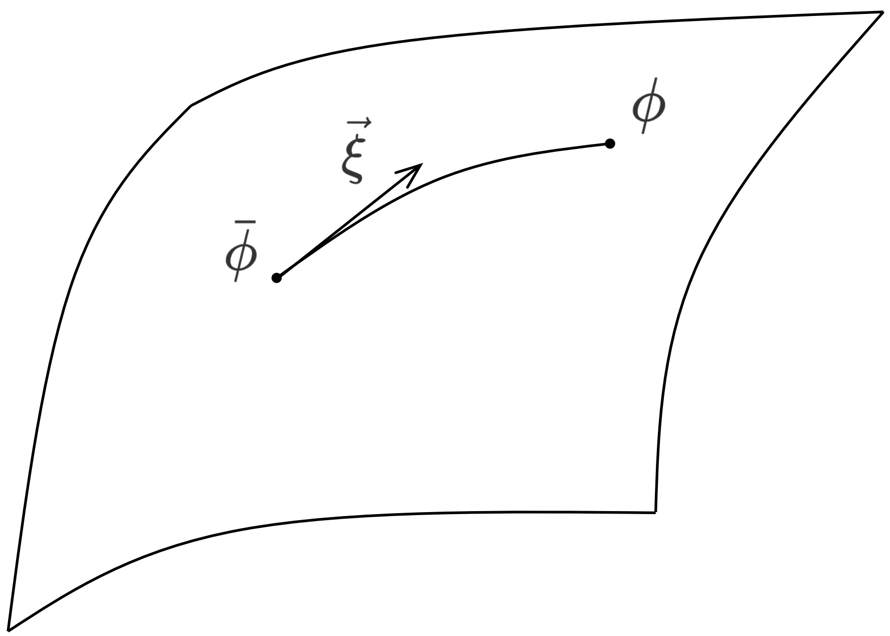
$$-i \log Z[J] = S[\bar{\phi}] + J\bar{\phi} + i\frac{1}{2}\mathrm{Tr}\log\left.\left(g^{-1}\frac{\delta^2 S}{\delta\phi^2}\right)\right|_{\phi=\bar{\phi}} + \cdots$$

$$\Gamma^{[0]}[\varphi] = S[\varphi]$$

$$\Gamma^{[1]}[\varphi] = i\frac{1}{2}\mathrm{Tr}\log\left.\left(g^{-1}\frac{\delta^2 S}{\delta\phi^2}\right)\right|_{\phi=\varphi}$$

Covariant Expansion of Action

- **Action = Function(al) on field space**



$$S(\phi) \sim S(\bar{\phi}) + \xi^i \nabla_i S(\bar{\phi}) + \frac{1}{2}\xi^i \xi^j \nabla_i \nabla_j S(\bar{\phi}) + \cdots$$

Covariant Expansion of Action

- Action = Function(al) on field space

Covariant expansion along geodesic from $\bar{\phi}$ to ϕ

$$\frac{\delta S}{\delta \phi^i} \to \frac{\nabla S}{\nabla \phi^i} = -g_{ij}(\phi)[\mathscr{D}_\mu(\partial^\mu \phi)]^j - \nabla_i V(\phi)$$

$$\frac{\delta^2 S}{\delta \phi^i \delta \phi^j} \to \frac{\nabla^2 S}{\nabla \phi^i \nabla \phi^j} = -g_{ij}(\phi)\mathscr{D}^2 - R_{ikjl}(\partial_\mu \phi^k)(\partial^\mu \phi^l) - \nabla_i \nabla_j V(\phi)$$

Alonso, Jenkins, Manohar [arXiv: 1605.03602]
Alonso, West [arXiv:2207.02050]

$$\mathscr{D}_\mu = \partial_\mu + (\partial_\mu \phi)^k \Gamma^i_{kj}(\phi) \sim (\partial_\mu \phi^i)\nabla_i$$

Covariant derivative on field space

- ⇨ Covariant 1-loop **Effective action**
  - **Divergent part ⇨ RGE**

    Alonso, Jenkins, Manohar [arXiv: 1511.00724]
    Alonso, Kanshin, Saa [arXiv: 1710.06848]
    Assi, Helset, Manohar, Pagès, Shen [arXiv:2307.03187]
  - **Hard region (div + finite) ⇨ EFT matching!**

    Fuentes-Martin, Portoles, Ruiz-Femenia [arXiv: 1607.02142]
    Zhang [arXiv: 1610.00710]

Geometrical Functional Matching

- **Top-down: find EFT from UV theory**

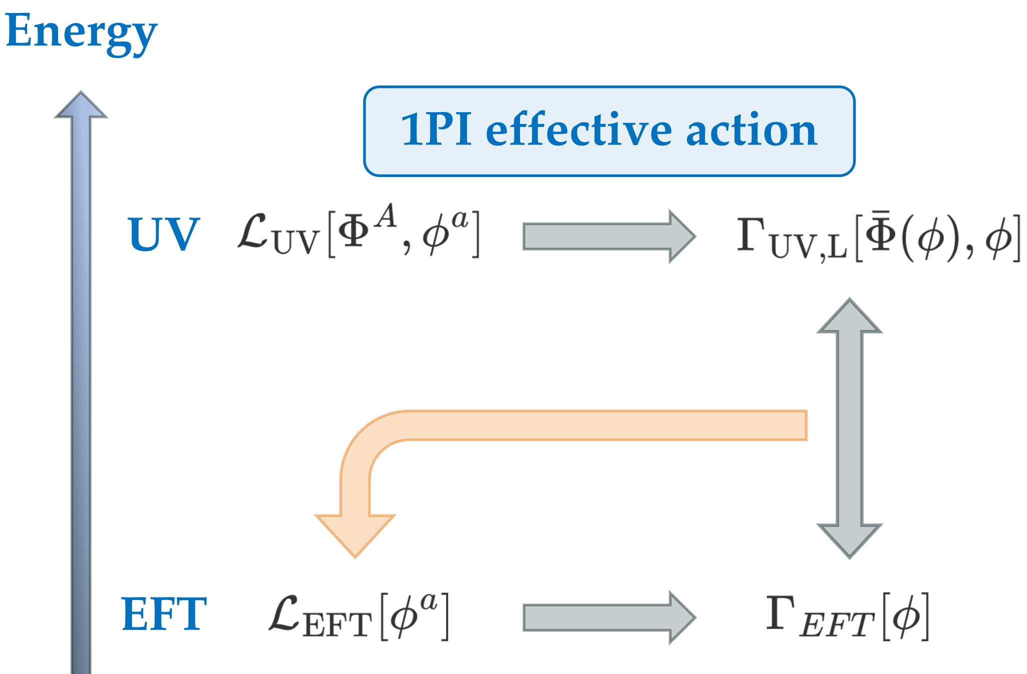
Energy

UV $\mathcal{L}_{\text{UV}}[\Phi^A, \phi^a]$

EFT $\mathcal{L}_{\text{EFT}}[\phi^a]$

Geometrical Functional Matching

- Top-down: find EFT from UV theory



Henning, Lu, Murayama [arXiv: 1412.1837]
Cohen, Lu, Zhang [arXiv: 2011.02484]
Fuentes-Martin, König, Pagès, Thomsen, Wilsch [arXiv:2212.04510]

- Tree level:

$$\int \mathrm{d}^4x \mathcal{L}_{EFT}^{[0]}[\phi] = \int \mathrm{d}^4x \mathcal{L}_{UV}[\bar{\Phi}(\phi), \phi]$$

Covariant!

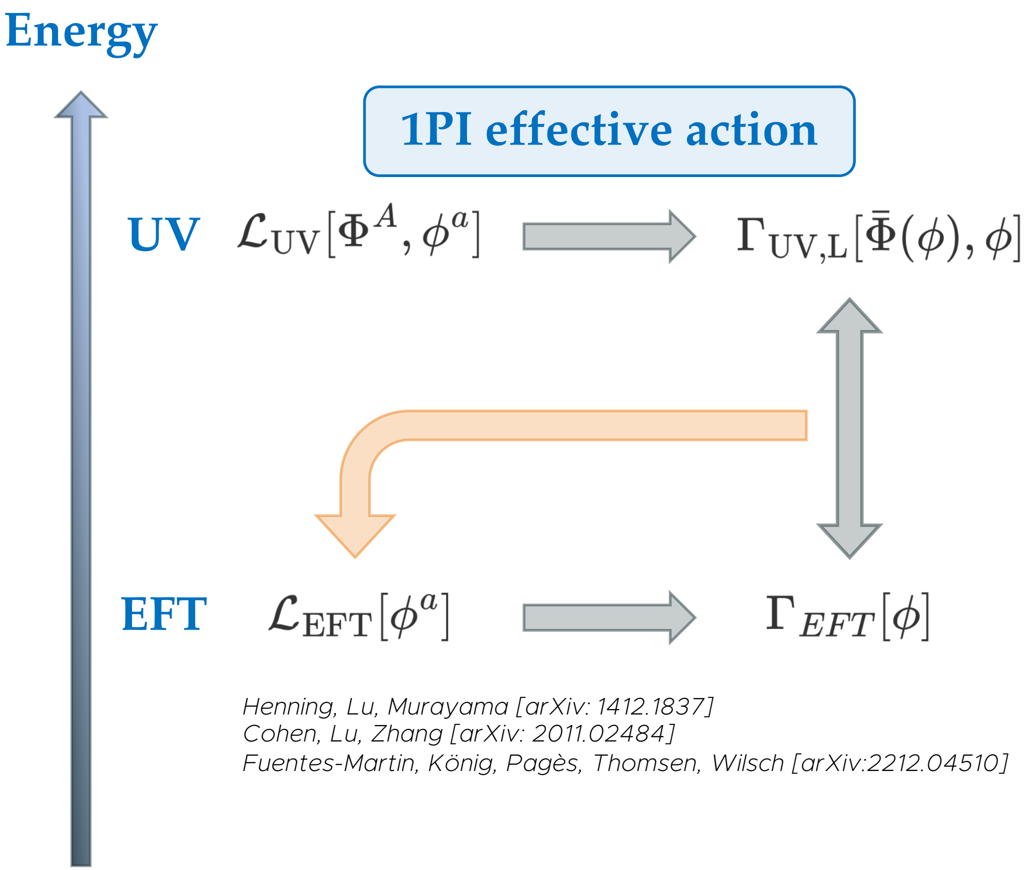
- One-loop level:

$$\int \mathrm{d}^4x \mathcal{L}_{EFT}^{[1]}[\phi] = i\frac{1}{2}\int \mathrm{d}^4x \int \frac{\mathrm{d}^4p}{(2\pi)^4} \mathrm{tr}\log\left(-g^{-1}\nabla^2 S_{UV}\right)\Big|_{i\partial_\mu \to i\partial_\mu - p_\mu, \Phi=\bar{\Phi}[\phi]}$$

Take hard region

Geometrical Functional Matching

- Top-down: find EFT from UV theory



Henning, Lu, Murayama [arXiv: 1412.1837]
Cohen, Lu, Zhang [arXiv: 2011.02484]
Fuentes-Martin, König, Pagès, Thomsen, Wilsch [arXiv:2212.04510]

- **Tree level:**

$$\int d^4x \mathcal{L}_{EFT}^{[0]}[\phi] = \int d^4x \mathcal{L}_{UV}[\bar{\Phi}(\phi), \phi]$$

~~Covariant!~~

- **One-loop level:**

$$\int d^4x \mathcal{L}_{EFT}^{[1]}[\phi] = i\frac{1}{2} \int d^4x \int \frac{d^4p}{(2\pi)^4} \mathrm{tr}\log\left(-g^{-1}\nabla^2 S_{UV}\right)\Big|_{i\partial_\mu \to i\partial_\mu - p_\mu, \Phi = \bar{\Phi}[\phi]}$$

Take hard region

only covariance of light components (submanifold)

Geometrical Functional Matching

$$\mathcal{L}^{[1]}_{EFT}[\phi] = i\frac{1}{2}\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{tr}\log\left[(\mathscr{D}_\mu + ip_\mu)^2 + R^i{}_{kjl}(\partial_\mu \varphi^k)(\partial^\mu \varphi^l) + \nabla^i\nabla_j V\right]\Bigg|_{\Phi=\bar{\Phi}[\phi]}$$

- Divide into covariant quantities of submanifold

intrinsic curvature

$$\mathscr{D}_\mu = \begin{pmatrix} \mathscr{D}^L_\mu & K^{LH}_\mu \\ K^{HL}_\mu & \mathscr{D}^H_\mu \end{pmatrix}$$

$$R_{abcd} = \hat{R}_{abcd} + K_{ac}K_{bd} - K_{ad}K_{bc}$$

extrinsic curvature

- Geometrical Covariant Derivative Expansion: keeping the covariance while evaluating the loop integral

$$\mathscr{D}_\mu = \partial_\mu + (\partial_\mu \phi)^k \Gamma^i_{kj}(\phi) \quad \Longleftrightarrow \quad D_\mu = \partial_\mu - igA^a_\mu T^a$$

use CDE trick with a little modification

Henning, Lu, Murayama [arXiv: 1412.1837]
Cohen, Lu, Zhang [arXiv: 2011.02484]
Fuentes-Martin, König, Pagès, Thomsen, Wilsch [arXiv:2212.04510]

Example: ‘Non’linear Sigma Model

- **Reparametrized linear sigma model:**

$$\mathcal{L}_{\mathrm{UV}} = \frac{1}{2}(\partial_\mu h)^2 + \frac{1}{2}\left(1+\frac{h}{v}\right)^2 \underbrace{g_{ab}(\pi)(\partial_\mu \pi^a)(\partial^\mu \pi^b)}_{\hat{g}} - \frac{1}{4}\lambda(h^2+2hv)^2$$

- ***h*: radial, *π*: spherical, field space is flat. *g* the meric on *π* sphere.**
- **O(N+1) symmetry breaking ⇨ massive *h***
- **Integrate out *h* ⇨ EFT of *π* (up to dim-8 under canonical dimension counting)**

potential → U

extrinsic curvature → $K_\mu K^\mu$

intrinsic curvature → $\mathscr{G}_{\mu\nu}$

$$\mathcal{L}^{[1]}_{EFT}[\pi] = \frac{1}{2}A(M^2)(U + K_\mu K^\mu)_{hh} + \frac{1}{d}B(0,M^2,0)(K^\mu \mathscr{G}_{\mu\nu} K^\nu)_{hh} + \cdots$$

$$= \frac{1}{16\pi^2}\left[\frac{1}{2}\lambda(-6L+7)\hat{g} + \frac{\lambda^2}{9M^4}(-102L+67)\hat{g}^2 + \frac{\lambda^2}{9M^4}(12L-34)(\partial_\mu\pi\cdot\partial_\nu\pi)^2\right]$$

Summary

- Geometrical Functional Matching: keep geometric covariance during EFT matching
  - Geometry on field space helps expand action covariantly
  - Hard region expansion breaks covariance, only covariance on submanifold survives
  - Geometric covariant derivative expansion: preserve the covaiance
  - EFT built by geometric quantities, e.g. extrinsic/intrinsic curvature

Thank you!