Momentum shift and on-shell constructible massive amplitudes

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Reference: *arXiv:2403.15538*

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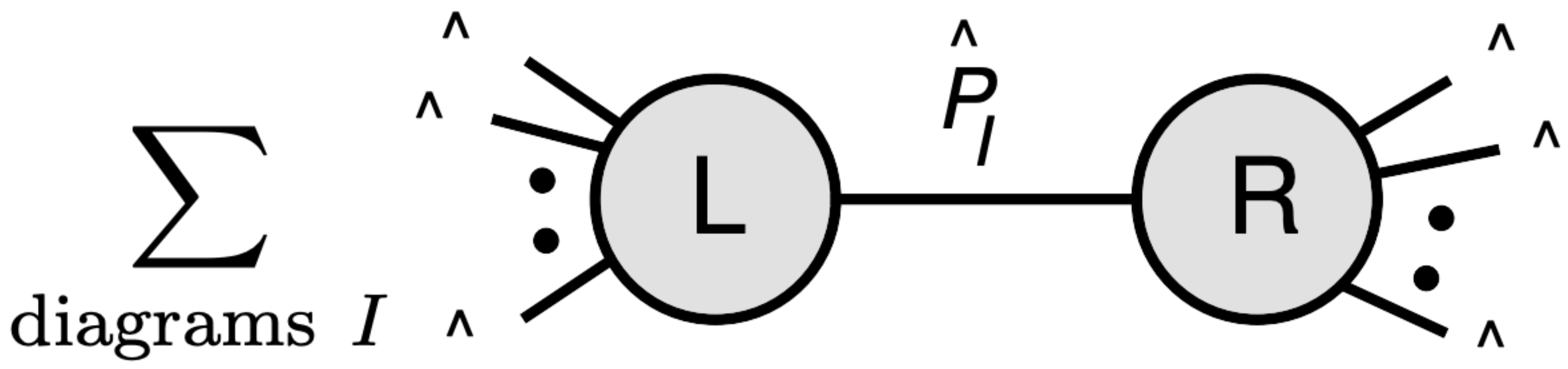
Outline

- Recursion relations and factorization
- Massive spinor variables
- Momentum shifts for *massive spinors*
  - All-line transverse shift
- Constructive methods for massive QED and massive spin-1 amplitudes
  - Resolve contact term ambiguity when “gluing” amplitudes
  - Catagorize depependent contact terms

Recursion Relations

On-shell recursion relation constructs higher-point amplitude from lower-point on-shell information

- Analytic contination of external momentum $\hat{p}_i(z) = p_i + zq_i.$
- Impose on-shell and momentum conservation $\sum_i \hat{p}_i(z) = 0, \quad \hat{p}_i^2(z) = p_i^2 = m_i^2,$
- Perform complex integral $A_n = \frac{1}{2\pi i}\oint_{z=0} \frac{dz}{z}\hat{A}_n(z) = -\sum_{\{z_I\}} \text{Res}\left[\frac{\hat{A}_n(z)}{z}\right] + B_\infty,$



Constructible if boundary term goes to zero

Massive Spinor Variables

SU(2) little group index

$$[\mathbf{i}|^{I}_{\dot{a}} \qquad |\mathbf{i}\rangle^{I}_{a}$$

Chiral Index

Spin index $I = 1{,}2$ represents the little group freedom (choice of spin-axis)

We work in the *helicity basis*

$$|\mathbf{i}\rangle^{I}_{a} = |i\rangle_{a}\delta^{I}_{-} + |\eta_i\rangle_{a}\delta^{I}_{+}$$

$$[\mathbf{i}|^{I}_{\dot{a}} = [i|_{\dot{a}}\delta^{I}_{+} + [\eta_i|_{\dot{a}}\delta^{I}_{-}$$

On-shell condition

$$\langle i\eta_i\rangle = [i\eta_i] = m_i$$

In the HE limit $\eta \to 0$

All-line Transverse Shift

- Shift external momentum by respective transverse polarization for spin-1/2 and spin-1

$$p_i \to \hat{p}_i = p_i + z\frac{c_i m_i}{\sqrt{2}}\epsilon_i^{(I_i)},$$

- “good” shift will only deform momentum and not external polarizations
- A “good” shift exists after fixing the spin-projection of external particles
- Define shift for spin-1/2 and spin-1 transverse modes

$$|i\rangle \to |\hat{i}\rangle = |i\rangle + zc_i|\eta_i\rangle \quad \text{for} \quad I_i = +,$$
$$|i] \to |\hat{i}] = |i] + zc_i|\eta_i] \quad \text{for} \quad I_i = -,$$

On-shell condition satisfied

$$\langle \hat{i}\eta_i\rangle = [\hat{i}\eta_i] = m_i$$

All-line Transverse Shift

- The shift is defined for spin-1 longitudinal modes

$$\begin{cases} |i\rangle \to |\hat{i}\rangle = |i\rangle + z\frac{c_i}{2}|\eta_i\rangle, \\ [\eta_i| \to [\hat{\eta}_i| = [\eta_i| - z\frac{c_i}{2}[i|, \end{cases} \quad \text{or} \quad \begin{cases} [i| \to [\hat{i}| = [i| + z\frac{c_i}{2}[\eta_i|, \\ |\eta_i\rangle \to |\hat{\eta}_i\rangle = |\eta_i\rangle - z\frac{c_i}{2}|i\rangle, \end{cases} \quad \text{for} \quad I_i = L,$$

- The definition can be extened for massless legs

- Shift defined in helicity basis and does not require specific spin-axis choice

- The shift exists for all possible spin configurations

Massive QED Amplitude

- Applied the shift to properly construct $ee\mu\mu$ and $ee\mu\mu\gamma$ amplitude

- Gluing results in different expressions for $ee\mu\mu$ calculations

$$A_4|_{\text{AHH}} = \tilde{e}^2 \frac{\langle \mathbf{12} \rangle \langle \mathbf{34} \rangle}{ms} (p_1 - p_2) \cdot p_3,$$

N. Arkani-Hamed, T. C. Huang, Y. Huang, arXiv:1709.04891

$$A_4|_{\text{CDFHM}} = \frac{\tilde{e}^2}{2m_e m_\mu s} \Big[\left(u - t + 2m_e^2 + 2m_\mu^2 \right) [\mathbf{12}][\mathbf{34}] + 2 \left([\mathbf{12}][\mathbf{3}|\mathbf{21}|\mathbf{4}] + [\mathbf{34}][\mathbf{1}|\mathbf{43}|\mathbf{2}] \right) \Big].$$

N. Christensen, H. Diaz-Quiroz, B. Field, J. Miles arXiv:2209.15018

- Correct amplitude: $$A_4 = \frac{\tilde{e}^2}{p_{12}^2} \left[\langle \mathbf{13} \rangle [\mathbf{24}] + \langle \mathbf{14} \rangle [\mathbf{23}] + \langle \mathbf{23} \rangle [\mathbf{14}] + \langle \mathbf{24} \rangle [\mathbf{13}] \right],$$

- All expressions agree when internal photon on-shell photon ($\hat{s} = 0$) but differ for off-shell photon.

H.Y. Lai, D. Liu, J. Terning, arXiv:2312.11621

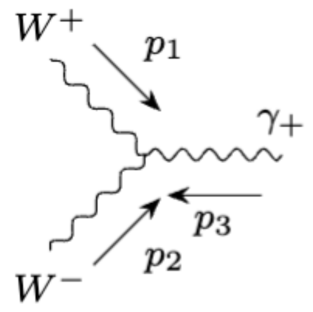
- This leads to contact term ambiguity when "gluing" amplitude

The shift resolves the ambiguity due to channels with $\hat{p}_I^2 = 0$

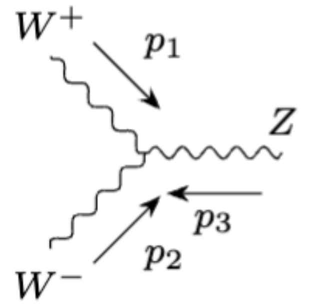
Massive Spin-1 Amplitude

- Consider $WWWW$ four-point tree level amplitude without Higgs

3-point input



$$= \frac{g_{WW\gamma}}{m_W} x_{12} \langle \mathbf{12} \rangle^2 = \frac{g_{WW\gamma}}{\sqrt{2} m_W^2 \langle 3\xi \rangle} \left[\langle \mathbf{12} \rangle [\mathbf{21}] \langle \mathbf{3} | p_1 - p_2 | \mathbf{3}] + \text{cycl.} \right]$$



$$= \frac{g_{WWZ}}{\sqrt{2} m_W^2 m_Z} \left[\langle \mathbf{12} \rangle [\mathbf{21}] \langle \mathbf{3} | p_1 - p_2 | \mathbf{3}] + \text{cycl.} \right]$$

- Apply All-line transverse shift to construct 4-point amplitude

$$A_4 = \frac{A_s^Z(0)}{p_{12}^2 - m_Z^2} + \frac{A_t^Z(0)}{p_{14}^2 - m_Z^2} + \frac{A_s^\gamma(0)}{p_{12}^2} + \frac{A_t^\gamma(0)}{p_{14}^2} + A_{contact}$$

Constructible if for at least one transeverse mode

Little group covariance allows us to find all longitudinal amplitude

$WWhh$, $ZZhh$ contact term not required as input as well

$WWWW$ Contact term not required as input

Conclusion

- We proposed an all-line transverse shift for on-shell construction of tree-level massive amplitudes
- The method gives an algorithm regarding when we can “unhat”
- Resolved any contact term ambiguity due to gluing massless internal legs
- Constructively built massive spin-1 amplitudes

Back up

Polarization tensors

Massless:

$$\epsilon^{(+)}_{a\dot{a}} = \sqrt{2}\frac{|\xi\rangle_a[i|_{\dot{a}}}{\langle i\xi\rangle}, \quad \epsilon^{(-)}_{a\dot{a}} = \sqrt{2}\frac{|i\rangle_a[\xi|_{\dot{a}}}{[i\xi]}$$

Massive:

$$\epsilon^{(+)}_{a\dot{a}} = \sqrt{2}\frac{|\eta_i\rangle_a[i|_{\dot{a}}}{m_i}, \quad \epsilon^{(-)}_{a\dot{a}} = \sqrt{2}\frac{|i\rangle_a[\eta_i|_{\dot{a}}}{m_i}, \quad \epsilon^{(L)}_{a\dot{a}} = \frac{|i\rangle_a[i|_{\dot{a}} + |\eta_i\rangle_a[\eta_i|_{\dot{a}}}{m_i}$$

Momentum:

$$(p_i)_{a\dot{a}} = |i\rangle_a[i|_{\dot{a}} - |\eta_i\rangle_a[\eta_i|_{\dot{a}}$$

Explicit form of spinors In helicity basis:

$$|i\rangle_a = \sqrt{E_i + p_i}\begin{pmatrix} -s_i^* \\ c_i \end{pmatrix}, \quad [i|_{\dot{a}} = \sqrt{E_i + p_i}\begin{pmatrix} -s_i & c_i \end{pmatrix}$$

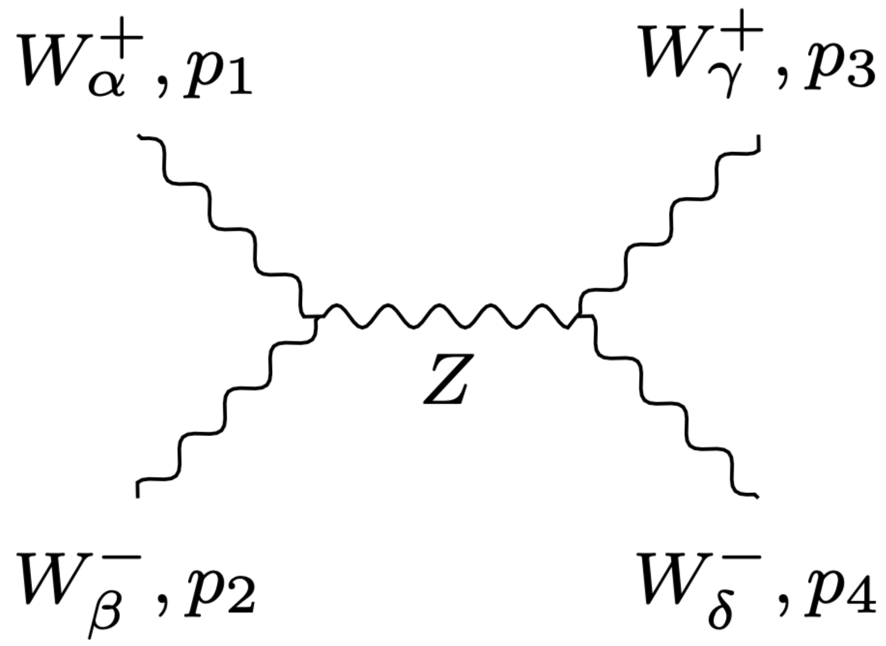
$$|\eta_i\rangle_a = \sqrt{E_i - p_i}\begin{pmatrix} c_i \\ s_i \end{pmatrix}, \quad [\eta_i|_{\dot{a}} = -\sqrt{E_i - p_i}\begin{pmatrix} c_i & s_i^* \end{pmatrix}$$

3-point QED:

$$A_3(\mathbf{12}3^+) = \tilde{e}x_{12}\langle\mathbf{12}\rangle$$
$$= -\tilde{e}\frac{\langle\mathbf{1}\xi\rangle[3\mathbf{2}] + \langle\mathbf{2}\xi\rangle[3\mathbf{1}]}{\langle\xi 3\rangle}$$

$WWWW$ Calculations

Example calculation:



$$A_s^Z(z)\Big|_{\hat{p}_{12}^2=m_Z^2} = \frac{g_{WWZ}^2}{m_W^4}\Big(2(\hat{p}_1-\hat{p}_2)\cdot(\hat{p}_3-\hat{p}_4)\langle\hat{\mathbf{1}}\hat{\mathbf{2}}\rangle[\hat{\mathbf{2}}\hat{\mathbf{1}}]\langle\hat{\mathbf{3}}\hat{\mathbf{4}}\rangle[\hat{\mathbf{4}}\hat{\mathbf{3}}]$$
$$-2\langle\hat{\mathbf{1}}\hat{\mathbf{2}}\rangle[\hat{\mathbf{2}}\hat{\mathbf{1}}]\langle\hat{\mathbf{3}}|\hat{p}_1-\hat{p}_2|\hat{\mathbf{3}}]\langle\hat{\mathbf{4}}|\hat{p}_3|\hat{\mathbf{4}}] + 2\langle\hat{\mathbf{1}}\hat{\mathbf{2}}\rangle[\hat{\mathbf{2}}\hat{\mathbf{1}}]\langle\hat{\mathbf{4}}|\hat{p}_1-\hat{p}_2|\hat{\mathbf{4}}]\langle\hat{\mathbf{3}}|\hat{p}_4|\hat{\mathbf{3}}] + (\hat{1},\hat{2})\leftrightarrow(\hat{3},\hat{4})$$
$$+4\langle\hat{\mathbf{1}}\hat{\mathbf{3}}\rangle[\hat{\mathbf{3}}\hat{\mathbf{1}}]\langle\hat{\mathbf{2}}|\hat{p}_1|\hat{\mathbf{2}}]\langle\hat{\mathbf{4}}|\hat{p}_3|\hat{\mathbf{4}}] - 4\langle\hat{\mathbf{1}}\hat{\mathbf{4}}\rangle[\hat{\mathbf{4}}\hat{\mathbf{1}}]\langle\hat{\mathbf{2}}|\hat{p}_1|\hat{\mathbf{2}}]\langle\hat{\mathbf{3}}|\hat{p}_4|\hat{\mathbf{3}}] - (\hat{1}\leftrightarrow\hat{2})\Big) \quad (3.4)$$

Need to sum over two poles at this factorization channel

Ward identity and simplification

$$A_4 = \epsilon^\mu_{1-}\epsilon^\nu_{2L}\epsilon^\lambda_{3L}\epsilon^\sigma_{4L}F_{\mu\nu\lambda\sigma}(p_1,p_2,p_3,p_4)$$

As $z\to\infty$, for ALT shift $\quad r_1^\mu F_{\mu\nu\lambda\sigma}(r_1,r_2,r_3,r_4)$ ⟵ Goes to zero due to **Ward identiy.** Thus leading z behavior vanishes

Massive BCFW-type Shift

[R. Franken, C. Schwinn, arXiv:1910.13407]

[C. Wu, S. H. Zhu, arXiv: 2112.12312]

Decompose momentum p_i, p_j as linear combination of two null-momentum

$$p_i = l_i + \frac{m_i^2}{2l_j \cdot l_i} l_j, \quad p_j = l_j + \frac{m_j^2}{2l_j \cdot l_i} l_i$$

Shift the spinors as

$$[\mathbf{i}, \mathbf{j}\rangle \text{ shift} : \begin{cases} [\hat{\mathbf{i}}|^2 = [\mathbf{i}|^2 - z[\mathbf{j}|^2, \\ |\hat{\mathbf{j}}\rangle^1 = |\mathbf{j}\rangle^1 + z|\mathbf{i}\rangle^1 \end{cases}$$

Good shift only exists for spin-projection $(s_i, s_j) = (1,2)$ or $(2,1)$

Need to use spin raising or lowering operator to construct amplitude

Boundary behavior

Study large-z behavior using dimensional analysis (for spin ≤ 1)

$$\hat{A}_n \sim z^{\gamma}, \quad \gamma = \gamma_N - [D] \leq [N] - [D] = 4 - n - [g] - \frac{N_F}{2}$$

n = number of external legs, [g] = dimension of coupling, N_F = number of external fermions

- Massive QED is on-shell constructible with the shift
- WWWW interaction scales as z^0, but Ward identity improves scaling for at least one transverse mode