A Generalization of Quantum Mechanics With a "Deformed" Interference Formula

Based on arXiv:2310.07457 and arXiv:24...(in preparation) with **D. Minic** and **T. Takeuchi**

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What this talk is about

- This talk is about a specific generalization of canonical quantum mechanics (QM) that only modifies the space over which "phases" of energy eigenstates evolve.
- In particular, I'll talk about the consequences it has for interference and oscillation phenomena.

Why consider generalizations?

- Better understanding: Relaxing the mathematical structure and generalizing QM can give insights into the aspects that were generalized.

- New phenomenology: It could describe physical phenomena not present in canonical QM.

- More parameters $\implies$ Wider testing: It could allow for a wider testing of certain aspects of QM.

Generalizations of quantum mechanics

- Canonical QM can be generalized in several distinct directions: Non-linear Schrödinger equation, replace $\mathbb{C}$ with $\mathbb{H}$, etc.
- QM has a rigid structure $\Longrightarrow$ Changes in dynamics can have unphysical consequences.
- For example, Weinberg's non-linear QM allows for FTL communication![1]
- In general, new parameters that quantify the deviation from QM should be strongly constrained.
- Our work generalizes QM through its geometric formulation.[2]

[1] See, however, Kaplan and Rajendran (PRD 105, 055002 (2022)) for a causal and non-linear framework.

[2] This formulation was developed in Kibble ('79), Heslot ('85), Ashtekar ('97), Brody ('99) and more.

Geometric quantum mechanics I

- Expand a state in terms of its energy eigenstates as $|\psi\rangle = \sum_n \psi_n |n\rangle$.

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle \implies \psi_n = N_n \, e^{-i\omega_n t},$$

where $H|n\rangle = \hbar\omega_n |n\rangle$.

- Write $\psi_n = q_n + ip_n$. Then $\vec{\psi}_n(t) = N_n \begin{bmatrix} \cos(\omega_n t) \\ -\sin(\omega_n t) \end{bmatrix}$, and

$$\frac{dq_n}{dt} = \omega_n p_n, \quad \frac{dp_n}{dt} = -\omega_n q_n.$$

- These are the classical Hamilton equations for coupled harmonic oscillators!

$$H = \sum_n \frac{1}{2}\omega_n (q_n^2 + p_n^2).$$

Geometric quantum mechanics II

- The complex inner product between $|\psi\rangle = \sum_n \psi_n |n\rangle$ and $|\phi\rangle = \sum_m \phi_m |m\rangle$ is given by

$$\begin{aligned} |\psi\rangle \; &= \; \sum_n \psi_n |n\rangle \; \mathit{and} \; |\phi\rangle \; = \; \sum_m \phi_m |m\rangle \\ \implies \langle\psi|\phi\rangle &= \left(\sum_n \psi_n^* \langle n|\right)\left(\sum_m \phi_m |m\rangle\right) \\ &= \underbrace{\sum_n (\vec{\psi}_n \cdot \vec{\phi}_n)}_{g(\psi,\phi)} + i \underbrace{\sum_n (\vec{\psi}_n \times \vec{\phi}_n)}_{\varepsilon(\psi,\phi)} . \end{aligned}$$

- So the probability amplitude is given by

$$|\langle\psi|\phi\rangle|^2 = g(\psi,\phi)^2 + \varepsilon(\psi,\phi)^2.$$

Generalization of geometric quantum mechanics

A generalization suggests itself: Replace the dynamics of the harmonic oscillator with a more complicated Hamiltonian. But not every arbitrary extension will be consistent and physically sensible!

- We extend this dynamics to that of an asymmetric top, with two conserved quantities

$$E = \frac{q_1^2}{2I_1} + \frac{q_2^2}{2I_2} + \frac{q_3^2}{2I_3}, \text{ and } L^2 = q_1^2 + q_2^2 + q_3^2.$$

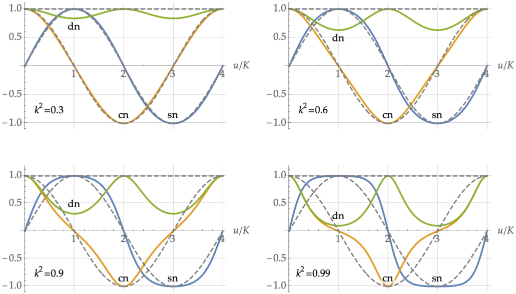
- The equations of motion are given by

$$\frac{dq_i}{dt} = \epsilon_{ijk} \left(\frac{1}{I_j} - \frac{1}{I_k} \right) q_j q_k.$$

Jacobi elliptic functions

- The solution of the above equations of motion is given in terms of Jacobi elliptic functions.

$$
\begin{aligned}
q_1(t) &= \quad N_1\,\mathrm{cn}(\Omega t, k),\\
q_2(t) &= -N_2\,\mathrm{sn}(\Omega t, k),\\
q_3(t) &= -N_3\,\mathrm{dn}(\Omega t, k).
\end{aligned}
$$



Consequences of the generalized dynamics

- The wavefunction is replaced by

$$\vec{\psi}_n \;=\; \underbrace{N_n \begin{bmatrix} \cos(\omega_n t) \\ -\sin(\omega_n t) \end{bmatrix}}_{|\vec{\psi}|^2 = N_n^2} \quad\rightarrow\quad \vec{\Psi}_n \;=\; \underbrace{A_n \begin{bmatrix} c_\xi \,\mathrm{cn}(\Omega_n t, k) \\ -\kappa_\xi \,\mathrm{sn}(\Omega_n t, k) \\ -s_\xi \,\mathrm{dn}(\Omega_n t, k) \end{bmatrix}}_{|\vec{\Psi}|^2 = A_n^2},$$

  where $c_\xi = \cos\xi,\; s_\xi = \sin\xi,\; \kappa_\xi = \sqrt{c_\xi^2 + k^2 s_\xi^2}$, and $0 \le k < 1$ and $-\frac{\pi}{2} \le \xi \le \frac{\pi}{2}$ are the deformation parameters.

- When $k = \xi = 0$, $\vec{\Psi}_n \rightarrow \vec{\psi}_n$ and the canonical QM limit is recovered.

- The inner product is generalized to

$$|\langle\Psi|\Phi\rangle|^2 \;=\; (\vec{\Psi}_n \cdot \vec{\Phi}_n)^2 + (\vec{\Psi}_n \times \vec{\Phi}_n) \cdot (\vec{\Psi}_n \times \vec{\Phi}_n).$$

"Phase" space of generalized quantum mechanics

The parameters $k \sim$ eccentricity and $\xi \sim$ size.

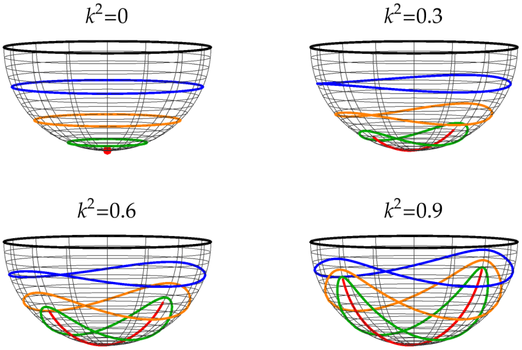


Figure: The colored lines in each figure indicate $\xi = 0$ (Black), $\xi = \pi/8$ (Blue), $\xi = \pi/4$ (Orange), $\xi = 3\pi/8$ (Green), and $\xi = \pi/2$ (Red). When $\xi = 0$, the trajectory always follows the equator regardless of the value of k.

Neutrino oscillation probability

- Flavor eigenstates of neutrinos, $|\alpha\rangle$ and $|\beta\rangle$, are superpositions of their mass eigenstates, $|1\rangle$ and $|2\rangle$.

$$\begin{aligned} |\alpha\rangle &= \quad \cos\theta\,|1\rangle + \sin\theta\,|2\rangle \\ |\beta\rangle &= -\sin\theta\,|1\rangle + \cos\theta\,|2\rangle \end{aligned}$$

- This causes the phenomena of interference and oscillation.

$$P(\alpha \to \beta) = \sin^2 2\theta \sin^2\left(\frac{\Delta m^2 L}{4E}\right).$$

In the current generalization upto $\mathcal{O}(k^2)$,

$$P_G(\alpha \to \beta) = \left(c_\xi^2 + \frac{k^2}{2} s_\xi^2\right) \sin^2 2\theta \, \sin^2\left(\frac{\Delta m^2 L}{4E}\right).$$

- Note that for $k = 0$, $\xi = 0$, $P_G \to P$.
- For $k = 0$, $\xi = \frac{\pi}{2}$, $P_G(\alpha \to \beta) = 0$, a classical behaviour!

Legget-Garg inequality: A measure of interference

- Consider a dichotomic observable $Q(t)$ that can only take on values ± 1.
- Then, for classical theories

$$K_3 := \langle Q(t_0)Q(t_1)\rangle + \langle Q(t_1)Q(t_2)\rangle - \langle Q(t_0)Q(t_2)\rangle \leq 1.$$

- Quantum mechanics violates this Legget-Garg[3] inequality ($K_3 > 1$) because of interference. (Contrast with Bell ineq.)

[3] A. J. Leggett and A. Garg PRL. 54, 857

Testing LG with neutrino oscillations

- Let $Q = +1$ if a neutrino is found in flavor α and $Q = -1$ if in β. Then,

$$\langle Q_i Q_j \rangle = 2P_{\alpha\alpha}(t_i, t_j) - 1$$

- Consequently,

$$K_3 = 2\left\{P_{\alpha\alpha}(t_0, t_1) + P_{\alpha\alpha}(t_1, t_2) - P_{\alpha\alpha}(t_0, t_2)\right\} - 1.$$ [4]

- LG quantifies interference. A theory that predicts a different interference pattern than canonical QM will give us a different value of K_3.

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[4] Also see D. S. Chattopadhyay and A Dighe (arXiv:2304.02475), where a different parameter is proposed as a measure of "quantumness".

Quantum to classical transition

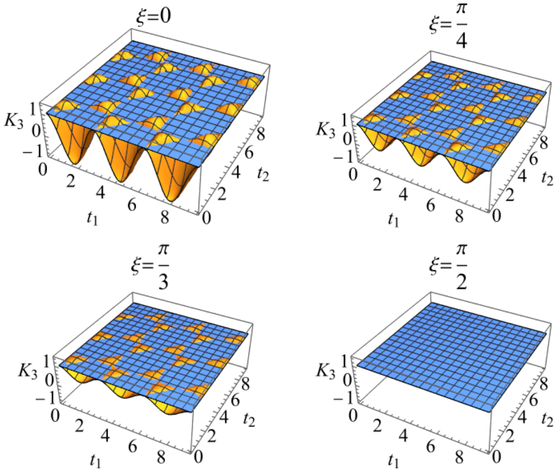
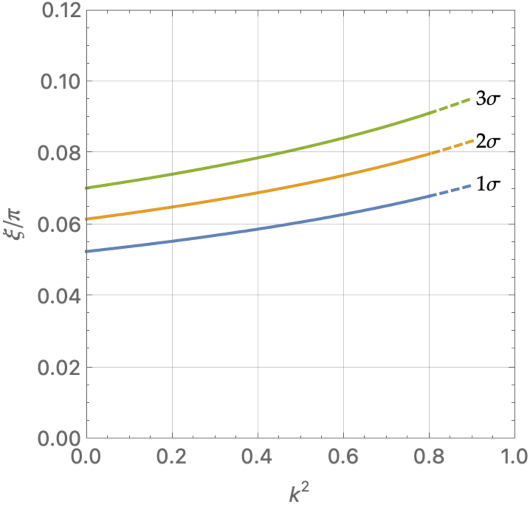


Figure: K_3 for different values of ξ for $k = 0$. Note that since $t_2 > t_1$, only the corresponding regions should be considered. LG inequality is violated for the pyramid regions above the plane $K3 = 1$

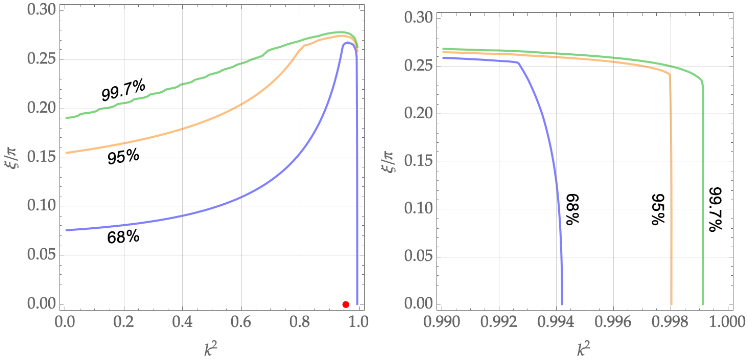
Summary

- The foundations of quantum mechanics can be confronted with experiments.
- Generalizing QM is one way to do it since it can help test old assumptions and provide new phenomenology.
- Such an exercise could give us a single framework encompassing classical theory, quantum theory, and possibly physics beyond QM.
- Neutrinos could potentially help us probe these issues experimentally.

Backup slide I (Atmospheric neutrino bounds)



Backup slide II (Bounds from $B^0 - \overline{B}^0$ oscillation)



Backup slide III (Nambu classical dynamics)

- The time evolution of a quantity f is given by:

$$\frac{df}{dt} = \{f, H\} = \varepsilon_{ij} \frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_j}.$$

- For a harmonic oscillator, $H = \omega \left(\frac{p^2}{2} + \frac{q^2}{2} \right), \ \frac{dp}{dt} = -\omega q.$

- For a system with two conserved quantities H_1 and H_2, an observable f evolves as:

$$\frac{df}{dt} = \{f, H_1, H_2\} = \varepsilon_{ijk} \frac{\partial f}{\partial q_i} \frac{\partial H_1}{\partial q_j} \frac{\partial H_2}{\partial q_k}.$$

- In a free asymmetric top,

$$H_1 = E = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3}, \ \ H_2 = \frac{L^2}{2} = \frac{1}{2} \left(L_1^2 + L_2^2 + L_3^2 \right)$$

are conserved.

Backup slide IV ("Deformation" of the interference formula)

- For two complex numbers $\psi_1 = A_1\, e^{i\theta_1} := A_1 \begin{bmatrix}\cos\theta_1\\ \sin\theta_1\end{bmatrix}$ and $\psi_2 = A_2\, e^{i\theta_2} := A_2 \begin{bmatrix}\cos\theta_2\\ \sin\theta_2\end{bmatrix}$,

$$|\psi_1 + \psi_2|^2 = A_1^2 + A_2^2 + 2\, A_1 A_2 \cos(\theta_1 - \theta_2).$$

- If we define "numbers" with a generalized phase

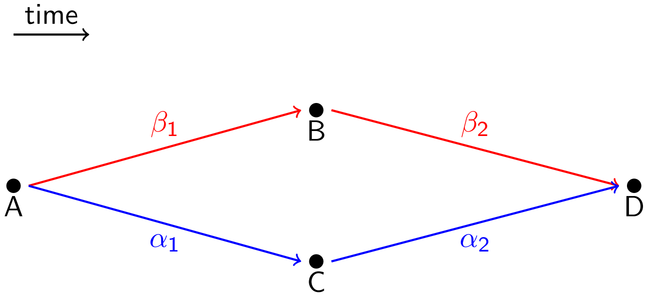
$$\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} \to \begin{bmatrix} c_\xi\, \mathrm{cn}(\theta,k)\\ \kappa_\xi\, \mathrm{sn}(\theta,k)\\ s_\xi\, \mathrm{dn}(\theta,k)\end{bmatrix}$$

so that

$$|\Psi_1+\Psi_2|^2 = A_1^2+A_2^2+2\, A_1 A_2 \Big(\cos\theta_1\cos\theta_2 + f(\xi,k)\sin\theta_1\sin\theta_2\Big),$$

where $-1 \le f(\xi,k) \le 1$.

A rose by any other name (Neutrino oscillation = Double slit experiment)



$$P_{(\alpha,\alpha)}(A \to D) = |\alpha_1 \alpha_2 + \beta_1 \beta_2|^2$$

$$= \underbrace{|\alpha_1|^2 |\alpha_2|^2}_{P_{ACD}} + \underbrace{|\beta_1|^2 |\beta_2|^2}_{P_{ABD}} + \underbrace{2\,\mathrm{Re}(\alpha_1^* \alpha_2^* \beta_1 \beta_2)}_{I_2(\alpha,\beta)}.$$