Full Prediction of the Bottom Quark Mass and Yukawa Coupling at $\mathcal{O}(\alpha\alpha_s)$

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1 Introduction

State of the Art

1 Introduction

m_b, Y_b, m_t, Y_t: QCD

- Four-loop up to $\mathcal{O}(\alpha_s^4)$ in perturbation theory [Marquard and al., 2016]

m_b, Y_b, m_t, Y_t: EW

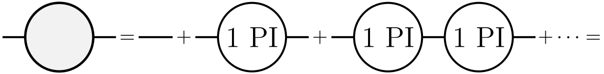
- Full two-loop correction at $\mathcal{O}(\alpha^2)$ is available [Kniehl and Veretin, 2014]

Mixed: Needs to be completed

- m_b and Y_b at $\mathcal{O}(\alpha\alpha_s)$ are expanded in terms of $\overline{m}_t$ and $\overline{m}_b$ itself [Kniehl, Piclum, and Steinhauser, 2004], quartic and quadratic respectively.
- **Y_b at this order is roughly one half of the one-loop EW correction.**
- Wavefunction renormalization constant at $\mathcal{O}(\alpha\alpha_s)$ for the bottom quark not available in literature. For the top quark only known in the $m_b = 0$ assumption [Eiras and Steinhauser, 2004].
- m_t and Y_t at $\mathcal{O}(\alpha\alpha_s)$ [Jegerlehner and Kalmykov, 2003].

Renormalization of the Fermion Masses

- To renormalize the masses, the propagators are expanded in terms of the self energies Σ
- For example, for the fermions:



$$= \frac{i}{\not{q} - m_{0,f} + \Sigma_f(\not{q})}$$

- The first term of the expansion explicitly reads

$$S_F^f(q) = \frac{i}{\not{q} - m_{0,f}} - \frac{i}{\not{q} - m_{0,f}} \Sigma_f(\not{q}) \frac{i}{\not{q} - m_{0,f}} - \ldots$$

- Σ can be expanded as vector Σ_V, and axial Σ_A and a scalar component Σ_S.

Overview: Mass

- The pole mass is defined as

$$0 = \left[\not{q} - m_{0,f} + \Sigma_f(\not{q})\right]\Big|_{\not{q}=M_f}.$$

- Expanding the self energy as

$$\Sigma_f(\not{q}) = \sum_l \left[\not{q}\Sigma_V^{(l)}\left(\frac{m_{0,f}^2}{q^2}\right) + \not{q}\gamma_5\Sigma_A^{(l)}\left(\frac{m_{0,f}^2}{q^2}\right) + m_{0,f}\Sigma_S^{(l)}\left(\frac{m_{0,f}^2}{q^2}\right)\right],$$

- The pole mass is given in terms of the bare mass as

$$M_f = m_{0,f}\left[1 - \Sigma_f^{(1)}(1) - \Sigma_f^{(2)}(1) + \Sigma_f^{(1)}(1)\left(\Sigma_V^{(1)}(1) - 2\Sigma_f^{(1)\prime}(1)\right)\right].$$

- We defined $\Sigma_f^{(l)} = \Sigma_V^{(l)} + \Sigma_S^{(l)}$. Axial term does not contribute to the correction.

Overview: Yukawa Coupling

1 Introduction

- The Yukawa coupling is defined as

$$Y_{f,0} = \frac{\sqrt{2}}{v_0} m_{f,0},$$

- The relationship between the $\overline{MS}$-Yukawa coupling and the pole mass M_f reads as

$$Y_f(\mu) = 2^{3/4} G_F^{1/2} M_f [1 + \delta_f(\mu)],$$

$$Y_{f,0} = Y_f(\mu) \left(1 + \delta_{f,CT}^{(1)} + \delta_{f,CT}^{(2)}\right).$$

- The Fermi constant has to be expressed in terms of SM parameters, exploiting its relation with muon lifetime [Sirlin, 1980].

$$G_F = \frac{1}{2\sqrt{2} v_0^2} \left[1 + \frac{\Pi_{WW}^{(1)}(0) + \Pi_{WW}^{(2)}(0)}{M_W^2} + \frac{T_{WW}^{(1)} + T_{WW}^{(2)}}{M_W^2} + E\right]$$

Overview: Wavefuntion Renormalization

1 Introduction

- The wavefunction renormalization constant Z_2 appears in the full renormalized inverse propagator as

$$iS_f^{-1}(\not{q}) = Z_{2,V}\left[\not{q} - m_{0,f} + \sum_l \left(\not{q}\Sigma_V^{(l)}\left(\frac{m_{0,f}^2}{q^2}\right) + m_{0,f}\Sigma_S^{(l)}\left(\frac{m_{0,f}^2}{q^2}\right)\right)\right] + Z_{2,A}\sum_l \not{q}\gamma_5\Sigma_A^{(l)}\left(\frac{m_{0,f}^2}{q^2}\right)$$

- Requiring that the residue for $\not{q} = M_F$ is one, one gets

$$Z_{2,V} = \sum_{l=1}^{2}\left[-\Sigma_V^{(l)}(1) + 2\Sigma_f^{(l)'}(1)\right] + 2\Sigma_f^{(1)}(1)\left(\Sigma_f^{(1)'}(1) + 2\Sigma_f^{(1)''}(1)\right),$$

$$Z_{2,A} = \sum_{l=1}^{2}\left[-\Sigma_A^{(l)}(1)\right] + 2\Sigma_f^{(1)}(1)\Sigma_A^{(1)'}(1).$$

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2 Master Integrals Approach

Diagrams and Topologies

2 Master Integrals Approach

- 24 diagrams for $\Sigma_f(\not{q})$ at $\mathcal{O}(\alpha\alpha_s)$. 3 topologies.

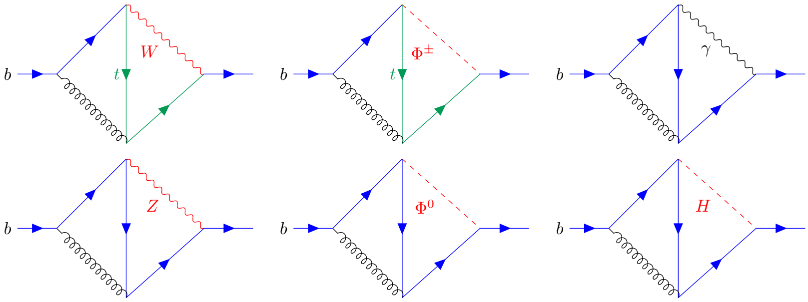


Figure: Top Sector Diagrams

Feynman Integrals and Differential Equations

2 Master Integrals Approach

- Feynman integrals belonging to the same topology may be expressed as linear combinations of the so called masters integrals (MIs).
- For a general basis $\vec{\mathcal{F}}$ we have

$$\frac{\partial}{\partial x_i}\vec{\mathcal{F}}(\{x_i\}) = A(\{x_i\},\epsilon)\vec{\mathcal{F}}(\{x_i\})$$

- Exploiting `Kira` [Maierhöfer, Usovitsch and Uwer, 2017] one can get the Laporta basis for the master integrals.
- However, in the canonical basis $\vec{\mathcal{G}}$, one can obtain [Henn,2014]

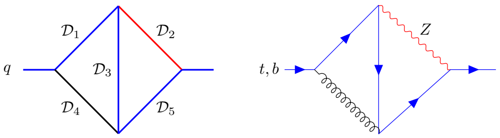
$$\frac{\partial}{\partial x_i}\vec{\mathcal{G}}(\{x_i\}) = \epsilon\mathcal{A}\{x_i\}\vec{\mathcal{G}}(\{x_i\})$$

The solution to the DEQ can be found iteratively [Chen, 1997; Caron-Huot and Henn, 2014]

$$\vec{\mathcal{G}}(\{x_i\}) = \left(\mathbb{I} + \epsilon\int_\gamma d\mathcal{A} + \epsilon^2\int_\gamma d\mathcal{A}d\mathcal{A} + \dots\right)\vec{\mathcal{G}}(\{x_i^0\})$$

Z & H Topology

2 Master Integrals Approach

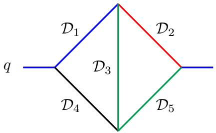


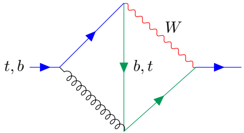
- **9 MIs**. All of them depend only on $x = \frac{M_Z^2}{m_b^2}$.
- To rationalize $\sqrt{-x(-4+x)}$ for values of $x > 4$, and solve as GPLs, one defines

$$x \to \frac{(1+m)^2}{m}$$

W Topology

2 Master Integrals Approach





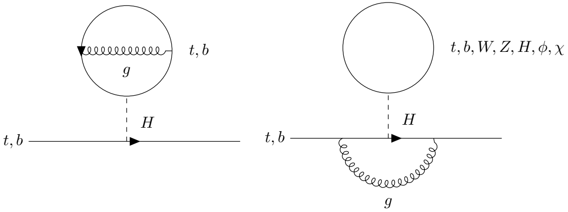
- **13 MIs**. Depend on $y = \frac{M_W^2}{m_b^2}$ and $z = \frac{m_t^2}{m_b^2}$. More complicated roots to rationalize to get the solution as GPL.
- To rationalise $\sqrt{y^2 + (z-1)^2 - 2y(z+1)}$ and $\sqrt{z}$ simultaneously, we define

$$y \to (1-v)\left(1 - \frac{w^2}{v}\right) \quad \text{and} \quad z \to w^2,$$

QED Topology & Tadpoles

2 Master Integrals Approach

- The diagrams that involve the photon as vector boson have no dynamical variable. An analytic expression can be easily obtained by direct integration
- Only the tadpoles involving the Higgs boson have a non-vanishing contribution to the 1-PI self-energy. Required for gauge and renormalization-group invariance.



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3 Results

Mass Correction: Results

3 Results

- We are now interested in the relation between the $\overline{MS}$-mass and the pole mass. The first step consists in finding $\frac{\overline{m}_b}{m_{b,0}}$ to cancel the poles in ϵ.

$$
\begin{aligned}
\frac{m_{b,0}}{\overline{m}_b} = 1 &- \frac{\alpha_s C_F}{\pi}\frac{3}{4}\frac{1}{\epsilon} + \frac{\alpha}{\pi}\Bigg[-\frac{3}{4}\left(Q_b^2+v_b^2-a_b^2\right) - \frac{3}{16\cos^2\theta_w\sin^2\theta_w}\frac{M_Z^2}{M_H^2} - \frac{3}{32\sin^2\theta_w}\frac{M_H^2}{M_W^2} \\
&- \frac{3}{8\sin^2\theta_w}\frac{M_W^2}{M_H^2} - \frac{3}{32\sin^2\theta_w}\frac{\overline{m}_t^2}{M_W^2} + \frac{3}{32\sin^2\theta_w}\frac{\overline{m}_b^2}{M_W^2} + \frac{N_C}{4\sin^2\theta_w}\frac{\overline{m}_t^4}{M_W^2M_H^2} + \frac{N_C}{4\sin^2\theta_w}\frac{\overline{m}_b^4}{M_W^2M_H^2}\Bigg]\frac{1}{\epsilon} \\
&+ \frac{\alpha\alpha_s}{\pi^2}C_F\Bigg\{\Bigg[\frac{9}{16}\left(Q_b^2+v_b^2-a_b^2\right) + \frac{9}{64\cos^2\theta_w\sin^2\theta_w}\frac{M_Z^2}{M_H^2} + \frac{9}{128\sin^2\theta_w}\frac{M_H^2}{M_W^2} + \frac{9}{32\sin^2\theta_w}\frac{M_W^2}{M_H^2} \\
&+ \frac{9}{64\sin^2\theta_w}\frac{\overline{m}_t^2}{M_W^2} - \frac{9}{64\sin^2\theta_w}\frac{\overline{m}_b^2}{M_W^2} - \frac{9}{16\sin^2\theta_w}\frac{\overline{m}_t^4}{M_W^2M_H^2} - \frac{9}{16\sin^2\theta_w}\frac{\overline{m}_b^4}{M_W^2M_H^2}\Bigg]\frac{1}{\epsilon^2} \\
&+ \Bigg[-\frac{3}{32}(Q_b^2+v_b^2) + \frac{21}{32}a_b^2 + \frac{9}{128\sin^2\theta_w} - \frac{3}{32\sin^2\theta_w}\frac{\overline{m}_t^2}{M_W^2} + \frac{3}{32\sin^2\theta_w}\frac{\overline{m}_b^2}{M_W^2} \\
&+ \frac{N_C}{8\sin^2\theta_w}\frac{\overline{m}_t^4}{M_W^2M_H^2} + \frac{N_C}{8\sin^2\theta_w}\frac{\overline{m}_b^4}{M_W^2M_H^2}\Bigg]\frac{1}{\epsilon}\Bigg\}
\end{aligned}
$$

Yukawa Coupling: Results

- The full result for the W self-energy is required [Djouadi and Gambino, 1993]
- One finds the counterterms as in [Kniehl et al., 2004], since quartic terms only appear in the tadpoles. The full dependence on $\overline{m}_b$ and $\overline{m}_t$ does affect the finite part of $\delta_b(\mu)$.

$$\delta^{(1)}_{f,CT} = -\frac{\alpha_s C_F}{\pi}\frac{3}{4}\frac{1}{\epsilon} + \frac{\alpha}{\pi}\Bigg[-\frac{3}{4}v_b^2 - \frac{3}{4}Q_b^2 - \frac{1}{\sin^2\theta_w}\left(\frac{3}{64\cos^2\theta_w} + \frac{3}{16}\right) + \frac{m_{t,0}^2}{M_W^2}\frac{1}{\sin\theta_w^2}\left(-\frac{3}{32} + \frac{N_C}{16}\right) + \frac{m_{b,0}^2}{M_W^2}\frac{1}{\sin\theta_w^2}\left(\frac{3}{32} + \frac{N_C}{16}\right)\Bigg]\frac{1}{\epsilon}$$

$$\delta^{(2)}_{f,CT} = -\frac{\alpha\alpha_s}{\pi}C_F\Bigg\{\left[\frac{9}{16}\left(Q_b^2 + v_b^2\right) + \frac{1}{\sin^2\theta_w}\left(\frac{9}{256\cos^2\theta_w} + \frac{9}{64}\right)\right]\frac{1}{\epsilon^2} + \Bigg[-\frac{3}{32}(Q_b^2 + v_b^2) + \frac{21}{32}a_b^2 + \frac{9}{128\sin^2\theta_w} + \frac{1}{\sin^2\theta_w}\frac{m_{t,0}^2}{M_W^2}\left(-\frac{3}{32} + \frac{5}{128}N_C\right) + \frac{1}{\sin^2\theta_w}\frac{m_{b,0}^2}{M_W^2}\left(\frac{3}{32} + \frac{5}{128}N_C\right)\Bigg]\frac{1}{\epsilon}\Bigg\}$$

Summary

- Original contributions at $\mathcal{O}(\alpha\alpha_s)$ for bottom quark mass and Yukawa coupling corrections, with full dependence on m_b and m_t.
- Reproduced counterterms for Yukawa coupling and obtained a complete expression for the poles of the mass. Preliminary results for the finite part.
- At the same order, the wavefunction renormalization constant is again an original contribution, not available at all for bottom quark and asymptotically expanded for top quark.

Full Prediction of the Bottom Quark Mass and Yukawa Coupling at $\mathcal{O}(\alpha\alpha_s)$

Thank you for listening!
Any questions?

Yukawa Coupling: More Details

- This gives an invariant quantity without involvement of tadpoles, using that

$$\frac{T^{(1)}_{WW}}{M^2_W} = T^{(1)}_f \quad \text{and} \quad \frac{T^{(2)}_{WW}}{M^2_W} = T^{(2)}_f + T^{(1)}_f \left[\Sigma^{(1)}_S(1) + 2\Sigma^{(1)\prime}_f(1)\right]$$

- Now, $\delta_f(\mu) = \delta^{(1)}_f(\mu) + \delta^{(2)}_f(\mu)$, where

$$\delta^{(1)}_f(\mu) = \Sigma^{(1)}_f - \frac{\Pi^{(1)}_{WW}(0)}{2M^2_W} - \frac{E}{2} - \delta^{(1)}_{f,CT};$$

$$\delta^{(2)}_f(\mu) = \Sigma^{(2)}_f - \frac{\Pi^{(2)}_{WW}(0)}{2M^2_W} + \Sigma^{(1)}_f(1)\left[\Sigma^{(1)}_S(1) + 2\Sigma^{(1)\prime}_f(1) - \frac{\Pi^{(1)}_{WW}(0)}{2M^2_W} - \frac{E}{2}\right]$$
$$- \delta^{(2)}_{f,CT} - \delta^{(1)}_{f,CT}\delta^{(1)}_f(\mu).$$

Yukawa Coupling: Plots

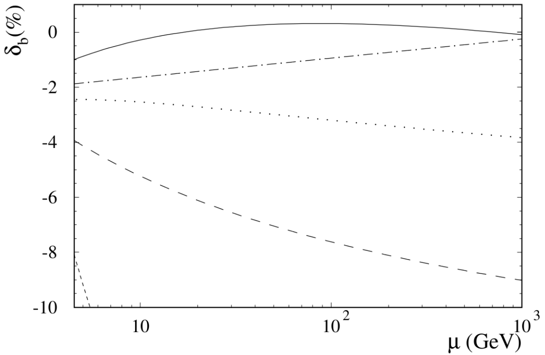

Figure: $\delta_b(\mu)$ as a function of μ for $M_H = 120 GeV$. The dash-dotted and full lines indicate the $\mathcal{O}(\alpha)$ and $\mathcal{O}(\alpha\alpha_s)$ contributions, respectively. For comparison, also the pure QCD contributions of $\mathcal{O}(\alpha_s)$ (short-dashed line), $\mathcal{O}(\alpha_s^2)$ (long-dashed line), and $\mathcal{O}(\alpha_s^3)$ (dotted line) are shown [Kniehl, Piclum, and Steinhauser, 2004].

Comments

4 Back-Up

- The axial term $Z_{2,A}$ has non-zero contribution only from W and Z diagrams, and depends on the QCD gauge parameter ξ [Eiras et al., 2005], while the former does not, similarly to the two loops result in QCD [Broadhurst, Gray and Schilcher, 1991].
- Off-shell, some topologies involve master integrals on elliptic curves [Broedel, Duhr, Dulat and Tancredi, 2017], [Hönemann, Tempest and Weinzierl, 2018], [Weinzierl, 2022]. In particular, the topology the involves the Z as heavy vector boson (Massive Kite) requires to solve elliptic integrals depending on two kinematic variables, whose solution is still not known in literature.