Is the Effective Potential, Effective for Dynamics?

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Outline

- The Usual Effective Potential—Static Case
- The Adiabatic Effective Potential—Dynamical Case
- Adiabatic Breakdown and Instabilities—Parametric Resonances
- Conclusions

The Effective Potential

- The effective potential origins: how do radiative corrections modify spontaneous symmetry breaking?
- Defined as the generating function of single particle irreducible Green's functions at zero momentum transfer.
- Useful in understanding phase transitions in quantum field theories.
- While originally computed using Feynman diagrams or functional methods, Symanzik (1970) gave a more expedient and intuitive Hamiltonian derivation (for zero temperature):

$$V_{eff}(\varphi) = \frac{1}{\mathcal{V}} \langle \Phi | H | \Phi \rangle$$

Volume

Coherent State

The Effective Potential—Static Case

- Consider a real scalar field Hamiltonian: $H = \int d^3x \left\{ \frac{\hat{\pi}^2}{2} + \frac{(\nabla\hat{\phi})^2}{2} + V(\hat{\phi}) \right\}$
- With the following conditions: $\varphi = \langle\Phi|\hat{\phi}(\vec{x},t)|\Phi\rangle \quad ; \quad \langle\Phi|\hat{\pi}(\vec{x},t)|\Phi\rangle = 0$
  - Therefore, the field is in a coherent state/condensate.
- Decompose into “classical”/mean field and fluctuation:

$$\hat{\phi}(\vec{x},t) = \underbrace{\varphi}_{\text{Spacetime constant mean field}} + \underbrace{\hat{\delta}(\vec{x},t)}_{\text{Quantum Fluctuation}} \quad ; \quad \hat{\pi}(\vec{x},t) \equiv \hat{\pi}_\delta(\vec{x},t)$$

The Effective Potential—Static Case

- Inserting into the Hamiltonian and expanding in δ, one readily obtains the effective potential:

$$V_{eff} = V(\varphi) + \frac{1}{\mathcal{V}} \int d^3x \, \langle\Phi|\left\{ \frac{\hat{\pi}_\delta^2}{2} + \frac{(\nabla\hat{\delta})^2}{2} + \frac{1}{2}\mathcal{M}^2(\varphi)\,\hat{\delta}^2 + \cdots \right\}|\Phi\rangle$$

- Where the fluctuation itself behaves as a free real scalar field with mass: $\mathcal{M}^2(\varphi) \equiv V''(\varphi)$
  - Note this mass is *time-independent* since the mean field/classical field is constant.

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- Solving the equations of motion for the fluctuation simplifies the expression:

$$\boxed{V_{eff}(\varphi) = V(\varphi) + \frac{\hbar}{2} \int \frac{d^3k}{(2\pi)^3}\, \omega_k(\varphi) + \mathcal{O}(\hbar^2) + \cdots}$$

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- Takeaways:
  1. The Hamiltonian approach works to extract the effective potential.
  2. The usual effective potential is a *static* quantity.

The Effective Potential—Dynamical Case

- Increasingly, phenomenologists have used the effective potential to describe the time evolution of the expectation values of homogeneous fields.
- The idea is to use the equation of motion:

$$\ddot{\varphi}(t) + \frac{d}{d\varphi} V_{eff}(\varphi(t)) = 0$$

- But is this ultimately justified?
- Consider the same Hamiltonian but with conditions: $\langle \Phi | \hat{\phi}(\vec{x}, 0) | \Phi \rangle = \varphi(0) \quad \langle \Phi | \hat{\pi}(\vec{x}, 0) | \Phi \rangle = \dot{\varphi}(0)$

$$\hat{\phi}(\vec{x}, t) = \underbrace{\varphi(t)}_{\text{Time-varying mean field}} + \underbrace{\hat{\delta}(\vec{x}, t)}_{\text{Quantum Fluctuation}} \quad ; \quad \hat{\pi}(\vec{x}, t) = \dot{\varphi}(t) + \hat{\pi}_\delta(\vec{x}, t)$$

The Effective Potential—Dynamical Case

- Inserting into the Hamiltonian one obtains the energy density:

$$\boxed{\mathcal{E} = \frac{\langle \Phi | \hat{H} | \Phi \rangle}{\mathcal{V}} = \frac{1}{2}\,\underline{\dot{\varphi}^2(t)} + \underline{V(\varphi(t))} + \underline{\mathcal{E}_f(t)}}$$

- Kinetic
- Classical Potential
- Fluctuation

- Where the one-loop fluctuation energy density:

$$\mathcal{E}_f(t) = \frac{\hbar}{2} \int \frac{d^3k}{(2\pi)^3} \left[|\dot{g}_k(t)|^2 + \omega^2(t)\, |g_k(t)|^2 \right]$$

  - The non-trivial time-dependence of $\varphi(t)$ means the fluctuation essentially has a time-dependent mass!

$$\mathcal{M}^2(\varphi) \equiv V''(\varphi)$$

  - The best one can do is express the result in terms of the mode functions of the fluctuation which satisfy:

$$\ddot{g}_k(t) + \omega_k^2(t) g_k(t) = 0 \;\; ; \;\; \omega_k^2(t) \equiv \left[k^2 + V''(\varphi(t)) \right]$$

Quasi-static/Adiabatic Approximation

- Often the dynamical situations of interest consider a slow evolution of the mean field.
- WKB Ansatz: $$g_k(t) = \frac{e^{-i\int_0^t W_k(t')dt'}}{\sqrt{2W_k(t)}}$$
- Adiabatic Expansion: $$W_k^2(t) = \omega_k^2(t)\left[1 - \frac{1}{2}\frac{\ddot{\omega}_k}{\omega_k^3} + \frac{3}{4}\left(\frac{\dot{\omega}_k}{\omega_k^2}\right)^2 + \cdots\right]$$

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- Insert into energy density; define the *adiabatic effective potential* (up to 2nd order adiabatic)*:*

$$V_{eff}^{(ad)}(\varphi) \equiv \underbrace{V(\varphi(t)) + \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\sqrt{k^2 + V''(\varphi(t))}}_{\text{usual 1-loop effective potential}} + \underbrace{\frac{\dot{\varphi}^2(t)}{64}(V'''(\varphi(t)))^2 \int \frac{d^3k}{(2\pi)^3}\frac{1}{(k^2 + V''(\varphi(t)))^{5/2}}}_{\text{2nd order adiabatic correction}}$$

Energy Conservation and Equation of Motion

- Energy is conserved for fields in Minkowski spacetime.
- Differentiating the energy density, one obtains the *true equation of motion* for the "classical" field:

$$\dot{\varepsilon} = 0 \quad \longrightarrow \quad \ddot{\varphi}(t) + V'(\varphi(t)) + \frac{\hbar}{2} V'''(\varphi(t)) \int \frac{d^3k}{(2\pi)^3} \, |g_k(t)|^2 = 0$$

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- Compare with the effective potential inspired equation: $\ddot{\varphi}(t) + \frac{d}{d\varphi} V_{eff}(\varphi(t)) = 0$
- Discrepancy: $U'(\varphi) - \frac{dV_{eff}^{(ad)}(\varphi)}{d\varphi} = \ddot{\varphi} \, \frac{\left(V'''(\varphi)\right)^2}{16} \int \frac{d^3k}{(2\pi)^3} \, \frac{1}{2\omega_k^5} + \cdots = \ddot{\varphi} \, \frac{\left(V'''(\varphi)\right)^2}{96\,\pi^2 \, V''(\varphi)} + \cdots$

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- Takeaways:
  1. Beyond zeroth adiabatic order, the equation of motion does NOT go with the effective potential!
  2. Insisting on using the effective potential entails violation of energy conservation!

How bad is this discrepancy really?

- Consider a simple tree level potential: $V(\varphi) = \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4}\varphi^4$
  - Where $m^2 > 0$
- In this case the discrepancy becomes:

$$U'(\varphi) - \frac{dV_{eff}^{(ad)}(\varphi)}{d\varphi} = \ddot{\varphi}(t)\,\frac{\lambda}{8\pi^2}\left[\frac{\left(3\lambda\varphi^2(t)/m^2\right)}{1+\left(3\lambda\varphi^2(t)/m^2\right)}\right]$$

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<u>Large Amplitude Limit:</u> $3\lambda\varphi^2(t)/m^2 \gg 1$

- Discrepancy seems perturbatively small.

- But for long wavelengths $k^2 \ll 3\lambda\varphi^2(t)$ adiabaticity is violated!

$$\frac{\ddot{\omega}_k(t)}{\omega_k^3(t)} \simeq \frac{\ddot{\varphi}(t)}{3\lambda\varphi^3} \simeq \mathcal{O}(1)$$

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Small Amplitude Limit: $3\lambda\varphi^2(t)/m^2 \ll 1$

- Discrepancy again seems perturbatively small.
- However, the classical potential will be mass dominated and feature oscillations around $\varphi = 0$.
- This leads to parametric resonance!

Parametric Resonances

- Consider mean field oscillations around minimum.

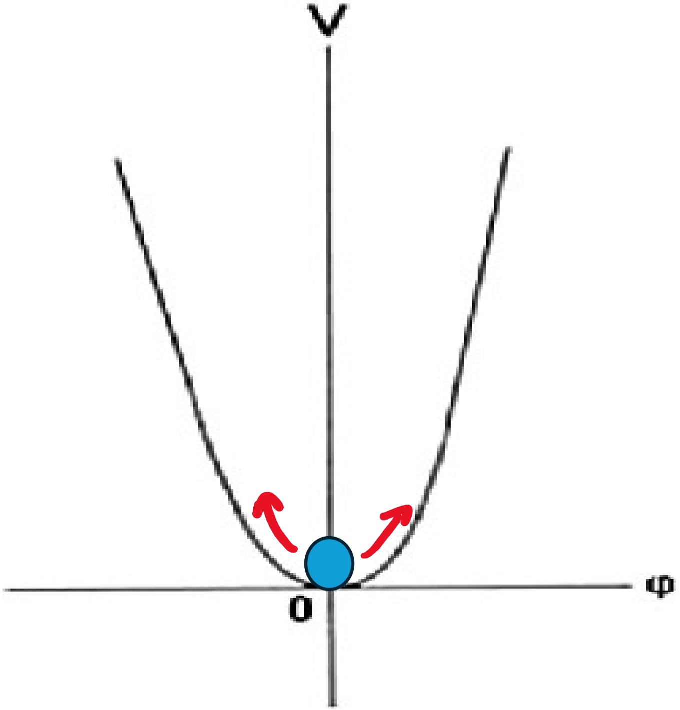
$$\varphi(t) = \varphi(0)\cos(mt)$$

- Floquet's Theory shows solutions have form:

$$g_k(\tau) = e^{i\nu_k\tau}\, P_k(\tau)$$

$$P_k(\tau+\pi) = P_k(\tau)$$

- The Floquet index, ν_k, becomes complex for certain ranges wavevectors → unstable modes!



Mode Function ODE → Mathieu's Equation

$$\frac{d^2}{d\tau^2}\, g_k(\tau) + \left[\eta_k - 2\alpha\, \cos(2\tau)\right] g_k(\tau) = 0$$

$$\alpha = 3\lambda\frac{\varphi^2(0)}{4\,m^2} \quad ; \quad \eta = 1+\kappa^2+2\alpha \quad ; \quad \kappa = \frac{k}{m}$$

Parametric Resonances

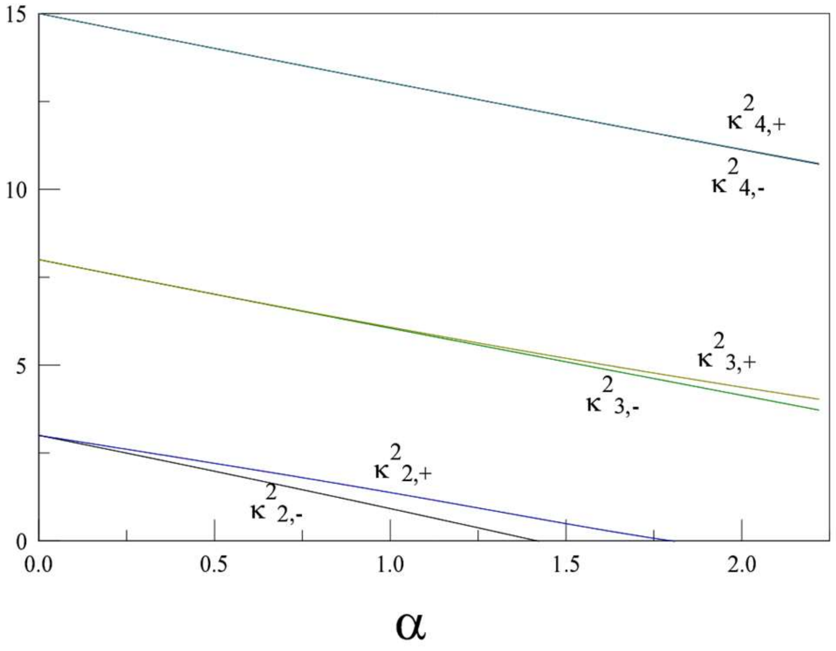
- Unstable modes correspond to bands of wavevectors.

- These bands narrow with
  - Small coupling
  - Small mean field amplitude
  - Large mass

$$\Delta k^2(n) = k^2_{n,+} - k^2_{n,-} = C_n \, \frac{(3\lambda\,\varphi^2(0)/4)^n}{m^{2(n-1)}} + \cdots$$

- Adiabatic modes are bounded in time. Unstable (growing) modes represent adiabatic breakdown!

Unstable Bands in the $\kappa - \alpha$ Plane



$$\kappa^2_{n,-} \leq \kappa^2 \leq \kappa^2_{n,+} \quad ; \quad \kappa^2 > 0 \quad ; \quad n = 0, 1, 2 \cdots$$

What do these instabilities represent?

- Energy is conserved, so these instabilities represent a “draining of energy” from “classical” energy density to the fluctuation term.

$$\mathcal{E} = \frac{\langle\Phi|\hat{H}|\Phi\rangle}{\mathcal{V}} = \underline{\frac{1}{2}\dot{\varphi}^2(t)} + \underline{V(\varphi(t))} + \underline{\mathcal{E}_f(t)}$$

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- This accumulation of energy in the fluctuation term can be viewed as a spontaneous production of *adiabatic particles*.
  - Can be shown via a time-dependent Bogoliubov transformation.
  - Asymptotically a stationary state fixed point should obtain.

- Note other sources of instability are possible. Recall: $\omega_k^2(t) \equiv \left[k^2 + V''(\varphi(t))\right]$
  - *Spinodal Instability*: If $V''\big(\varphi(t)\big) < 0, \rightarrow \omega(t)_k^2 < 0$ (for $k < |V''(\varphi)|$)
  - Imaginary frequencies possible when above the inflection point of the classical potential → unstable modes!
  - Example: This will occur for potentials with spontaneous symmetry breaking

Conclusions

- The usual effective potential **does not** correctly capture the dynamics of a *dynamical mean field*.
- Extending the effective potential concept to these situations via a *quasi-static/adiabatic approximation* to be implemented in the equations of motion is highly dubious:
  - Energy conservation is violated.
  - Adiabaticity is easily violated: 1.) Long wavelength modes, 2.) Instabilities
- In particular, both parametric resonance and spinodal instabilities can be viewed through lens of particle production.

Conclusions

- The usual effective potential **does not** correctly capture the dynamics of a *dynamical mean field*.
- Extending the effective potential concept to these situations via a *quasi-static/adiabatic approximation* to be implemented in the equations of motion is highly dubious:
  - Energy conservation is violated.
  - Adiabaticity is easily violated: 1.) Long wavelength modes, 2.) Instabilities
- In particular, both parametric resonance and spinodal instabilities can be viewed through lens of particle production.
- How to properly handle the dynamical case? Use true energy-conserving equations of motion:

$$\ddot{\varphi}(t) + V'(\varphi(t)) + \frac{\hbar}{2} V'''(\varphi(t)) \int \frac{d^3k}{(2\pi)^3} |g_k(t)|^2 = 0$$

$$\ddot{g}_k(t) + \omega_k^2(t) g_k(t) = 0 \;\; ; \;\; \omega_k^2(t) \equiv \left[k^2 + V''(\varphi(t))\right]$$

- Closed set of equations.
- Can renormalize away UV divergences.
- Can be solved numerically via appropriate initial conditions on $\varphi, \dot{\varphi}, g_k, \dot{g}_k$