Gauging the Maximal Compact Subgroup of a Simple Lie Group

Gauge Theories with an Infinite Multiplet of Fermions

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Non-Compact Simple Lie Groups

- Faithful unitary Irreducible representations of a non-compact Lie group are always infinite dimensional
- They have been of limited utility in particle physics so far
- A free particle can be thought of as an unitary irrep of the Poincare group
- A massless particle is a unitary irrep of the conformal group $SO(4,2) \sim SU(2,2)$
- Before QCD, non-compact groups were considered as internal symmetries to explain the apparently infinite number of hadrons
- Such symmetries arise naturally is supergravity and (super)-string theory
- Of some use in Quantum Mechanics to understand the spectrum of the harmonic oscillator- $SU(1,1)$-or the scattering states of the hydrogen atom -$SO(3,1)$).

Gauge Groups Have Positive Inner Product

- The Yang-Mills action is

$$L_{YM} = -\frac{1}{4} g_{ab} F^a_{\mu\nu} F^{b\mu\nu}$$

where $a, b = 1, \cdots d$ label a basis in the Lie algebra $\underline{K}$ of the gauge group K

$$[e_a, e_b] = f^c_{ab} e_c$$

- The symmetric matrix g_{ab} must be an invariant inner product on $\underline{K}$:

$$f^d_{ab} g_{dc} + g_{ad} f^d_{bc} = 0$$

for the action to be gauge invariant

- To have a positive inner product in the quantum Hilbert space we need g_{ab} to be a positive matrix ("BRS No Ghosts Theorem")

Gauge Lie Algebras Are of Compact Type

- A Lie algebra $\underline{K}$ with a positive invariant inner product can exponentiate to a compact Lie group K
- So we will say that such Lie algebras of "compact type"
- They are direct sums of compact simple Lie algebras and some abelian Lie algebra
- E.g., The standard model: $\underline{K} = \underline{U}(1) \oplus \underline{SU}(2) \oplus \underline{SU}(3)$ exponentiates to $S\left(U(2) \times U(3)\right)$.
- The coupling constants of a gauge theory parametrize solutions for g_{ab}; e.g., for the standard model there is a three parameter family of invariant inner products
- Recall that there are many Lie groups for the same Lie algebra.
- Subtle point: Some of these groups may not be compact even if the Lie algebra is of compact type
- Think of $U(1)$ vs its universal cover $\mathbb{R}$

Gauge Theories With Fermionic Matter

- The fermions are in a (possibly reducible) unitary representation of $\underline{K}$
- The usual perturbative renormalization procedure works only if the fermionic representation is finite dimensional: otherwise some of the Casimirs (that appear in the beta function for example) can diverge
- Asymptotic freedom imposes strict constraints on the size of the fermion representation
- A tiny window of theories are of special interest as they are both asymptotically free and have an IR stable fixed point (e.g., Banks-Zaks for QCD)

QCD-like Composite Models

- There is a natural way to get bosons of small mass (compared to the compositeness scale) as bound states: spontaneously break some internal symmetry
- There is no simple way to get fermions of small mass; spontaneous breaking of supersymmetry has been tried but found wanting
- One idea is that fundamental theory is a gauge theory with massless fermions (much like QCD)
- It is asymptotically free and has a non-trivial IR stable fixed point
- Near this fixed point there can be mass less bound states: some scalars and some fermions
- Perhaps these can be Higgs and quarks/leptons
- With a finite number of fundamental fermions the possibilities are tightly constrained

Infinite Multiplets of Fermions

- A natural origin for an infinite multiplet of fermions is an approximate symmetry under a non-compact Lie Group G
- The gauge group $K \subset G$ is a compact subgroup; for example, the maximal compact subgroup
- The reduction of a unitary representation of G will give an infinite sum over irreducible representations of K
- We would need a regularization and renormalization in the sum over these representations in addition to the usual regularization and renormalization of momenta in Feynman diagrams
- Need to give a meaning to the Casimirs; zeta function regularization is natural

Discrete Series Representations

- Especially nice mathematical theory in the case of "Discrete Series": each element of the representation matrices are square integrable functions on the group
- A beautiful branch of mathematics with deep connections to number theory (Langlands Program)

Theorem

(Harish-Chandra) $\underline{G}$ *has Discrete Series representations iff its rank is the same as that of its maximal compact subalgebra* $\underline{K}$

An Example: $\underline{U}(1) \subset \underline{SU}(1,1)$

- The Lie algebra $\underline{SU}(1,1) \approx \underline{SL(2,R)}$ has commutation relations $[e_-, e_+] = e_0, [e_0, e_\pm] = \pm 2 e_\pm$
- e_0 spans the maximal compact subalgebra $\underline{U}(1)$
- The simple harmonic oscillator provides an example of a unitary representation
- With $[a, a^\dagger] = 1$ we have

$$\left[\frac{a^2}{2}, \frac{a^{\dagger 2}}{2}\right] = \left(a^\dagger a + \frac{1}{2}\right)$$

$$\left[a^\dagger a + \frac{1}{2}, \frac{a^{\dagger 2}}{2}\right] = 2\frac{a^{\dagger 2}}{2}, \quad \left[a^\dagger a + \frac{1}{2}, \frac{a^2}{2}\right] = -2\frac{a^2}{2}$$

An Example: $\underline{U}(1) \subset \underline{SU}(1,1)$ (contd.)

- The states with even eigenvalues for $a^\dagger a$ form one unitary irreducible representation: $|\,0\rangle, |\,2\rangle, \cdots$; those with odd eigenvalues $|\,1\rangle, |\,3\rangle, \cdots$ form another
- **Subtle point**: these exponentiate to representations of the **double cover** ("Metaplectic Group") of the group $SU(1,1)$: because $a^\dagger a + \frac{1}{2}$ has half-integer eigenvalues
- We can gauge the $\underline{U}(1)$ subalgebra, with massless Dirac fermions in the "even" (resp. "odd") multiplet above
- The particles have charges $e_n = \frac{1}{2}, 2\frac{1}{2}, 4\frac{1}{2} \cdots$ for the "even" multiplet; for the "odd" multiplet $e_n = 1\frac{1}{2}, 3\frac{1}{2}, 5\frac{1}{2}, \cdots$.
- More generally, discrete series representations have charges in an arithmetic sequence

$$e_n = j + 2n, \quad n = 0, 1, \cdots$$

- The lowest weight j can be **any positive real number** if we allow the Lie group to be the universal cover $\widetilde{SU}(1,1)$. (For rational j it will be a finite index cover.)

Abelian Gauge Theory

- The perturbative beta function of QED with a single fermion of unit charge is, up to fourth order in charge vertices[DeRafael and Rosner,Itzykson and Zuber]

$$\beta(\alpha) = \frac{2}{3}\frac{\alpha}{\pi} + \frac{1}{2}\left(\frac{\alpha}{\pi}\right)^2 + \mathrm{O}(\alpha^3)$$

- If we have a multiplet of charges e_n instead we would have

$$\beta(\alpha) = \frac{2}{3}\frac{\alpha}{\pi}\sum_n e_n^2 + \frac{1}{2}\left(\frac{\alpha}{\pi}\right)^2 \sum_n e_n^4 + \mathrm{O}(\alpha^3)$$

- To this order there are no fermion loops; also the beta function is "scheme" independent (i.e., unchanged under changes of coupling constant $\alpha \mapsto f(\alpha)$)
- For a finite multiplet this gives the familiar result that QED beta function is positive: the two coefficients are separately positive.

Abelian Gauge Theory (contd.)

- In the next order (six vertices) the diagrams with one fermion loop would give $\sum_n e_n^6$
- Those with two fermion loops will have a factor of $\sum_n e_n^4 \sum_n e_n^2$
- The beta function is known to very high order (five?); but the qualitative UV/IR behavior don't change, I believe.
- Still, it would be good to make a more detailed study beyond what I describe below including higher orders

Abelian Gauge Theory with an Infinite Multiplet of Fermions

- For infinite multiplets (such as those arising from a discrete series representation of $SU(1,1)$) the sums $z(k) = \sum_n e_n^k$ can diverge.
- For the discrete series representations of $\underline{U}(1) \subset \underline{SU}(1,1)$:

$$z(k,j) = \sum_{n=0}^{\infty} (j+2n)^k$$

- A further regularization/renormalization is needed to make sense of these "Casimir sums"
- A method which works in many contexts is to consider the sum $\sum_{n=0}^{\infty} (j+2n)^{-s}$. For $\mathrm{Re}\, s > 1$ this converges and can be related to the Hurwitz Zeta function $\zeta_H(s,a) = \sum_{n=0}^{\infty}(n+a)^{-s}$:

$$\sum_{n=0}^{\infty} (j+2n)^{-s} = 2^{-s}\zeta_H\left(s, \frac{j}{2}\right)$$

The Hurwitz Zeta Function

- T. M. Apostol *Introduction to Analytic Number Theory* Springer (1972)
- The Riemann zeta function is the special case $\zeta_H(s, 1)$.
- The only singularity of $\zeta_H(s, a)$ is also is a simple pole at $s = -1$
- So we can get by analytic continuation

$$z(k, j) = \sum_{n=0}^{\infty} (j + 2n)^k = 2^k \zeta_H \left(-k, \frac{j}{2} \right)$$

- Of particular interest to us are the values at $s = -k = -1, -2, \cdots$

$$\zeta_H(-k, a) = -\frac{B_{k+1}(a)}{k + 1}$$

and

$$z(k, j) = -2^k \frac{B_{k+1}\left(\frac{j}{2}\right)}{k + 1}$$

where $B_k(a)$ are the Bernoulli polynomials

Bernoulli Polynomials

- There is a generating function
$$\sum_{k=0}^{\infty} B_k(a)\frac{t^k}{k!} = \frac{te^{at}}{e^t-1}$$
- A finite sum
$$B_k(a) = \sum_{r=0}^{k}\left[\frac{1}{r+1}\sum_{l=0}^{r}(-1)^l \left(\begin{array}{c} r \\ l \end{array}\right)(a+l)^k\right]$$
- If $a=\frac{1}{2}$ we have $\frac{te^{\frac{1}{2}t}}{e^t-1} = \frac{t}{e^{\frac{1}{2}t}-e^{-\frac{1}{2}t}}$ is an even function so that $B_k\left(\frac{1}{2}\right) = 0$ for all odd k
- If $a = 1$we have
$$\frac{te^t}{e^t-1} = \frac{t}{2} + \frac{1}{2}\frac{t}{e^{\frac{t}{2}}-e^{\frac{t}{2}}}\left(e^{\frac{t}{2}}+e^{\frac{t}{2}}\right)$$
so that $B_1(a) = \frac{1}{2}$, and $B_k\left(a\right) = 0$ for all odd $k > 1$

Explicit Values for $z\left(k, j\right)$

- So the values we need for $z(k, j) = \sum_{n=0}^{\infty} (j + 2n)^k$ are known explicitly

$$z(0, j) = \frac{1 - j}{2}, \quad \text{"virtual dimension"}$$

$$z(2, j) = -\frac{1}{6} j(j - 1)(j - 2)$$

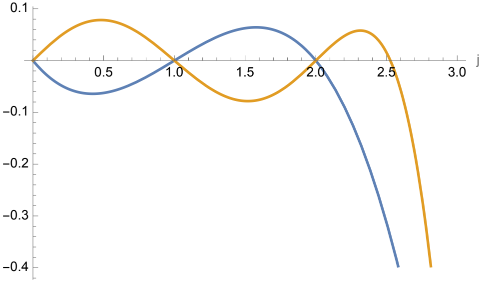
$$z(4, j) = -\frac{1}{30} j(j - 1)(j - 2)(3j^2 - 6j - 4)$$

- Higher orders will need $z(6, j)$ etc. which are also known explicitly
- Also, $z\left(k, \frac{1}{2}\right) = 0 = z\left(k, 1\right)$ for all even $k = 2, 4, 6, \cdots$
- Sum over odd powers of charges won't appear because of Furry's theorem (γ_5 invariance)
- Incidentally, $z(2, j)$ and $z(4, j)$ are related to the finite sums $\sum_{\text{even } m}^{j} m^2, \sum_{\text{even } m}^{j} m^4$ by changing $j \to -j$. Mysterious.

Abelian Gauge Theory with a Discrete Series of Fermions

- So we have

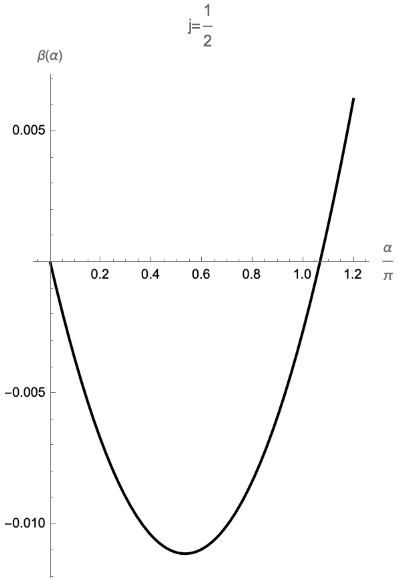
$$\beta(\alpha) = -\frac{1}{9}j(j-1)(j-2)\frac{\alpha}{\pi} + \frac{1}{60}j(j-1)(j-2)(3j^2-6j-4)\left(\frac{\alpha}{\pi}\right)^2 + \mathrm{O}(\alpha^3)$$



Asymptotically Free Abelian Gauge Theory

- So we have asymptotic freedom for $0 < j < 1$ and $j > 2$
- In addition, there may be a fixed point for $0 < j \lesssim 2.52753$ at $\frac{\alpha_0}{\pi} \approx \frac{20}{3(3j^2-6j-4)}$; which varies between $\frac{20}{21} \lesssim \frac{\alpha_0}{\pi} \lesssim \frac{5}{3}$ as j varies from 1 to 0
- It is IR stable for $0 < j < 1$ and $2 < j \lesssim 2.52753$
- At the fixed point are $\frac{\alpha}{\pi} \gtrsim 1$; outside the range of perturbation theory
- **Whether there really is such a fixed point can only be verified by lattice simulations**
- For $1 < j < 2$ this theory is not asymptotically free; but could be "asymptotically safe": the coupling constant $\frac{\alpha}{\pi}$ tends at high energies to a constant of order one
- **How does one regularize the fermion multiplet for lattice simulations?**

Asymptotically Free Abelian Gauge Theory (contd.)



Perturbatively Finite Abelian Gauge Theory

- For $j = 1, 2$ both coefficients $z(2, j)$ and $z(4, j)$ vanish.
- In fact all the sums of even powers of charged $z(k, 1)$ and $z(k, 2)$ vanish for $k = 2, 4, 6, \cdots$; suggests that the beta function is identically zero when $j = 1, 2$
- Does this mean that abelian gauge theory of this type is a CFT, analogous to $N = 4$ Super Yang-Mills?
- Do they have Gravity Duals?
- Or could be they be free field theories?

$\underline{G} = \underline{SU}(p, q)$

- The maximal compact sub-algebra of $\underline{G} = \underline{SU}(p, q)$ is $\underline{K} = \underline{U}(1) \oplus \underline{SU}(p) \oplus \underline{SU}(q))$
- The groups are $SU(p, q)$ and $S\left(U(p) \times U(q)\right)$ or any of their covers
- Of special interest is the case $p = 2, q = 3$ as the standard model gauge group is $S\left(U(2) \times U(3)\right)$
- $K$and G have the same rank
- So there are discrete series unitary representations of G (Harish-Chandra Amer. J. Math. 77 (1955), 743-777)
- They can be constructed one the space of complex analytic functions of a $p \times q$ matrix Z satisfying $Z^\dagger Z < 1$ (This "Segal disk" is G/K)
- Similar to the way the even states of the harmonic oscillator can be thought as analytic functions on the unit disk in the complex plane (the case $p = 1 = q$)

$\underline{K} = \underline{U}(1) \oplus \underline{SU}(p) \oplus \underline{SU}(q) \subset \underline{SU}(p,q)$

- This gauge theory will have three coupling constants, much like the standard model
- All the information needed to calculate the beta functions is in the character functions of the representations, for which there are explicit formulas (i.e., explicit construction of the representations is possible but not needed).
- At two loops (which is enough to see if there are fixed points) the basic calculations have been a long time ago. e.g., D.R.T.Jones, Phys. Rev. D25 (1982) 581.
- Some work is needed to adapt these results to a generic Lie algebra of compact type
- But indications are that there are again new classes of gauge theories with non-trivial fixed points

Other Discrete Series

- Another example is $\underline{SO}(m) \oplus \underline{SO}(n) \subset \underline{SO}(m,n)$
- These are also holomorphic discrete series: they can be obtained as analytic functions on $SO(m,n)/S\left(O(m) \times O(n)\right)$
- There are quaternionic discrete series related to symplectic groups: $\underline{U}(n) \subset \underline{Sp}(n)$
- These are obtained as solutions to a higher dimensional Dirac equation
- There is a unified approach to all of the discrete series:M. Atiyah and W. Schmid Inv. Math. **42,** 1-62 (1977)
- Thanks to Charles Nash for the last reference