Towards QCD splitting functions at four-loop order

Tong-Zhi Yang

with Thomas Gehrmann, Andreas von Manteuffel and Vasily Sotnikov

Based on JHEP 04 (2023) 041, JHEP 01 (2024) 029, Phys.Lett.B 849 (2024) 138427

QCD@LHC 2024
University of Freiburg, Freiburg, Oct. 7-11

Universität Zürich[UZH]

European Research Council
Established by the European Commission

Swiss National Science Foundation

Parton densities and splitting functions

- Quark parton density

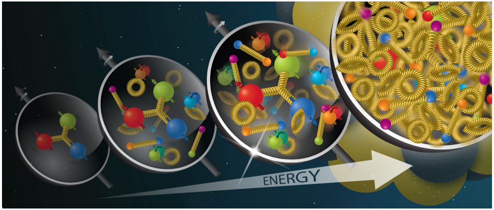
$$f_{q|N}(x_B) = \int \frac{dt}{2\pi} e^{-i\, x_B\, t\Delta\cdot p} \langle N(P)|\, \left[\bar{\psi} W\right](t\Delta) \frac{\not{\Delta}}{2} \left[W^\dagger \psi\right](0) |N(P)\rangle$$

- Quark collinear Wilson line

$$W^\dagger(x) = \bar{\mathcal{P}} \exp\Big(-i\, g_s \int_{-\infty}^{0} ds\, \Delta\cdot A(x + s\,\Delta)\Big)\,, \Delta^2 = 0$$

- Splitting functions (SFs) govern the DGLAP evolutions of PDFs

$$\frac{df_{i|N}}{d\ln\mu} = 2\sum_k P_{ik} \otimes f_{k|N}$$



Splitting functions & Anomalous dimensions (A.D.)

- Mellin transformation

$$f_q(n) = -\int_0^1 dz\, z^{n-1} f_q(z)\,, \quad \gamma_{ij}(n) = -\int_0^1 dz\, z^{n-1} P_{ij}(z)$$

- DGLAP evolution in n-space

$$\frac{d}{d\ln\mu} f_q(n,\mu^2) = -2\sum_j \gamma_{qj}(n)\, f_j(n,\mu^2)$$

- PDFs in n-space are hadronic operator matrix elements (OMEs)

$$f_q(n) \sim \langle N(P)|\, \bar\psi \not\Delta (\Delta\cdot D)^{n-1}\psi\, |N(P)\rangle = \langle N(P)|\, O_q\, |N(P)\rangle$$

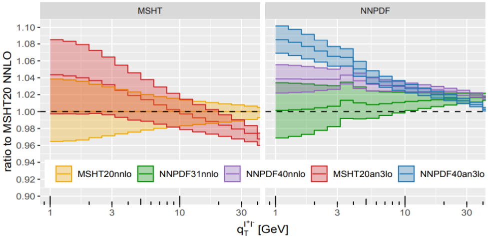
$$f_g(n) \sim \langle N(P)| \Delta_{\mu_1} G^{\mu_1}_{a,\mu} (\Delta\cdot D)^{n-2}_{ab} \Delta_{\mu_n} G^{\mu_n \mu}_{b}\, |N(P)\rangle = \langle N(P)|\, O_g\, |N(P)\rangle$$

- Twist-two operators, twist = [mass] - spin

$$O_q = \mathcal{O}_q^{\mu_1\cdots\mu_n} J_{\mu_1\cdots\mu_n} = \left[\bar\psi \gamma^{\mu_1} D^{\mu_2}\cdots D^{\mu_n}\psi\right]\left[\Delta_{\mu_1}\cdots\Delta_{\mu_n}\right]$$

Motivation: Required by high-precision physics

- (HL-)LHC and EIC will generate precise experimental data
- Several matching coefficients are available at N3LO $\longrightarrow$ N3LO PDFs
- The effect of N3LO PDFs can be large [talk by T. Neumann from LL2024]



- The fields in fitting N3LO PDFs are active
  - MSHT20 aN3LO[McGowana,Cridgea,Harland-Langb,Thorne, 2022]
  - NNPDF aN3LO[R. D. Ball et al. 2024]
  - CTEQ is planning talk by P. Nadolsky from DIS2024
- $P_{ij}^{(3)}$ is a crucial ingredient in N^{4}LL resummation talk by T. Becher

Motivation: Theoretical interest

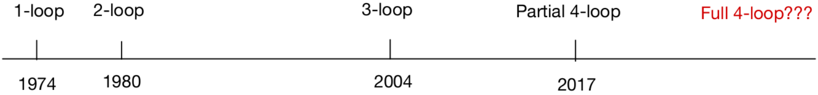
- Universal A.D. in planar $\mathcal{N} = 4$ to seven loops[Marboe, Velizhanin, 2016]
- Reciprocity relation between space-like and time-like A.D.
  - In CFT: $2\gamma^S(n) = 2\gamma^T(n + 2\gamma^S(n))$[Basso, Korchemsky, 2006]
  - Non-singlet in QCD $2\gamma^S(n) = 2\gamma^T(n + 2\gamma^S(n))$[Dokshitzer, Marchesini,Salam, 2006;Basso, Korchemsky, 2006; Mitov,Moch,Vogt,2006]
  - Singlet in QCD: $2\gamma^S_\pm(n) = 2\gamma^T_\pm(n + 2\gamma^S_\pm(n))$ with $\gamma_\pm$ being eigenvalues[Chen, **TZY**, Zhu, Zhu, 2020]
- Simple mathematical structures
  - Only HPLs[Remiddi and Vermaseren,1999] or Harmonic Sums[Vermaseren 1998,Blumlein and Kurth,1998] up to three loops in QCD
  - Non-planar universal anomalous dimensions in $\mathcal{N} = 4$, only Harmonic Sums[B.A. Kniehl, V.N. Velizhanin, 2021]
  - Strong hints: only Harmonic Sums in four-loop QCD

Splitting functions to three loops

- One-loop[Gross, Wilczek, 1973, 1974]
- Two-loop[Curci,Furmanski,Petronzio,1980ab]
- Three-loop
  - DIS[Moch,Vermaseren, Vogt,2004,2004]
  - Massive OMEs[Ablinger,Behring,Blumelein, Freitas,Klein, Manteuffel, Schneider, Schönwald, Wissbrock,2010, 2014abc, 2017]
  - Hadronic cross sections[Anastasiou,Duhr, Dulat,Herzog,Mistlberger,2015;Duhr,Dulat,Mistlberger,2020]
  - Beam functions[Luo,**TZY**,Zhu,Zhu,2019, 2020; Ebert,Mistlberger,Vita,2020ab; Baranowski,Behring,Melnikov,Tancredi,Wever,2022]
  - Non-singlet from off-shell OMEs[Blümlein,Marquard,Schneider,Schönwald, 2021]
  - Singlet from off-shell OMEs This talk

Towards four-loop splitting functions

- Timeline of the calculation of splitting functions

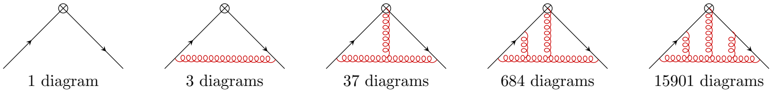


- Fixed moments
  - Non-singlet $\gamma_{\rm ns}^{(3)}$ with $n \leq 16$[Moch,Ruijl,Ueda,Vermaseren,Vogt,2017]
  - Singlet $\gamma_{\rm ps}^{(3)}$, $\gamma_{qg}^{(3)}$and $\gamma_{gq}^{(3)}$ with $n \leq 20$[Falcioni,Herzog,Moch,Pelloni,A. Vogt,2023, 2024]
  - $\gamma_{gg}^{(3)}$ with $n \leq 10$[Moch,Ruijl,Ueda,Vermaseren,Vogt,2023]
- Exact results with all-n dependence
  - All $\gamma_{ij}^{(3)}$ in the large-N_f limit[Gracey 1994,1996; Davies,Vogt, Ruijl,Ueda,Vermaseren,2016]
  - $\gamma_{\rm ns}^{(3)}$ with leading color[Moch,Ruijl,Ueda,Vermaseren,Vogt,2017]
  - N_f^2 term for $\gamma_{qg}^{(3)}$ [Falcioni,Moch,Ruijl,Ueda,Vermaseren,Vogt,2023]
  - N_f^2 term for $\gamma_{\rm ps}^{(3)}$ and $N_f C_F^3$ term for $\gamma_{\rm ns}^{(3)}$ This talk

OME method

- Partonic off-shell OME

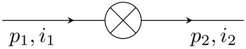
$$f_{qj}(n) \sim \langle j(p)|\bar{\psi}\not{\Delta}(\Delta \cdot D)^{n-1}\psi|j(p)\rangle \text{ , with } p^2 < 0$$



- OME method contains non-redundant information to extract SFs
- Off-shell OMEs are not gauge invariant, physical operators mix with gauge-variant (GV) operators
- Main goal: find all GV operators or their Feynman rules

Computation of off-shell OMEs with all-n dependence

- Non-standard terms appearing in the Feynman rules
- Example: Feynman rules for O_q at lowest order



$$\to \not{\Delta}(\Delta \cdot p_1)^{n-1}$$

- How to retain all-n dependence?
  - Sum non-standard term into a linear propagator using a tracing parameter t[Ablinger,Bluemlein,Hasselhuhn,Klein,Schneiderm,Wissbrock,2012]

$$(\Delta \cdot p)^{n-1} \to \sum_{n=1}^{\infty} t^n (\Delta \cdot p)^{n-1} = \frac{t}{1 - t\Delta \cdot p}$$

  - Parameter-t space in HPLs $\to$ n-space in Harmonic Sums

$$H(1,1;t) = \sum_{n=1}^{\infty} t^n \left(-\frac{1}{n^2} + \frac{S(1,n)}{n} \right)$$

Significant efforts in deriving GV operators

- [Gross, Wilczek, 1974] pointed out mixing with GV operators
- [Dixon and Taylor, 1974] constructed order g_s GV operators, not clear how to generalize to higher order
  - Used by [R. Hamberg and W. L. van Neerven, 1992] to extract 2-loop singlet splitting functions All flaws from omitting GV operators resolved
- [Joglekar and Lee, 1975] gave a general theorem about the renormalization of gauge invariant operators No explicit results were given
- [G. Falcioni and F. Herzog, 2022] constructed the GV operators for fixed n
  - Applied to compute fixed moments to $n \leq 20$ talk by S. Moch
  - Leading GV all-n F.R to five legs[Falcioni, Herzog, Moch, Thurenhout, 24]
- This talk: A systematic framework to derive all-n GV counterterm Feynman rules (C.F.R) to any loop orders[Gehrmann, Manteuffel, Yang, 23]
  - Applied to extract 3-loop splitting functions
  - Applied to extract the exact result of N_f^2 term for $\gamma_{\text{ps}}^{(3)}$
  - Leading GV all-n F.R to five legs[Gehrmann, Manteuffel, Yang, 24]

A new framework of deriving GV operators

Generalize the 2×2 mixing matrix

$$\begin{pmatrix} O_q \\ O_g \end{pmatrix}^{\text{R, naive}} = \begin{pmatrix} Z_{qq} & Z_{qg} \\ Z_{gq} & Z_{gg} \end{pmatrix} \begin{pmatrix} O_q \\ O_g \end{pmatrix}^{\text{B}}$$

to

$$\begin{pmatrix} O_q \\ O_g \\ O_{ABC} \end{pmatrix}^{\text{R}} = \begin{pmatrix} Z_{qq} & Z_{qg} & Z_{qA} \\ Z_{gq} & Z_{gg} & Z_{gA} \\ 0 & 0 & Z_{AA} \end{pmatrix} \begin{pmatrix} O_q \\ O_g \\ O_{ABC} \end{pmatrix}^{\text{B}} + \begin{pmatrix} [ZO]_q^{\text{GV}} \\ [ZO]_g^{\text{GV}} \\ [ZO]_A^{\text{GV}} \end{pmatrix}^{\text{B}}$$

$$Z_{gA} = \mathcal{O}(a_s),\ Z_{qA} = \mathcal{O}(a_s^2),\ [ZO]_{g,q}^{\text{GV}} = \sum_{l=2,3}^{\infty} a_s^l\ [ZO]_{g,q}^{\text{GV},\,(l)}$$

- $O_{ABC} = O_A + O_B + O_C$, O_A(gluon fields only), O_B(quark+gluon fields), O_C (ghost + gluon fields)
- $[ZO]_{g,q}^{\text{GV},\,(l)}$: collection of counterterms

Derive Feynman rules from off-shell OMEs

- Idea: derive Feynman rules instead of GV operators themselves
- Consider all-off-shell OMEs with $2j + m$-gluon external states

$$\langle j|O_g^{\mathsf{R}}|j+m\,g\rangle_{1\mathsf{PI}}^{\mu_1\cdots\mu_m} = \langle j|(Z_{gq}O_q^{\mathsf{B}} + Z_{gg}O_g^{\mathsf{B}})|j+m\,g\rangle_{1\mathsf{PI}}^{\mu_1\cdots\mu_m}$$
$$+ \langle j|Z_{gA}O_{ABC}^{\mathsf{B}}|j+m\,g\rangle_{1\mathsf{PI}}^{\mu_1\cdots\mu_m} + \langle j|\,[ZO]_g^{\mathrm{GV}}\,|j+m\,g\rangle_{1\mathsf{PI}}^{\mu_1\cdots\mu_m}\,,\; j = q, g \text{ or c}$$

- Expand OMEs order by order in loops and legs

$$\langle j|O|j+m\,g\rangle^{\mu_1\cdots\mu_m} = \sum_{l=1}^{\infty}\left[\langle j|O|j+m\,g\rangle^{\mu_1\cdots\mu_m,\,(l),\,(m)}\right]\left(\frac{\alpha_s}{4\pi}\right)^l g_s^m$$

- Left: UV renormalized and IR finite $\rightarrow$ no poles in ϵ
- Right: Each term is UV divergent, but the sum is finite
- Requirement of finiteness $\longrightarrow$ counterterm Feynman rules order by order in α_s

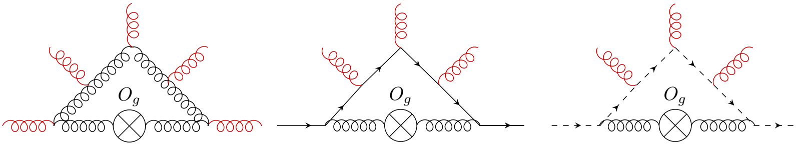
Example: determine Feynman rules for O_{ABC}

- As an example, consider two ghosts + m-gluon external states and expand to one-loop order

$$\langle c|O_C|c + m\, g\rangle_{\text{1PI}}^{\mu_1\cdots\mu_m,\,(0),\,(m)} = -\frac{1}{Z_{gA}^{(1)}}\left[\langle c|O_g|c + m\, g\rangle_{\text{1PI}}^{\mu_1\cdots\mu_m,\,(1),\,(m),\,\text{B}}\right]_{1/\epsilon}$$

- $Z_{gA}^{(1)}$ is a m-independent constant and can be determined from $m = 0$

$$Z_{gA}^{(1)} = \frac{-C_A}{\epsilon}\frac{1}{n(n-1)}$$



Sample digrams to extract Feynman rules for O_{ABC} with $m = 3$

OMEs required to derive four-loop splitting functions

Loops \ Legs	2	3	4	5	6
0	A.D.	$[ZO]_g^{\mathrm{GV},\,(3)}$	$[ZO]_g^{\mathrm{GV},\,(2)}$	O_{ABC}	O_q, O_g
1	$[ZO]_g^{\mathrm{GV},\,(3)}$	$[ZO]_g^{\mathrm{GV},\,(2)}$	O_{ABC}	O_g	
2	$[ZO]_g^{\mathrm{GV},\,(2)}$	O_{ABC}	O_g		
3	O_{ABC}	O_g			
4	O_q, O_g				

- 3-loop splitting functions (done)
- 1-loop five-point OMEs to extract Feynman rules of O_{ABC} (done)
- 3-loop 2-point OMEs with O_{ABC} insertion (in progress)
- 2-loop 4-point OMEs to extract two-loop C.F.R (in progress)
- 4-loop two-point OMEs: focus on $\langle q|O_q|q\rangle^{(4)}$

Complexities from multi-loop multi-leg OMEs

- Besides the parameter t, there are many scales
  - 9 for four-point OMEs

$$6 \text{ Mandelstam variables} + \Delta \cdot p_1,\, \Delta \cdot p_2,\, \Delta \cdot p_3$$

  - 14 for five-point OMEs

$$10 \text{ Mandelstam variables} + \Delta \cdot p_1,\, \Delta \cdot p_2,\, \Delta \cdot p_3,\, \Delta \cdot p_4$$

- Many Lorentz structures for pure gluon OMEs
  - 5 for two-gluon OMEs:

$$\Delta^\mu \Delta^\nu,\, \Delta^\mu p^\nu,\, p^\mu \Delta^\nu,\, g^{\mu\nu},\, p^\mu p^\nu$$

  - 36 for three-gluon OMEs
  - 353 for four-gluon OMEs
  - 4400 for five-gluon OMEs

Evaluations of multi-loop multi-point OMEs

- Constrain Lorentz structures of F.R. based on dimensional analysis
  - Feynman rules involving quarks or ghosts has one structure only

$$\langle j|O|j+m\,g\rangle_{\text{1PI}}^{\mu_1\cdots\mu_m,\,(0),\,(m)} = [c_m(\Delta\cdot p_i)]\;\Delta^{\mu_1}\cdots\Delta^{\mu_m} \text{ with } j=q,c$$

  - 31 Lorentz structures for 5-gluon F.R, instead of naively expected 4400
  - $a_m\Delta^{\mu_1}\cdots \to a_m$ is linear in or has no dependence on $p_i^2, p_i\cdot p_j$
- One-loop: only two types of integrals are needed, others are finite

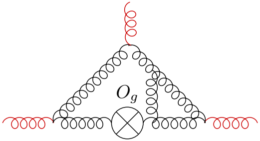
$$I_1 = \int \frac{d^d l}{i\pi^{d/2}}\frac{1}{(l-q_1)^2 l^2}, \quad I_2 = \int \frac{d^d l}{i\pi^{d/2}}\frac{1}{(l-q_1)^2 l^2\big(1-t\Delta\cdot(l+q_2)\big)}$$

- Two-loop: difficult to evaluate the master integrals analytically
  - Derive differential equation (DE)[Gehrmann,Remiddi,1999] in t
  - Solve DE in $t\to 0$ limit $\longrightarrow$ fixed moments for the OMEs
  - Reconstruct the all-n counterterm Feynman rules

Two-loop C.F.R from two-loop three-leg OMEs

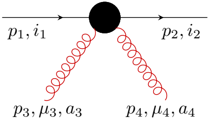
- Set all Mandelstam variables p_1^2, p_2^2 $\cdots$ to numerical numbers and

$$\Delta \cdot p_1 = 1, \, \Delta \cdot p_2 = z_1$$



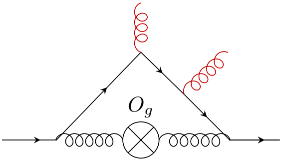
- Derive DE with respect to t
- Solve DE in $t \to 0$ limit with boundary conditions[Birthwright, Glover, Marquard, 2004]
- Determine two-loop counterterm Feynman rules to $n = 100$
- For a fixed n, the result is a polynomial in z_1
- Reconstruct the all-n two-loop C.F.R

Sample results: Feynman rules for O_B



$$\rightarrow \frac{1+(-1)^n}{-8} g_s^2 \Delta^{\mu_3} \Delta^{\mu_4} \left(T^{a_3} T^{a_4} - T^{a_4} T^{a_3}\right)_{i_2 i_1} \not{\Delta} \sum_{j_1=0}^{n-3} \Big(3\left(\Delta \cdot (p_1+p_2)\right)^{-j_1+n-3} \left[(-\Delta \cdot p_3)^{j_1} - (-\Delta \cdot p_4)^{j_1} \right] - (-\Delta \cdot p_4)^{j_1} (\Delta \cdot p_3)^{-j_1+n-3} \Big)$$

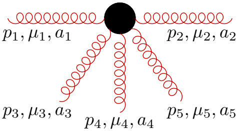
extracted from



$$\sum_n^\infty t^n (\text{all-}n \text{ Feynman rules}) \rightarrow \text{linear propagator in } t\text{-space}$$

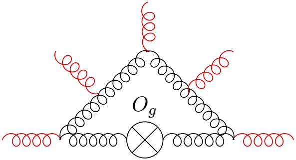
Feynman rules for O_{ABC} with five legs

[Falcioni, Herzog, Moch, Thurenhout, 24; Gehrmann, Manteuffel, Yang, 24]



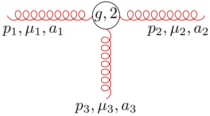
$$\rightarrow \frac{1+(-1)^n}{2} i\, g_s^3 \Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3}\Delta^{\mu_4} p_1^{\mu_5}\Bigg[\frac{1}{C_A} f^{aa_1a_2} d_4^{aa_3a_4a_5}\Bigg\{ \frac{3}{32}\sum_{j_1=0}^{n-4}\sum_{j_2=0}^{j_1}\sum_{j_3=0}^{j_2}(-\Delta\cdot p_1)^{n-4-j_1}(-\Delta\cdot(p_1+p_2))^{j_1-j_2} \times (\Delta\cdot(p_4+p_5))^{j_2-j_3}(\Delta\cdot p_5)^{j_3}+\cdots\Bigg\} + 11 \text{ color structures}\Bigg] + 30 \text{ Lorentz Structures}$$

extracted from



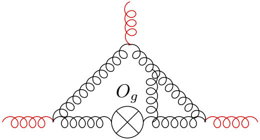
$$\sum_n^{\infty} t^n(\text{all-}n \text{ Feynman rules}) \rightarrow \text{linear propagator in } t\text{-space}$$

Complexities from two-loop counterterm Feynman rules



$$\to 2ig_s C_A^2 f^{a_1 a_2 a_3} \frac{1+(-1)^n}{256n(n-1)} \frac{(\Delta\cdot p_1)^{n-2}}{\Delta\cdot p_2} \Big(\Delta^{\mu_2}\Delta^{\mu_3} p_1^{\mu_1} \Delta\cdot p_1 + \cdots \Big) \left\{ \frac{F_{-2,0}(\xi, z_1, n)}{\epsilon^2} + \frac{F_{-1,0}(\xi, z_1, n)}{\epsilon} \right\},\ z_1 = \frac{\Delta\cdot p_2}{\Delta\cdot p_1}$$

extracted from



- $F_{-1,0}$ contains generalized harmonic sums[Moch,Uwer,Weinzierl,2002]

$$S_1\,(z_1+1;n) = \sum_{x_1=1}^{n} \frac{(1+z_1)^{x_1}}{x_1},\ S_1(z_1+1;2) = \frac{z_1^2}{2} + 2z_1 + \frac{3}{2}$$

$$\sum_n^{\infty} t^n (\text{all-}n \text{ Feynman rules}) \to \text{polylogarithms in } t\text{-space}$$

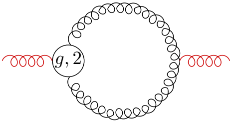
- IBP reductions with polylogarithms? (not feasible) *Need new idea*

Two-point OMEs with two-loop counterterm insertions

- Consider insertions of two-loop counterterms with 3-gluon vertex

$$
i g_s f^{a_1 a_2 a_3} C_A^2 \Delta^{\mu_1} \Delta^{\mu_2} \Delta^{\mu_3} p_1^2 \times \sum_{m=0}^{n-3} a_{mn} (\Delta \cdot p_1)^m (\Delta \cdot p_2)^{n-3-m} + \cdots
$$

a_{mn} is rational number for any fixed m, n



- *New idea*: replace a_{mn} by another tracing parameter t_1

$$
h(t, t_1) = \sum_{n=3}^{\infty} t^n \sum_{m=0}^{n-3} t_1^m (\Delta \cdot p_1)^m (\Delta \cdot p_2)^{n-3-m} = \frac{t^3}{(1 - t\, t_1 \Delta \cdot p_1)(1 - t \Delta \cdot p_2)}
$$

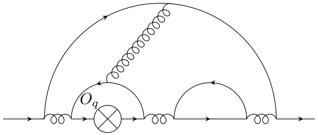
- Insert h into two-point diagrams: $\langle g | h(x, t) | g \rangle = \sum_{n=3}^{\infty} t^n \sum_{m=0}^{n-3} t_1^m c_{mn}$
- $\langle g | \sum_{m=0}^{n-3} a_{mn} (\Delta \cdot p_1)^m (\Delta \cdot p_2)^{n-3-m} | g \rangle = \sum_{m=0}^{n-3} a_{mn}\, c_{mn}$
- Evaluate OMEs to any fixed n efficiently
- Compute OMEs to $n = 500$ and reconstruct the all-n result to ϵ^0

Computations of the four-loop two-point OMEs

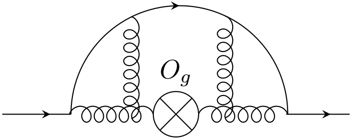
- Focus on specific color structures of four-loop $\langle q|O_q^{\mathsf{B}}|q\rangle$
  - Non-singlet: N_f^3, N_f^2, $N_f C_F^3$
  - Pure-singlet: N_f^3, N_f^2
- Working in Feynman gauge with $\xi = 1$
- IBP reductions and derivation of DEs in parameter-t space
  - Syzygy + finite-field sampling + denominator guessing + function reconstruction implemented in `Finred`
  - Reconstruct the rational number for the coefficient of HPLs
- All DEs can be turned into canonical form[J. Henn, 2013] by `CANONICA` and `Libra`

Sample Feynman diagrams for N_f^2 contributions

- OMEs with physical operator insertions

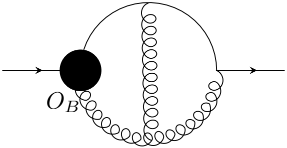


$\langle q|O_q^{\mathsf{B}}|q\rangle^{(4)}$

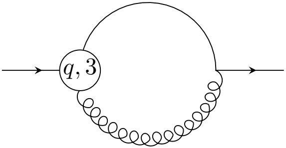


$\langle q|O_g^{\mathsf{B}}|q\rangle^{(3)}$

- OMEs with GV operator or counterterm insertion



$\langle q|O_B^{\mathsf{B}}|q\rangle^{(2)}$



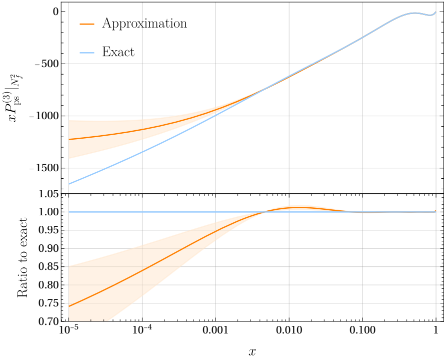
$\langle q|\,[ZO]_q^{\mathrm{GV},\,(3)}\,|q\rangle^{(1)} = 0$

Results for N_f^2 pure-singlet contributions

- First exact result, agree with the $n \le 20$ results[G. Falcioni et al. 2023]
- Extract small-x_B result

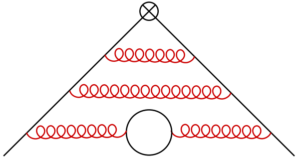
$$P_{\rm ps}^{(3)}(x_B)\big|_{N_f^2} = \frac{\log(x_B)^2}{x_B} \times 0 + \frac{\log(x_B)}{x_B}(\text{New results}) + \cdots$$

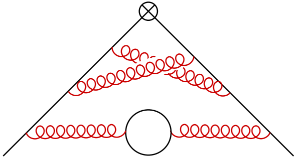
- Approximation: fitted from $n \le 20$ results and previously known limits
- Large x_B-region: agree well with the exact result
- $x \sim 10^{-4}$, derivation $\sim$15%



$N_f\, C_F^3$ contributions to 4-loop P_{ns}

- No mixture with non-physical operators
- Sample Feynman diagrams





- IBP reduction: 54 thousand integrals $\longrightarrow$ 658 master integrals
- Solve the master integrals by DE method analytically
- First exact resut, agree with $n \leq 16$ results from[S. Moch et al. 2017]
- The results are written in terms of HPLs only
- $x_B \to 1$ limit $\to$ a new analytic result for 4-loop rapidity A.D. $\gamma_3^{\text{R}}\big|_{N_f C_F^3}$

Summary

- For off-shell OMEs, renormalization of physical operators mix with unidentified GV operators
- Developed a systematic framework to infer splitting functions
  - Two-point OMEs are used to extract splitting functions
  - Multi-point OMEs are required to determine C.F.R of the GV operators
- Proof of concept: get *exact results* for $\gamma_{\mathsf{ps}}^{(3)}\Big|_{N_f^2}$ and $\gamma_{\mathsf{ns}}^{(3)}\Big|_{N_f C_F^3}$
- Feynman rules for O_{ABC} with 5 legs (done)
- 3-loop 2-point OMEs with O_{ABC} insertion (in progress)
- 2-loop 4-point OMEs to extract two-loop C.F.R (in progress)

Thanks for your attention!

Why 4-loop SFs for the evolutions of N3LO PDFs?

- Expand PDFs and SFs with $a_s = \alpha_s/(4\pi)$

$$f_{i|N} = f_{i|N}^{(0)} + f_{i|N}^{(1)} a_s + \cdots + f_{i|N}^{(3)} a_s^3 + \cdots$$
$$P_{ij} = P_{ij}^{(0)} a_s + \cdots + P_{ij}^{(3)} a_s^4 + \cdots$$

- Evolution of a_s

$$\frac{da_s}{d\ln\mu} = -2(a_s^2\beta_0 + a_s^3\beta_1 + \cdots)$$

- A consistent evolution of N3LO PDFs requires 4-loop SFs

$$f_{i|N}^{(3)} \frac{da_s^3}{d\ln\mu} = f_{i|N}^{(3)}(-6a_s^4\beta_0 + \cdots) = \sum_k P_{ik}^{(3)} a_s^4 \otimes f_{k|N}^{(0)} + \cdots$$

Decomposition of splitting functions

- The general structure of quark splitting functions,

$$P_{q_i q_k} = \delta_{ik} P^V_{qq} + P^S_{qq}, \quad P_{q_i \bar{q}_k} = \delta_{ik} P^V_{q\bar{q}} + P^S_{q\bar{q}}$$

- Non-singlet and singlet splitting functions

$$\text{Non-singlet: } P^{\pm}_{\text{ns}} = P^V_{qq} \pm P^V_{q\bar{q}}, \quad P^V_{\text{ns}} = P^- + \overbrace{n_f(P^S_{qq} - P^S_{q\bar{q}})}^{P^s_{\text{ns}}}$$

$$\text{Singlet: } P_{qq} = P^+ + \underbrace{n_f(P^S_{qq} + P^S_{q\bar{q}})}_{P_{\text{ps}}}, \; P_{qg}, \; P_{gq}, \; P_{gg}$$

- Evolution of PDFs

$$\frac{dT^{\pm}_i}{d\ln\mu} = 2P^{\pm}_{\text{ns}} \otimes T^{\pm}_i, \; \frac{d\sum_{k=1}^{n_f} q^-_k}{d\ln\mu} = 2P^V_{\text{ns}} \otimes \sum_{k=1}^{n_f} q^-_k, \; i = 3, 8, \cdots n_f^2 - 1$$

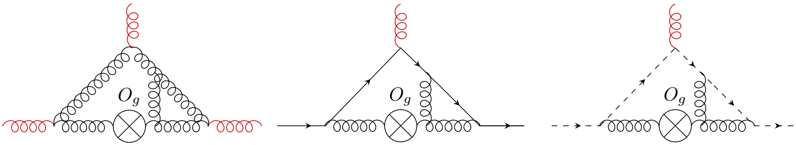
$$T^{\pm}_3 = u^{\pm} - d^{\pm}, T^{\pm}_8 = u^{\pm} + d^{\pm} - 2s^{\pm}, \cdots, q^{\pm}_k = q_k \pm \bar{q}_k,$$

$$\frac{d}{d\ln\mu}\begin{pmatrix} \Sigma \\ g \end{pmatrix} = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} \Sigma \\ g \end{pmatrix}, \quad \Sigma = \sum_{k=1}^{n_f} q^+_k$$

Determin Feynman rules for $[ZO]_g^{\mathrm{GV},\,(2)}$

- As an example, consider two ghosts + m-gluon external states and expand to two-loop order

$$
\begin{aligned}
\langle c|\,[ZO]_g^{\mathrm{GV},\,(2)}\,|c+m\,g\rangle_{\mathrm{1PI}}^{\mu_1\cdots\mu_m,\,(0),\,(m)} = -\Bigg\{\Bigg[&\langle c|O_g|c+m\,g\rangle_{\mathrm{1PI}}^{\mu_1\cdots\mu_m,\,(2),\,(m),\,\mathrm{B}}\\
&+\left(Z_c^{(1)}+\frac{mZ_g^{(1)}}{2}+Z_{gg}^{(1)}-\frac{\beta_0(m+2)}{2\epsilon}\right)\langle c|O_g|c+m\,g\rangle_{\mathrm{1PI}}^{\mu_1\cdots\mu_m,\,(1),\,(m),\,\mathrm{B}}\\
&+\left(Z_c^{(1)}Z_{gA}^{(1)}+\frac{1}{2}mZ_g^{(1)}Z_{gA}^{(1)}-\frac{\beta_0 mZ_{gA}^{(1)}}{2\epsilon}+Z_{gA}^{(2)}\right)\langle c|O_C|c+m\,g\rangle_{\mathrm{1PI}}^{\mu_1\cdots\mu_m,\,(0),\,(m),\,\mathrm{B}}\\
&+Z_{gA}^{(1)}\,\langle c|O_{AC}|c+m\,g\rangle_{\mathrm{1PI}}^{\mu_1\cdots\mu_m,\,(1),\,(m),\,\mathrm{B}}+\cdots\Bigg]_{\mathrm{div}}\Bigg\}
\end{aligned}
$$



Sample digrams to extract Feynman rules for $[ZO]_g^{\mathrm{GV},\,(2)}$ with $m=1$

Renormalization of O_q to four loops in $q \to q$ channel

- Renormalization of two-point OMEs

$$\begin{aligned}\langle q|O_q^{\mathrm{R}}|q\rangle = Z_{qq}\,\langle q|O_q^{\mathrm{B}}|q\rangle + Z_{qg}\,\langle q|O_g^{\mathrm{B}}|q\rangle \\ + Z_{qA}\,\langle q|O_{ABC}^{\mathrm{B}}|q\rangle + \langle q|\,[ZO]_q^{\mathrm{GV}}\,|q\rangle, \\ Z_{qg} = \mathcal{O}(a_s), Z_{qA} = \mathcal{O}(a_s^2), [ZO]_q^{\mathrm{GV}} = \mathcal{O}(a_s^3)\end{aligned}$$

- $\langle q|\,[ZO]_q^{\mathrm{GV}}\,|q\rangle = \mathcal{O}(a_s^5)$
  - Only $\langle q|\,[ZO]_q^{\mathrm{GV},\,(4)}\,|q\rangle^{(0)}$ and $\langle q|\,[ZO]_q^{\mathrm{GV},\,(3)}\,|q\rangle^{(1)}$ are relevant
  - Other operators (O_q, O_g, O_A, O_B) give all possible Lorentz structures of $q\bar{q}$, gg, $q\bar{q}g$ vertex Feynman rules
  - $\longrightarrow \langle q|\,[ZO]_q^{\mathrm{GV}}\,|q\rangle^{(0)} = 0,\ \langle g|\,[ZO]_q^{\mathrm{GV}}\,|g\rangle^{(0)} = 0,$

$$\langle q|\,[ZO]_q^{\mathrm{GV}}\,|qg\rangle^{(0)} = 0$$

Lorentz structures of a twist-two operator

- Based on the following two properties
  - A twist-two operator has spin-n and mass dimension $n+2$
  - Propagator-type Feynman rules like $1/p^2$ can not appear in a vertex
- A twist-2 operator involving quarks or ghosts has one Lorentz structure only

$$\langle q|O|q+m\,g\rangle_{\text{1PI}}^{\mu_1\cdots\mu_m,\,(0),\,(m)} = c_m \Delta^{\mu_1}\cdots\Delta^{\mu_m}$$

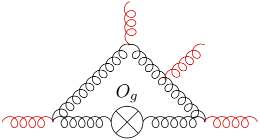
- A twist-two operator involving only gluons
  - Only $1+3/2m(m-1)$ Lorentz structures for m-gluon Feynman rules
  - $m=3$: $a_1\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3} + a_2\Delta^{\mu_1}\Delta^{\mu_2}p_1^{\mu_3} + \cdots + a_{10}\Delta^{\mu_3}g^{\mu_1\mu_2}$
  - 19 for $m=4$ and 31 for $m=5$
- Count the mass dimension of a_i: $[a_i] = x_i[\Delta\cdot p_j] + y_i[p_j\cdot p_k](y_i\geq 0)$
  $[a_1] = n-3+y_1[p_j\cdot p_k] = n+2-3 \rightarrow y_1 = 1$(Linear in $p_1^2,\ p_1\cdot p_2\cdots$)
  $[a_2]+[p_1^{\mu_3}] = n-2+y_2[p_j\cdot p_k]+1 = n+2-3 \rightarrow y_2 = 0$
- Why not $a_{11}\Delta^{\mu_1}p_1^{\mu_2}p_2^{\mu_3}$

$$[a_{11}]+2+3 \geq n-1+y_{11}[p_j\cdot p_k]+2+3 = n+4(\text{if } y_{11}=0)$$

where 3 is mass dimension of the external 3 gluons. Twist-4 operators

Computations of single pole for one-loop multi-leg OMEs

- Set all Mandelstam variables $p_1^2,\, p_2^2\, \cdots$ to numerical numbers and reconstruct their linear dependence



- Only two types of integrals are needed, other integrals are finite

$$I_1 = \int \frac{d^d l}{i\pi^{d/2}} \frac{1}{(l-q_1)^2 l^2}, \quad I_2 = \int \frac{d^d l}{i\pi^{d/2}} \frac{1}{(l-q_1)^2 l^2 \big(1 - x\Delta\cdot(l+q_2)\big)}$$

- At most x-dependent logarithms appear in the single pole

$$I_2 = \frac{1}{\epsilon}\left[\frac{\ln(1-x\Delta\cdot q_1 - x\Delta\cdot q_2) - \ln(1-x\Delta\cdot q_2)}{-x\Delta\cdot q_1}\right] + \mathcal{O}(\epsilon^0)$$

- Logarithms in x-space $\to$ n-space

$$\ln(1-x\Delta\cdot p_1 - x\Delta\cdot p_2) = \sum_{n=1}^{\infty} x^n \left[\frac{-1}{n}(\Delta\cdot p_1 + \Delta\cdot p_2)^n\right]$$

- Factoring out the overall factor $Z_{gA}^{(1)} = -\frac{C_A}{\epsilon}\frac{1}{n(n-1)}$

Reconstruct two-loop counterterm Feynman rules

- Obtain two-loop three-leg OMEs to x^{96} or $n = 96$
- For a fixed n, the result is a polynomial in z_1
- Construct full-x or full-n results from data to $n = 76$ based on ansatz
- Polylogarithms to weight-3, generalized Harmonic sums to weight-2

$$G(1,1,1/(1+z_1);x) = \sum_{n=1}^{\infty} x^n \left[\frac{S_1(z_1+1;n)}{n^2} + \frac{S_2(z_1+1;n)}{n} - \frac{S_{1,1}(1,z_1+1;n)}{n} - \frac{(z_1+1)^n}{n^3} \right]$$

where $S_{1,1}(1, z_1+1; n) = \sum_{t_1=1}^{n} \frac{1}{t_1} \sum_{t_2=1}^{t_1} \frac{(1+z_1)^{t_2}}{t_2}$

- Due to the generalized Harmonic sums, impossible to disentangle
  - renormalization constants (no z_1 dependence)
  - operator Feynman rules (no high-weight (≥ 1) functions)

A counterterm Feynman rule & infinite operator Feynman rules ($N_2 = \infty$)