Progress in the calculation of N^{3}LO splitting functions

Sven-Olaf Moch

Universität Hamburg

Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

European Research Council
Established by the European Commission

Workshop *QCD@LHC*, Freiburg, Oct 07, 2024

Present: Work at four loops:

- *Constraints for twist-two alien operators in QCD*
  G. Falcioni, F. Herzog, S. M., and S. Van Thurenhout `arXiv:2409.02870`
- *Four-loop splitting functions in QCD – The quark-to-gluon case –*
  G. Falcioni, F. Herzog, S. M., A. Pelloni and A. Vogt `arXiv:2404.09701`
- *Additional moments and x-space approximations of four-loop splitting functions in QCD*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt `arXiv:2310.05744`
- *The double fermionic contribution to the four-loop quark-to-gluon splitting function*
  G. Falcioni, F. Herzog, S. M., J. Vermaseren and A. Vogt `arXiv:2310.01245`
- *Four-loop splitting functions in QCD – The gluon-to-quark case –*
  G. Falcioni, F. Herzog, S. M., and A. Vogt `arXiv:2307.04158`
- *Four-loop splitting functions in QCD – The quark-quark case –*
  F. Herzog, G. Falcioni, S. M., and A. Vogt `arXiv:2302.07593`
- *Low moments of the four-loop splitting functions in QCD*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt `arXiv:2111.15561`
- *On quartic colour factors in splitting functions and the gluon cusp anomalous dimension*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt `arXiv:1805.09638`
- *Four-Loop Non-Singlet Splitting Functions in the Planar Limit and Beyond*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt `arXiv:1707.08315`

Past: Work at three loops:

- Many papers of MVV and friends . . . 2001 – ...

Future: Work at five loops:

- *Five-loop contributions to low-N non-singlet anomalous dimensions in QCD*
  F. Herzog, S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt
  arXiv:1812.11818

- More papers to come . . . QCD@LHC 2026

Parton evolution

- Evolution equations for parton distributions
  - non-singlet valence PDFs $q_{\rm ns}^{\rm v} = \sum_f (q_f - \bar{q}_f)$
  - flavor asymmetries $q_{{\rm ns},ff'}^{\pm} = (q_f \pm \bar{q}_f) - (q_{f'} \pm \bar{q}_{f'})$

$$\frac{d}{d\ln\mu^2} q_{\rm ns}^{\pm,{\rm v}} \quad = \quad P_{\rm ns}^{\pm,{\rm v}} \otimes q_{\rm ns}^{\pm,{\rm v}}$$

  - quark-flavor singlet PDFs $q_{\rm s} = \sum_f (q_f + \bar{q}_f)$ and gluon PDF g
  - 2x2 matrix equation

$$\frac{d}{d\ln\mu^2} \begin{pmatrix} q_{\rm s} \\ g \end{pmatrix} \quad = \quad \begin{pmatrix} P_{\rm qq} & P_{\rm qg} \\ P_{\rm gq} & P_{\rm gg} \end{pmatrix} \otimes \begin{pmatrix} q_{\rm s} \\ g \end{pmatrix}$$

- Splitting functions P up to N^{3}LO (work in progress)

$$P_{\rm ij} \quad = \quad \underbrace{\alpha_s\, P_{\rm ij}^{(0)} + \alpha_s^2\, P_{\rm ij}^{(1)} + \alpha_s^3\, P_{\rm ij}^{(2)}}_{\text{NNLO: standard approximation}} + \alpha_s^4\, P_{\rm ij}^{(3)} + \ldots$$

- Anomalous dimensions (Mellin transform)

$$\gamma_{\rm ij}(N) = -\int_0^1 dx\, x^N\, P_{\rm ij}(x) = \alpha_s\, \gamma_{\rm ij}^{(0)} + \alpha_s^2\, \gamma_{\rm ij}^{(1)} + \alpha_s^3\, \gamma_{\rm ij}^{(2)} + \alpha_s^4\, \gamma_{\rm ij}^{(3)} + \ldots$$

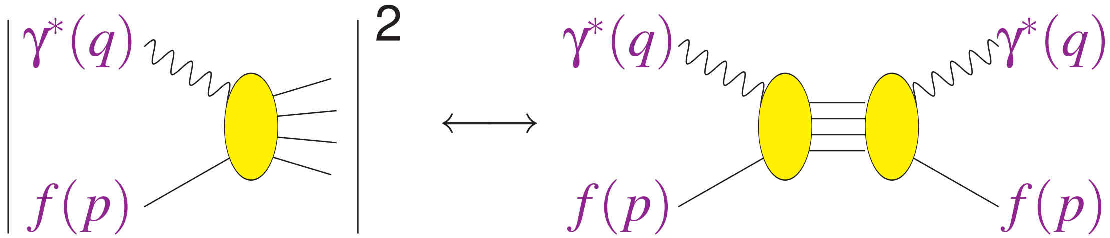
Research methodology

Operator product expansion (I)

- Direct computation of physical observable
  - structures functions in deep-inelastic scattering (DIS)

Optical theorem

- Total cross section related to imaginary part of Compton amplitude
  - Bjorken variable $x = Q^2/(2p \cdot q)$ and momentum transfer $Q^2 = -q^2$



- Optical theorem relates hadronic tensor $W_{\mu\nu}$ to imaginary part of Compton amplitude $T_{\mu\nu} = \mathrm{i} \int d^4z \, \mathrm{e}^{\mathrm{i}q\cdot z} \langle \mathrm{P} | \, T\left(j_\mu^\dagger(z) j_\nu(0)\right) | \mathrm{P} \rangle$

$$W_{\mu\nu} = e_{\mu\nu} \frac{1}{2x} F_L(x, Q^2) + d_{\mu\nu} \frac{1}{2x} F_2(x, Q^2) + \mathrm{i}\epsilon_{\mu\nu\alpha\beta} \frac{p^\alpha q^\beta}{p \cdot q} F_3(x, Q^2)$$

- OPE of $T_{\mu\nu}$ for short distances $z^2 \simeq 0$ in Bjorken limit $Q^2 \to \infty$, x fixed

  Wilson ‘72; Christ, Hasslacher, Mueller ‘72

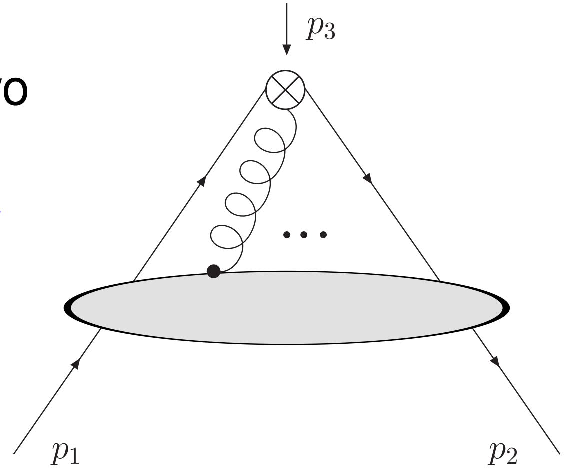
Operator product expansion (II)

- OPE for parton states gives coefficient functions in Mellin space $C^N_{a,i}$

$$T_{\mu\nu,\mathrm{k}} \;=\; \sum_{N,\mathrm{j}} \left(\frac{1}{2x}\right)^N \Bigg[e_{\mu\nu}\, C^N_{L,\mathrm{j}} \left(\frac{Q^2}{\mu^2}, \alpha_s\right) + d_{\mu\nu}\, C^N_{2,\mathrm{j}} \left(\frac{Q^2}{\mu^2}, \alpha_s\right)$$
$$+\mathrm{i}\epsilon_{\mu\nu\alpha\beta}\frac{p^\alpha q^\beta}{p\cdot q} C^N_{3,\mathrm{j}} \left(\frac{Q^2}{\mu^2}, \alpha_s\right)\Bigg] A^N_{\mathrm{jk}}\left(\mu^2\right) \;+\; \text{higher twists}$$

- Operator matrix elements $A^N_{\mathrm{ij}} = \langle j | O^N_{\mathrm{i}} | j \rangle$ in parton state
- Anomalous dimensions $\gamma_{\mathrm{ij}}(N)$ from collinear singularities of Compton amplitude $T_{\mu\nu}$ after mass factorization
  - established computational approach through four loops
    one loop Buras '80; two loops Kazakov, Kotikov '90; S.M., Vermaseren '99; three loops S.M., Vermaseren, Vogt '04; four loops Davies, Vogt, Ruijl, Ueda, Vermaseren '17; S.M., Ruijl, Ueda, Vermaseren, Vogt to appear
- Versatile calculation method
  - photon-DIS $\longrightarrow \gamma_{\mathrm{qq}}\,, \gamma_{\mathrm{qg}}$
  - Higgs (scalar)-DIS $\longrightarrow \gamma_{\mathrm{gq}}\,, \gamma_{\mathrm{gg}}$
  - graviton-DIS $\longrightarrow \Delta\gamma_{\mathrm{ij}}$ (polarized quantities) S.M., Vermaseren, Vogt '14

Operator matrix elements



- Scalar singlet operators of spin-N and twist two from contraction with light-like vector Δ_μ
  - quarks ψ, field strenghth $F^{\mu;a} = \Delta_\nu F^{\mu\nu;a}$
  - N covariant derivatives $D = \Delta_\mu D^\mu$

$$
\begin{aligned}
O_{\rm q} &= \overline{\psi}\, \not{\Delta}\, D^{N-1}\, \psi \\
O_{\rm g} &= F_\nu{}^a\, D_{ab}^{N-2}\, F^{\nu;b}
\end{aligned}
$$

- Direct computation of OMEs $A_{\rm ij}^N = \langle j | O_{\rm i}^N | j \rangle$ in parton state
  - anomalous dimensions $\gamma_{\rm ij}(N)$ from renormalization of operators
- Physical operators mix under renormalization with alien operators
  Dixon, Taylor ‘74; Kluberg-Stern, Zuber ‘75; Joglekar, Lee ‘76

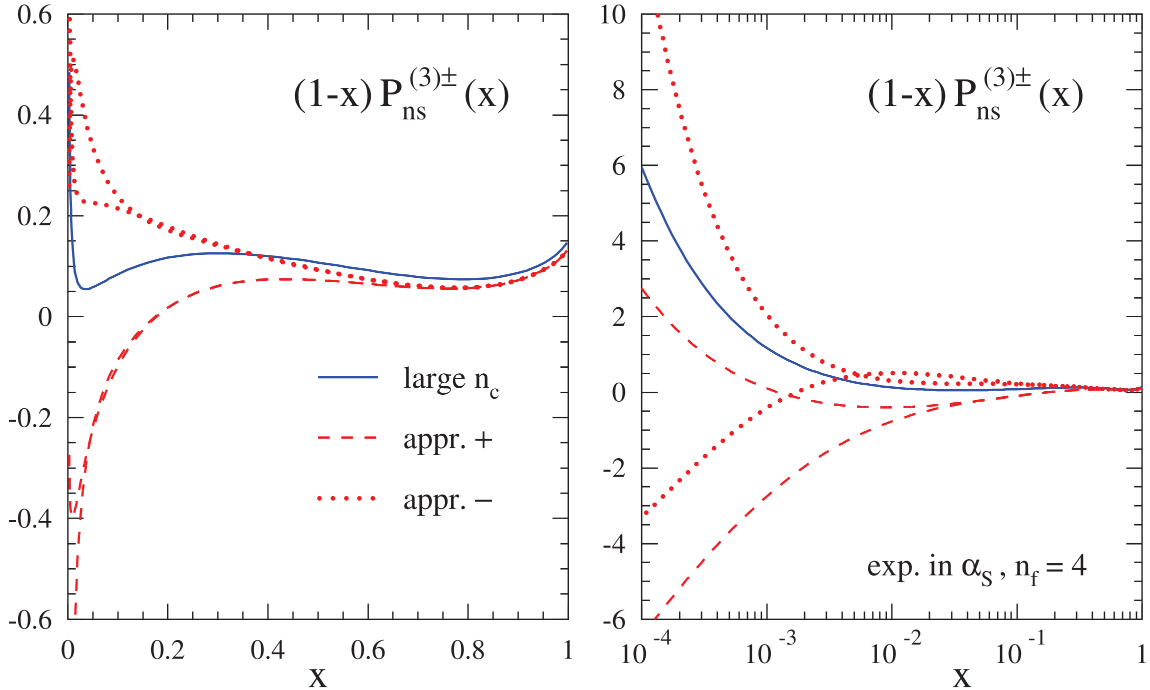
Workflow

- Zero-momentum transfer through operator gives 2-point functions
- Feynman diagrams generation with **Qgraf** Nogueira ‘91
- Four-loop IBP reduction with **Forcer** Ruijl, Ueda, Vermaseren ‘17
- Symbolic manipulations with **Form** Vermaseren ‘00; Kuipers, Ueda, Vermaseren, Vollinga ‘12 and **TForm** Tentyukov, Vermaseren ‘07

Non-singlet splitting functions $P_{\rm ns}^{\pm,\rm v}$

Four-loop non-singlet splitting functions

- Four-loop $P_{\rm ns}^{(3)\pm}(x)$ and uncertainty bands beyond large-n_c limit with $n_f = 4$



Analytic results

- contributions to non-singlet splitting functions
  - n_f-terms (n_f^3 Gracey '94; n_f^2 Davies, Vogt, Ruijl, Ueda, Vermaseren '16)
  - leading n_c terms S.M., Vogt, Ruijl, Ueda, Vermaseren '17
  - $n_f C_F^3$ terms Gehrmann, von Manteuffel, Sotnikov, Yang '23

Outlook

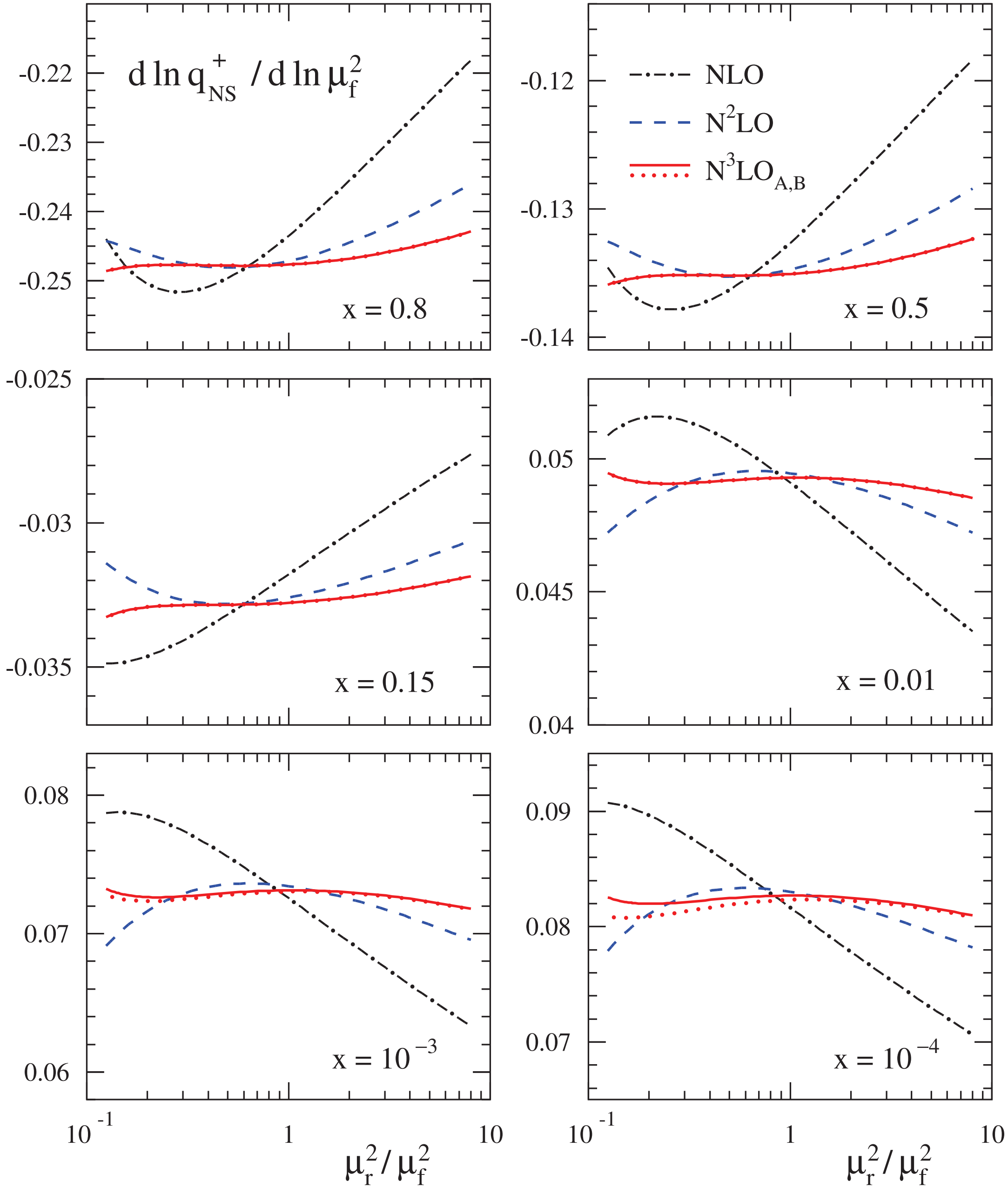
- $P_{\rm ns,x\to 1}^{(n)\pm} = A^{(n)}/(1-x)_+ + B^{(n)}\delta(1-x) + \ldots$ (known $B^{(4)}$ Das, S.M. Vogt '19)
- Higher moments $N = 21, 22, \ldots$ to be published
- Improved approximations to be done

Scale stability of evolution

- Renormalization scale dependence of evolution kernel $d \ln q_{\rm ns}^{+} / d \ln \mu_f^2$
  - non-singlet shape

$$xq_{\rm ns}^{+}(x, \mu_0^2) \;=\; x^{\,0.5}(1-x)^3$$

- NLO, NNLO and N^{3}LO predictions
  - remaining uncertainty of four-loop splitting function $P_{\rm ns}^{(3)+}$ almost invisible



Quark pure-singlet splitting function $P_{\mathrm{qq}} = P_{\mathrm{ns}}^{+} + P_{\mathrm{ps}}$

$$\begin{pmatrix} P_{\mathrm{qq}} & P_{\mathrm{qg}} \\ P_{\mathrm{gq}} & P_{\mathrm{gg}} \end{pmatrix}$$

Moments of pure-singlet splitting function

- Moments $N = 2, \ldots 20$ for pure-singlet anomalous dimension $\gamma_{\rm ps}^{(3)}(N)$

$$\begin{aligned}
\gamma_{\rm ps}^{(3)}(N=2) &= -691.5937093\, n_f + 84.77398149\, n_f^2 + 4.466956849\, n_f^3\,, \\
\gamma_{\rm ps}^{(3)}(N=4) &= -109.3302335\, n_f + 8.776885259\, n_f^2 + 0.306077137\, n_f^3\,, \\
&\ldots \\
\gamma_{\rm ps}^{(3)}(N=20) &= -0.442681568\, n_f + 0.805745333\, n_f^2 - 0.020918264\, n_f^3\,.
\end{aligned}$$

- Results $N \leq 8$ agree with inclusive DIS S.M., Ruijl, Ueda, Vermaseren, Vogt '21 (also for $N = 10$ and $N = 12$)
- Quartic color terms $d_R^{abcd} d_R^{abcd}$ agree with S.M., Ruijl, Ueda, Vermaseren, Vogt '18
- Large-n_f parts agree with all-N results Davies, Vogt, Ruijl, Ueda, Vermaseren '17;
- ζ_4 terms in $\gamma_{\rm ps}^{(3)}(N)$ agree with Davies, Vogt '17 based on no-π^2 theorem Jamin, Miravitllas '18; Baikov, Chetyrkin '18
- Checked by n_f^2 terms at all-N Gehrmann, von Manteuffel, Sotnikov, Yang '23

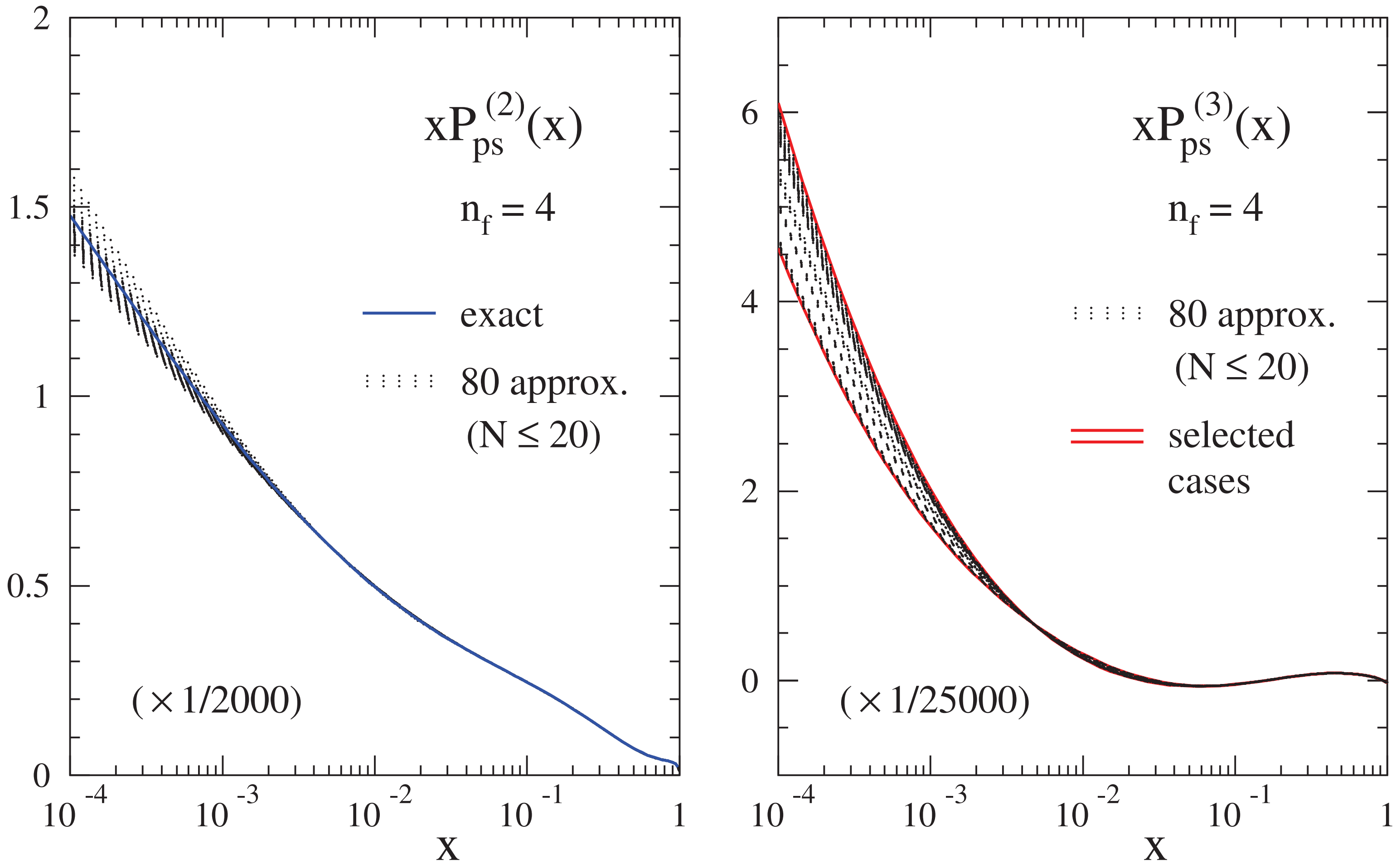
Outlook

- Higher moments $N = 22, \ldots$ to be published

Approximations in x-space

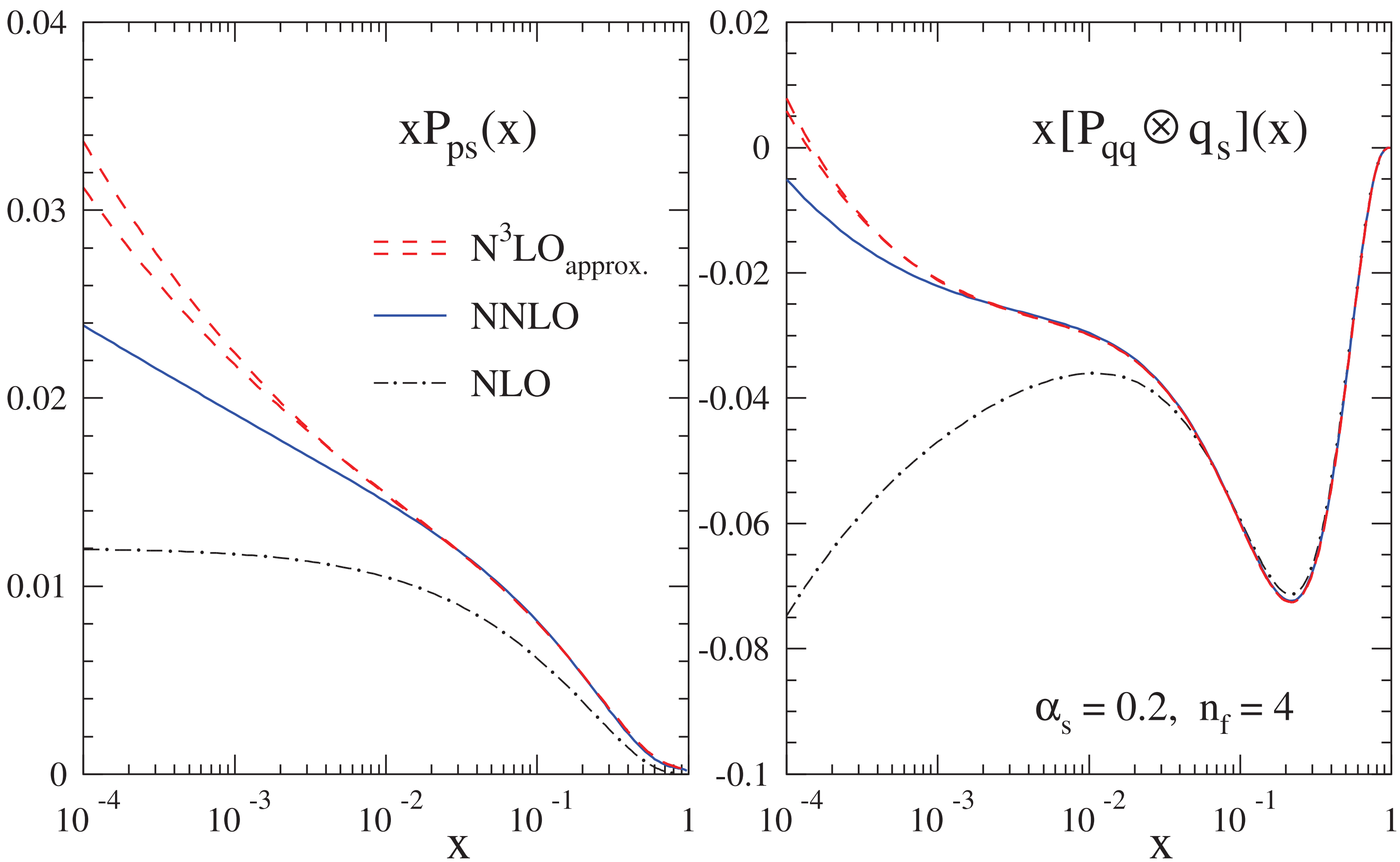
- Large- and small-x information about four-loop splitting function $P_{\rm ps}^{(3)}(x)$
  - leading logarithm $(\ln^2 x)/x$ Catani, Hautmann '94
  - sub-dominant logarithms $\ln^k x$ with $k = 6, 5, 4$ Davies, Kom, S.M., Vogt '22
  - leading large-x terms $(1-x)^j \ln^k(1-x)$ with $j \geq 1$ and $k \leq 4$ with $k = 4, 3$ known Soar, S.M., Vermaseren, Vogt '09
- Approximation of four-loop splitting function $P_{\rm ps}^{(3)}(x)$ with suitable ansatz
  - unknown leading small-x terms: $(\ln x)/x$, $1/x$
  - unknown sub-dominant logarithms: $\ln^k x$ with $k = 3, 2, 1$
  - two remaining large-x terms $(1-x)\ln^k(1-x)$ with $k = 2, 1$
  - different two-parameter polynomials together one function (dilogarithm $\mathrm{Li}_2(x)$ or $\ln^k(1-x)$ with $k = 2, 1$, suppressed as $x \to 1$)
- Approximations for phenomenology with fixed $n_f = 3, 4, 5$
  - easy-to-use
  - no correlations between different n_f dependenct terms accounted for

Pure-singlet splitting function



- Approximations to pure-singlet splitting function $P_{\rm ps}^{(n)}(x)$ at $n_f = 4$ with 80 trial functions
  - left: three-loops ($n = 2$) with comparison to known result
  - right: four-loops ($n = 3$) with remaining uncertainty

Pure-singlet splitting function



- Left: results for $P_{\rm ps}(x)$ up to N^{3}LO; $\alpha_s \;=\; 0.2$ fixed, $n_f \;=\; 4$
- Right: contribution to evolution kernel $d \ln q_{\rm s} / d \ln \mu_f^2$ up to N^{3}LO for typical quark-singlet shape

$$x q_{\,\rm s}(x, \mu_0^2) \;=\; 0.6\, x^{\,-0.3} (1-x)^{3.5} \,(1 + 5.0\, x^{\,0.8})$$

Gluon-to-quark splitting function P_{qg}

$$\begin{pmatrix} P_{\mathrm{qq}} & P_{\mathrm{qg}} \\ P_{\mathrm{gq}} & P_{\mathrm{gg}} \end{pmatrix}$$

Moments of gluon-to-quark splitting function

- Moments $N = 2, \ldots 20$ for gluon-to-quark anomalous dimension $\gamma^{(3)}_{\mathrm{qg}}(N)$

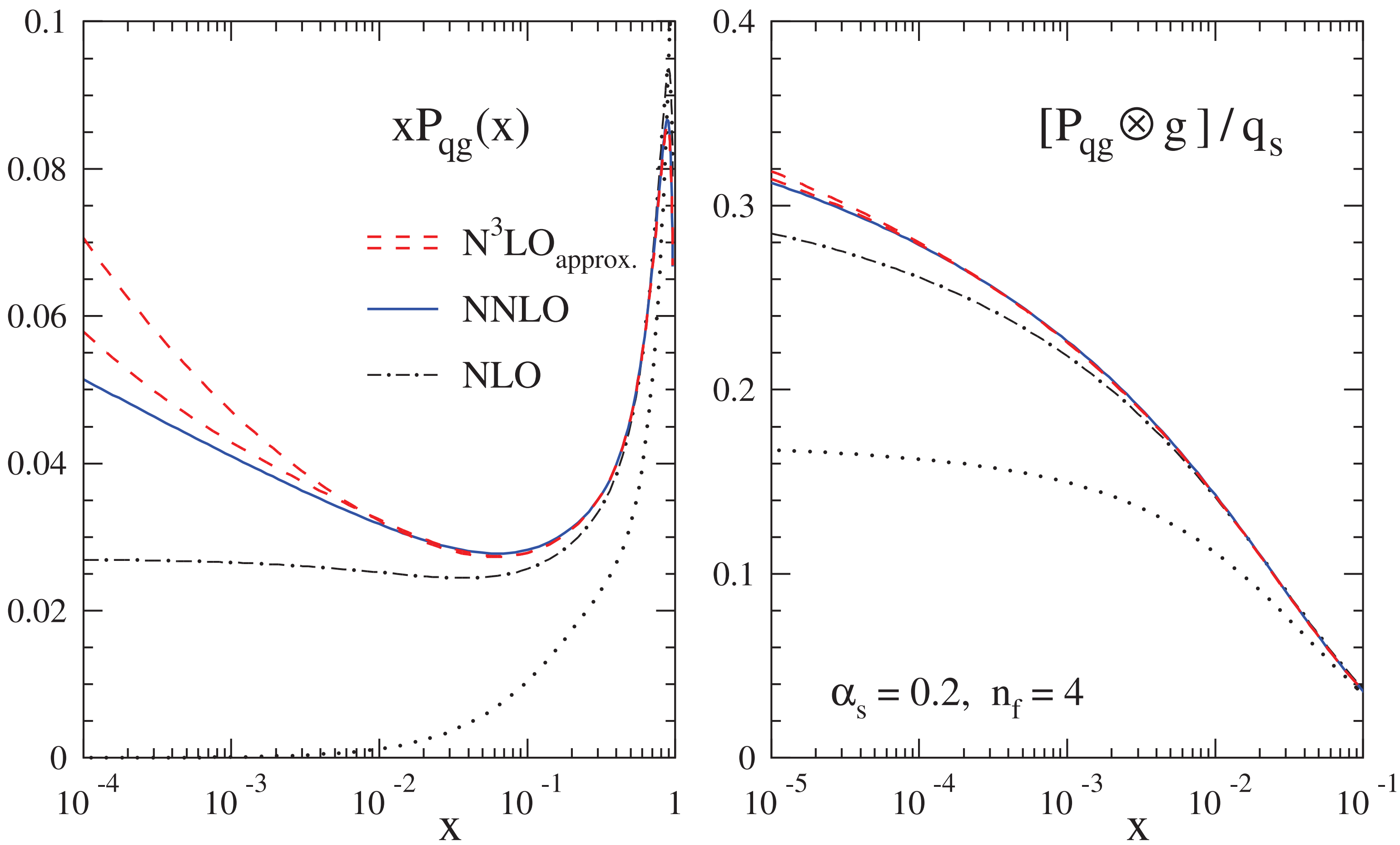
$$
\begin{aligned}
\gamma^{(3)}_{\mathrm{qg}}(N=2) &= -654.4627782\, n_f + 245.6106197\, n_f^2 - 0.924990969\, n_f^3 \,, \\
\gamma^{(3)}_{\mathrm{qg}}(N=4) &= 290.3110686\, n_f - 76.51672403\, n_f^2 - 4.911625629\, n_f^3 \,, \\
\gamma^{(3)}_{\mathrm{qg}}(N=6) &= 335.8008046\, n_f - 124.5710225\, n_f^2 - 4.193871425\, n_f^3 \,, \\
\gamma^{(3)}_{\mathrm{qg}}(N=8) &= 294.5876830\, n_f - 135.3767647\, n_f^2 - 3.609775642\, n_f^3 \,, \\
&\ldots \\
\gamma^{(3)}_{\mathrm{qg}}(N=20) &= 52.24329555\, n_f - 109.3424891\, n_f^2 - 2.153153725\, n_f^3 \,.
\end{aligned}
$$

- Approximation of four-loop splitting function $P^{(3)}_{\mathrm{qg}}(x)$ again with known large- and small-x information and suitable ansatz

Outlook

- Higher moments $N = 22, \ldots$ to be published

Gluon-to-quark splitting function



- Left: results for $P_{qg}(x)$ up to N^{3}LO; $\alpha_s = 0.2$ fixed, $n_f = 4$
- Right: contribution to evolution kernel $d \ln g / d \ln \mu_f^2$ up to N^{3}LO for typical gluon shape

$$xg(x, \mu_0^2) = 1.6\, x^{-0.3} (1-x)^{4.5} \left(1 - 0.6\, x^{0.3}\right)$$

Quark-to-gluon splitting function P_{gq}

$$\begin{pmatrix} P_{\mathrm{qq}} & P_{\mathrm{qg}} \\ P_{\mathrm{gq}} & P_{\mathrm{gg}} \end{pmatrix}$$

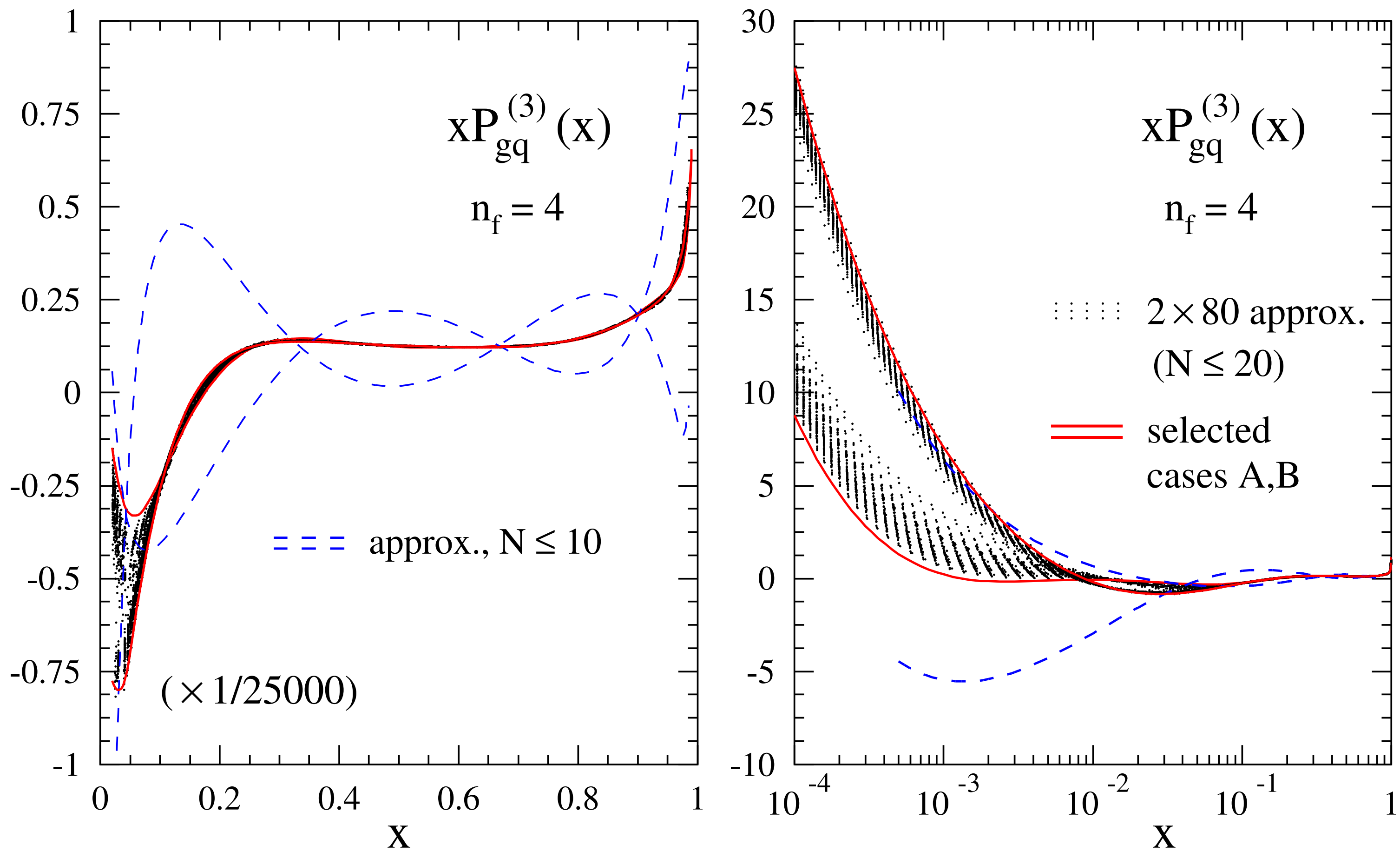
Moments of quark-to-gluon splitting function

- Moments for quark-to-gluon anomalous dimension $\gamma_{\rm gq}^{(3)}(N)$
  - moments $N = 2, \dots 10$ S.M., Ruijl, Ueda, Vermaseren, Vogt '23
  - moments $N = 12, \dots 20$ Falcioni, Herzog, S.M., Pelloni, Vogt '24

$$
\begin{aligned}
\gamma_{\rm gq}^{(3)}(N\!=\!2) &= -16663.2255 + 4439.14375\, n_f - 202.555479\, n_f^{\,2} - 6.37539072\, n_f^{\,3}\,, \\
\gamma_{\rm gq}^{(3)}(N\!=\!4) &= -6565.73145 + 1291.06746\, n_f - 16.1461902\, n_f^{\,2} - 0.83976340\, n_f^{\,3}\,, \\
\gamma_{\rm gq}^{(3)}(N\!=\!6) &= -3937.47937 + 679.718506\, n_f - 1.37207753\, n_f^{\,2} - 0.13979433\, n_f^{\,3}\,, \\
\gamma_{\rm gq}^{(3)}(N\!=\!8) &= -2803.64411 + 436.393057\, n_f + 1.81494624\, n_f^{\,2} + 0.07358858\, n_f^{\,3}\,, \\
&\dots \\
\gamma_{\rm gq}^{(3)}(N\!=\!20) &= -1054.26140 + 105.497994\, n_f + 2.39223577\, n_f^{\,2} + 0.19938504\, n_f^{\,3}\,.
\end{aligned}
$$

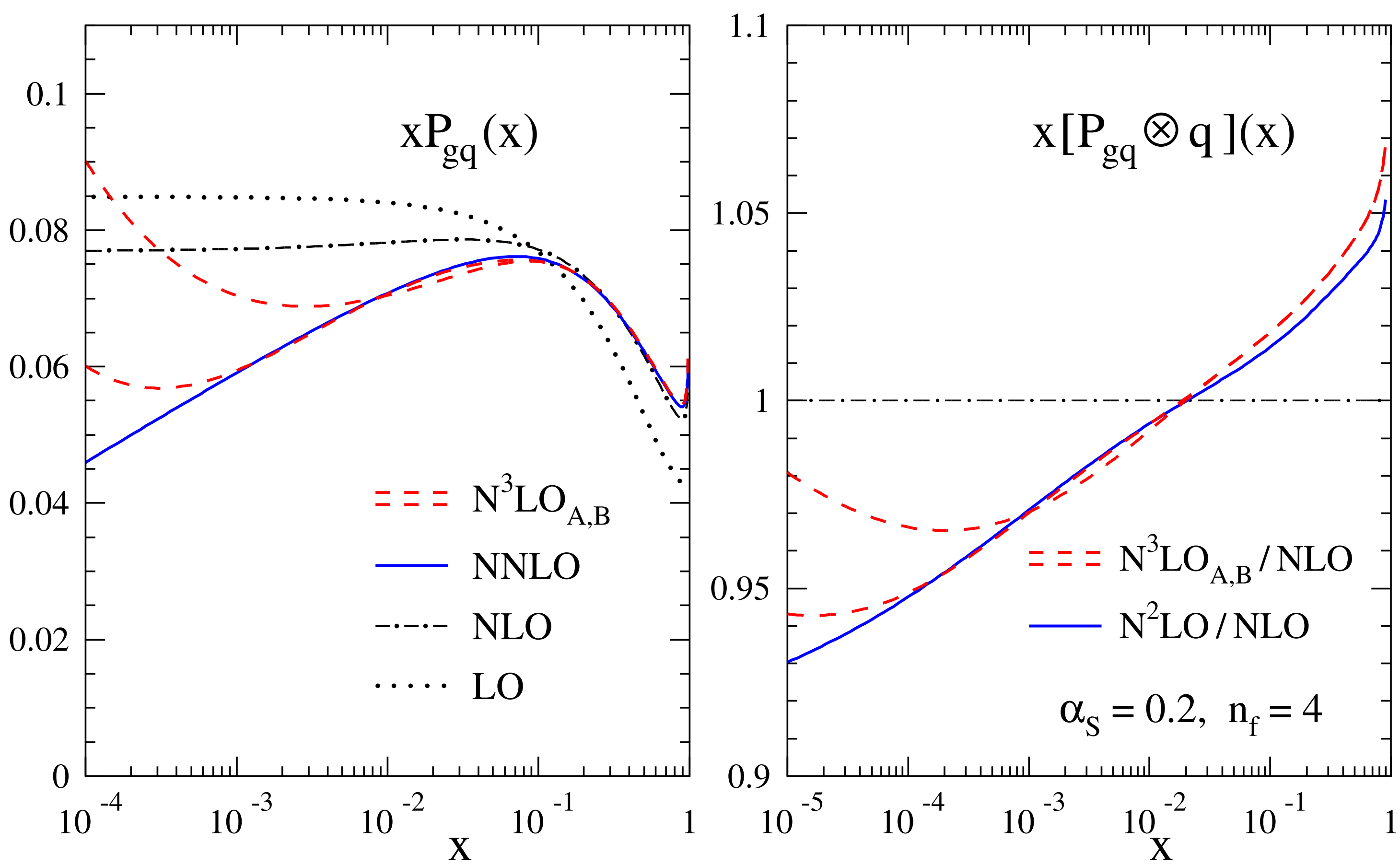
- Approximation of four-loop splitting function $P_{\rm gq}^{(3)}(x)$ again with known large- and small-x information and suitable ansatz

Quark-to-gluon splitting function (I)



- Approximations for $P^{(3)}_{\rm gq}(x)$ based on moments $N \leq 10$ vs. $N \leq 20$
  - clear improvements at large-x (left) and small-x (right)

Quark-to-gluon splitting function (II)



- Left: results for $P_{\rm gq}(x)$ up to N^{3}LO; $\alpha_s \;=\; 0.2$ fixed, $n_f \,=\, 4$
- Right: contribution to evolution kernel $d \ln q_{\rm s} / d \ln \mu_f^2$ up to N^{3}LO for typical quark-singlet shape

$$xq_{\,\rm s}(x, \mu_0^2) \;=\; 0.6\, x^{\,-0.3}(1-x)^{3.5}\,\left(1 + 5.0\, x^{\,0.8}\right)$$

Gluon-gluon splitting function P_{gg}

$$\begin{pmatrix} P_{\mathrm{qq}} & P_{\mathrm{qg}} \\ P_{\mathrm{gq}} & P_{\mathrm{gg}} \end{pmatrix}$$

Moments of gluon-gluon splitting function

- Moments for gluon–gluon anomalous dimension $\gamma_{\rm gg}^{(3)}(N)$
  - moments $N = 2, \ldots 10$ S.M., Ruijl, Ueda, Vermaseren, Vogt '23
  - *New:* moments $N = 12, \ldots 20$ Falcioni, Herzog, S.M., Pelloni, Vogt '24

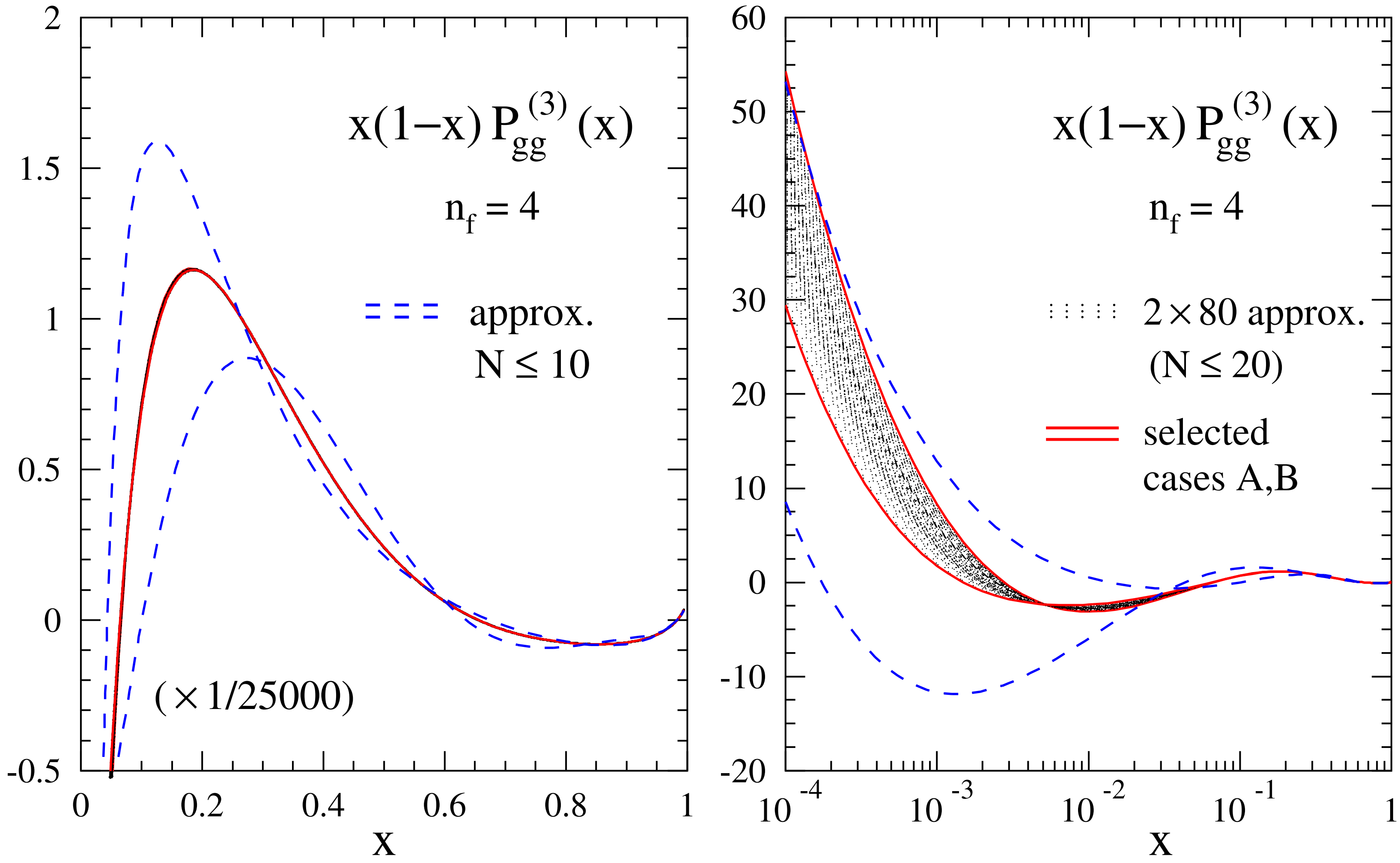
$$
\begin{aligned}
\gamma_{\rm gg}^{(3)}(N\!=\!2) &= \phantom{39876.1233 - {}} 654.462778\, n_f - 245.610620\, n_f^{\,2} + 0.92499097\, n_f^{\,3}\,, \\
\gamma_{\rm gg}^{(3)}(N\!=\!4) &= 39876.1233 - 10103.4511\, n_f + 437.098848\, n_f^{\,2} + 12.9555655\, n_f^{\,3}\,, \\
\gamma_{\rm gg}^{(3)}(N\!=\!6) &= 53563.8435 - 14339.1310\, n_f + 652.777331\, n_f^{\,2} + 16.6541037\, n_f^{\,3}\,, \\
\gamma_{\rm gg}^{(3)}(N\!=\!8) &= 62279.7438 - 17150.6968\, n_f + 785.880613\, n_f^{\,2} + 18.9331031\, n_f^{\,3}\,, \\
&\ldots \\
\gamma_{\rm gg}^{(3)}(N\!=\!20) &= 90499.2530 - 26132.2983\, n_f + 1178.50283\, n_f^{\,2} + 25.6433278\, n_f^{\,3}\,.
\end{aligned}
$$

- Known large- and small-x limits and suitable ansatz approximate $P_{\rm gg}^{(3)}(x)$

Outlook

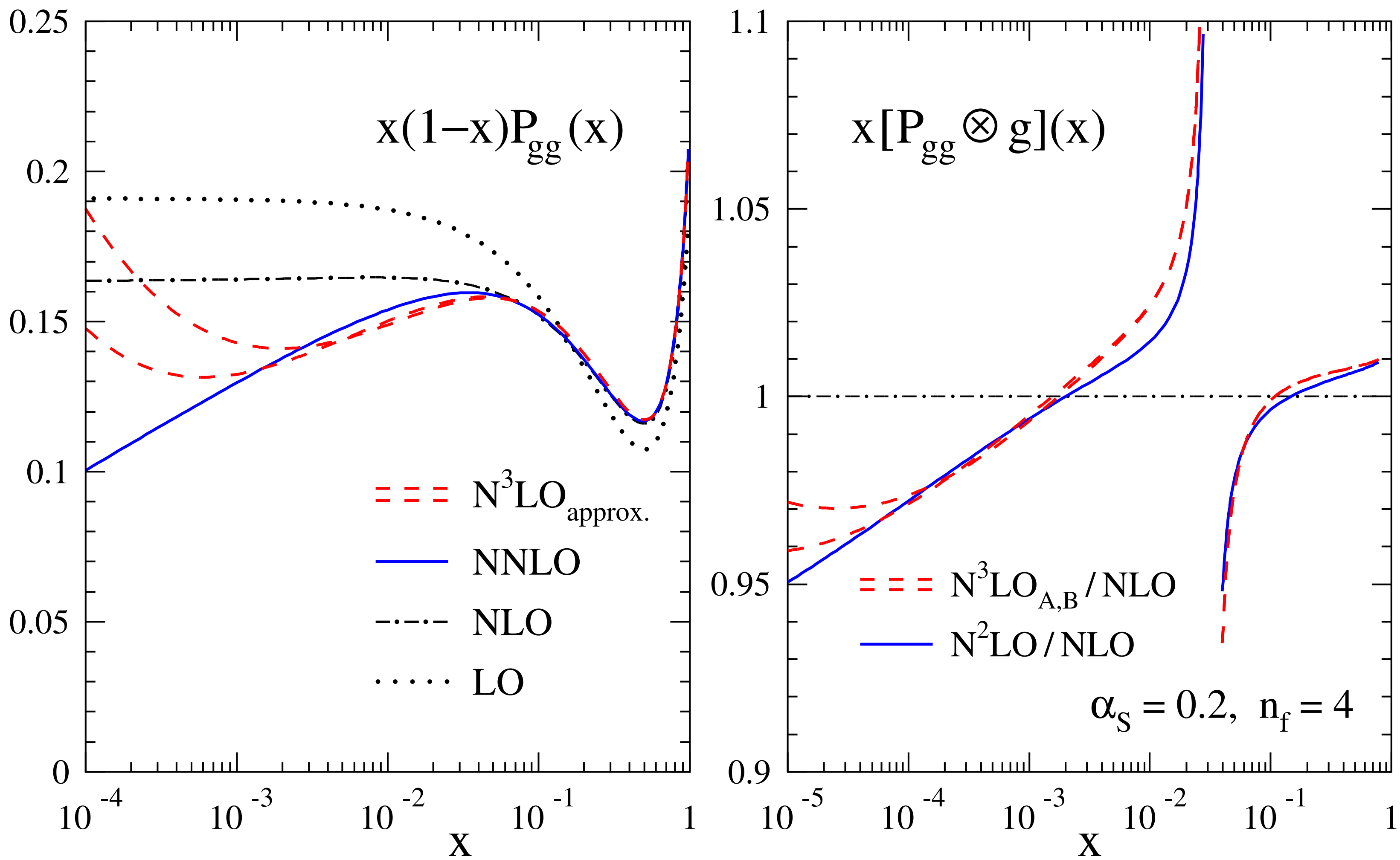
- Comparison to other approximations for $P_{\rm gg}^{(3)}$
  McGowan, Cridge, Harland-Lang, Thorne '22; NNPDF collaboration '24
- Benchmark N^{3}LO evolution
  Cooper-Sarkar, Cridge, Harland-Lang, Hekhorn, Huston, Magni, S.M., Thorne '24

Gluon-gluon splitting function (I)



- Approximations for $P_{gg}^{(3)}(x)$ based on moments $N \leq 10$ vs. $N \leq 20$
  - clear improvements at large-x (left) and small-x (right)

Gluon-gluon splitting function (II)



- Left: results for $P_{gg}(x)$ up to N^{3}LO; $\alpha_s = 0.2$ fixed, $n_f = 4$
- Right: contribution to evolution kernel $d \ln g / d \ln \mu_f^2$ up to N^{3}LO for typical gluon shape

$$xg(x, \mu_0^2) = 1.6\, x^{-0.3} (1-x)^{4.5} \left(1 - 0.6\, x^{0.3}\right)$$

Scale stability of evolution (I)

- PDF evolution
  - splitting functions enter PDF evolution via convolution

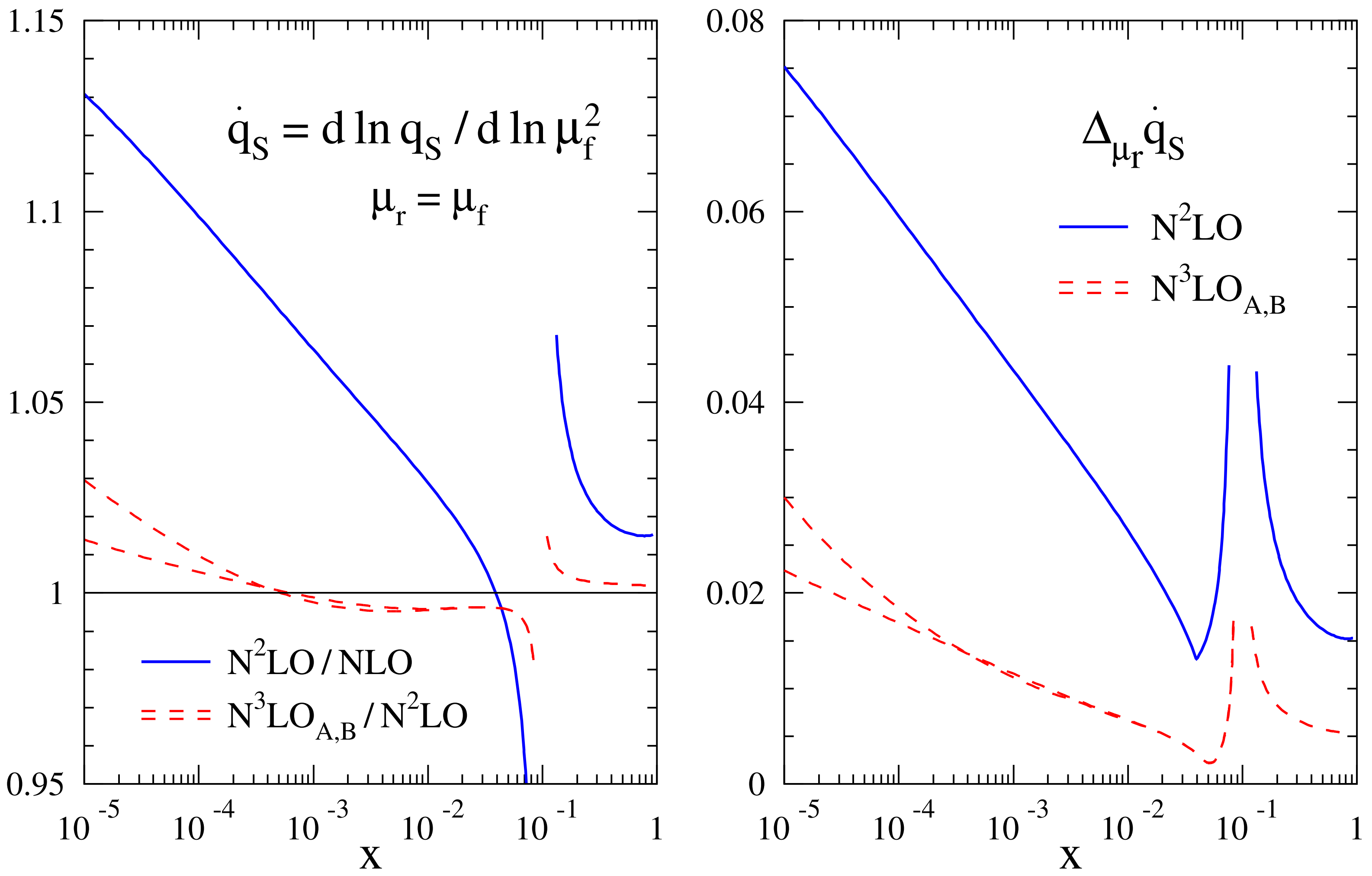
$$\frac{d}{d\ln\mu^2}\, f_{\rm i}(x) \quad = \quad \sum_{\rm j} \int_x^1 \frac{dz}{z}\, P_{\rm ij}(z)\, f_{\rm j}\left(\frac{x}{z}\right)$$

- Interplay between $P(z \sim x \to 0)$ and $f(\frac{x}{z} \to 1)$
  - $P(\mathbf{z} \sim \mathbf{x} \to \mathbf{0})$ has largest uncertainty
  - $f(\frac{\mathbf{x}}{\mathbf{z}} \to \mathbf{1})$ is suppressed
- Model singlet PDFs

$$\begin{aligned} x\mathrm{q}_{\,\mathrm{s}}(x,\mu_0^2) &= 0.6\, x^{\,-0.3}(\mathbf{1}-\mathbf{x})^{\mathbf{3.5}}\,\left(1+5.0\, x^{\,0.8}\right) \\ x\mathrm{g}(x,\mu_0^2) &= 1.6\, x^{\,-0.3}(\mathbf{1}-\mathbf{x})^{\mathbf{4.5}}\,\left(1-0.6\, x^{\,0.3}\right) \end{aligned}$$

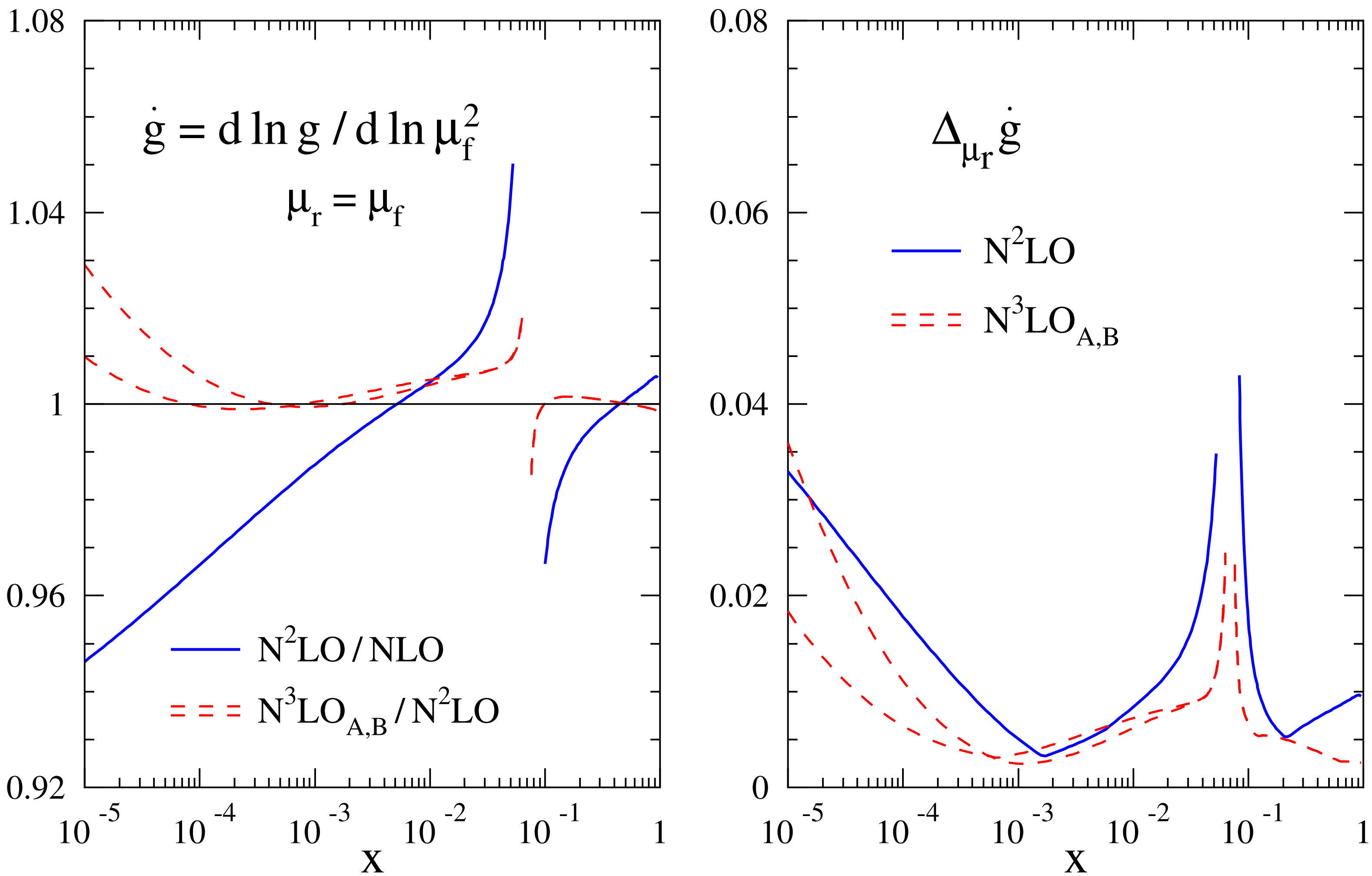
- Residual small-x uncertainty in four-loop splitting functions at $x \sim \mathcal{O}(10^{-4})$ affects PDFs only at $x \sim \mathcal{O}(10^{-5})$
  - edge of LHC parton kinematics (low scales, forward region)
  - $x \sim 10^{-5}$ corresponds to $y \sim 4$ and $Q \sim 10$ GeV

Scale stability of evolution (II)



- Relative NNLO and N^3LO corrections to scale derivative of the quark PDF q_s for $\alpha_s = 0.2$ fixed, $n_f = 4$
- Renormalization scale dependence of evolution kernel $d \ln q_s / d \ln \mu_r^2$

Scale stability of evolution (III)



- Relative NNLO and N^{3}LO corrections to scale derivative of the quark PDF g for $\alpha_s = 0.2$ fixed, $n_f = 4$
- Renormalization scale dependence of evolution kernel $d \ln \mathrm{g} / d \ln \mu_r^2$

All-N results

Analytic reconstruction (I)

- Sufficiently many Mellin moments allow for reconstruction of analytic all-N expressions through solution of Diophantine equations
  Lenstra, Lenstra, Lovász '82
- Harmonic sums define basis in space of functions for $\gamma_{\rm ij}(N)$

$$S_{\pm m_1,\ldots,m_k}(N) = \sum_{i=1}^{N} \frac{(\pm 1)^i}{i^{m_1}} S_{m_2,\ldots,m_k}(i)$$

  - at weight w there are $2 \cdot 3^{w-1}$ harmonic sums
  - l-loop $\gamma_{\rm ij}^{(l-1)}(N)$ contains harmonic sums up to weight $2l-1$
    $\longrightarrow$ numbers grow quickly: $2, 18, 162, 1458$ sums for $l = 1, 2, 3, 4$
- Some applications in QCD
  - three-loop non-singlet transversity $\gamma_{\rm tr}^{(2)}$ Velizhanin'12
  - three-loop polarized $\Delta\gamma_{\rm ij}^{(2)}$ S.M., Vermaseren, Vogt '14
  - four-loop non-singlet $\gamma_{\rm ns}^{(3)\pm}$ (large-n_c) S.M., Vogt, Ruijl, Ueda, Vermaseren '17
  - four-loop non-singlet DIS $C_{\rm ns}^{(4)}$ (large-n_f)
    Basdew-Sharma, Pelloni, Herzog, Vogt '22
  - ...

Analytic reconstruction (II)

Conformal symmetry and integrability

- Gribov-Lipatov reciprocity relation (RR)
  - diagonal splitting functions $P_{\rm ii}^{(0)}(x)$ invariant under mapping $x \to \frac{1}{x}$
- RR realized for universal $\gamma_{\rm u}(N)$ in $N = 4$ SYM theory
  - uniform transcendentality sums with $w = 2l - 1$ only at l-loops
- RR in N-space for QCD implies $\gamma(N) = \gamma_{\rm u}\left(N + \gamma(N) - \beta(\alpha_s)\right)$
- RR constraints for $\gamma_{\rm u}$ reduce number to 2^{w-1} sums at weight w for $\gamma_{\rm u}$
  - $2^{w+1} - 1$ objects with denominators $1/(N+1)$ added (255 at $w = 7$)

Example

- Large-n_c limit of $\gamma_{\rm ns}^{(3)\pm}$ only needs harmonic sums with positive index
  - weight w RR sums given by Fibonacci number $F(w)$
  - total number of unknowns (including powers $1/(N+1)$) amount to $F(w+4) - 2$ (87 at $w = 7$)
- Additional 46 constraints from large-x/small-x ($N \to \infty$/$N \to 0$) limit
- Solution becomes feasible with 18 Mellin moments for $\gamma_{\rm ns}^{(3)\pm}$

Analytic reconstruction (III)

- Mellin moments suffice to determine all-N result for parts of $\gamma_{\rm ps}^{(3)}(N)$
  - harmonic sums and Riemann ζ_n terms up to total weight $w = 7$
- Terms proportional to ζ_5 are particularly simple
  - N-dependent terms respect RR
  - RR implies invariance under mapping $N \to -N-1$
- Combinations of denominators $\eta = \dfrac{1}{N} - \dfrac{1}{N+1}$ and $\nu = \dfrac{1}{N-1} - \dfrac{1}{N+2}$

$$
\begin{aligned}
\gamma_{\rm ps}^{(3)}(N)\Big|_{\zeta_5} \quad = \quad & 160\, n_f\, C_F^3\, \Big(9\,\eta + 6\,\eta^2 - 4\,\nu\Big) + 80/3\, n_f\, C_A\, C_F^2\, \Big(-9\,\eta - 6\,\eta^2 + 4\,\nu\Big) \\
& +40/9\, n_f\, C_A^2\, C_F\, \Big(-1 - 214\,\eta - 144\,\eta^2 + 104\,\nu\Big) \\
& +320/3\, n_f\, \frac{d_R^{abcd} d_R^{abcd}}{n_c}\, \Big(-1 + 56\,\eta + 36\,\eta^2 - 16\,\nu\Big)
\end{aligned}
$$

- Inverse Mellin transformation generates additional terms with ζ_n
  - ζ_n in N-space $\neq$ ζ_n in x-space

Analytic reconstruction (IV)

- Quartic Casimir terms at four loops are effectively 'leading-order'
  - $d_{xy}^{(4)} \;\equiv\; d_x^{abcd} d_y^{abcd}$ for representations labels x, y with generators T_r^a

$$d_r^{abcd} \;=\; \frac{1}{6}\,\mathrm{Tr}\,(\,T_r^a\,T_r^b\,T_r^c\,T_r^d \;+\; \text{five } bcd \text{ permutations}\,)$$

  - anomalous dimensions fulfil relation for $\mathcal{N} = 1$ supersymmetry
    - '$\stackrel{Q}{=}$' equivalence restricted to quartics

$$\gamma_{\,\mathrm{qq}}^{(3)}(N) + \gamma_{\,\mathrm{gq}}^{(3)}(N) - \gamma_{\,\mathrm{qg}}^{(3)}(N) - \gamma_{\,\mathrm{gg}}^{(3)}(N) \;\stackrel{Q}{=}\; 0$$

- Eigenvalues of singlet splitting functions (conjectured to be) composed of reciprocity-respecting sums
  - quartic Casimir terms fulfil stronger condition   Belitsky, Müller, Schäfer '99

$$\gamma_{\,\mathrm{qg}}^{(0)}(N)\,\gamma_{\,\mathrm{gq}}^{(3)}(N) \;\stackrel{Q}{=}\; \gamma_{\,\mathrm{gq}}^{(0)}(N)\,\gamma_{\,\mathrm{qg}}^{(3)}(N)$$

- Moments $N \leq 22$ for quartic Casimir terms at four loops known for all singlet anomalous dimensions $\gamma_{\mathrm{qq}}, \gamma_{\mathrm{qg}}, \gamma_{\mathrm{gq}}$ and γ_{gg} to be published

Analytic reconstruction (V)

- Reconstruction of analytic all-N expressions for ζ_5 terms from solution of Diophantine equations
  - example for $\gamma_{\rm gg}^{(3)}$ with $\eta = \frac{1}{N} - \frac{1}{N+1}$ and $\nu = \frac{1}{N-1} - \frac{1}{N+2}$

$$
\begin{aligned}
\gamma_{\rm gg}^{(3)}(N)\Big|_{\zeta_5\, d_{AA}^{(4)}/n_A} \;=\; & \frac{64}{3}\Big(30\left(12\,\eta^{\,2} - 4\,\nu^{\,2} - S_1(4\,S_1 + 8\,\eta - 8\,\nu - 11) - 7\nu\right) \\
& + 188\,\eta - \frac{751}{3} - \frac{1}{6}\,N\,(N+1)\Big)
\end{aligned}
$$

- Recall large-N limit of anomalous dimensions

$$
\gamma_{\rm ii}^{(k)}(N)\Big|_{N\to\infty} \;=\; A_{n,\rm i}\,\ln(N) + \mathcal{O}(\text{const}_N)
$$

- Terms $S_1(N)^2 \sim \ln(N)^2$ and $N(N+1)$ proportional to ζ_5 must be compensated in large-N limit

Universal anomalous dimension

- Universal anomalous dimension γ_{u} in $N = 4$ SYM
  - one-loop $\gamma_{\mathrm{u}}^{(0)}(N) = n_c 4S_1$ emerges from

$\gamma_{\mathrm{qq}}^{(0)}(N) = C_F\left(-3 - 2\eta + 4S_1\right)$ or $\gamma_{\mathrm{gg}}^{(0)}(N) = C_A\left(4\eta - 4\nu + 4S_1\right) - \beta_0$

  - two & three loops Kotikov, Lipatov, Onishchenko, Velizhanin '04
- Starting at four loops wrapping corrections to complement asymptotic Bethe ansatz
  - four-loop Bajnok, Janik, Lukowski '08, five-loop Lukowski, Rej, Velizhanin '09, six-loop [...], ...
- $\gamma(N)^{\mathrm{wrap},(4)} \simeq S_1(N)^2\, f^{\mathrm{wrap}}(N)$

$$f^{\mathrm{wrap}}(N) \;=\; 5\zeta_5 - 2S_{-5} + 4S_{-2}\zeta_3 - 4S_{-2,-3} + 8S_{-2,-2,1} + 4S_{3,-2} - 4S_{4,1} + 2S_5$$

- Three-loop QCD coefficient functions $c_{\mathrm{ns}}^{(3)}(N)$ S.M., Vermaseren, Vogt '05
  - $c_{\mathrm{ns}}^{(3)}(N) \simeq C_F\left(C_F - \frac{C_A}{2}\right)^2 \{N(N+1)\, f^{\mathrm{wrap}}(N)\}$
- Planar $N = 4$ SYM: quantum spectral curve Gromov, Kazakov, Leurent, Volin '13
- Non-planar $N = 4$ SYM: γ_{u} at four loops Kniehl, Velizhanin '21

Summary

- Experimental precision of $\lesssim 1\%$ motivates computations at higher order in perturbative QCD
  - theoretical predictions at NNLO in QCD nowadays standard
- Push for theory results at N^{3}LO (and even N^{4}LO)
  - evolutions equations expected to achieve percent-level
  - massive use of computer algebra
- Four-loop splitting functions approximated from moments $N = 2, \ldots 20$
  - residual uncertainties negligible in wide kinematic range of x probed at current and future colliders
  - $P_{\rm qq} = P^{+}_{\rm ns} + P_{\rm ps}$, $P_{\rm qg}$ $P_{\rm gq}$ and $P_{\rm gg}$ all done
- All-N results to come
- Novel structural insights into QCD from integrabilty and conformal symmetry
  - Key parts of QCD inherited from $N = 4$ Super Yang-Mills theory
  - Conformal symmetry in parts of QCD evolution equations