Developing an amplitude level parton shower: CVolver

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With Jeff Forshaw and Simon Plätzer

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The University of Manchester

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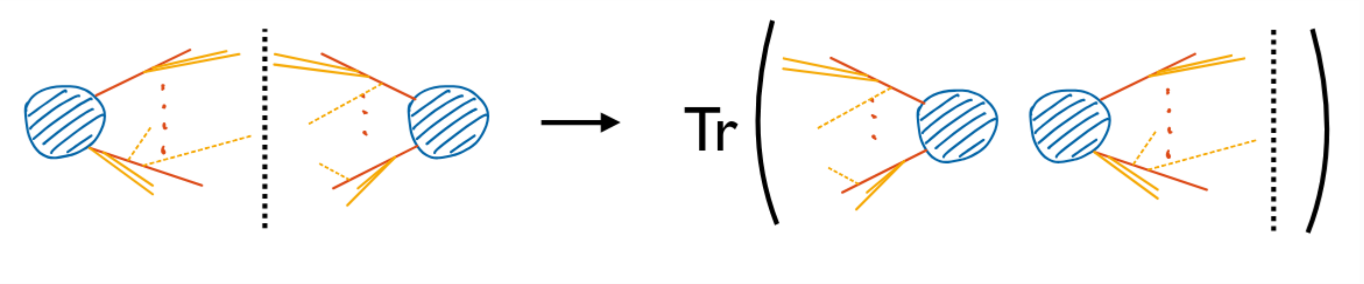
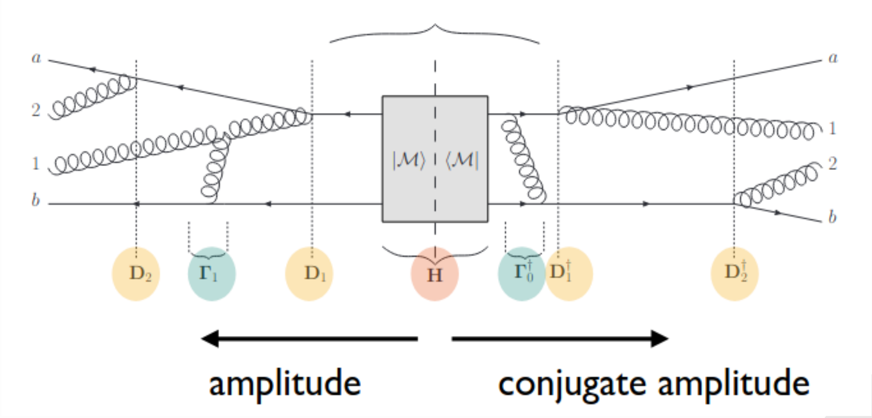
Recap from Plätzer's plenary talk

Why amplitude evolution?



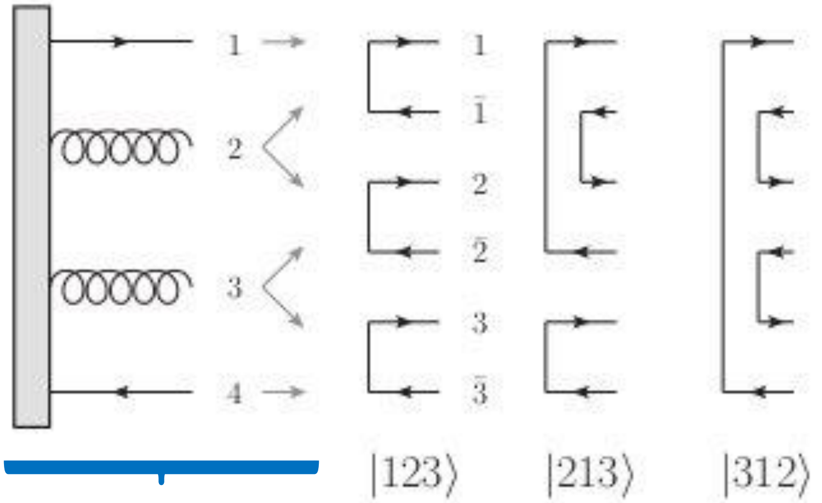
Amplitude evolution is a recursive algorithm which iterates gluon exchanges and emissions.

Different evolution of amplitude and conjugate amplitude necessary to include interference.



The colour flow basis

$$\sum_a t^a_{ij} t^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{kj} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right)$$



The colour lines that each parton carries.

Possible ways of connecting the colour flows. Each permutation corresponds to a different state in colour space.

Gluons carry a colour and an anti-colour.
Quarks carry colour and anti-quarks carry anti-colour.

A possible colour state is given by connecting each colour line with an anti-colour line.

If there are *n* colour flows (colour lines connected to anti-colour lines), in total there must be *n!* possible colour states, as this is the number of permutations.

Graphic from Angeles Martinez, de Angelis, Forshaw, Plätzer, Seymour [1802.08531]

Plätzer [1312.2448] and surrounding work

The colour flow basis

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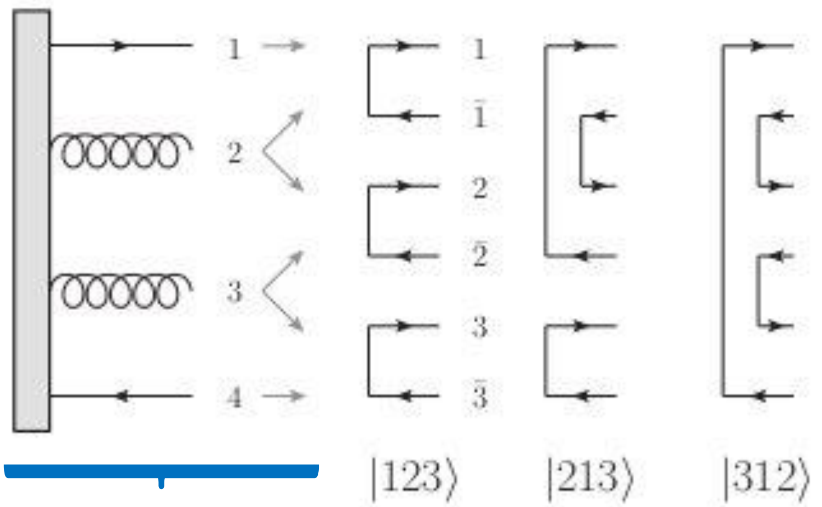

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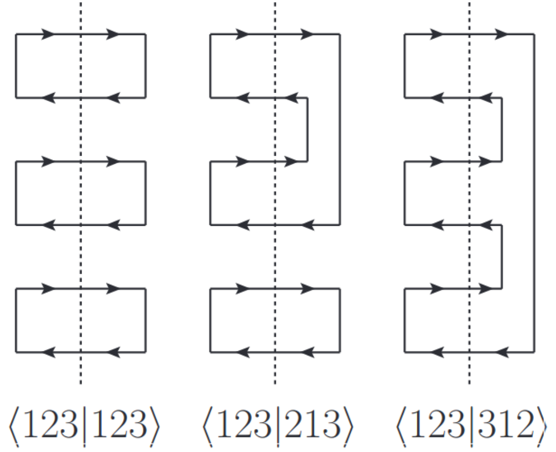


number of colour flows

$$\langle \sigma | \tau \rangle = N_c^{n - \mathrm{T}(\sigma,\tau)}$$

number of permutations between the colour states

Graphic from Angeles Martinez, de Angelis, Forshaw, Plätzer, Seymour [1802.08531]

More details in Forshaw, Plätzer et al. [1802.08531]

The colour flow basis: real emissions

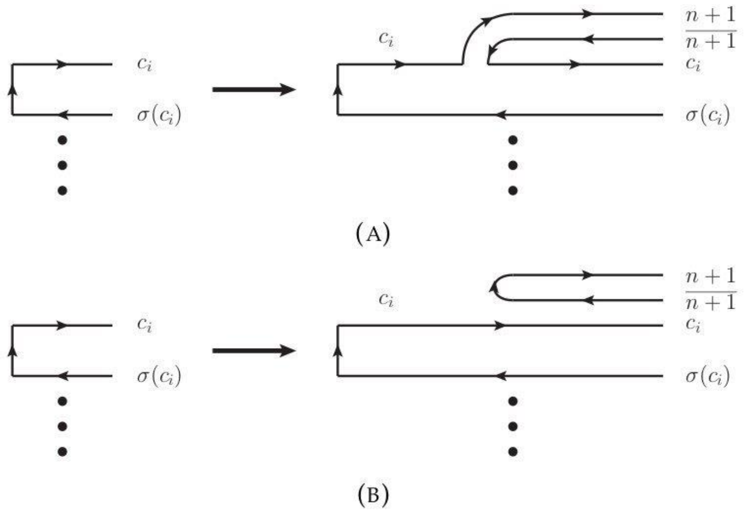
i	c_i	$\bar{c}_i$	λ_i	$\bar{\lambda}_i$
1	1	0	$\sqrt{T_R}$	0
2	2	1	$\sqrt{T_R}$	$\sqrt{T_R}$
3	3	2	$\sqrt{T_R}$	$\sqrt{T_R}$
4	0	3	0	$\sqrt{T_R}$

$$\mathbf{T}_i = \lambda_i \mathbf{t}_{c_i} + \bar{\lambda}_i \bar{\mathbf{t}}_{\bar{c}_i} - \frac{1}{N_c}\left(\lambda_i - \bar{\lambda}_i\right)\mathbf{s},$$

These are always rectangular matrices, of dimensions *(n+1)!xn!*

$\mathbf{t}_{c_i}$ or $\bar{\mathbf{t}}_{\sigma(c_i)}$

$\mathbf{s}$



More details in Forshaw, Plätzer et al. [1802.08531]

The colour flow basis: real emissions

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4	0	3	0	$\sqrt{T_R}$

These are always rectangular matrices, of dimensions *(n+1)!xn!*

For a $e^+e^- \to$ qqbar hard process, emitting from the 1st quark every time:

0 to 1 emission

$$\begin{pmatrix} -\frac{1}{\sqrt{2}\,n} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

1 to 2 emissions

$$\begin{pmatrix} -\frac{1}{\sqrt{2}\,n} & 0 \\ 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}\,n} \\ 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

2 to 3 emissions

$$\begin{pmatrix} -\frac{1}{\sqrt{2}\,n} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}\,n} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}\,n} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}\,n} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}\,n} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}\,n} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \end{pmatrix}$$

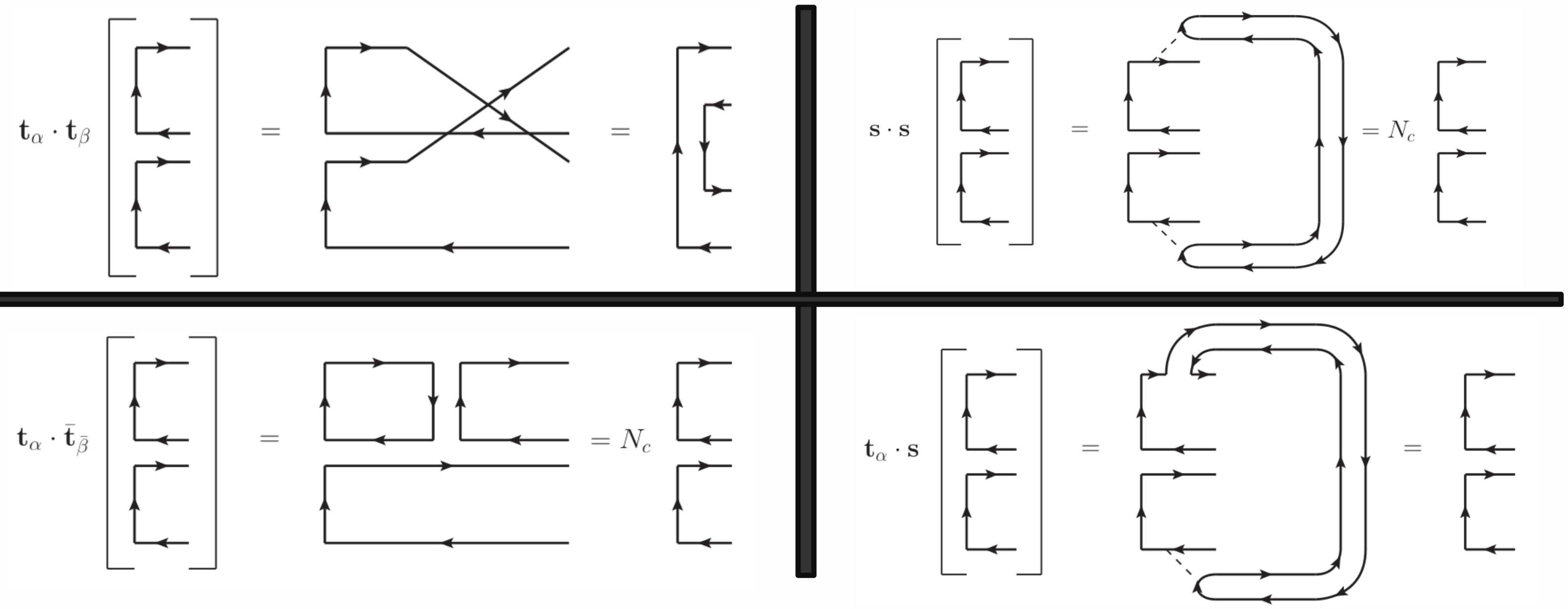
In these matrices, *n* is the number of colours.

More details in Forshaw, Plätzer et al. [1802.08531]

The colour flow basis: virtual exchanges

$$\mathbf{T}_i = \lambda_i \mathbf{t}_{c_i} + \bar{\lambda}_i \bar{\mathbf{t}}_{\bar{c}_i} - \frac{1}{N_c}\left(\lambda_i - \bar{\lambda}_i\right)\mathbf{s},$$

The $\boldsymbol{T}_i \bullet \boldsymbol{T}_j$ matrices are square, of *n!xn! dimensions.*



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The colour flow basis: virtual exchanges

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The $\boldsymbol{T}_i \cdot \boldsymbol{T}_j$ matrices are square, of *n!xn! dimensions.*

For a $e^+e^- \rightarrow$ qqbar hard process, doing a virtual exchange between the two hard quarks:

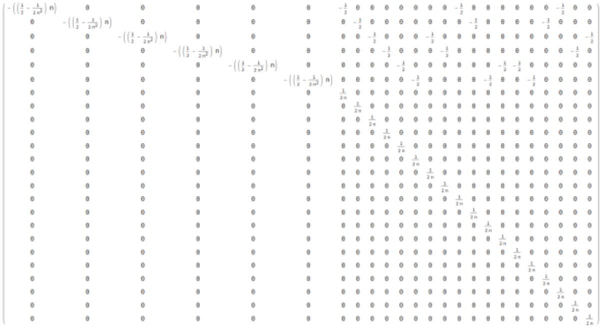
after 1 emission

$$\begin{pmatrix} -\left(\left(\frac{1}{2} - \frac{1}{2\,n^2}\right) n\right) & -\frac{1}{2} \\ 0 & \frac{1}{2\,n} \end{pmatrix}$$

after 2 emissions

$$\begin{pmatrix} -\left(\left(\frac{1}{2} - \frac{1}{2\,n^2}\right) n\right) & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & -\left(\left(\frac{1}{2} - \frac{1}{2\,n^2}\right) n\right) & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2\,n} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2\,n} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2\,n} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2\,n} \end{pmatrix}$$

after 3 emissions



In these matrices, n is the number of colours.

Plätzer [1312.2448]

The colour flow basis: virtual exchanges

$$\mathbf{T}_i = \lambda_i \mathbf{t}_{c_i} + \bar{\lambda}_i \bar{\mathbf{t}}_{\bar{c}_i} - \frac{1}{N_c}\left(\lambda_i - \bar{\lambda}_i\right) \mathbf{s},$$

$\boldsymbol{T}_i \bullet \boldsymbol{T}_j$ leads to multiple ways of connecting the emission operators:

$$\mathbf{t}_\beta \cdot \mathbf{t}_\alpha = 1 \text{ transposition}$$

$$\mathbf{t}_\beta \cdot \bar{\mathbf{t}}_{\bar{\alpha}} = N_c \mathbb{1} \text{ or } 1 \text{ transposition}$$

$$\mathbf{t}_\alpha \cdot \mathbf{s} = \mathbb{1}$$

$$\mathbf{s} \cdot \mathbf{s} = N_c \mathbb{1}$$

Sigma contains terms with one colour flow swap

Rho contains all the singlet exchanges

Gamma contains the diagonal, leading colour part

We can separate the anomalous dimension in three terms by colour structure:

$$[\tau|\,\mathbf{\Gamma}\,|\sigma\rangle = -N_c \delta_{\tau\sigma} \Gamma_\sigma + \Sigma_{\tau\sigma} + \frac{1}{N_c} \delta_{\tau\sigma} \rho$$

Studied at two loops by Simon Plätzer and Ines Ruffa [2012.15215]

Plätzer [1312.2448]

Exponentiating the anomalous dimension

$$[\tau|\,\mathbf{\Gamma}\,|\sigma\rangle = -N_c\delta_{\tau\sigma}\Gamma_\sigma + \Sigma_{\tau\sigma} + \frac{1}{N_c}\delta_{\tau\sigma}\rho$$

We can exponentiate the anomalous dimension as an infinite series in the number of swaps:

$$[\tau|e^{\Gamma}|\sigma] = \delta_{\tau\sigma}e^{-N\Gamma_\sigma}\left(1+\frac{\rho}{N}\right) - \frac{1}{N}\Sigma_{\tau\sigma}\frac{e^{-N\Gamma_\tau} - e^{-N\Gamma_\sigma}}{\Gamma_\tau - \Gamma_\sigma} + \mathrm{NNLC}$$

Plätzer [1312.2448]

Exponentiating the anomalous dimension

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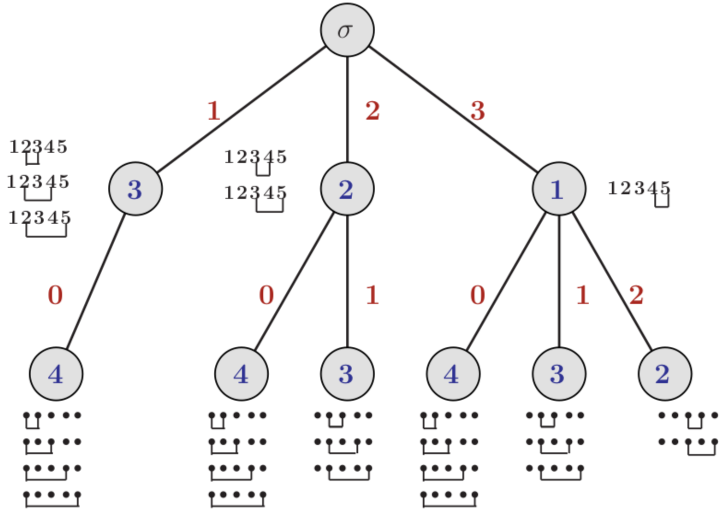
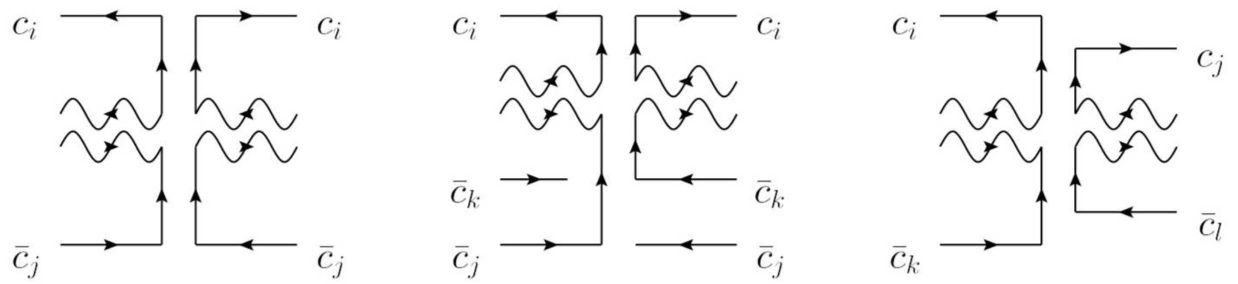
Which we can use to sample directly from during Monte Carlo evolution. We simply need to select at each step how many swaps we want to introduce, and select a permutation that number of swaps away.



Graphic from de Angelis's talk at PSR19, showing the Level Swap Algorithm

Revisiting the real emissions: rings and strings

We can distinguish three different cases, depending on the choice of colour flows in the amplitude versus the conjugate amplitude:



Both colour lines are the same

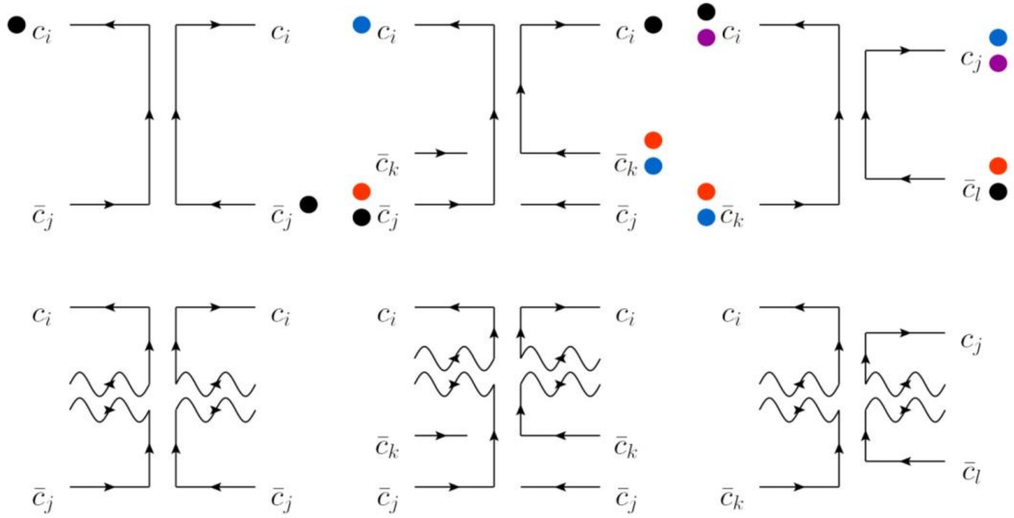
They share one colour line

Both colour lines are different

Forshaw, Holguin, Plätzer [2112.13124]

Revisiting the real emissions: rings and strings

$$\omega_{ij}(q_n) = \frac{q_i \cdot q_j}{q_n \cdot q_i \; q_n \cdot q_j}.$$



If both colour lines are shared: only one dipole contributes

ω_{ij}

If one colour line is shared: three dipoles contributes

$\omega_{ij} + \omega_{ik} - \omega_{jk}$

If no colour line is shared: four dipoles contribute

$\omega_{il} + \omega_{ik} - \omega_{kl} - \omega_{ij}$

Forshaw, Holguin, Plätzer [2112.13124]

Revisiting the real emissions: rings and strings

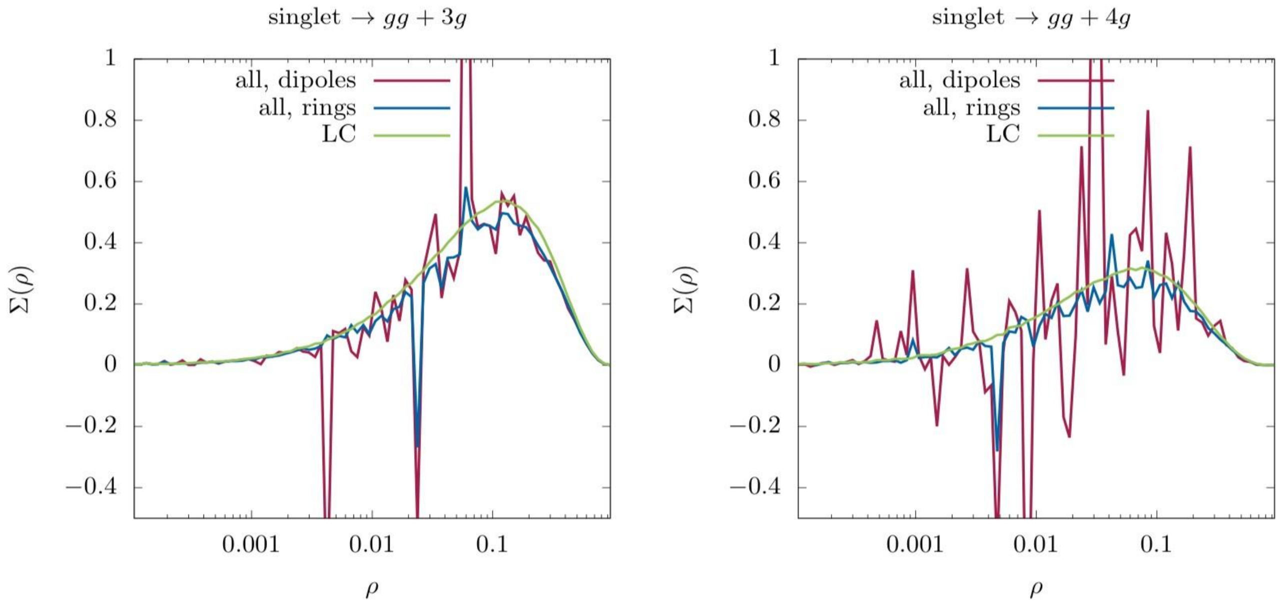
$$\omega_{ij}(q_n) = \frac{q_i \cdot q_j}{q_n \cdot q_i \; q_n \cdot q_j}.$$

Dipole, two uncancelled poles	LC, Γ term	If both colour lines are shared: only one dipole contributes ω_{ij}
String, one uncancelled pole	Subleading colour, Σ term Can only happen in off-diagonal evolution	If one colour line is shared: three dipoles contribute $\omega_{ij} + \omega_{ik} - \omega_{jk}$
Ring, no poles	Subleading colour, Σ term Can happen in diagonal or off-diagonal evolution	If no colour line is shared: four dipoles contribute $\omega_{il} + \omega_{ik} - \omega_{kl} - \omega_{ij}$

Forshaw, Holguin, Plätzer [2112.13124]

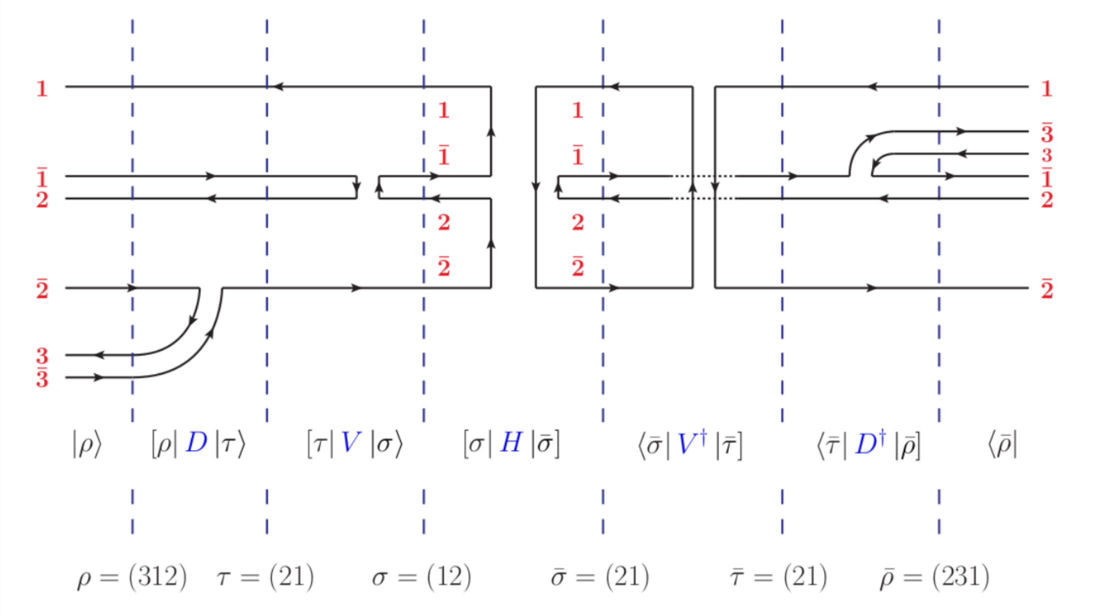
Revisiting the real emissions: rings and strings

In CVolver, after we select the after emission colour flows, we can reweight using the corresponding dipole, string or ring. This makes collinear cancellations explicit and helps computationally.



More details in Forshaw, Plätzer et al. [1802.08531]

The soft evolution algorithm



$$\mathbf{A}_n(E) = \mathbf{V}_{E,E_n}\mathbf{D}_n^{\mu}\,\mathbf{A}_{n-1}(E_n)\,\mathbf{D}_{n\mu}^{\dagger}\mathbf{V}_{E,E_n}^{\dagger}\,\Theta(E \leq E_n)$$

Graphic from de Angelis, Forshaw, Plätzer: 2007.09648

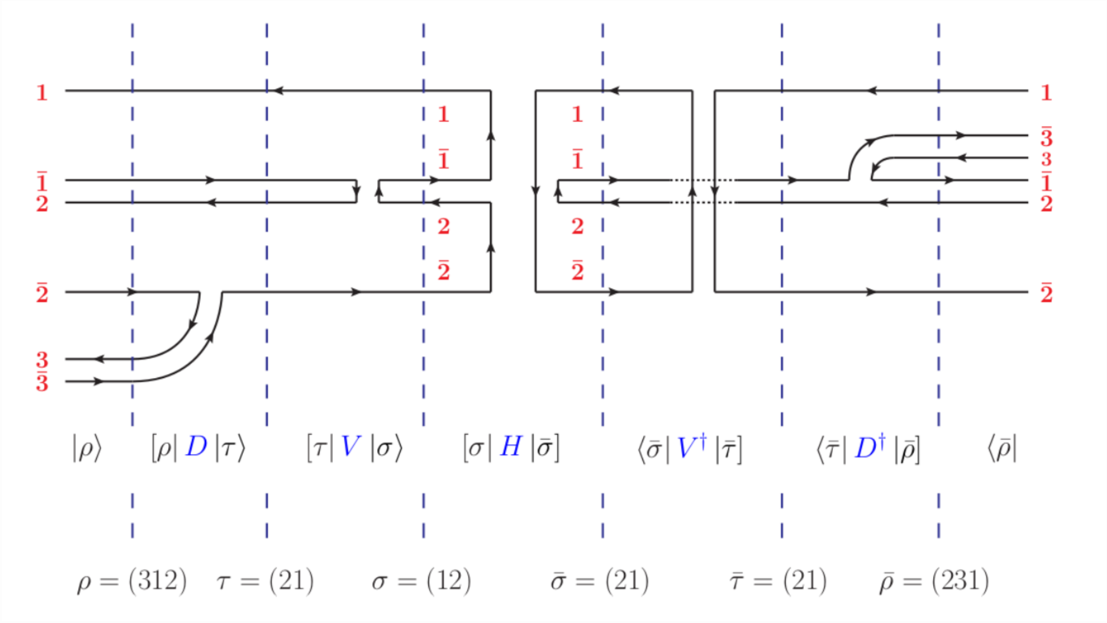
We dress the hard process density matrix with iterative real and virtual operators:

$$\begin{aligned}
\sigma_0 &= \mathrm{Tr}\left(\mathbf{V}_{\mu,Q}\mathbf{H}(Q)\mathbf{V}_{\mu,Q}^{\dagger}\right) \equiv \mathrm{Tr}\,\mathbf{A}_0(\mu) \\
\mathrm{d}\sigma_1 &= \mathrm{Tr}\left(\mathbf{V}_{\mu,E_1}\mathbf{D}_1^{\mu}\mathbf{V}_{E_1,Q}\mathbf{H}(Q)\mathbf{V}_{E_1,Q}^{\dagger}\mathbf{D}_{1\mu}^{\dagger}\mathbf{V}_{\mu,E_1}^{\dagger}\right)\mathrm{d}\Pi_1 \\
&\equiv \mathrm{Tr}\,\mathbf{A}_1(\mu)\mathrm{d}\Pi_1, \\
\mathrm{d}\sigma_2 &= \mathrm{Tr}\left(\mathbf{V}_{\mu,E_2}\mathbf{D}_2^{\nu}\mathbf{V}_{E_2,E_1}\mathbf{D}_1^{\mu}\mathbf{V}_{E_1,Q}\mathbf{H}(Q)\mathbf{V}_{E_1,Q}^{\dagger}\mathbf{D}_{1\mu}^{\dagger}\mathbf{V}_{E_2,E_1}^{\dagger}\mathbf{D}_{2\nu}^{\dagger}\mathbf{V}_{\mu,E_2}^{\dagger}\right)\mathrm{d}\Pi_1\mathrm{d}\Pi_2 \\
&\equiv \mathrm{Tr}\,\mathbf{A}_2(\mu)\mathrm{d}\Pi_1\mathrm{d}\Pi_2 \\
&\vdots \\
\mathrm{d}\sigma_n &= \mathrm{Tr}\,\mathbf{A}_n(\mu)\prod_{i=1}^{n}\mathrm{d}\Pi_i
\end{aligned}$$

$$\mathrm{Tr}\,\mathbf{A}_n = \mathrm{Tr}[\mathcal{A}\mathcal{S}] = \sum_{\sigma,\tau}[\tau|\,\mathbf{A}_n\,|\sigma]\,\langle\sigma|\,\tau\rangle$$

$$\mathrm{d}\sigma_n = \mathrm{Tr}\,\mathbf{A}_n(\mu)\prod_{i=1}^{n}\mathrm{d}\Pi_i$$

The soft evolution algorithm



$$\mathbf{A}_n(E) = \mathbf{V}_{E,E_n} \mathbf{D}_n^{\mu} \, \mathbf{A}_{n-1}(E_n) \, \mathbf{D}_{n\mu}^{\dagger} \mathbf{V}_{E,E_n}^{\dagger} \, \Theta(E \leq E_n)$$

At each step in the evolution, the colour state after the action of the real and virtual operators is sampled. This defines each event as a trajectory in colour space.

We count every factor of $1/N_c$ included at each step. There are four possible sources: the reals, the virtuals, the scalar product of the final colour states, and the hard process.

Thus, we can expand the cross-section in this way:

$$\sum_m N_c^m g_m \left(C_{0,m} + \frac{C_{1,m}}{N_c} + \frac{C_{2,m}}{N_c^2} + \ldots \right)$$

Graphic from de Angelis, Forshaw, Plätzer: 2007.09648

Keeping track of colour

$$
\begin{pmatrix}
n^3 & n^2 & n^2 & n & n & n^2 \\
n^2 & n^3 & n & n^2 & n^2 & n \\
n^2 & n & n^3 & n^2 & n^2 & n \\
n & n^2 & n^2 & n^3 & n & n^2 \\
n & n^2 & n^2 & n & n^3 & n^2 \\
n^2 & n & n & n^2 & n^2 & n^3
\end{pmatrix}
$$

Scalar product matrix for 3 colour flows

We have four possible sources of subleading colour:

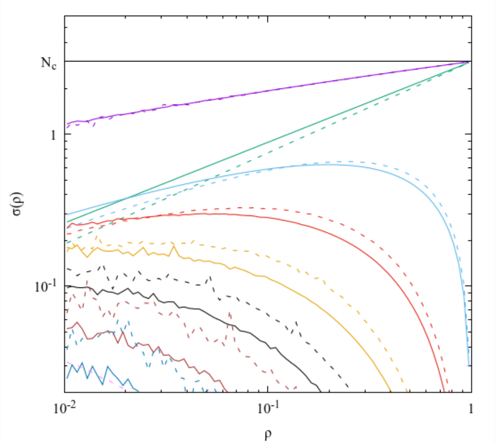
- **The hard process matrix:** the colour structure of the process we are evolving.
- **The virtual evolution:** every V operator can produce any number of swaps on either side of the density matrix. Each swap comes with a factor of $1/N_c$.
- **The real emissions:** the only explicit factors of $1/N_c$ come from the s operator, singlet gluon emissions.
- **The scalar product matrix:** the inner product of the colour permutations of the amplitude and conjugate amplitude.

$$\mathrm{Tr}\, \mathbf{A}_n = \mathrm{Tr}[\mathcal{A}\mathcal{S}] = \sum_{\sigma,\tau} [\tau|\, \mathbf{A}_n \,|\sigma]\, \underbrace{\langle\sigma|\, \tau\rangle}$$

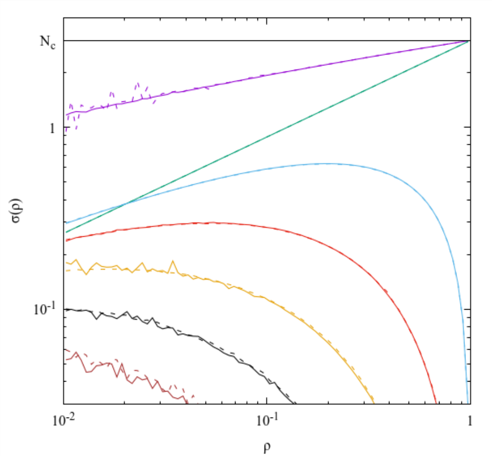
Scalar product matrix

Cross checks

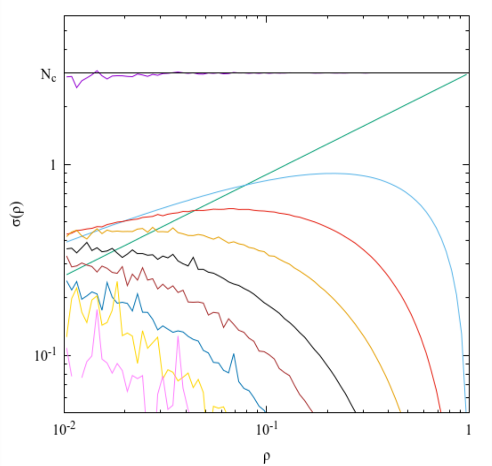
Collinear cutoff independence



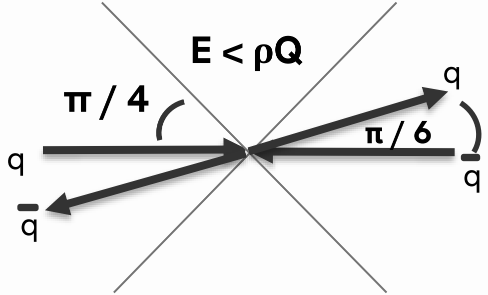
Lorentz Invariance



Unitarity

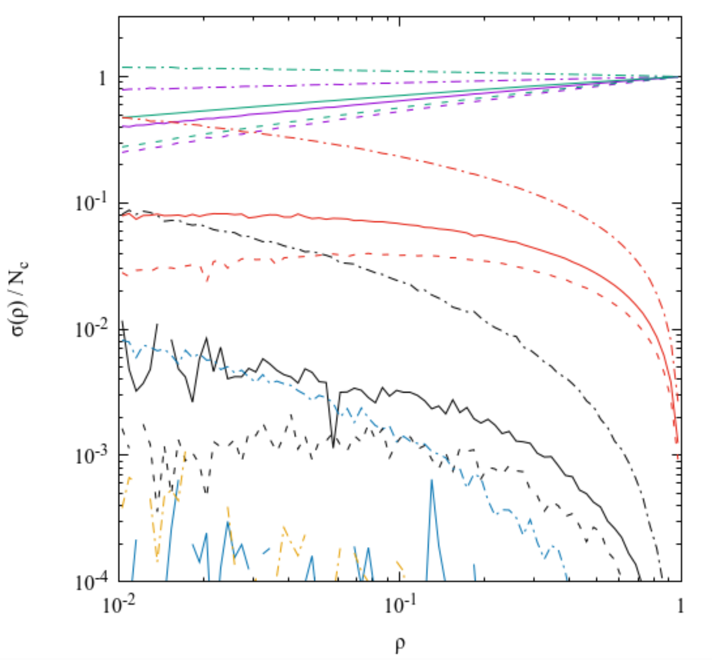


We have also compared against Hatta and Ueda's results [2011.04154] performed using Langevin evolution, which is a completely orthogonal method.



$e^+e^- \to q\bar{q}$ | gaps between jets

Breakdown in powers of $1/N_c$



Full colour

LC'

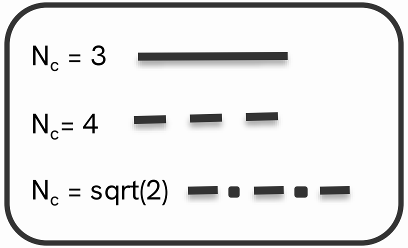
-(1 / N_c^2) : NLC'

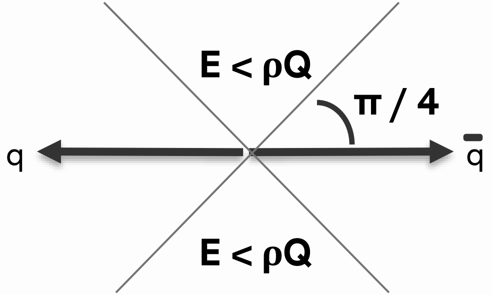
(1 / N_c^4) : NNLC'

-(1 / N_c^6) : N3LC'

(1 / N_c^8) : N4LC'

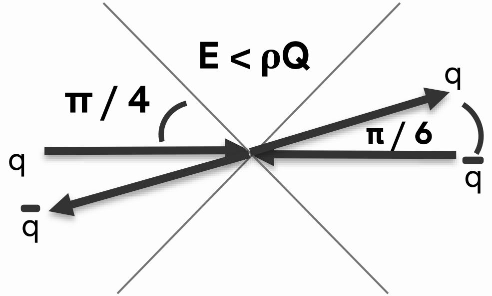
$$\sum_m N_c^m g_m \left(C_{0,m} + \frac{C_{1,m}}{N_c^2} + \frac{C_{2,m}}{N_c^4} + \frac{C_{3,m}}{N_c^6} + \frac{C_{4,m}}{N_c^8} + \dots \right)$$

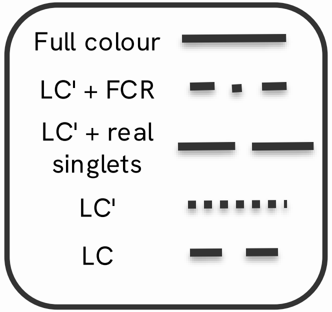


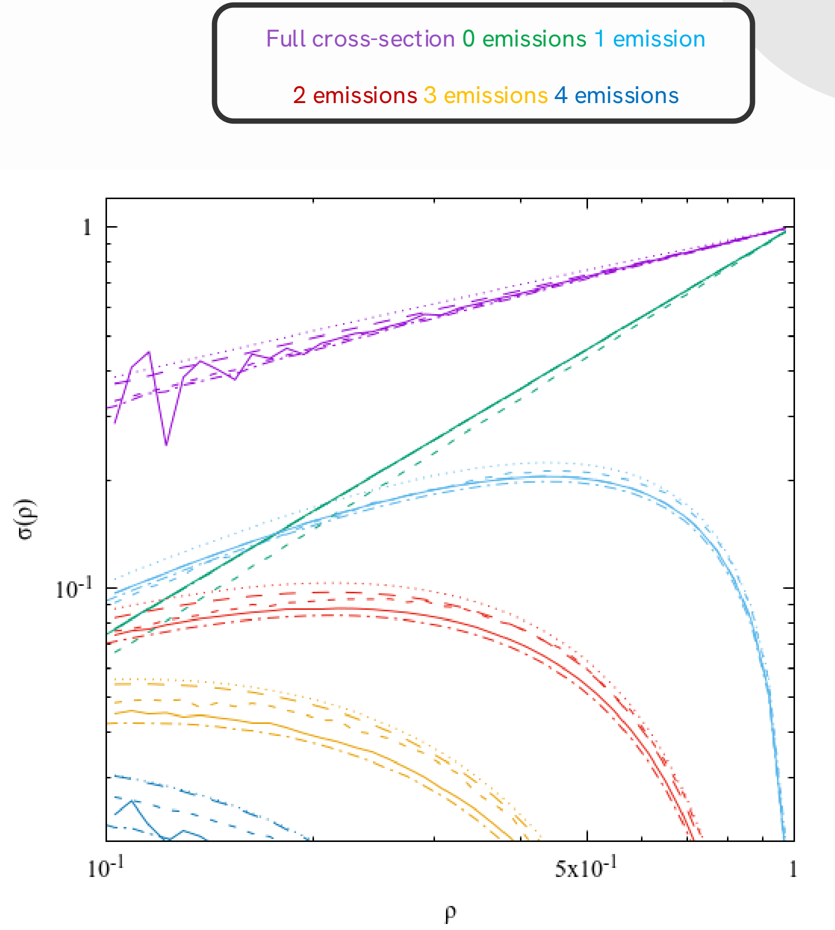


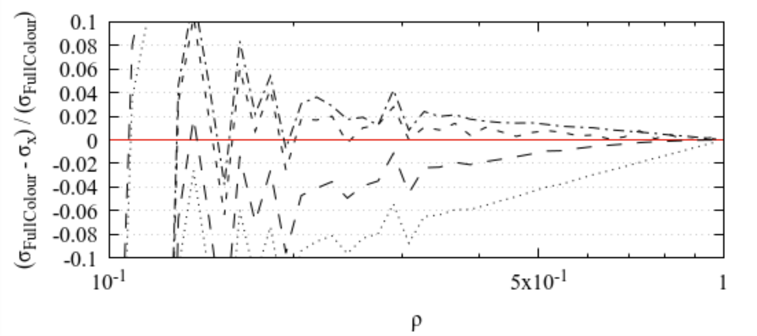
$q\bar{q} \to q\bar{q}$ | colour study

Back-to-back configuration

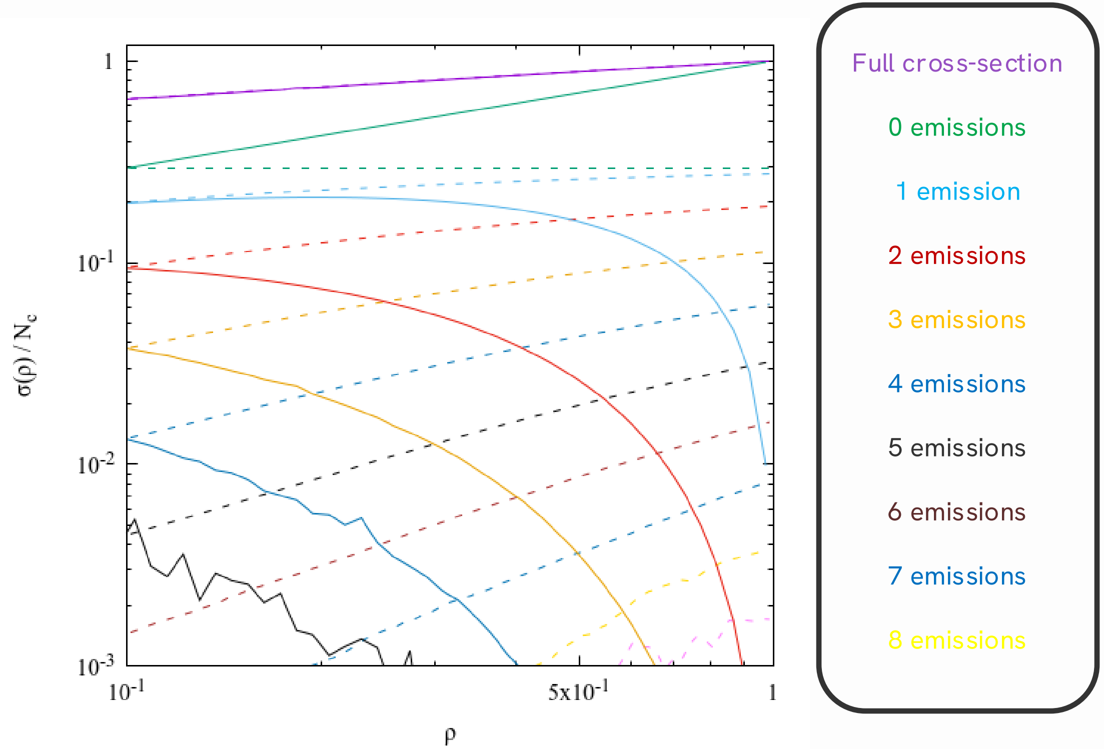






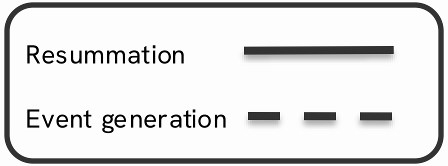


$e^+e^- \to q\bar{q}$ | event generator mode



Until now we have only discussed using CVolver as a resummation tool for gaps between jets.

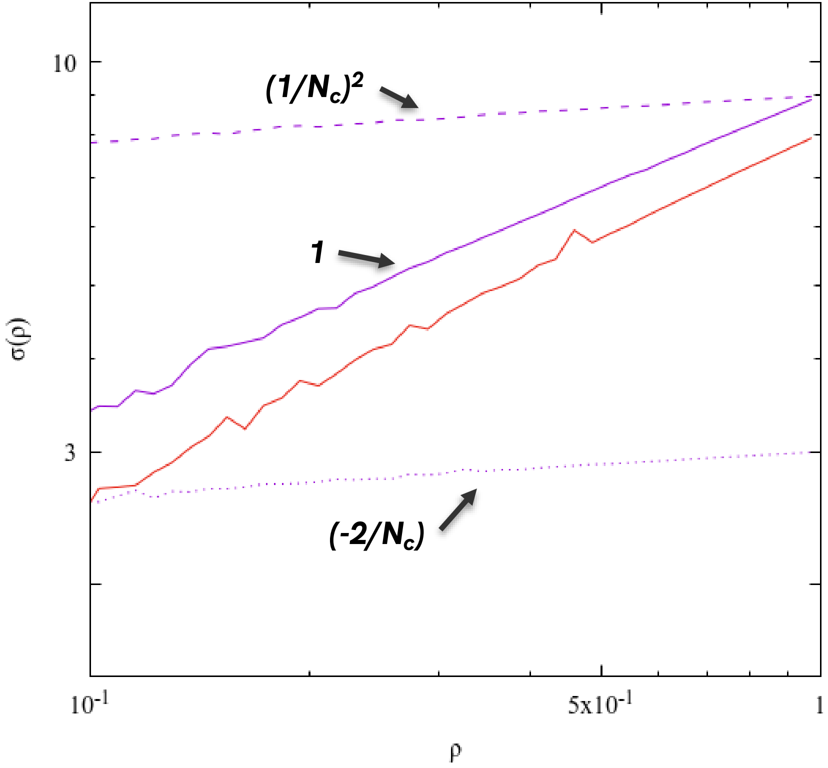
But it can also be run as an event generator -- evolving all events down to some infra-red scale μ, and then applying the measurement function to generate a differential cross-section.



$q\bar{q} \to q\bar{q}$ | evolving interference terms

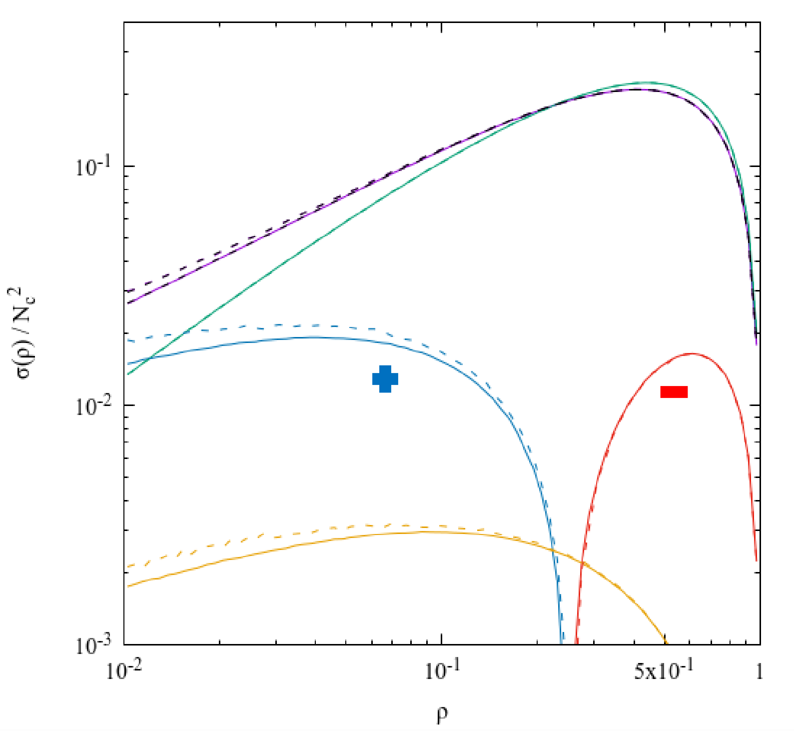
Back-to-back configuration

Initial states colour connected and final states colour connected	solid purple line
Initial states connected with final states	dashed purple line
Interference	dotted purple line
Full hard process t-channel gluon exchange	solid red line



$q\bar{q} \to q\bar{q}$ | Coulomb exchanges

1 emission, broken down in powers of $1/N_c$



Full colour

LC'

$-(1 / N_c^2)$: NLC'

$+(1 / N_c^2)$: NLC'

$+(1 / N_c^4)$: NNLC'

No coulomb gluons	solid line
With coulomb gluons	dashed line

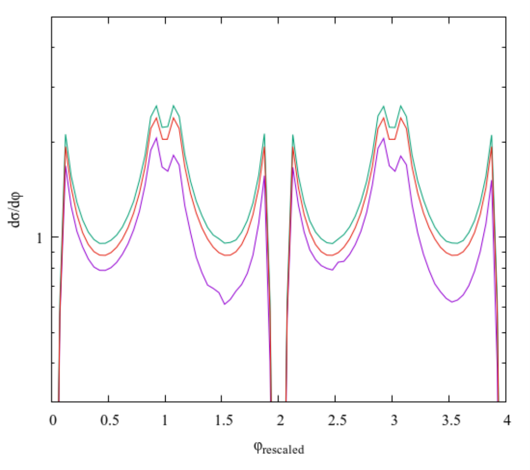
We can simply insert the Coulomb/Glauber phases in the anomalous dimension, and the algorithm is able to resum them.

Perfect agreement with independent calculation.

$$\mathbf{V}_{a,b} = \exp\left(-\frac{\alpha_s}{\pi}\ln\left(\frac{b}{a}\right)\sum_{i<j}(-\mathbf{T}_i\cdot\mathbf{T}_j)\left(\int\frac{d\Omega_k}{4\pi}\omega_{ij}(\hat{k}) - i\pi\tilde{\delta}_{ij}\right)\right)$$

Summary and next steps

- We have implemented a full colour soft evolution algorithm that keeps track of every source of subleading colour in a systematic way.
- We have performed every cross-check we can think of.
- We are studying the subleading colour structures for different processes and kinematic configurations.
- It is ready to produce lots of other interesting physics -- for example we can study differential observables in event generator mode (like colour reconnection) and Glauber effects.
- The algorithm design is such that we can "plug in" new features, which are under development:
  - Full kinematics parton shower, with hard process elements
  - k_t ordering
  - Hard-collinear physics



Colour reconnection in e^+e^- to four jets