The Structure of Perturbative Renormalization Group Functions

Anders Eller Thomsen

Applications of Field Theory to Hermitian and non-Hermitian Systems
King's College London, 10-13 September 2024

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How I Learned to Stop Worrying and Tolerate Divergence

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Parametrizing β-functions

Most general 4D renormalizable theory (ignoring relevant couplings):

$$\begin{aligned}\mathcal{L} = &+ \tfrac{1}{2}(\partial^\mu \phi - i A_\mu^A T_\phi^A \phi)_a^2 + i\psi_i^\dagger \bar{\sigma}^\mu(\partial_\mu \psi - i A_\mu^A T_\psi^A \psi)^i \\ &- \tfrac{1}{4} a_{AB}^{-1} F_{\mu\nu}^A F^{B\mu\nu} - \tfrac{1}{2}\left(y_{aij}\psi^i\psi^j + \mathrm{H.c.}\right)\phi_a - \tfrac{1}{24}\lambda_{abcd}\phi_a\phi_b\phi_c\phi_d\end{aligned}$$

Parametrize RG functions with monomials of the couplings, e.g.,

$$\beta_{aij} = \frac{\mathrm{d}y_{aij}}{\mathrm{d}t}, \qquad \beta_{aij}^{(\ell)} = \sum_n \underbrace{\mathbf{y}_n^{(\ell)}}_{\text{real-valued coefficients}} [Y_n^{(\ell)}(a,\, y,\, \lambda,\, T_\psi,\, T_\phi)]_{aij}$$

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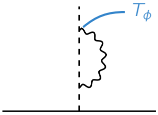
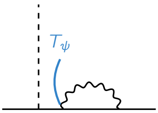
Parametrization of **1-loop Yukawa β-functions**:

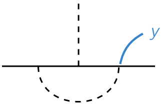
$$\beta_{aij}^{(1)} = \mathbf{y}_1^{(1)} y_b [T_\phi^A a_{AB} T_\phi^B]_{ba} + \mathbf{y}_2^{(1)} y_a [T_\psi^A a_{AB} T_\psi^B] + \mathbf{y}_3^{(1)} y_b y_a^* y_b + \mathbf{y}_4^{(1)} y_b y_b^* y_a + \mathbf{y}_5^{(1)} y_b \operatorname{Tr} \tilde{y}_b y_a$$



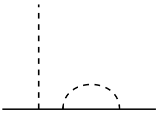
$\mathbf{y}_1^{(1)}$



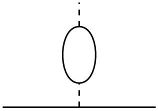
$\mathbf{y}_2^{(1)}$



$\mathbf{y}_3^{(1)}$



$\mathbf{y}_4^{(1)}$



$\mathbf{y}_5^{(1)}$

General β-functions in 4D QFTs

- General formulas for β-functions are long known to order 3–2–2 ($\overline{\text{MS}}$).

  Macachek, Vaughn '83,'84; Jack, Osborn '84; Pickering, Gracey, Jones [hep-ph/0104247]

- Computer packages with implementation of the general formulas: **SARAH 4**, **PyR@TE 3**, **ARGES**, and **RGBeta**.

- Recent results for the general 4–3–3 β-functions

  Bednyakov, Pikelner [2105.09918]; Davies, Herren, **AET** [2110.05496]; Steudtner, **AET** [arXiv:2408.05267]

  - Loops are hard!
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  - γ_5-related ambiguity when using NDR
  - Flavor becomes non-trivial at 3-loop order

    Flavor is present when multiple matter fields have the same quantum numbers; it manifests as a continuous symmetry G_F

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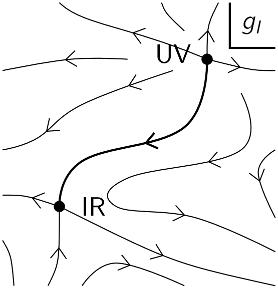
  - Loops are hard!
  - γ_5-related ambiguity when using NDR
  - Flavor becomes non-trivial at 3-loop order
  - Some calculations have resulted in seemingly nonsensical divergent RG functions, e.g.,

$$\gamma_u \supset \frac{1}{\epsilon} \frac{1}{(4\pi)^6} \frac{1}{16} y_u^\dagger \big[y_d y_d^\dagger ,\, y_u y_u^\dagger \big] y_u$$

CFTs: a showcase that flavor matters

Fixed Points



Traditionally CFTs were understood to be RG fixed points:

$$[T^{\mu}{}_{\mu}] = \beta_I [\mathcal{O}^I] = 0$$

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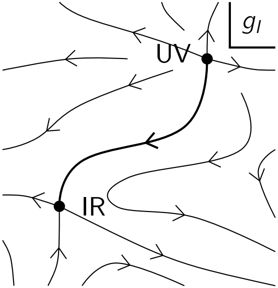

Traditionally CFTs were understood to be RG fixed points: ignores J_F^μ

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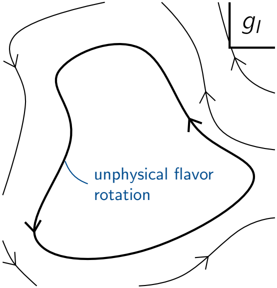
Limit Cycles



unphysical flavor rotation

Limit cycles can be (are?) CFTs

Fortin, Grinstein, Stergiou [1206.2921] [1208.3674]

$$[T^\mu{}_\mu] = B_I[\mathcal{O}^I] = 0$$

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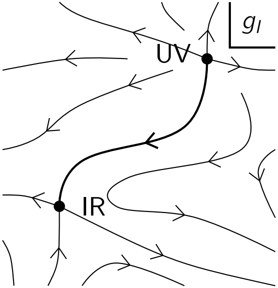

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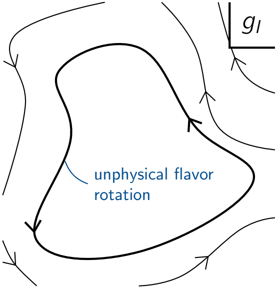

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$$[T^\mu{}_\mu] = B_I[\mathcal{O}^I] = 0$$

$B_I = \beta_I - (\upsilon\, g)_I$ is a more physical β-function

Four-dimensional QFT

Most general renormalizable theory in 4D (ignoring relevant couplings):

$$\begin{aligned}\mathcal{L} = &+\tfrac{1}{2}(D_\mu\phi)_a(D^\mu\phi)_a + i\psi_i^\dagger\bar{\sigma}^\mu(D_\mu\psi)^i + \mathcal{L}_{\mathrm{gh}} + \mathcal{L}_{\mathrm{gf}} \\ &-\tfrac{1}{4}a^{-1}_{AB}F^A_{\mu\nu}F^{B\mu\nu} - \tfrac{1}{2}\left(Y_{aij}\psi^i\psi^j + \mathrm{H.c.}\right)\phi_a - \tfrac{1}{24}\lambda_{abcd}\phi_a\phi_b\phi_c\phi_d\end{aligned}$$

The vacuum functional

$$e^{i\mathcal{W}[\mathcal{J}]} = \int[\mathcal{D}\Phi]\, e^{iS[\Phi,\,\mathcal{J}]}, \qquad S = S_{\mathrm{kin}}[\Phi] + \int \mathrm{d}^d x\left(\underbrace{g_I}_{\text{set of all marginal couplings}}\mathcal{O}^I(x) + \overbrace{\mathcal{J}_\alpha}^{\text{all field sources}}\Phi^\alpha\right)$$

generates all the connected *n*-point functions

G_F is the largest continuous symmetry group of S_{kin}

For instance, in the SM $G_F = \mathrm{U}(3)_q \times \mathrm{U}(3)_u \times \mathrm{U}(3)_d \times \mathrm{U}(3)_\ell \times \mathrm{U}(3)_e$

Weyl symmetry

The **local renormalization group** introduces new sources:

Shore '87; Osborn 89'; Jack, Osborn '90; Osborn '91; Fortin, Grinstein, Stergiou [1208.3674]; Jack, Osborn [1312.0428]; Baume *et al.* [1401.5983]

$$\underbrace{T_{\mu\nu}}_{\text{stress-energy tensor}} : \quad \eta_{\mu\nu} \to \gamma_{\mu\nu}(x) \qquad \mathcal{O}^I : \quad g_I \to g_I(x) \qquad \underbrace{J^\mu_F}_{\text{flavor current; } J^\mu_F \in \mathfrak{g}_F} : \quad D_\mu \to D_\mu - a_\mu(x)$$

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The generator of Weyl symmetry—local scale invariance:

$$\Delta^W_\sigma = \int \mathrm{d}^d x \bigg(2\overset{\text{infinitesimal parameter}}{\sigma} \gamma^{\mu\nu} \frac{\delta}{\delta\gamma^{\mu\nu}}$$

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flavor current; $J^\mu_F \in \mathfrak{g}_F$

The generator of Weyl symmetry—local scale invariance:

infinitesimal parameter

field anomalous dimension

$$\Delta^W_\sigma = \int \mathrm{d}^d x \Bigl(2\sigma\gamma^{\mu\nu} \frac{\delta}{\delta\gamma^{\mu\nu}} - \sigma\beta_I \frac{\delta}{\delta g_I} + \sigma \mathcal{J}_\beta \big[(d - \Delta_\alpha) \delta^\beta{}_\alpha - \gamma^\beta{}_\alpha \big] \frac{\delta}{\delta \mathcal{J}_\alpha}$$

β-function

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$$+ \big[\partial_\mu \sigma\, \upsilon - \sigma\, D_\mu g_I\, \rho^I \big] \cdot \frac{\delta}{\delta a_\mu} \bigg)$$

β-function

RG functions of the G_F current; $\upsilon, \rho^I \in \mathfrak{g}_F$

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RG functions of the G_F current; $\upsilon, \rho^I \in \mathfrak{g}_F$

Δ_σ^W generates the the trace anomaly equation (modulo anomaly)

$$[T^\mu{}_\mu] = \beta_I[\mathcal{O}^I] + \upsilon \cdot \partial_\mu [J_F^\mu] \qquad \text{(FSCC)}$$

Flat-space constant-coupling limit:
$\gamma_{\mu\nu}(x) = \eta_{\mu\nu},\ g_I(x) = g_I,\ a_\mu = 0$

RG symmetry

Accounting identity for mass dimension:

$$\Delta^{\mu}\mathcal{W} = 0, \qquad \Delta^{\mu} = \mu\frac{\partial}{\partial\mu} + \int \mathrm{d}^{d}x \left(2\gamma^{\mu\nu}\frac{\delta}{\delta\gamma^{\mu\nu}} + (d-\Delta_{\alpha})\mathcal{J}_{\alpha}\frac{\delta}{\delta\mathcal{J}_{\alpha}}\right)$$

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The RG is a symmetry of the theory generated by $\Delta^{\mathrm{RG}}=\Delta^{\mu}-\Delta^{W}_{\sigma=1}$, which gives rise to the **Callan–Symanzik equation**:

$$0=\Delta^{\mathrm{RG}}\mathcal{W}=\left(\frac{\partial}{\partial t}+\beta_{I}\partial^{I}+\int \mathrm{d}^{d}x\,\mathcal{J}_{\beta}\gamma^{\beta}{}_{\alpha}\frac{\delta}{\delta\mathcal{J}_{\alpha}}\right)\mathcal{W} \qquad \text{(FSCC)}$$

Exactly what we would get from $\frac{\mathrm{d}\mathcal{W}}{\mathrm{d}t}=0$:

$$\left(\frac{\partial}{\partial t}+\beta_{g}\frac{\partial}{\partial g}+n\gamma\right)G^{(n)}(\{p\})=0$$

Flavor symmetry

G_F is a symmetry of S with generator

$$\Delta^F_\omega = \int \mathrm{d}^d x \left(D_\mu \omega \cdot \frac{\delta}{\delta a_\mu} - (\omega\, g)_I \frac{\delta}{\delta g_I} - (\omega\, \mathcal{J})_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha} \right), \qquad \omega \in \mathfrak{g}_F$$

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There is a **class of equivalent Weyl symmetries**:

$$\Delta^{W'}_\sigma = \Delta^W_\sigma + \Delta^F_{\sigma\alpha}, \qquad \alpha(g) \in \mathfrak{g}_F,$$

Ambiguity in RG functions defined by the Weyl transformation:

$$\beta'_I = \beta_I + (\alpha\, g)_I, \qquad \upsilon' = \upsilon + \alpha, \qquad \rho'^I = \rho^I - \partial^I \alpha, \qquad \gamma'^\alpha{}_\beta = \gamma^\alpha{}_\beta - \alpha^\alpha{}_\beta$$

The RG flow has a flavor ambiguity

Ambiguity in the RG

Flavor-improved RG functions are invariant w.r.t. the flavor ambiguity:

$$B_I = \beta_I - (\upsilon\, g)_I, \qquad \Gamma^{\alpha}{}_{\beta} = \gamma^{\alpha}{}_{\beta} + \upsilon^{\alpha}{}_{\beta}, \qquad P^I = \rho^I + \partial^I \upsilon$$

We can choose a ‘gauge’ where $\upsilon = 0$:

$$\widehat{\Delta}^W_\sigma = \Delta^W_\sigma + \Delta^F_{-\sigma\,\upsilon} = \int \mathrm{d}^d x \bigg(2\sigma\gamma^{\mu\nu} \frac{\delta}{\delta\gamma_{\mu\nu}} - \sigma B_I \frac{\delta}{\delta g_I} \\ + \sigma \mathcal{J}_\beta \big[(d - \Delta_\alpha)\delta^{\beta}{}_{\alpha} - \Gamma^{\beta}{}_{\alpha} \big] \frac{\delta}{\delta \mathcal{J}_\alpha} - \sigma\, D_\mu g_I\, P^I \cdot \frac{\delta}{\delta a_\mu} \bigg)$$

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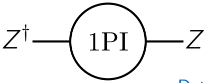
- β_I and B_I appear in two equally valid versions of the Weyl symmetry and the CS equation
- $B_I = 0$ indicates that the theory is a CFT
- $\beta_I = \dfrac{\mathrm{d}g_I}{\mathrm{d}t}$ whereas $B_I \neq \dfrac{\mathrm{d}g_I}{\mathrm{d}t}$; the coupling counterterms determines β_I rather than B_I

γ-pole at 3-loop order

Renormalization condition for 2-point functions: ($\overline{\text{MS}}$, $d = 4 - 2\epsilon$)



$$= \text{finite}, \qquad Z = \mathbb{1} + \sum_{n=1}^{\infty} \frac{z^{(n)}}{\epsilon^n}$$

Determines $Z^\dagger$ Z

The field anomalous dimension

$$\gamma = Z^{-1} \frac{\mathrm{d}}{\mathrm{d}t} Z = \sum_{n=0}^{\infty} \frac{\gamma^{(n)}}{\epsilon^n} \quad \Longrightarrow \quad \gamma^{(0)} = -\zeta z^{(1)}, \qquad \zeta = k_l g_l \partial^l$$

loop-counting operator

γ-pole at 3-loop order

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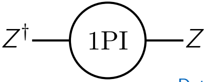

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In the SM $\gamma^{(1)} \neq 0$ at 3-loop order for $Z^\dagger = Z$:

Bednyakov, Pikelner, Velizhanin [1406.7171] Herren, Mihaila, Steinhauser [1712.06614]

$$(4\pi)^6 \gamma_q^{(1)} = \frac{g_1^2}{96} \big[y_u y_u^\dagger,\, y_d y_d^\dagger \big] + \frac{1}{32} \big[y_u y_u^\dagger y_u y_u^\dagger,\, y_d y_d^\dagger \big] + \frac{1}{32} \big[y_d y_d^\dagger y_d y_d^\dagger,\, y_u y_u^\dagger \big]$$

$$(4\pi)^6 \gamma_u^{(1)} = \frac{1}{16} y_u^\dagger \big[y_d y_d^\dagger,\, y_u y_u^\dagger \big] y_u$$

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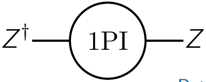

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Bednyakov, Pikelner, Velizhanin [1406.7171] Herren, Mihaila, Steinhauser [1712.06614]

$$\gamma^{(1)} - \gamma^{(1)\dagger} = \left[z^{(1)}, \zeta z^{(1)}\right] \qquad \text{for} \quad Z^\dagger = Z$$

$\gamma^{(1,2)}$ can be made to vanish by choosing a non-Hermitian Z

The RG with divergent RG functions

The evolution of *renormalized* amplitudes is governed by the CS Eq.:

$$0 = \Delta^{\mathrm{RG}} \mathcal{W} = \left(\frac{\partial}{\partial t} + \beta_I \partial^I + \int \mathrm{d}^d x \, \mathcal{J}_\beta \gamma^\beta{}_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha} \right) \mathcal{W} \qquad \text{(FSCC)}$$

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$$\Big(\underbrace{\frac{\partial}{\partial t} + \Big(\epsilon \beta_I^{(-1)} + \beta_I^{(0)}\Big)\partial^I + \int \mathrm{d}^d x\, \mathcal{J}_\beta \gamma^{(0)\beta}{}_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha}}_{\text{finite}}\Big)\mathcal{W} = -\sum_{n=1}^{\infty} \frac{1}{\epsilon^n}\Big(\beta_I^{(n)}\partial^I + \int \mathrm{d}^d x\, \mathcal{J}_\beta \gamma^{(n)\beta}{}_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha}\Big)\mathcal{W}$$

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The Ward identity for the flavor symmetry group G_F (FSCC):

$$0 = \Delta_\omega^F \mathcal{W} = \Bigg((\omega\, g)_I \partial^I - \int \mathrm{d}^d x\, \mathcal{J}_\beta \omega^\beta{}_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha} \Bigg)\mathcal{W}, \qquad \omega \in \mathfrak{g}_F$$

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The RG flow is finite due to

RG Finiteness *(theorem)*

$$\gamma^{(n)} \in \mathfrak{g}_F \quad \text{and} \quad \beta_I^{(n)} = -\big(\gamma^{(n)}\, g\big)_I\,, \qquad n \geq 1$$

Herren, **AET** [2104.07037]

RG finiteness in the SM

3-loop RG divergences in the SM:

using counterterms from Herren, Mihaila, Steinhauser [1712.06614]

$$(4\pi)^6 \gamma_q^{(1)} = \underbrace{\frac{g_1^2}{96}\left[y_u y_u^\dagger,\, y_d y_d^\dagger\right]} + \underbrace{\frac{1}{32}\left[y_u y_u^\dagger y_u y_u^\dagger,\, y_d y_d^\dagger\right]} + \underbrace{\frac{1}{32}\left[y_d y_d^\dagger y_d y_d^\dagger,\, y_u y_u^\dagger\right]}$$

$$(4\pi)^6 \gamma_u^{(1)} = \underbrace{\frac{1}{16} y_u^\dagger \left[y_d y_d^\dagger,\, y_u y_u^\dagger\right] y_u}$$

$$\begin{aligned}(4\pi)^6 \beta_{y_u}^{(1)} = &-\underbrace{\frac{g_1^2}{96}\left[y_u y_u^\dagger,\, y_d y_d^\dagger\right] y_u} - \underbrace{\frac{1}{32}\left[y_u y_u^\dagger y_u y_u^\dagger,\, y_d y_d^\dagger\right] y_u} \\ &-\underbrace{\frac{1}{32}\left[y_d y_d^\dagger y_d y_d^\dagger,\, y_u y_u^\dagger\right] y_u} + \underbrace{\frac{1}{16} y_u y_u^\dagger \left[y_d y_d^\dagger,\, y_u y_u^\dagger\right] y_u}\end{aligned}$$

$$(\omega\, y_u)^i{}_j = \omega_q{}^i{}_k y_u{}^k{}_j - y_u{}^i{}_k \omega_u{}^k{}_j + \omega_h y_u{}^i{}_j$$

$\beta_{y_u}^{(1)} = -(\gamma^{(1)}\, y_u)$, $\beta_{y_u}^{(2)} = -(\gamma^{(2)}\, y_u)$, *etc.* in the SM

SM RG functions are RG finite at 3-loop order

Renormalization ambiguity

$\mathcal{W}$ is invariant under flavor rotations $R \in G_F$: e.g., $y_u \longrightarrow R_q y_u R_u^\dagger$ in the SM

$$\mathcal{W}[\gamma,\, g,\, \mathcal{J},\, a] = \mathcal{W}[\gamma,\, Rg,\, R\mathcal{J},\, a^R] =$$
$$\mathcal{W}_0[\gamma,\, g_0,\, \mathcal{J}_0,\, a_0] = \mathcal{W}_0[\gamma,\, Rg_0,\, R\mathcal{J}_0,\, a_0^R], \quad (R\, g_0)_I = g_{0,I}(R\, g)$$

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Take instead a divergent rotation of $\mathcal{W}_0$:

$$U = \exp\Big[-\sum_{n=1}^{\infty} \frac{1}{\epsilon^n} u^{(n)}(g)\Big], \qquad u^{(n)} \in \mathfrak{g}_F$$

$$\mathcal{W}[\gamma,\, g,\, \mathcal{J},\, a] = \mathcal{W}_0[\gamma,\, g_0,\, \mathcal{J}_0,\, a_0] = \mathcal{W}_0[\gamma,\, Ug_0,\, U\mathcal{J}_0,\, a_0^U]$$

Leads to **change of counterterms w/o change in g_I**: ambiguity in taking $\sqrt{Z^\dagger Z}$

Jack, Osborn '90; Fortin *et al.* [1208.3674]

$$(U\mathcal{J}_0)_\alpha = \mathcal{J}_{0,\beta} U^{\dagger\beta}{}_\alpha = \mathcal{J}_\beta (Z^{-1}U^\dagger)^\beta{}_\alpha \implies Z^{U\alpha}{}_\beta = U^\alpha{}_\gamma Z^\gamma{}_\beta$$

Ambiguity in RG the functions

$\mathcal{W}_0[\gamma,\, g_0,\, \mathcal{J}_0,\, a_0] = \mathcal{W}_0[\gamma,\, U g_0,\, U\mathcal{J}_0,\, a_0^U]$ *but produce different RG functions!*

Herren, AET [2104.07037]

$$\Delta\gamma \equiv \gamma^U - \gamma = -\beta_I U \partial^I U^\dagger \in \mathfrak{g}_F$$

$$\Delta\beta_I \equiv \beta_I^U - \beta_I = -(\Delta\gamma\, g)_I$$

$$\Delta\upsilon \equiv \upsilon^U - \upsilon = -\Delta\gamma$$

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i) By choosing U, one can engineer any $\Delta\gamma = \alpha(g) \in \mathfrak{g}_F$
   - The ct. ambiguity reproduces the ambiguity in defining the Weyl symmetry:

$$\Delta_\sigma^W \to \Delta_\sigma^W + \Delta_{\sigma\alpha}^F$$

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ii) RG-finiteness is conserved $\qquad \beta_I^{(n)} = -\big(\gamma^{(n)}\, g\big)_I, \quad \gamma^{(n)} \in \mathfrak{g}_F, \quad \forall n \geq 1$

- Either *all or none* of the RG functions are RG finite

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iii) If $(\beta_I,\, \gamma)$ are RG finite, U can be chosen to make them finite

iv) One can choose counterterms such that the RG functions coincide with the flavor-improved $(B_I,\, \Gamma) = \big(\beta_I - (\upsilon\, g)_I,\, \gamma + \upsilon\big)$
  - Choosing $\Delta\gamma = \upsilon$, $(\beta_I^U,\, \gamma^U) = (B_I,\, \Gamma)$
  - $(B_I,\, \Gamma)$ are invariant under the renormalization ambiguity

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v) Finiteness of $(B_I,\, \Gamma)$ ensures RG-finiteness of $(\beta_I,\, \gamma)$
   - Their finiteness follows from the trace (scale) anomaly

Summary

i) The occurrence of certain ϵ poles in the RG functions is consistent with the Callan–Symanzik equation

ii) The flavor symmetry causes an ambiguity in the choice of renormalization constants

iii) Using the ambiguity, it is always possible to recover finite RG functions (β_I, γ)

iv) The flavor-improved RG functions (B_I, Γ) are unambiguous, finite, and more physical: they are the preferred RG functions

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The general 4-3-3 β-functions are implemented in

RGBeta v1.2.0

`https://github.com/aethomsen/RGBeta`

Backup

Why flavor becomes important at 3-loop order

- Elements of $\mathfrak{g}_F$ are anti-Hermitian
- $\upsilon(g) \in \mathfrak{g}_F$ is a flavor-covariant polynomial in the couplings
- It is not possible to construct 2-loop anti-Hermitian contractions of the marginal couplings

The **leading 1PI contributions to $\boldsymbol{\upsilon}$** are 3-loop order: