High-density QCD: a paradigm for PT symmetry in field theory

Michael C. Ogilvie,
Washington University

Applications of Field Theory to Hermitian and Non-Hermitian Systems
KIngs College, London, September 2024

Collaborators

Washington University in St. Louis


Leandro Medina


Zohar Nussinov


Moses Schindler


JUSTUS-LIEBIG-UNIVERSITÄT GIESSEN

Fabian Rennecke

UNIVERSITÄT BIELEFELD

Laurin Pannullo

BROOKHAVEN NATIONAL LABORATORY


Rob Pisarski

MIT → Los Alamos National Laboratory EST. 1943

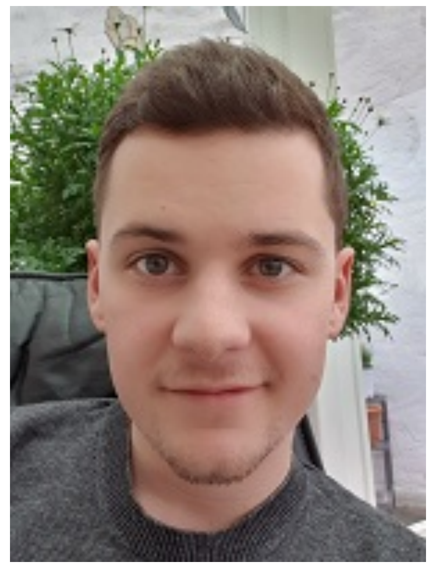

Stella Schindler & Frodo

IOWA STATE UNIVERSITY

Semeon Valgushev

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

Marc Winstel



Outline

- QCD at nonzero temperature and density
- The sign problem at nonzero density
- Exotic dispersion relations and inhomogeneous phases
- Experimental signatures

The Phase Diagram of QCD: Here be dragons

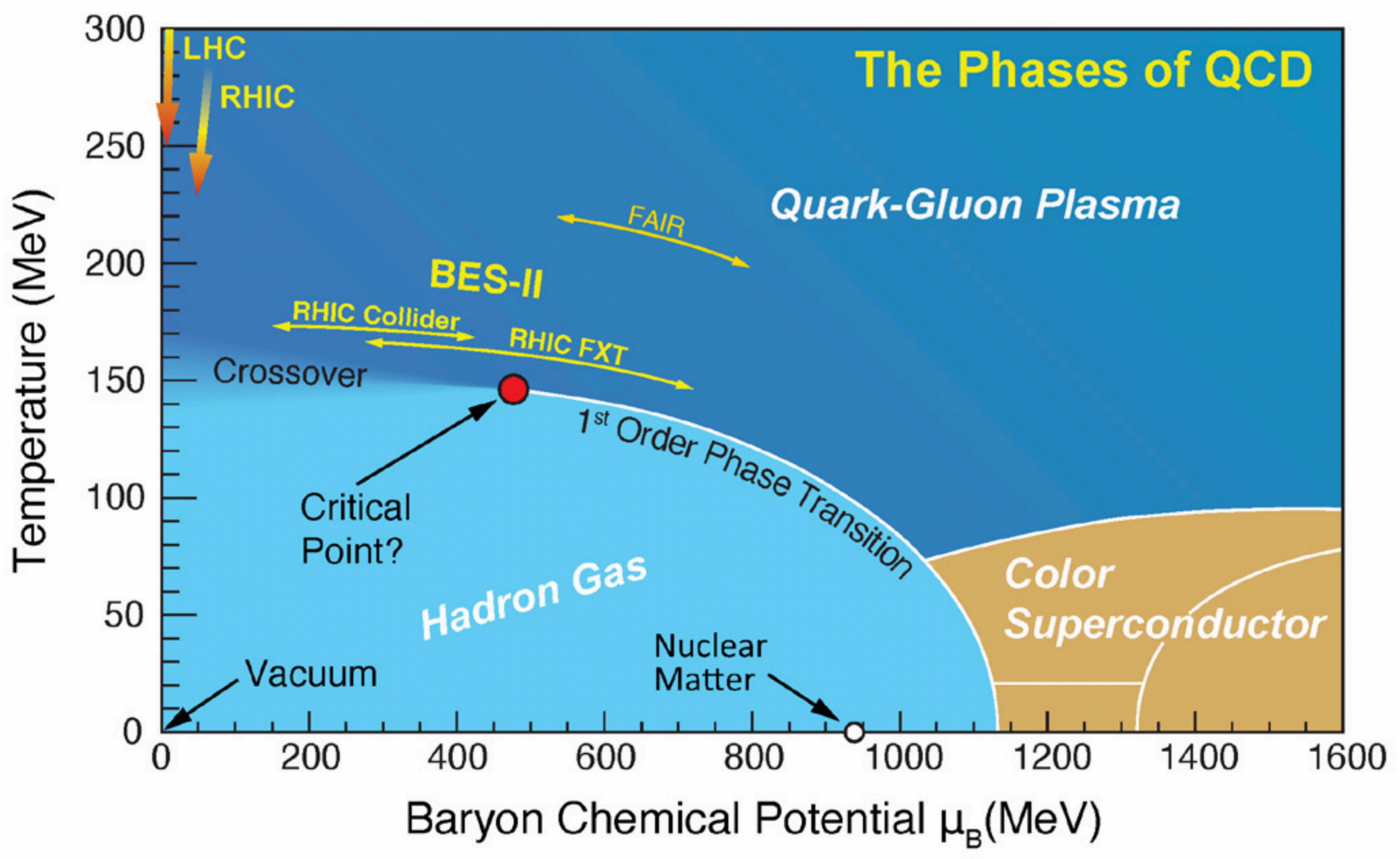


Figure: NSAC Long-Range Plan 2023.

The Phases of QCD: relevant areas of physics

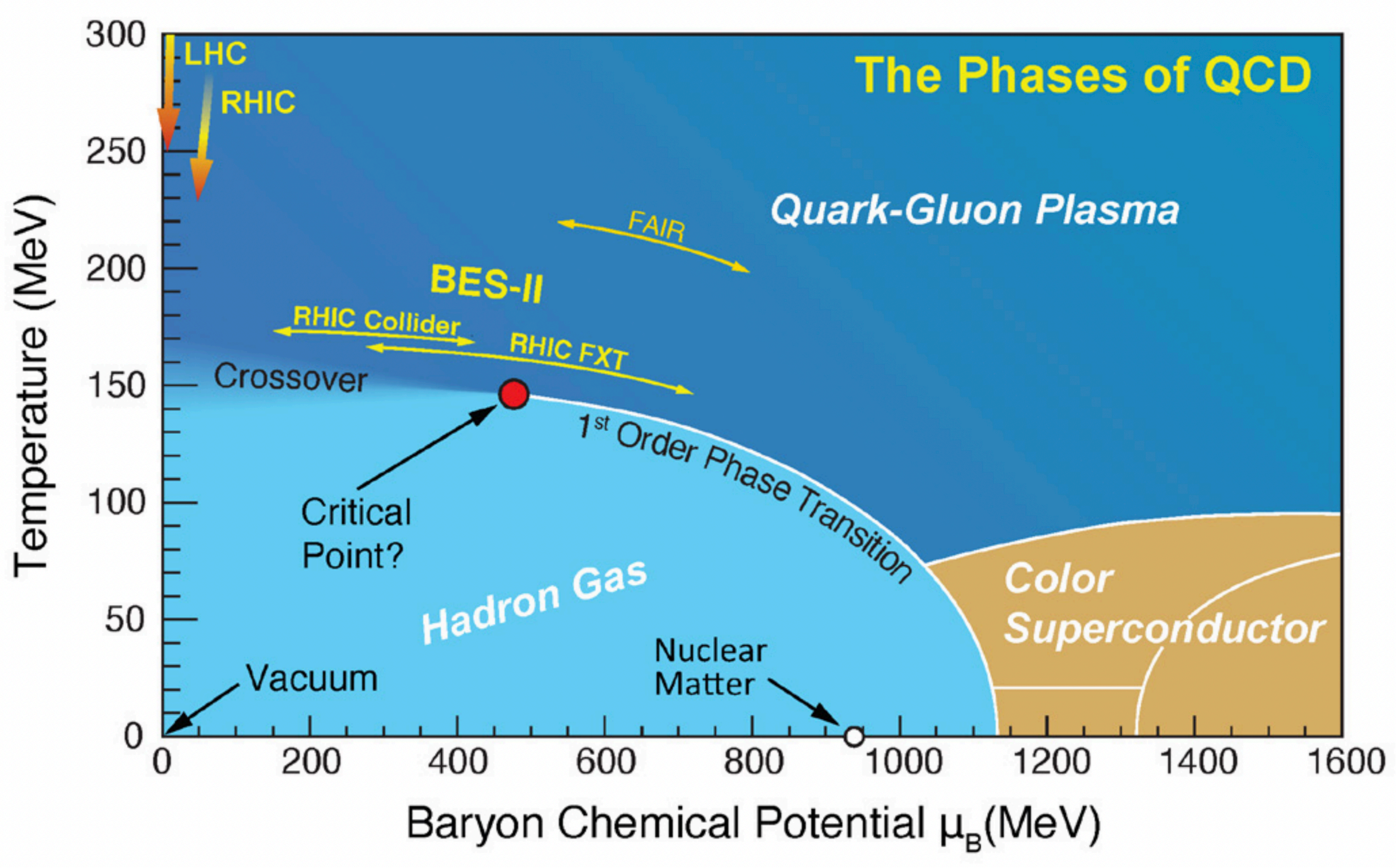


Figure: NSAC Long-Range Plan 2023.

- **Nuclear Physics**
- **Particle Physics**
- **Astrophysics**
- *Many Body Physics*
- *Condensed Matter*

The Phases of QCD: research synergy

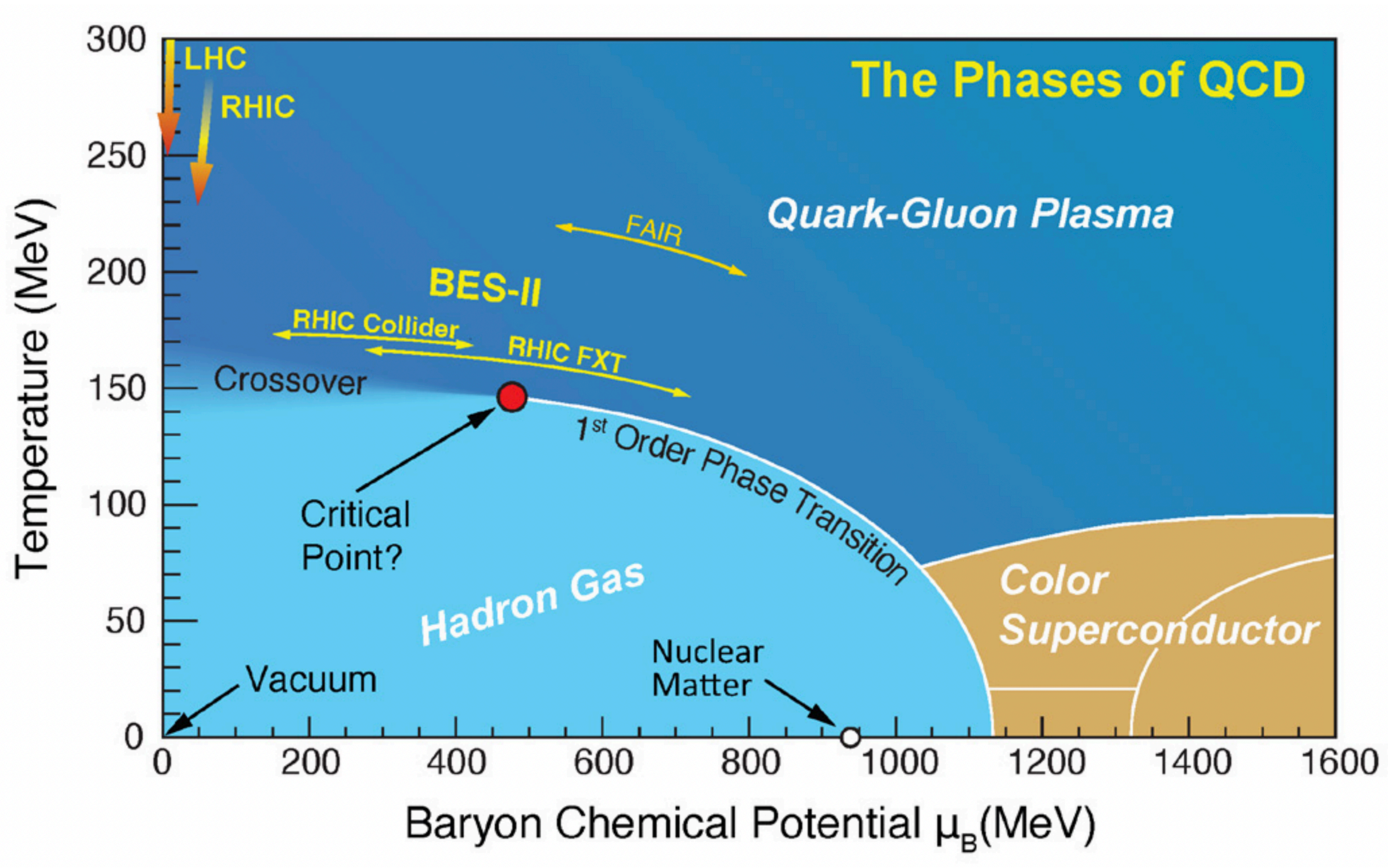
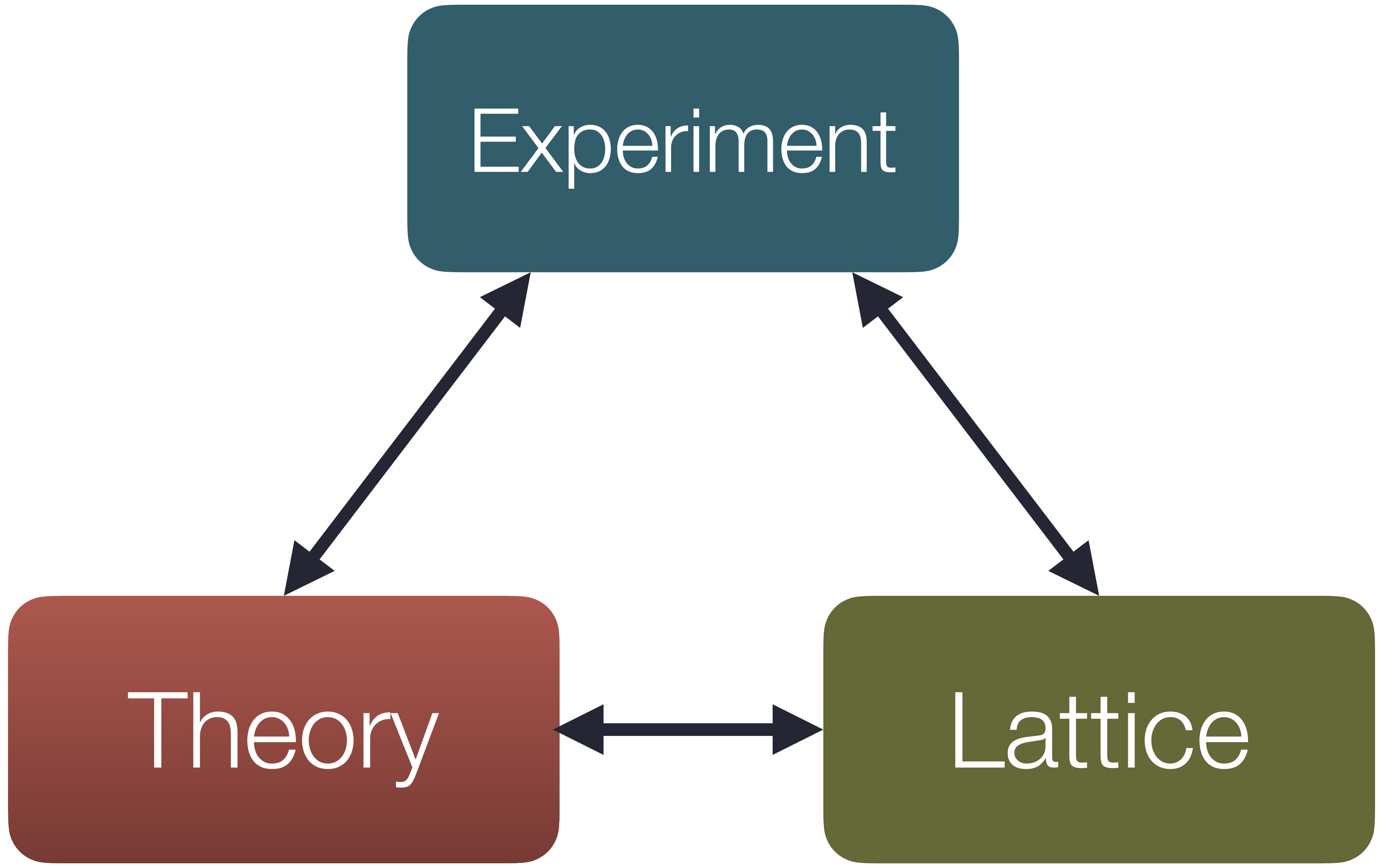
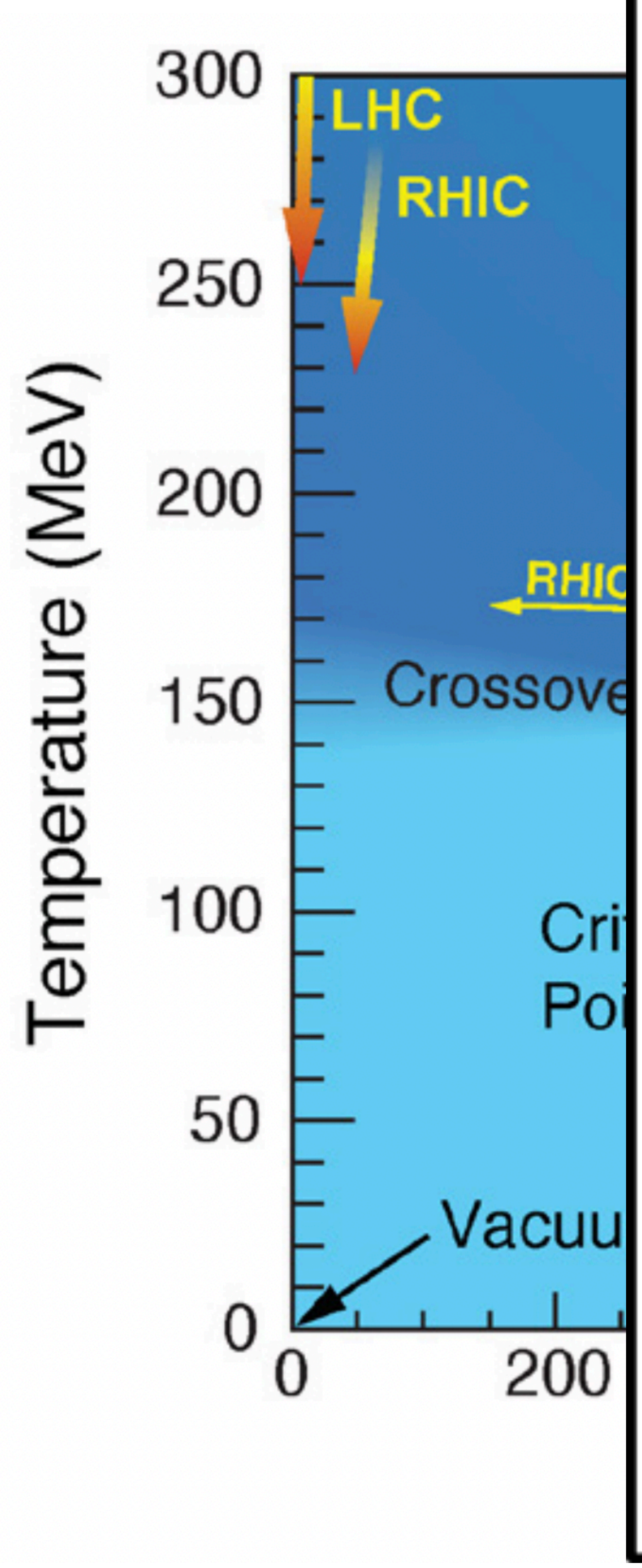
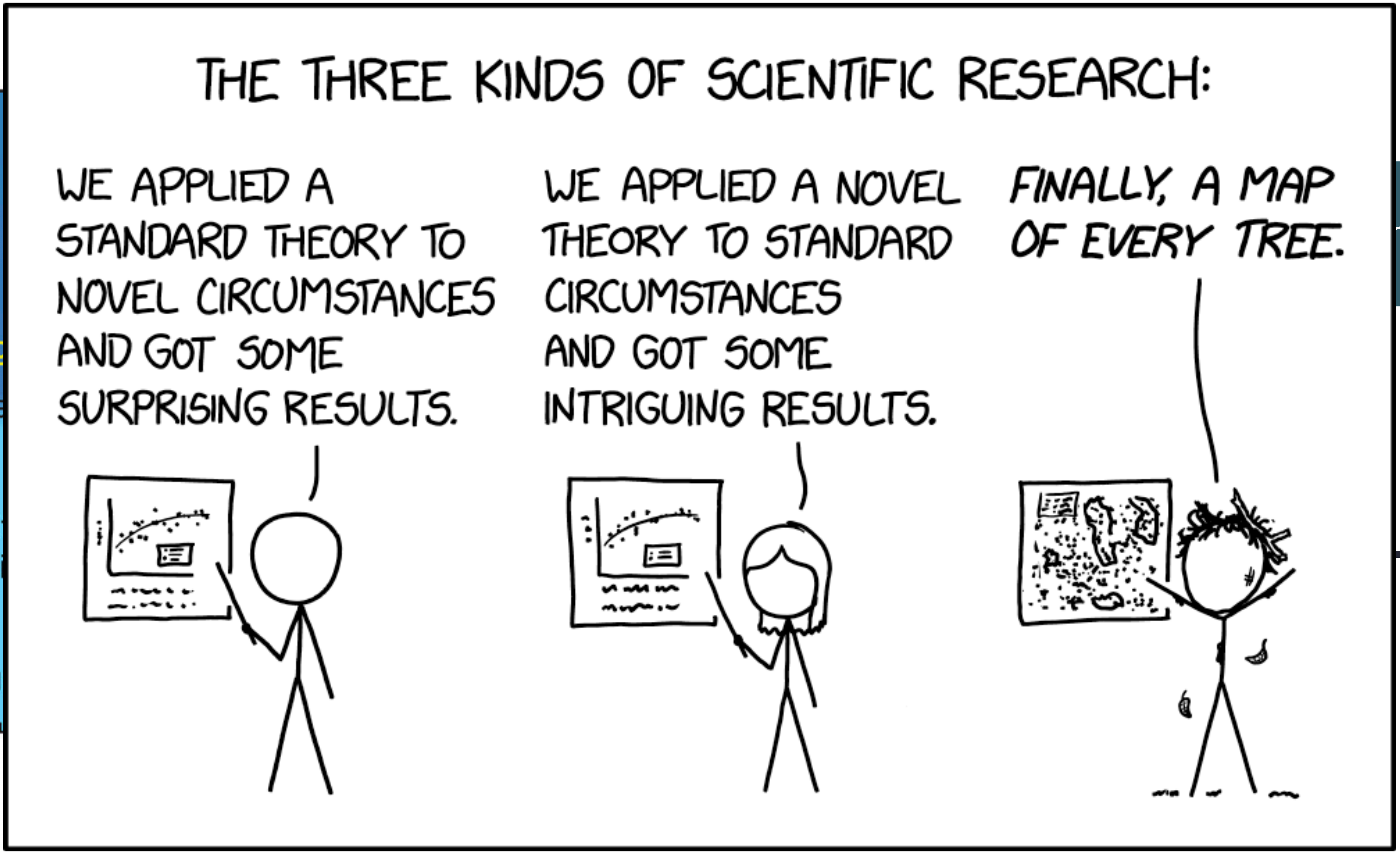


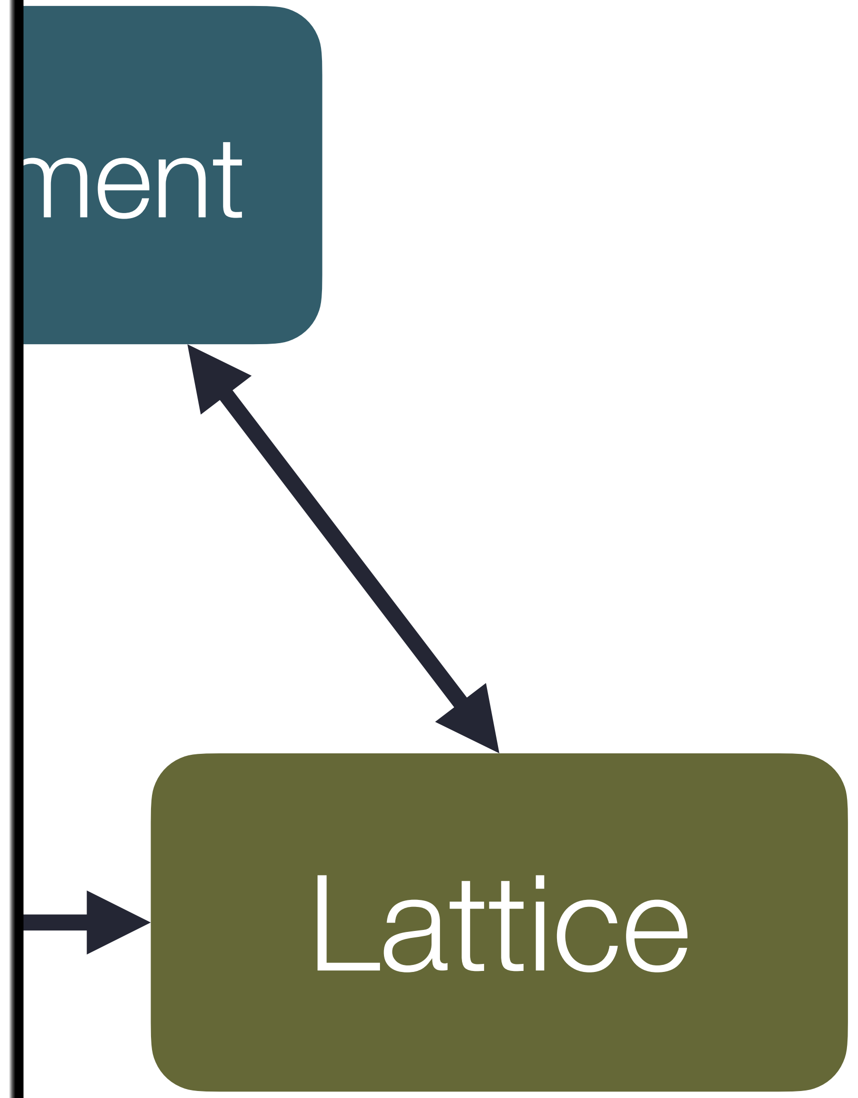
Figure: NSAC Long-Range Plan 2023.



The Phases of QCD: a synergistic research area







The Phases of QCD: difficult problems in fundamental physics

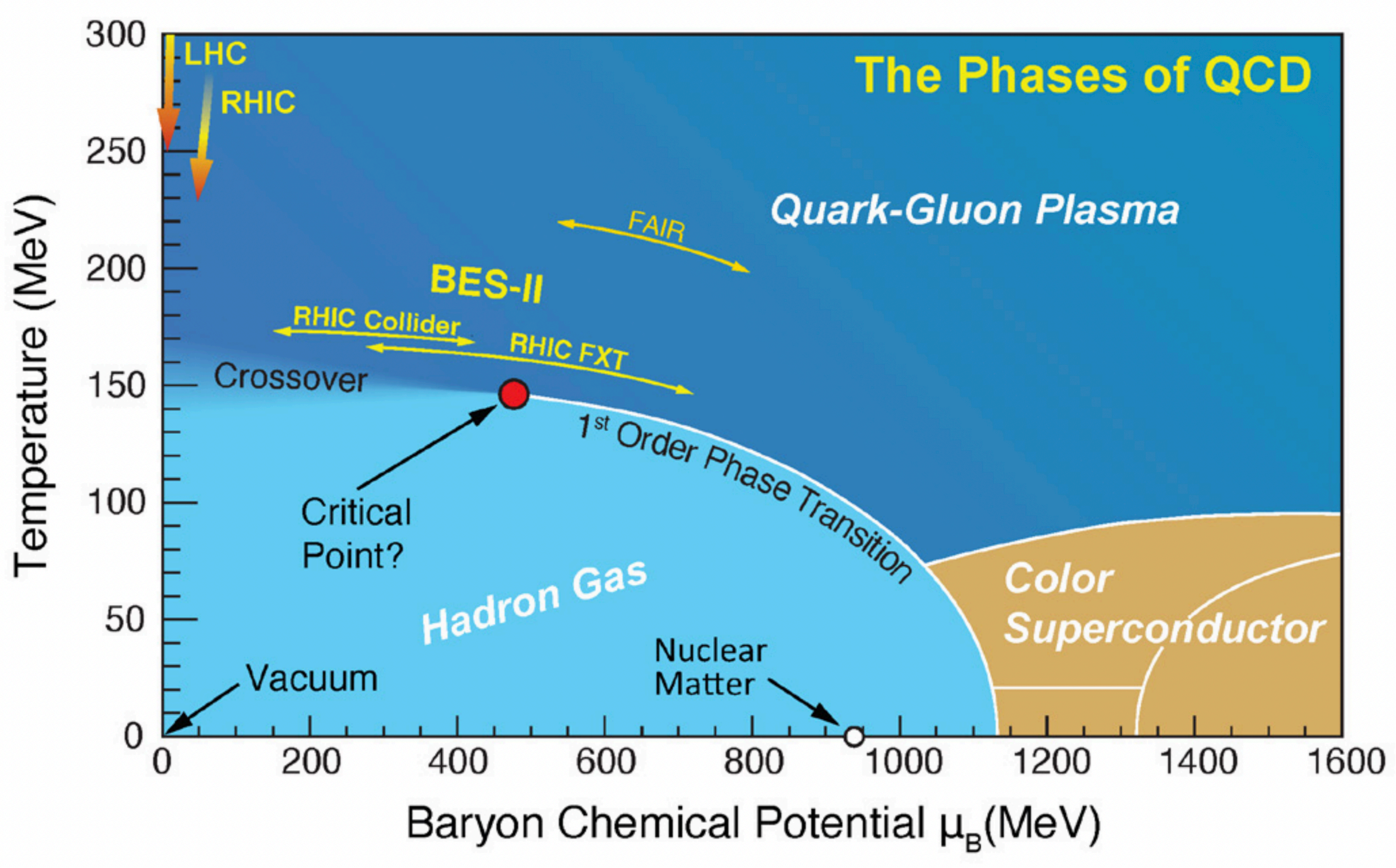


Figure: NSAC Long-Range Plan 2023.

- Quark confinement
- Chiral symmetry breaking
- Quark-hadron duality
- Non-hermiticity and sign problems
- Exotic dispersion relations and inhomogeneous phases

The sign problem in QCD at $\mu_B \neq 0$: introduction

Covariant derivative is non-Hermitian when $\mu_q \neq 0$ but sign problem occurs when gauge field $A_\nu \neq 0$ as well

$$D_\nu = \partial_\nu + iA_\nu + \mu_q \delta_{4\nu}$$

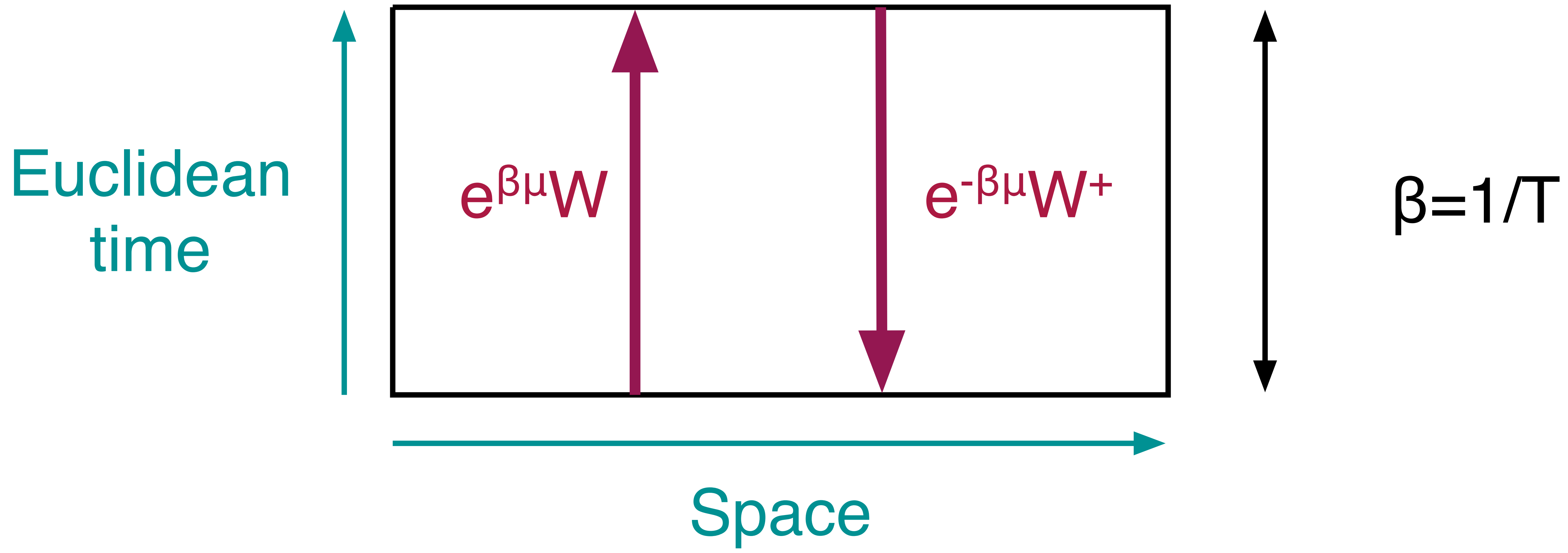
Functional determinants become complex.

$$\det\left(-\mu, A\right) = \det\left(\mu, A\right)^*$$

The Wilson line, or Polyakov loop, measures the free energy of a static quark, and is an order parameter for the deconfinement transition

$$W(\vec{x}) = tr\,\mathscr{P}\exp\left[i\int_0^\beta dx_\nu A_\nu(t,\vec{x})\right]$$

The essence of the sign problem: when $\mu_q \neq 0$ the imaginary part of nontrivially winding quark paths are not cancelled by antiquarks



Note: This sign problem may be NP

The sign problem in QCD at $\mu_B \neq 0$: a formal proof

$$D_\nu = \partial_\nu + iA_\nu + \mu_q \delta_{4\nu}$$

$$Tr \log\left[-D^2 + m^2\right] = -\int_0^\infty \frac{dT}{T} e^{-Tm^2} \int [dx_\nu] \exp\left[-\int_0^T d\tau\, \dot{x}_\nu^2\right] tr\, \mathscr{P} \exp\left[\left[\int dx_\nu \left(iA_\nu + \delta_{4\nu}\mu_q\right)\right]\right]$$

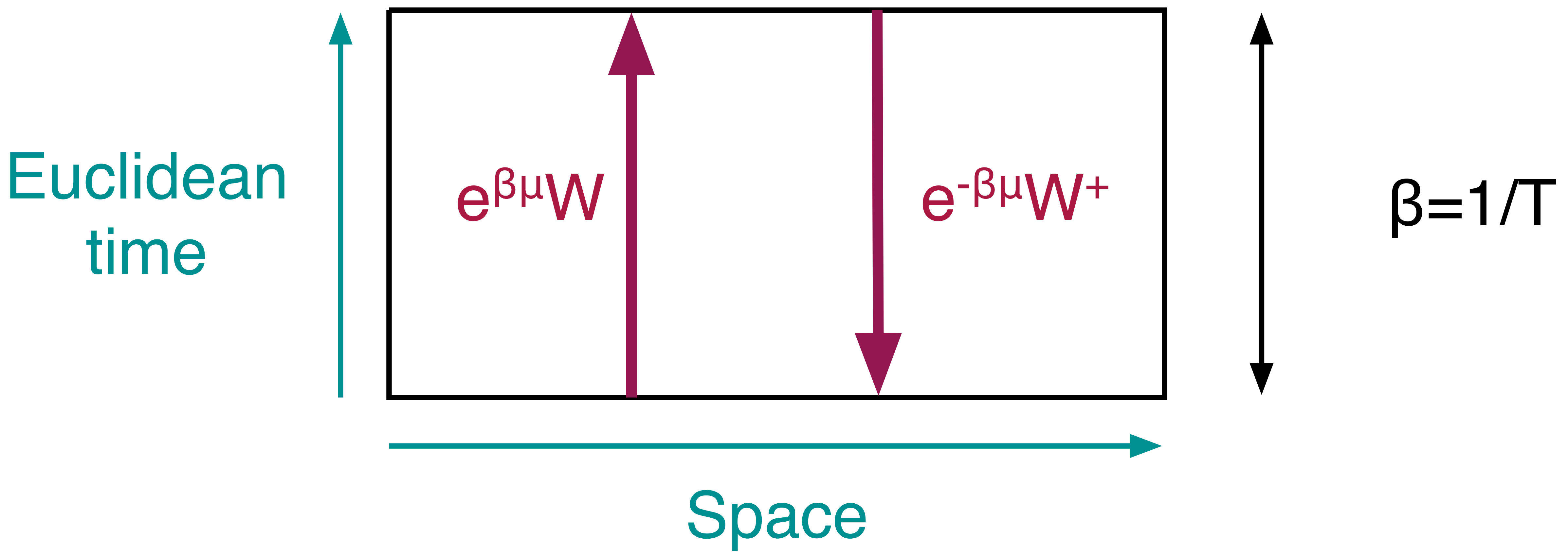
$$Tr \log\left[-D^2 + m^2\right] = -\int_0^\infty \frac{dT}{T} e^{-Tm^2} \int [dx_\nu] \exp\left[-\int_0^T d\tau\, \dot{x}_\nu^2\right] e^{n\beta\mu_q} W\left[x_\nu\right]$$

n = winding number of the path $x_\nu(\tau)$

$$e^{n\beta\mu_q}\, tr\, W[x_\nu] \rightarrow \frac{1}{2}\left[e^{n\beta\mu_q} W[x_\nu] + e^{-n\beta\mu_q} W^*[x_\nu]\right]$$

$$\det\left(-\mu_q, A\right) = \det\left(\mu_q, A\right)^*$$

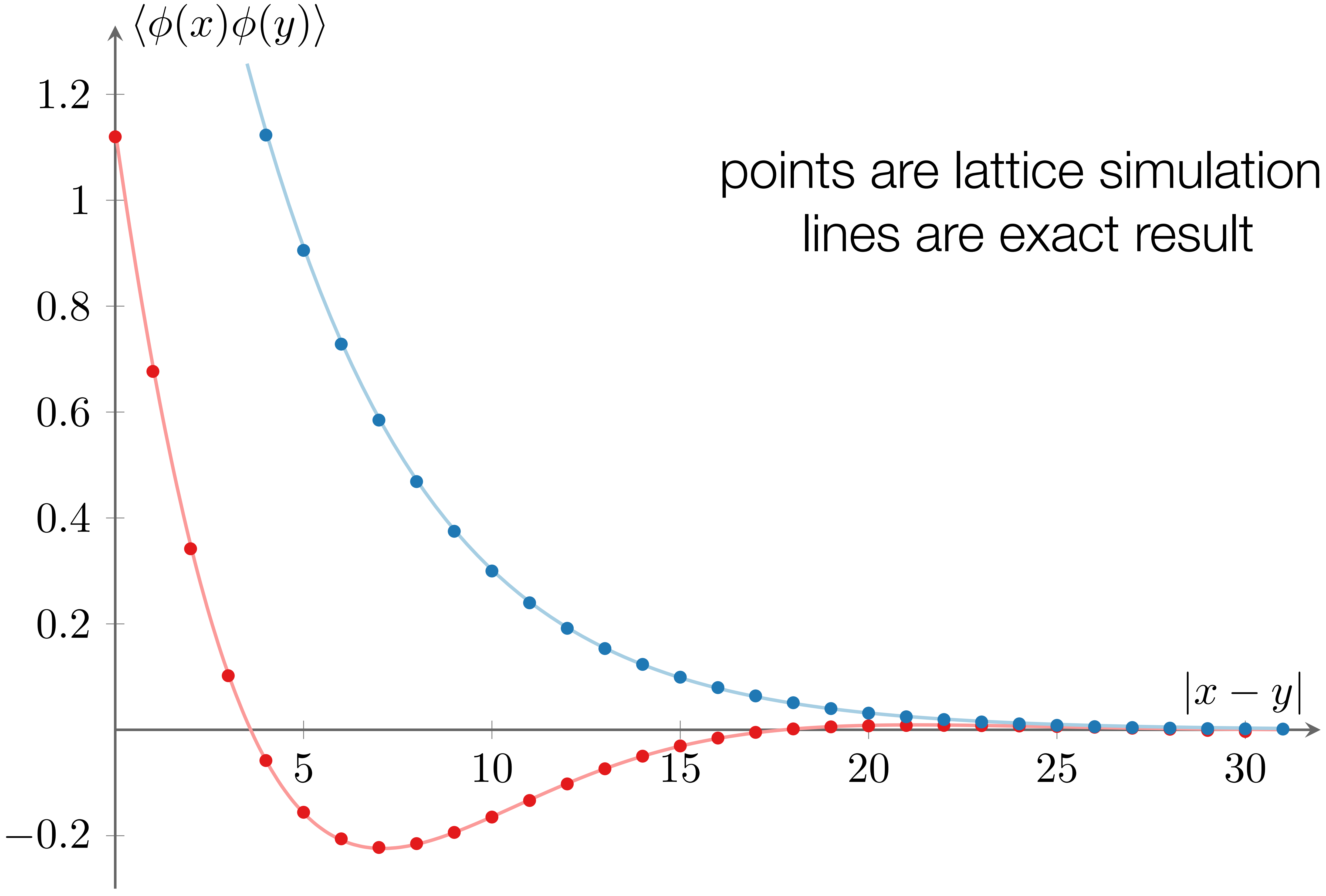
Sign problem for complex representations $SU(N)$ with $N \geq 3$



Back to basics: a simple non-Hermitian mass mixing model

$$L_E(\phi,\chi) = \frac{1}{2}\left(\nabla\phi\right)^2 + \frac{1}{2}\left(\nabla\chi\right)^2 + \frac{1}{2}m_\phi^2\phi^2 + \frac{1}{2}m_\chi^2\chi^2 - ig\phi\chi$$

mass matrix $\mathcal{M} = \begin{pmatrix} m_\phi^2 & -ig \\ -ig & m_\chi^2 \end{pmatrix}$ $\mathcal{M} = \sigma_3 \mathcal{M}^* \sigma_3$



points are lattice simulation; lines are exact result

A novel algorithm for scalar fields related to Kramers-Wannier duality for the Ising model can be used to simulate this model.

Disorder lines mark the boundary between exponential decay of propagators and sinusoidally-modulated exponential decay. The appearance of disorder lines and regions of sinusoidal modulation follows directly in PT-symmetric theories from the existence of conjugate eigenvalue pairs. Spectral positivity is violated in both cases.

d=1: transition between 2 real eigenvalues and a complex conjugate pair

Ogilvie and Medina, Lattice 2018, 1811.11112

A ϕ^4 mass mixing model with $Z(2)$ and PT symmetry

$$S(\phi,\chi) = \sum_x \left[\frac{1}{2}(\nabla_\mu \phi)^2 + U(\phi) + h\phi + \frac{1}{2}(\nabla_\mu \chi)^2 + \frac{1}{2}m_\chi^2\chi^2 - ig\phi\chi\right]$$

where $U(\phi) = \lambda(\phi^2 - v^2)^2$

$Z(2)$ symmetry for $h = 0$: $(\phi,\chi) \to (-\phi, -\chi)$

PT symmetry: $\chi \to -\chi$ and $i \to -i$

This is a model of a scalar field ϕ in a double well mixing with another particle χ.

Assuming constant vev's ϕ_0 and χ_0, we can solve the tree-level equation to find the phase diagram. Stability of any solution is determined from the mass matrix, which is PT symmetric. The eigenvalues are either both real or form a conjugate pair.

$$\mathcal{M} = \begin{pmatrix} U''(\phi_0) & -ig \\ -ig & m_\chi^2 \end{pmatrix}$$

$$\mathcal{M} = \sigma_3 \mathcal{M}^* \sigma_3$$

Schindler, Schindler and Ogilvie 1906.07288; PRD 102 (2020)

Equivalent forms of the ϕ^4 mass-mixing model

- **Original complex form with manifest PT symmetry**

$$S(\phi,\chi) = \sum_x \left[\frac{1}{2}(\nabla_\mu\phi)^2 + U(\phi) + h\phi + \frac{1}{2}(\nabla_\mu\chi)^2 + \frac{1}{2}m_\chi^2\chi^2 - ig\phi\chi\right]$$

- **Nonlocal real action ("attractive vs. repulsive" forces): Yukawa-frustrated ϕ^4**

$$S_{\text{eff}} = \sum_x \left[\frac{1}{2}(\partial_\mu\phi(x))^2 + U(\phi) + h\phi\right] + \frac{g^2}{2}\sum_{x,y}\phi(x)\Delta(x-y)\phi(y)$$

Integration over ϕ gives a term which acts to restore symmetry

- **Derivative expansion of S_{eff}**

$$S_{\text{eff}} \approx \sum_x \left[\frac{1}{2}(\partial_\mu\phi(x))^2 + U(\phi) + h\phi\right] + \frac{g^2}{2m_\chi^2}\sum_x \left[\phi(x)^2 - \frac{1}{m_\chi^2}(\partial_\mu\phi(x))^2 + \ldots\right]$$

Derivative expansion shows **Lifshitz instability** for large g

- **Local real action**

$$\tilde{S} = \sum_x \left[\frac{1}{2}\left(\nabla_\mu\phi(x)\right)^2 + U(\phi) + h\phi + \frac{1}{2}\pi_\mu^2(x) + \frac{\left(\nabla\cdot\pi - g\phi\right)^2}{2m_\chi^2}\right]$$

This local real form can be simulated using standard methods

Stability of homogeneous phases

Nonlocal real action ("attractive vs. repulsive" forces): Yukawa-frustrated ϕ^4 with χ integrated out

$$S_{\text{eff}} = \sum_x \left[\frac{1}{2}(\partial_\mu \phi(x))^2 + U(\phi) + h\phi\right] + \frac{g^2}{2}\sum_{x,y} \phi(x)\Delta(x-y)\phi(y)$$

where $U(\phi) = \lambda(\phi^2 - v^2)^2$

$$G^{-1}_{\phi\phi}(q) = q^2 + U''(\phi_0) + \frac{g^2}{q^2+m^2} > 0$$

required for all q for stability of homogeneous solution ϕ_0

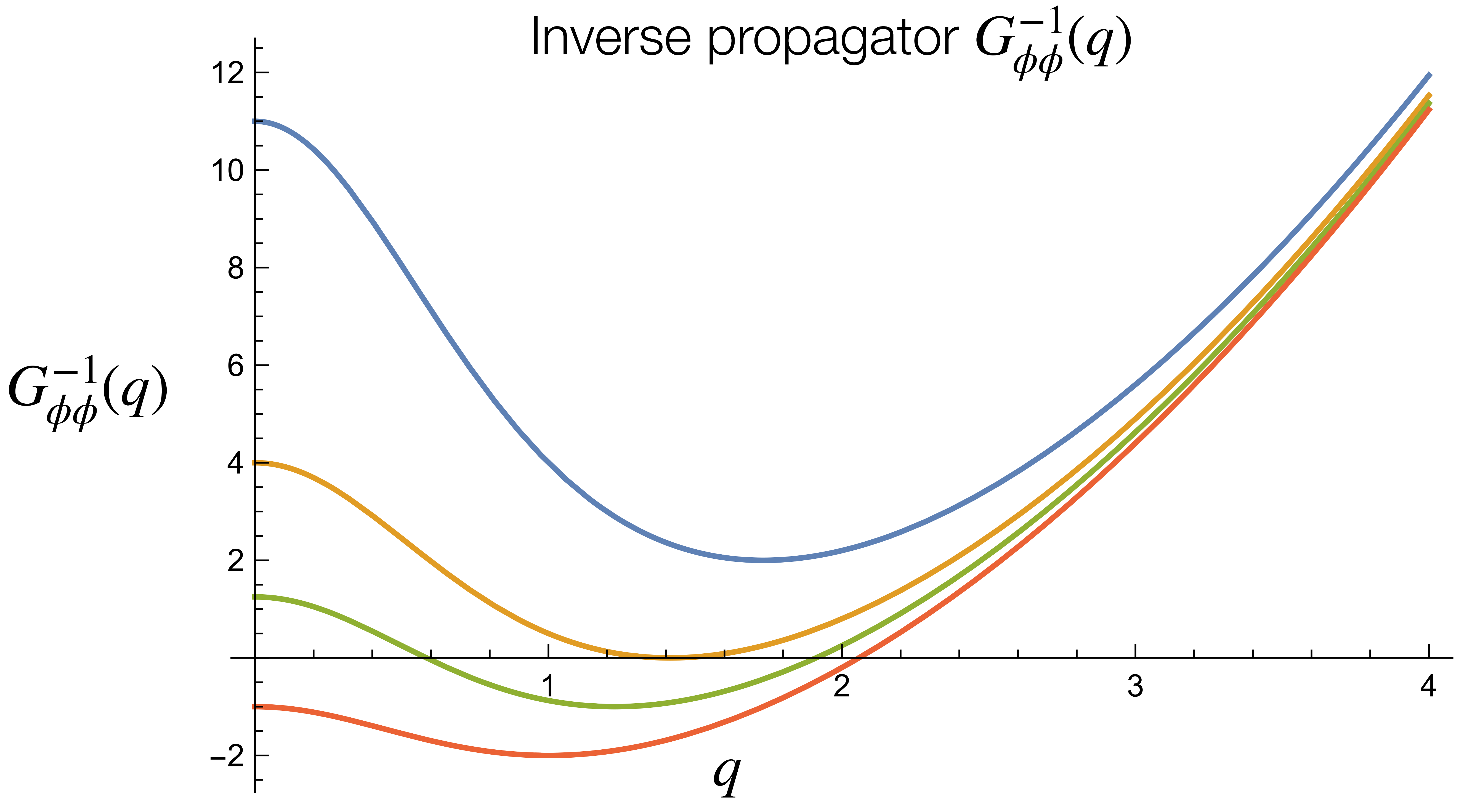
Parameters $U''(\phi_0) = -5 \quad m^2 = 1$

$g = 4$ (blue) stable

$g = 3$ (orange) critical

$g = 2.5$ (green) unstable

$g = 2$ (red) unstable



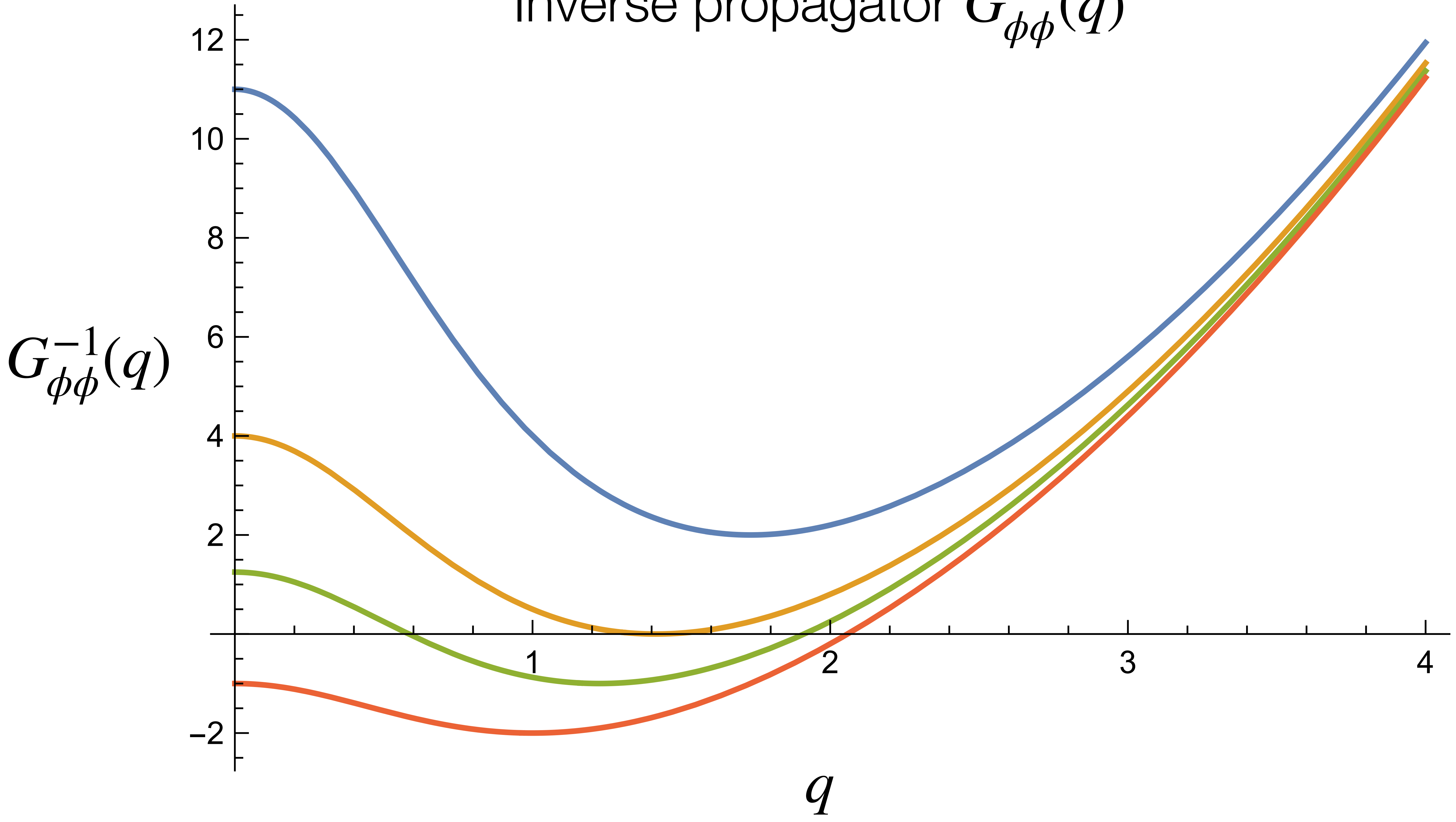
Moatons

Moat

Bodiam Castle in East Sussex

Moatons

Inverse propagator $G^{-1}_{\phi\phi}(q)$



Stability of homogeneous phases: another approach

The vev's ϕ_0 and χ_0 are determined at tree level from the static field equations. The stability of a solution against fluctuations is determined by the eigenvalues of $q^2 + \mathcal{M}$.

$$\mathcal{M} = \begin{pmatrix} U''(\phi_0) & -ig \\ -ig & m_\chi^2 \end{pmatrix}$$

In particular, the stability at $q = 0$ is determined by the sign of $\det(\mathcal{M})$

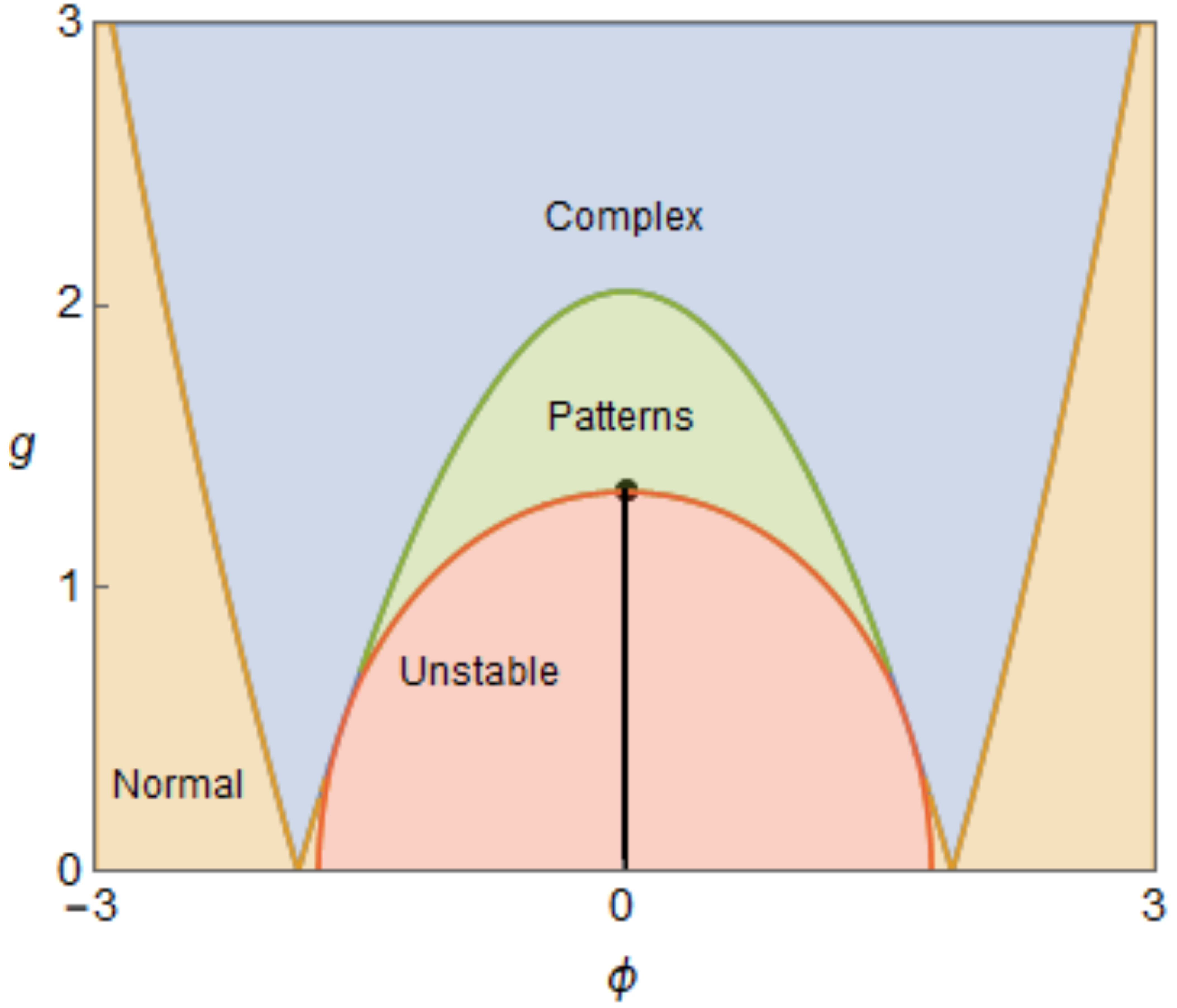
$$\mathcal{M} = \sigma_3 \mathcal{M}^* \sigma_3$$

PT symmetry of the mass matrix $\mathcal{M}$

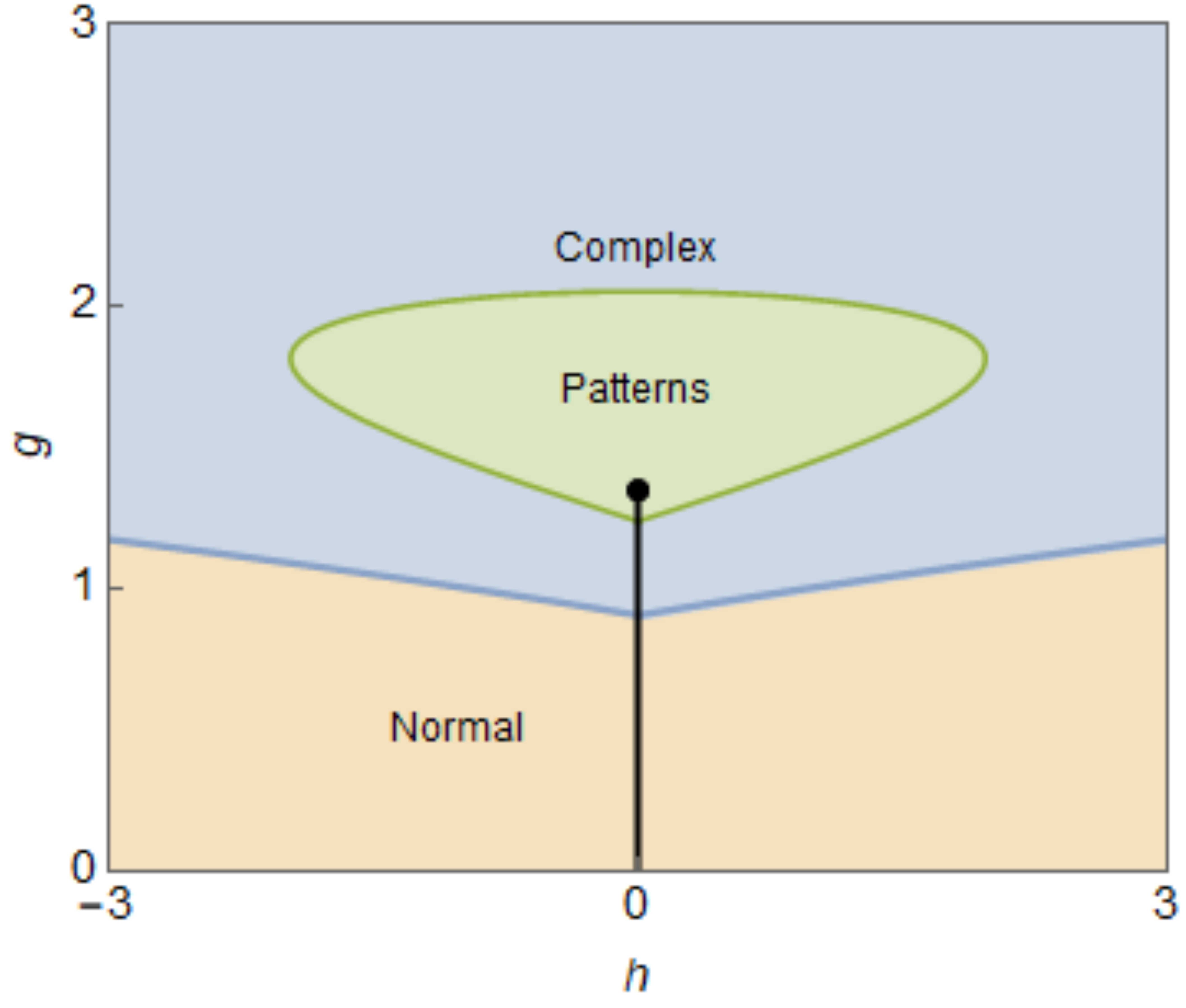
Eigenvalues of $\mathcal{M}$	Propagators	Phase
$\lambda_1 > 0$ and $\lambda_2 > 0$	Exponential Decay	Normal
$\lambda_1 < 0$ and $\lambda_2 > 0$	Exponential Growth of ϕ_0 mode	Unstable
$\lambda_1 = \lambda_2^*$	Sinusoidally Modulated Exponential Decay	PT Broken
$\lambda_1 < 0$ and $\lambda_2 < 0$	Exponential Growth of some $p \neq 0$ modes	Inhomogeneous Lifshitz Phase

Phase diagram of the ϕ^4 mass-mixing model

(ϕ, g) plane



(h, g) plane

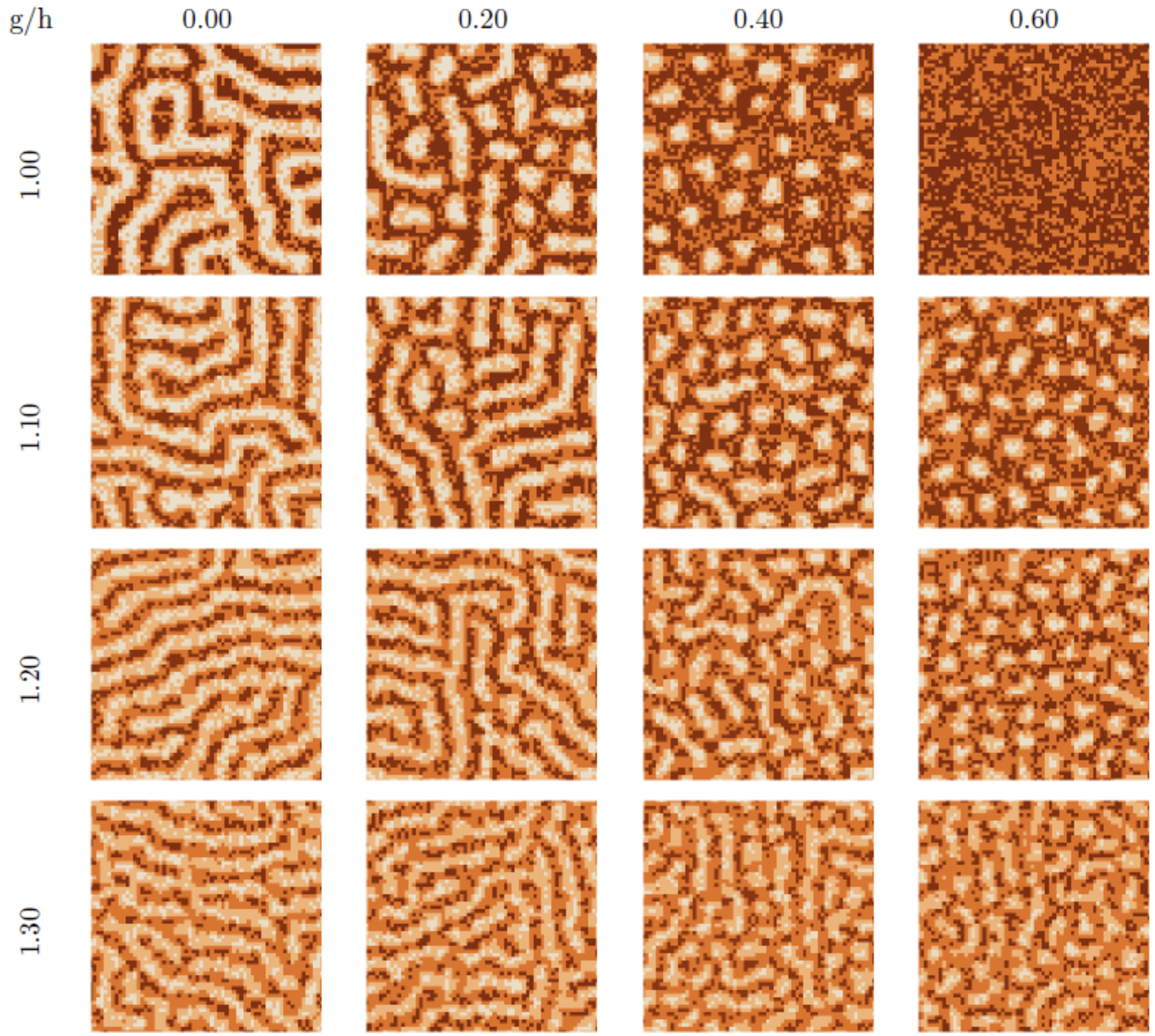


The normal to complex boundary is a disorder line, not a phase transition

Schindler, Schindler, Ogilvie, J. Phys. CS 2038 (2021, 2106.0709

Inhomogeneous behavior in simulations of the ϕ^4 mass-mixing model

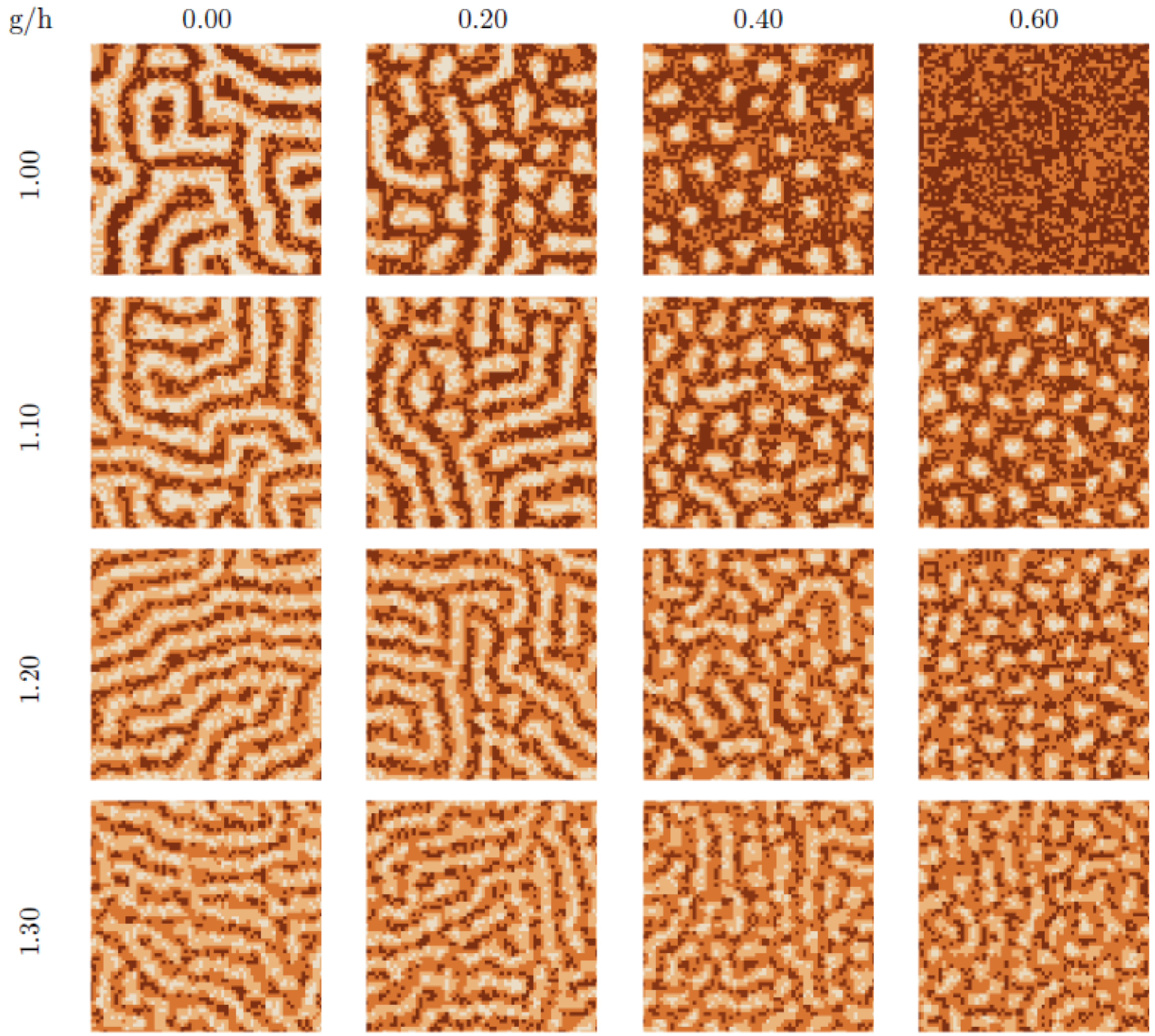
$\phi(x)$ in coordinate space



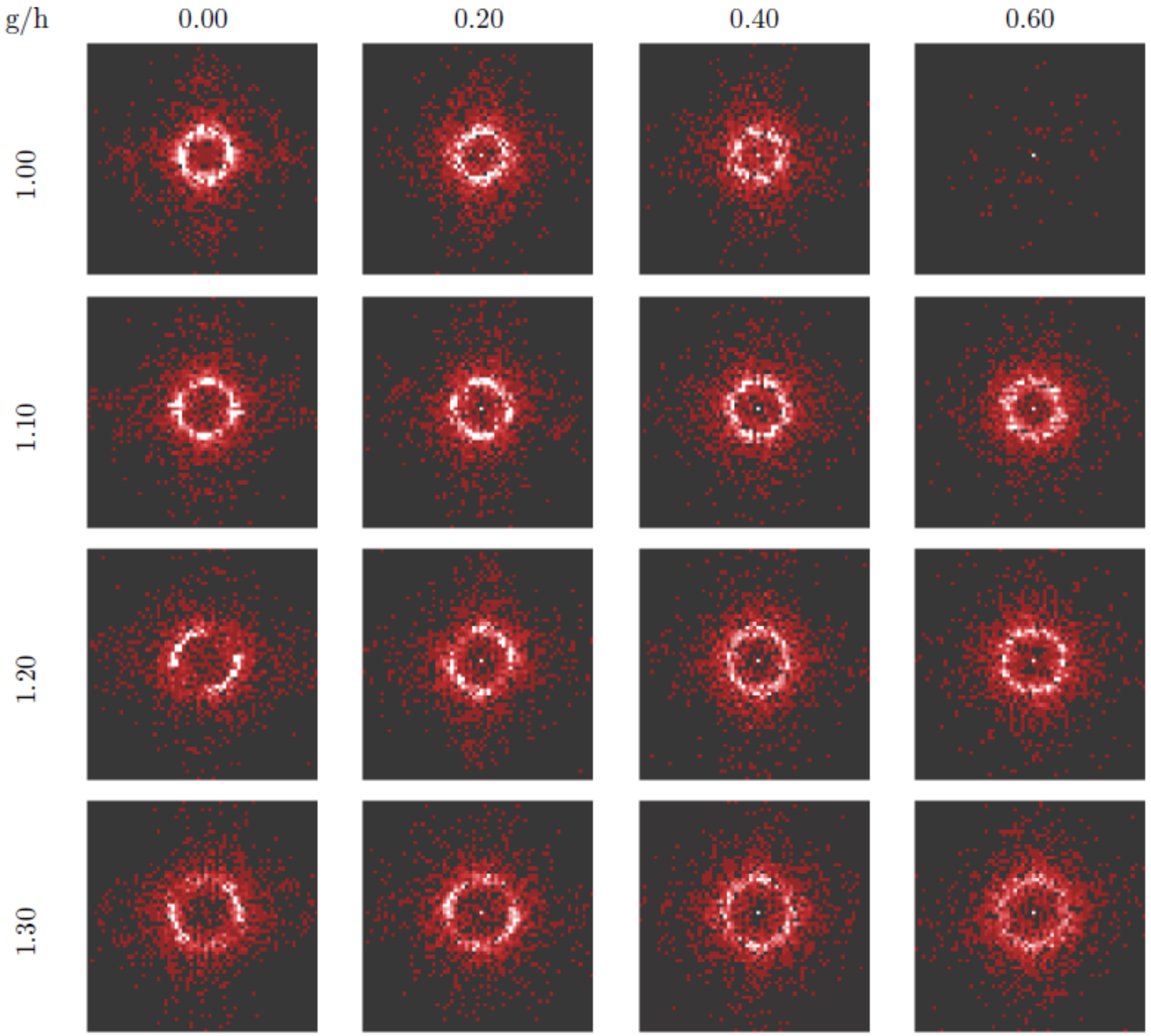
Schindler, Schindler, Ogilvie, J. Phys. CS 2038 (2021), 2106.0709; see e.g. Muratov, PRE 66 (2002) for the Coulomb case ($m_\chi = 0$)

Inhomogeneous behavior in simulations of the ϕ^4 mass-mixing model

$\phi(x)$ in coordinate space



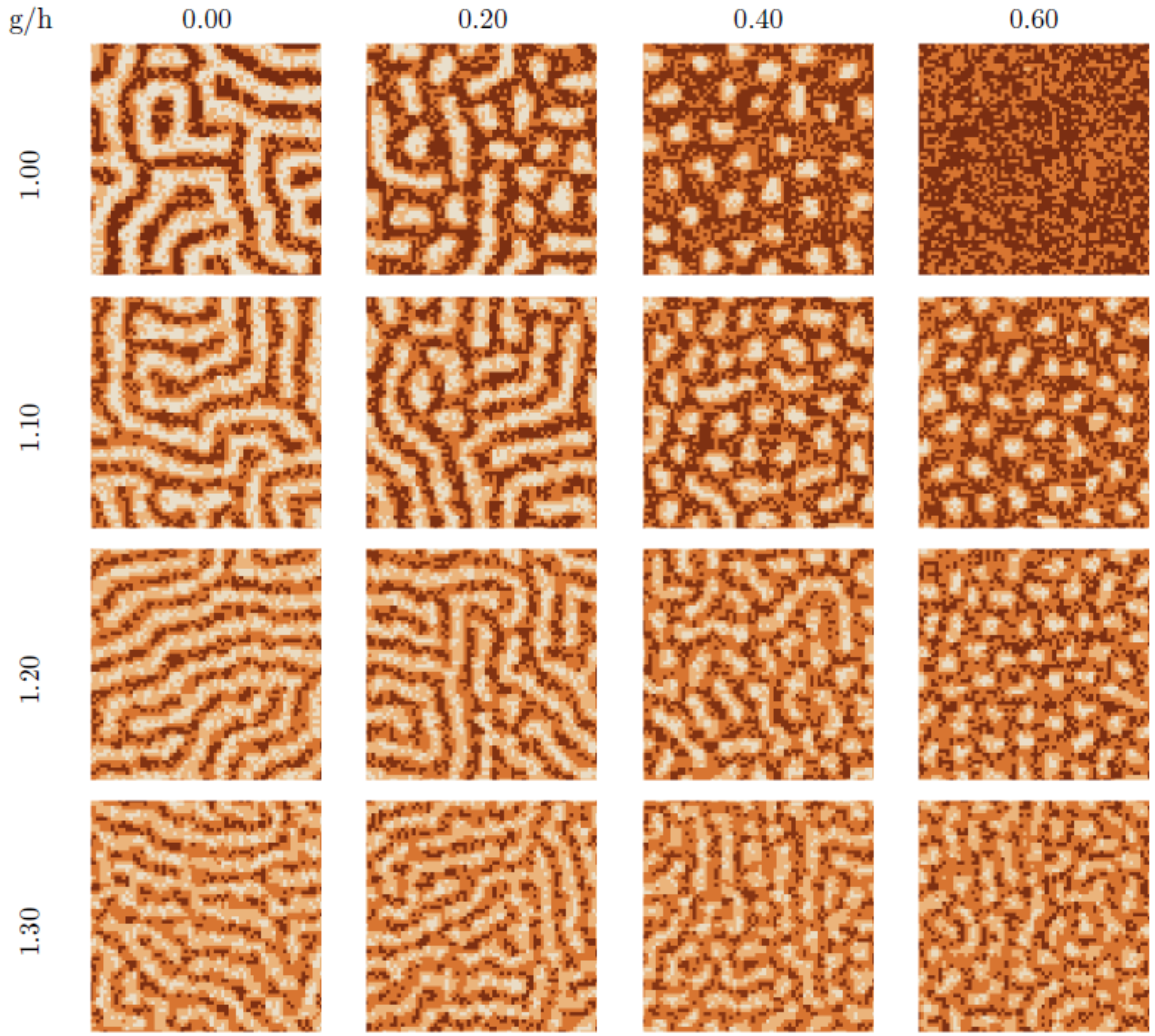
$\tilde{\phi}(p)$ in momentum space



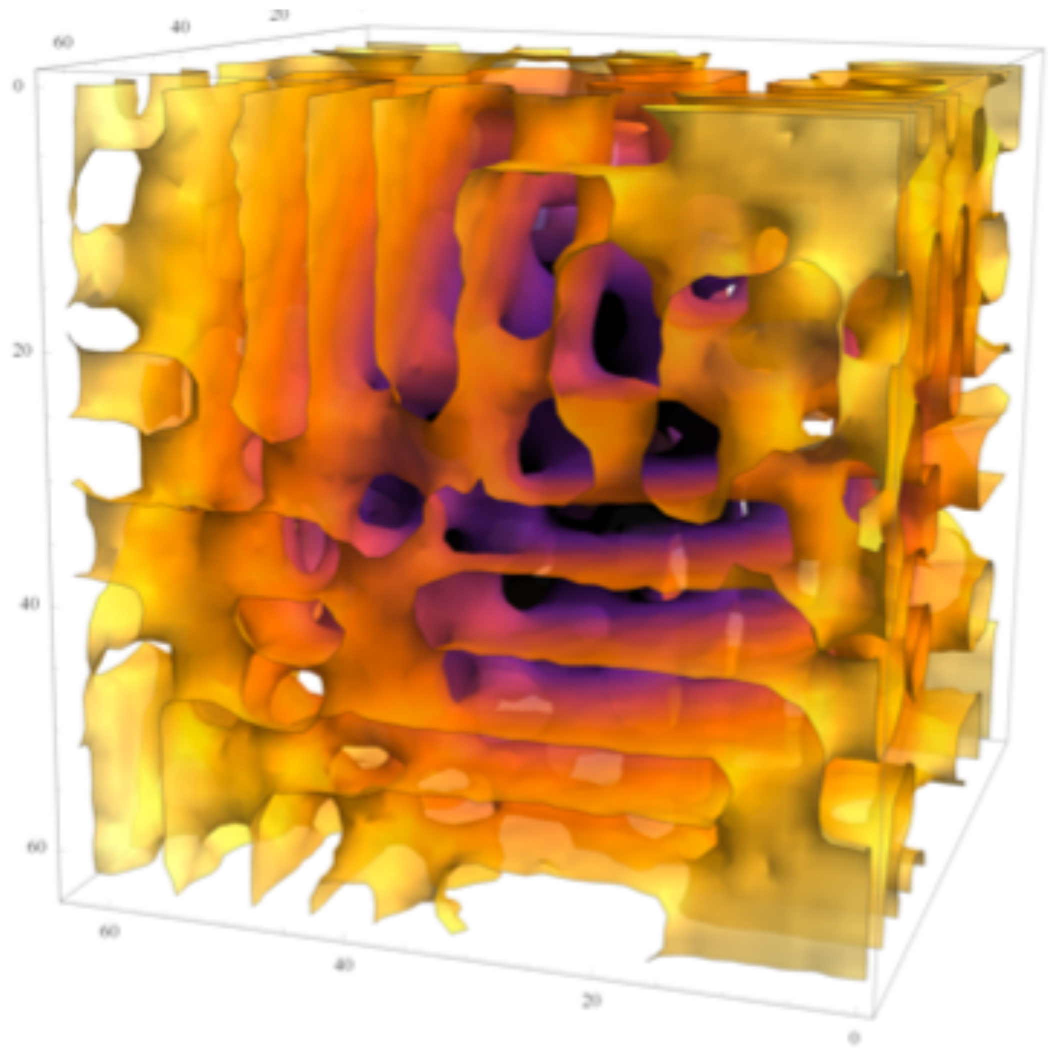
Schindler, Schindler, Ogilvie, J. Phys. CS 2038 (2021), 2106.0709; see e.g. Muratov, PRE 66 (2002) for the Coulomb case ($m_\chi = 0$)

Inhomogeneous behavior in simulations of the ϕ^4 mass-mixing model

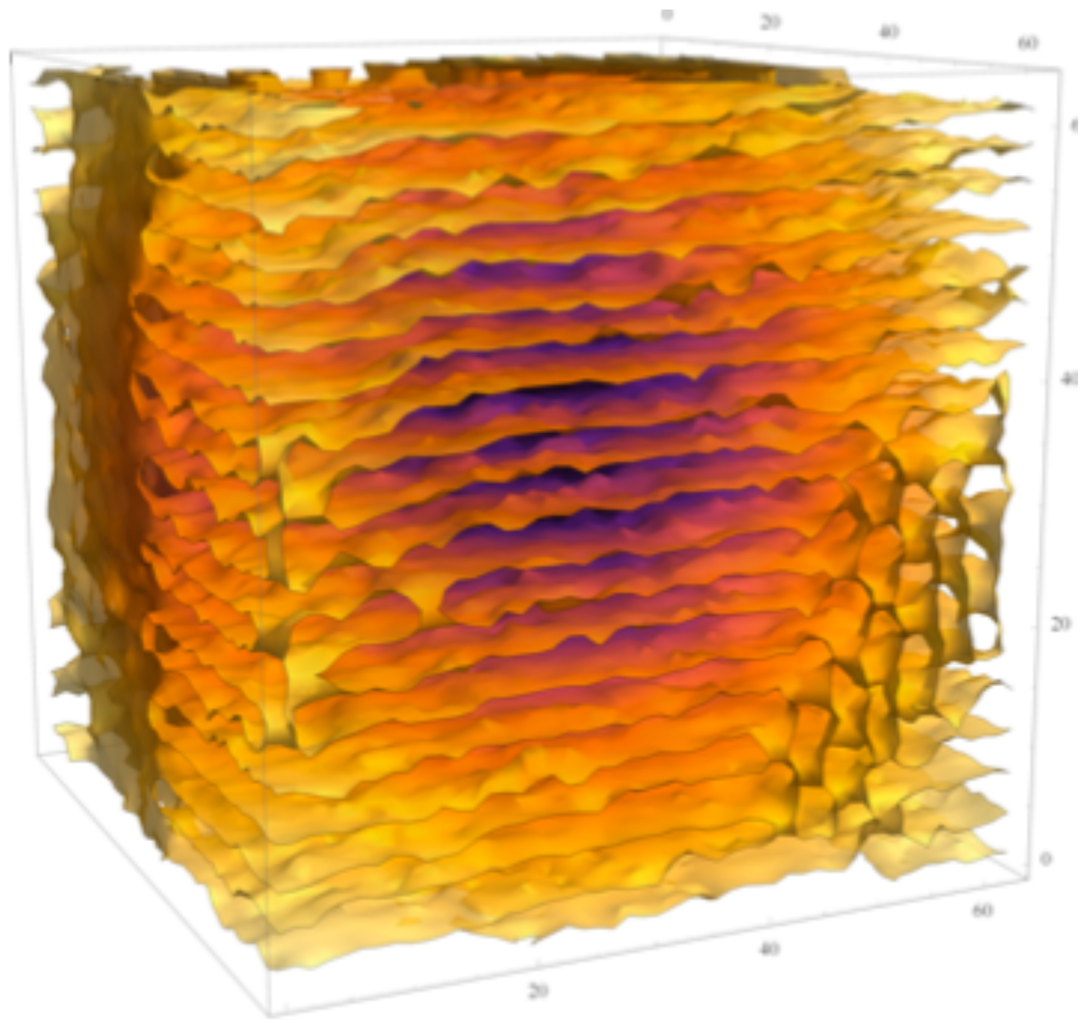
d=2



d=3



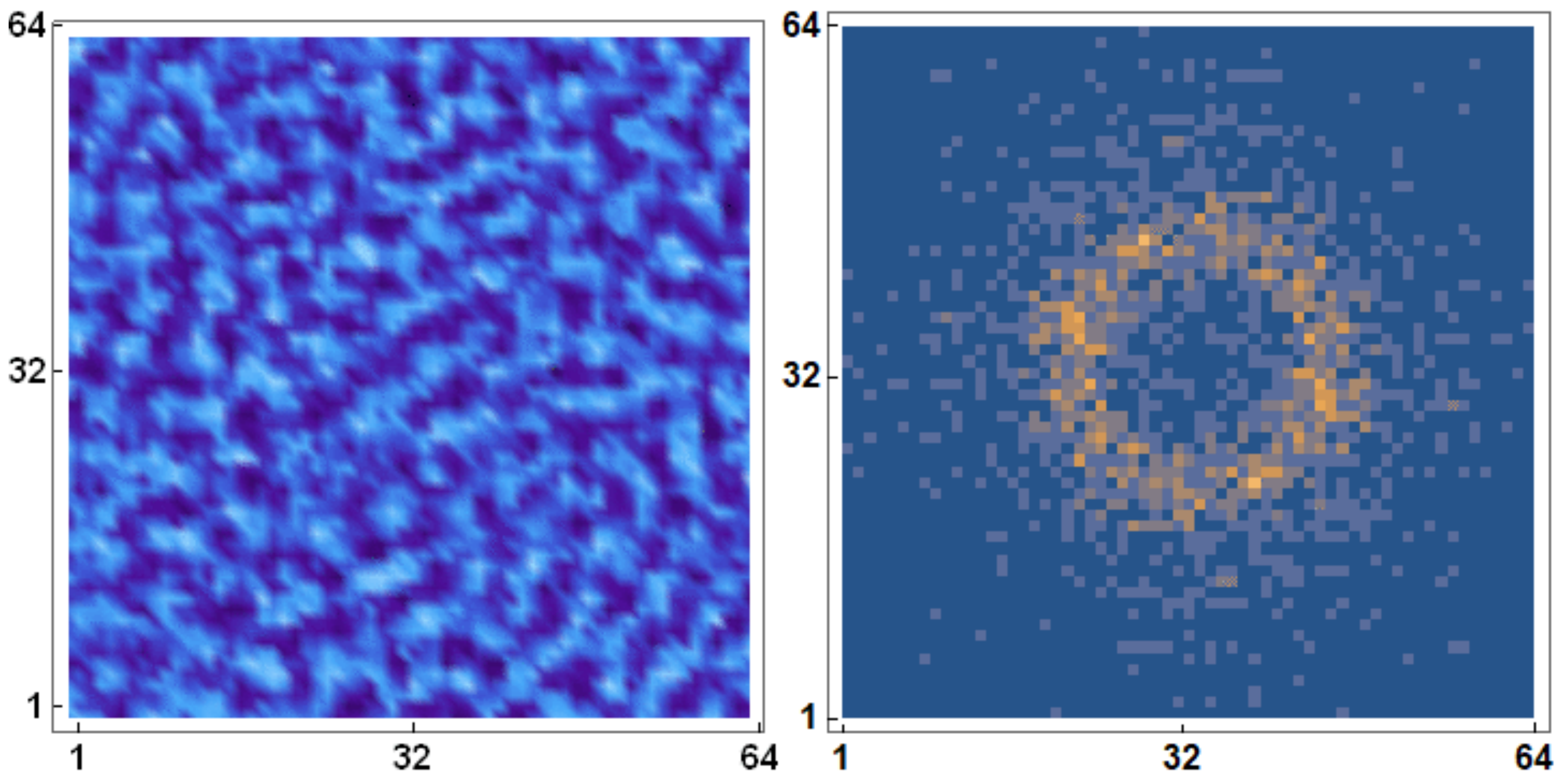
g=1.0

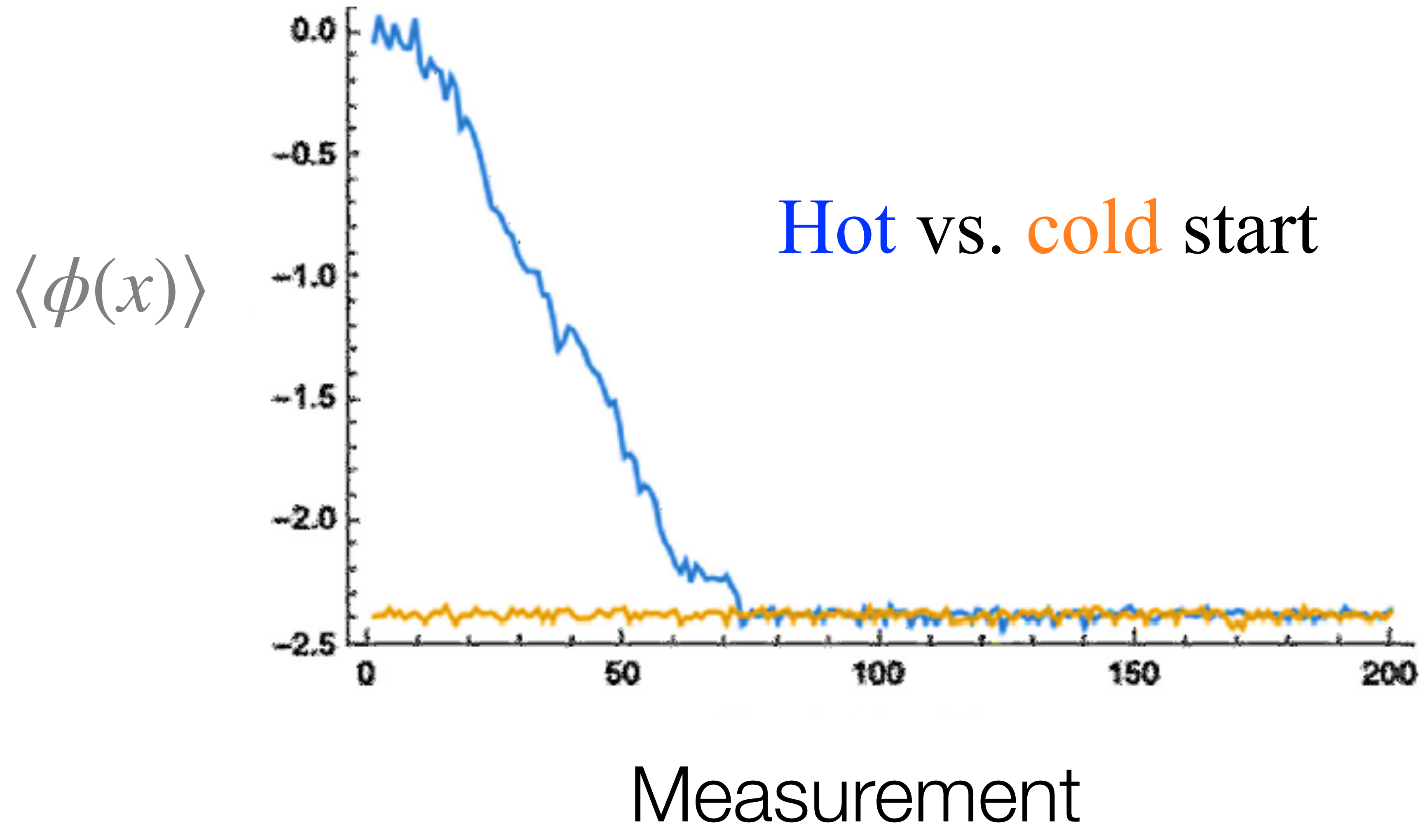


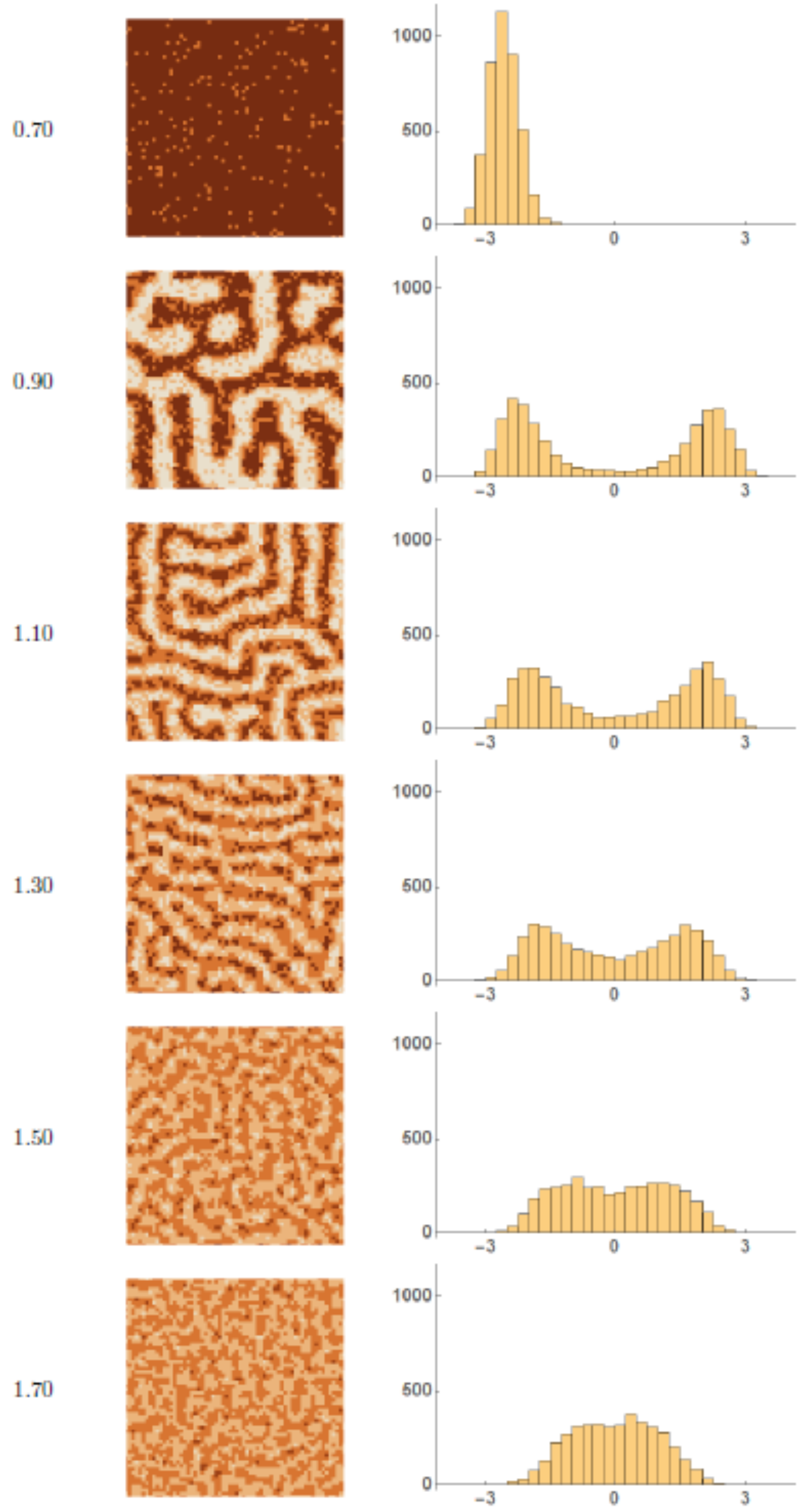
g=1.4

Stability of inhomogeneous phases in the ϕ^4 mass-mixing model

The inhomogeneous phases represent stable equilibrium behavior







Heavy fermions at nonzero temperature and density

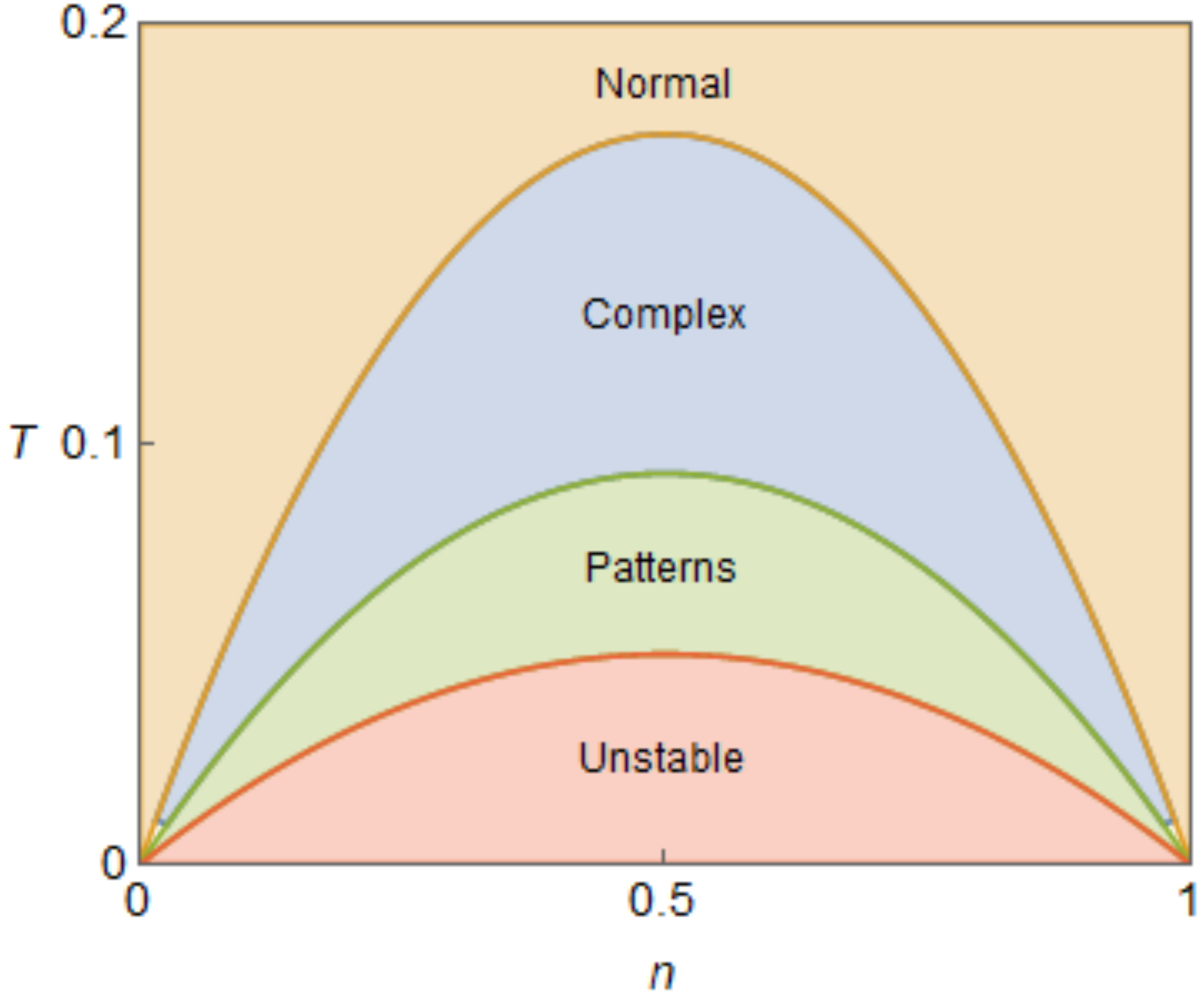
$$S = \frac{1}{2\beta}\sum_x \left\{ \frac{1}{g_\chi^2}\left[(\nabla\chi)^2 + m_\chi^2\chi^2\right] + \frac{1}{g_\phi^2}[(\nabla\phi)^2 + m_\phi^2\phi^2] \right\} - \sum_x \log\left[1 + z e^{\beta\chi(x) + i\beta\phi(x)}\right]$$

- First used by Fisher and Park (1999) to study an $i\phi^3$ transition for $z < 0$
- Same symmetries as the mass-mixing model
- Model of a gas of fermions interacting via attractive and repulsive Yukawa interactions mediated by χ and ϕ respectively; equivalent to a generalized Ising model
- The log term is essentially a heavy quark determinant, with $z = \exp\left(\beta\mu - \beta M\right)$. The role of the Polyakov loop in the fermion determinant is played by $\exp\left(\beta\chi(x) + i\beta\phi(x)\right)$

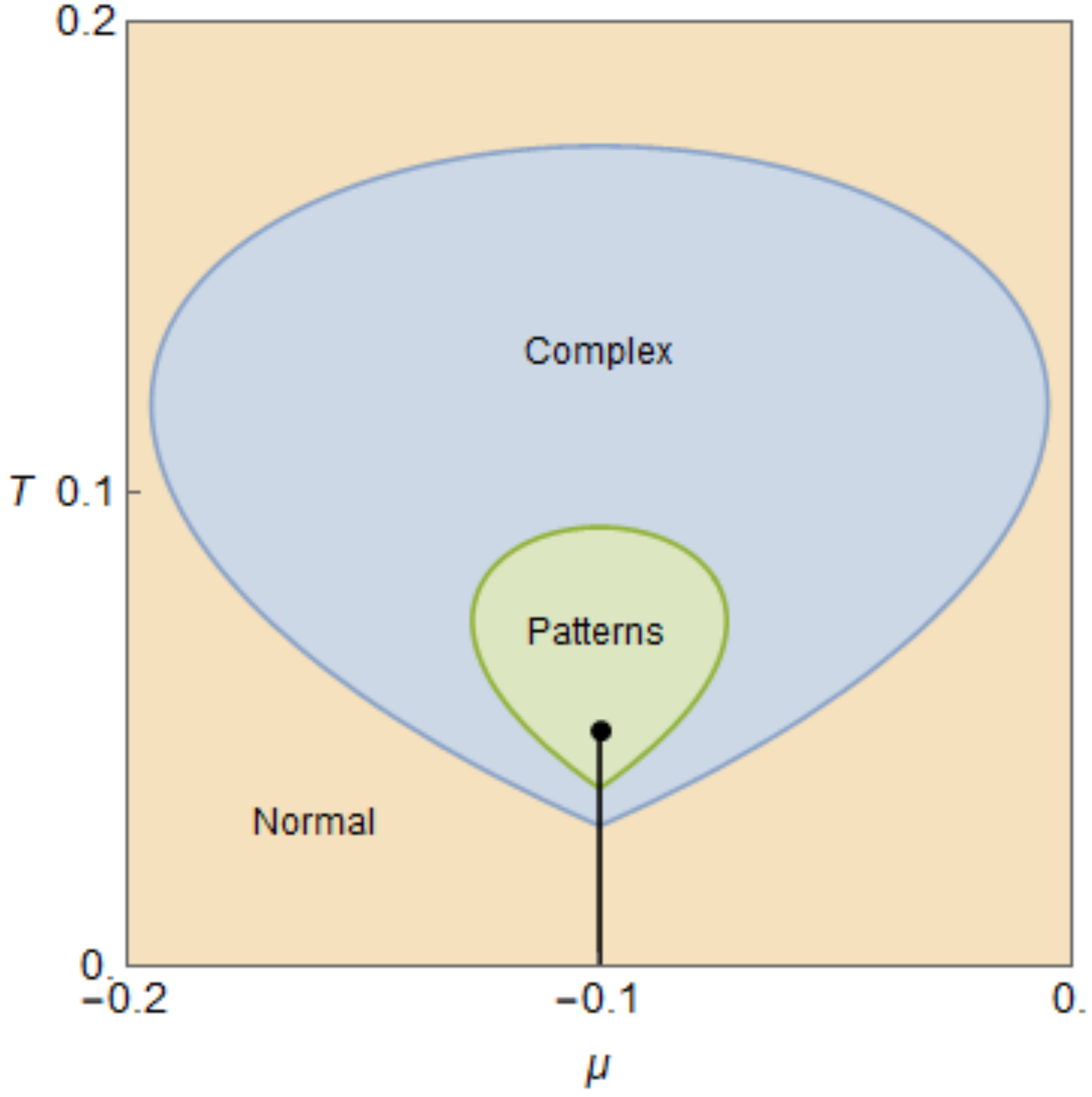
Fisher and Park, PR E 60 (1999); Glaser et al., EPL, 78 (2007); Shin et al. Soft Matter, 2009; Nishimura, Ogilvie, Pangeni, Phys. Rev. D 95, 076003 (2017), 1612.09575; Schindler, Schindler, Ogilvie, J. Phys. CS 2038 (2021, 2106.0709

Phase diagram of heavy fermion model

(n, T) plane



(μ, T) plane



Criterion for stability against patterns valid in lattice model and continuum:

$$1 + \beta\chi_Q \tilde{V}_{qq}(k) > 0$$

Schindler, Schindler, Ogilvie, J. Phys. CS 2038 (2021, 2106.0709

Complex $Z(3)$ lattice field theories: phase structure

Complex $Z(3)$ spin model with a real chemical potential

$$Z_c = \sum_{\{z_j\}} \exp\left[\sum_{\langle jk\rangle_{\parallel}} J\left(e^{\mu} s_j s_k^* + e^{-\mu} s_j^* s_k\right) + \sum_{\langle jk\rangle_{\perp}} J\left(s_j s_k^* + s_j^* s_k\right)\right]$$

Chiral $Z(3)$ spin model with an imaginary chemical potential

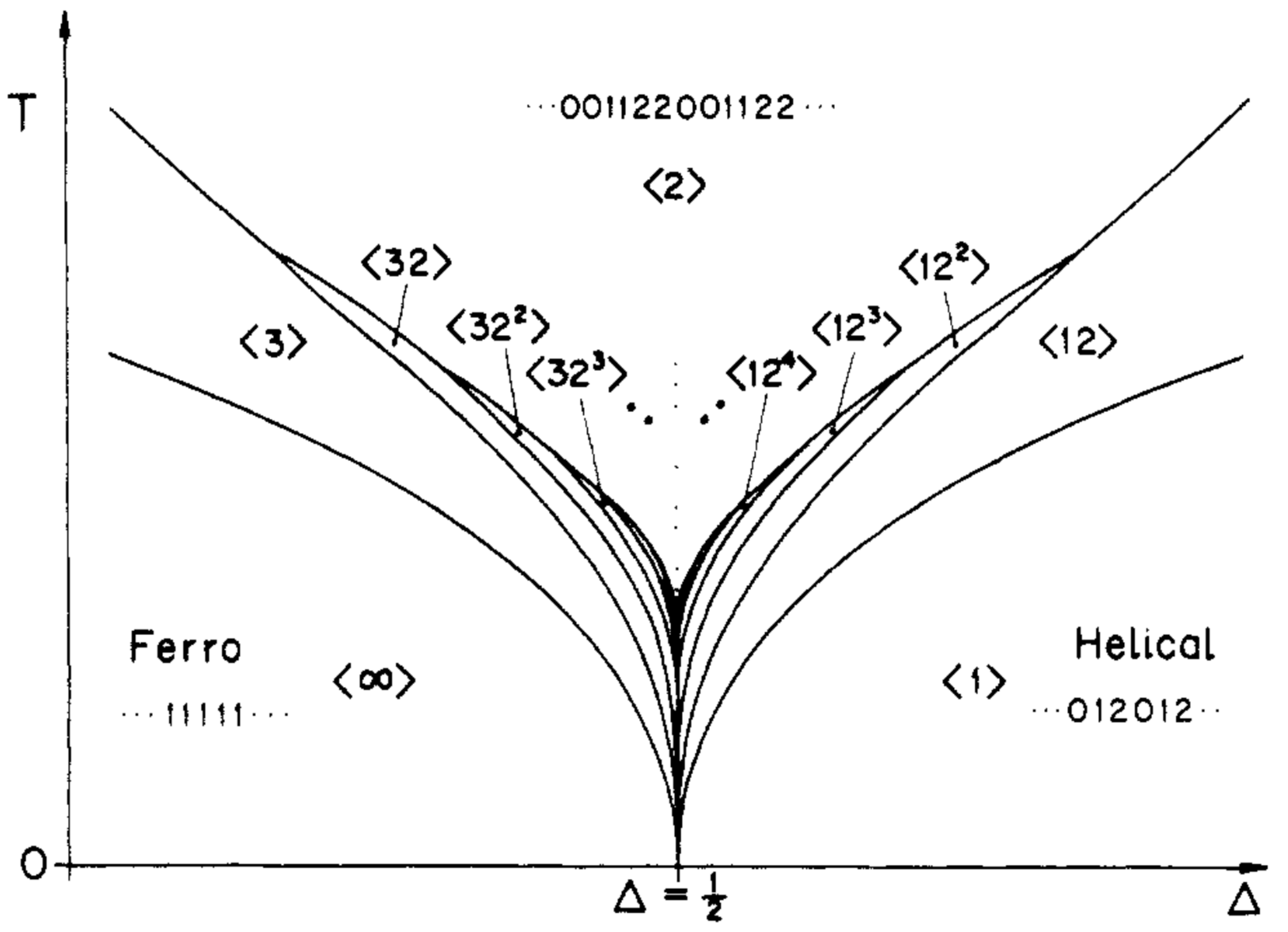
$$Z_\chi = \sum_{\{z_j\}} \exp\left[\sum_{\langle jk\rangle_{\parallel}} \tilde{J}\left(e^{i\tilde{\mu}} s_j s_k^* + e^{-i\tilde{\mu}} s_j^* s_k\right) + \sum_{\langle jk\rangle_{\perp}} \tilde{J}\left(s_j s_k^* + s_j^* s_k\right)\right]$$

Complex and chiral are dual to one another $(J,\mu) \leftrightarrow (\tilde{J},\tilde{\mu})$, in an extension of Kramer-Wannier duality: spin-spin in d=2, spin-gauge in d=3, gauge-gauge in d=4. This duality maps a complex lattice model into a real one.

Meisinger and Ogilvie, PoS Lattice2013

In $d \geq 3$ $Z(3)$ *chiral spin models* have an intricate phase structure, the **Devil's Flower** with an infinite number of inhomogeneous phases with repeating structures such as $\ldots 012012\ldots$ or $\ldots 021021\ldots$along the chiral direction similar to chiral spirals. This behavior is closely related to the Devil's Staircase of the Frenkel-Kontorova model.

Yeomans and Fisher, J. Phys. C 14 (1981)



Complex $Z(3)$ lattice field theories: the Migdal-Kadanoff real-space renormalization group

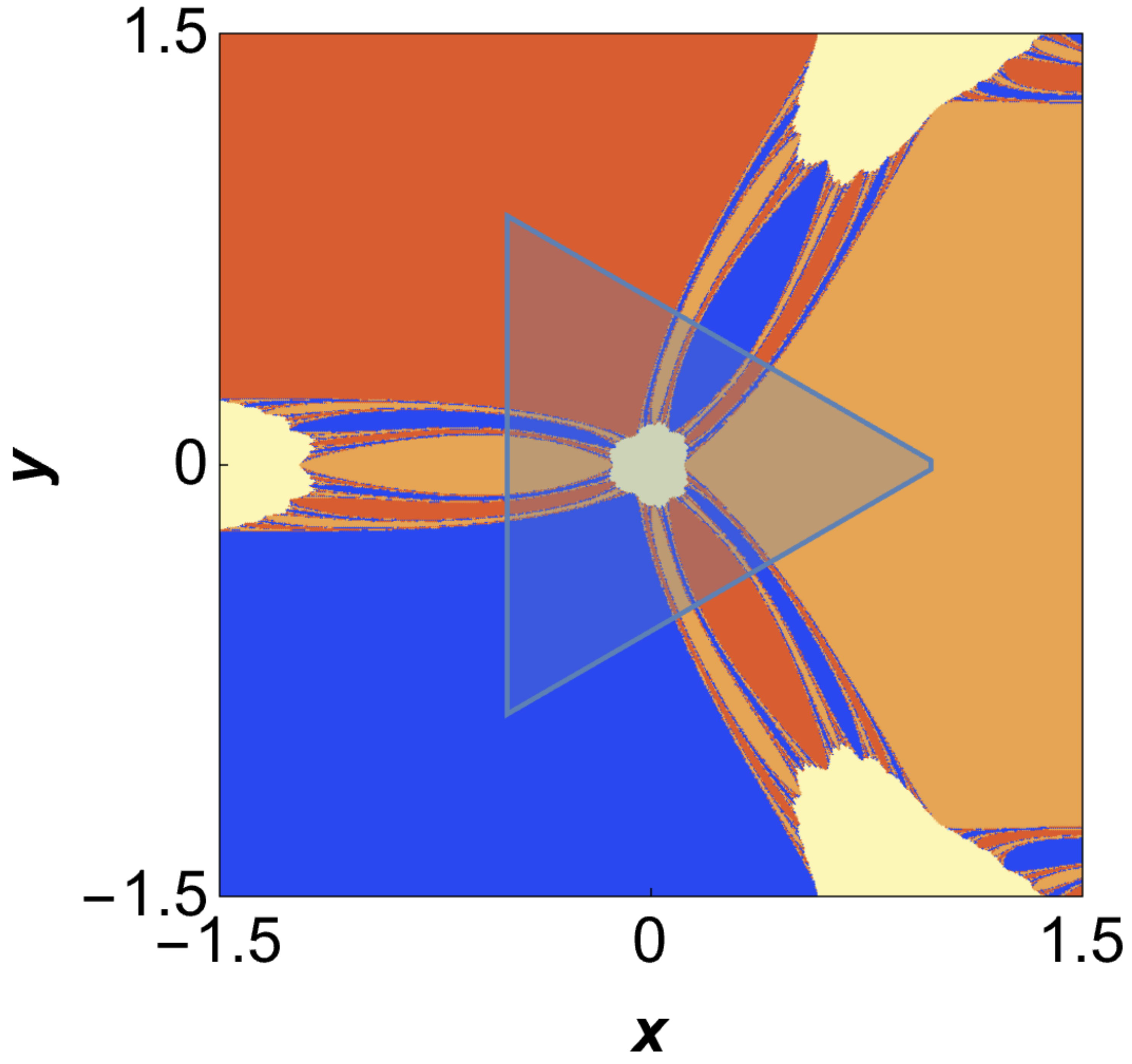
The MKRG shows the devil's flower structure for chiral Z(3) spin models.

$$z' = R(z) \qquad z = \exp(-3J/2 + i\theta)$$

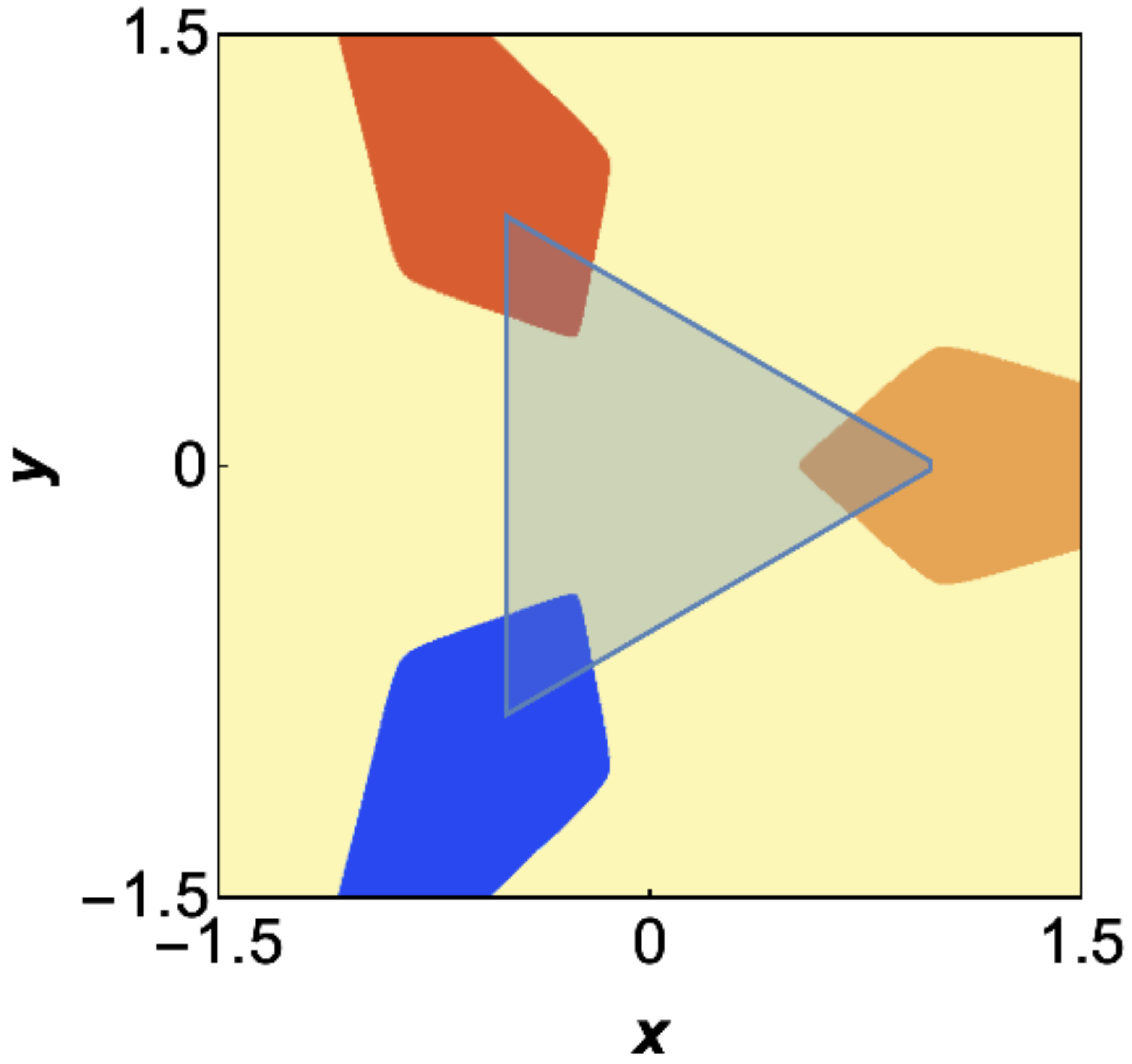
$$R(\omega z) = \omega^p R(z) \quad \omega \in Z(3) \qquad R(z^*) = R^*(z)$$

The MKRG respects duality, and can be applied to all chiral and complex Z(3) models

3d Z(3) complex gauge theory



3d Z(3) complex spin theory



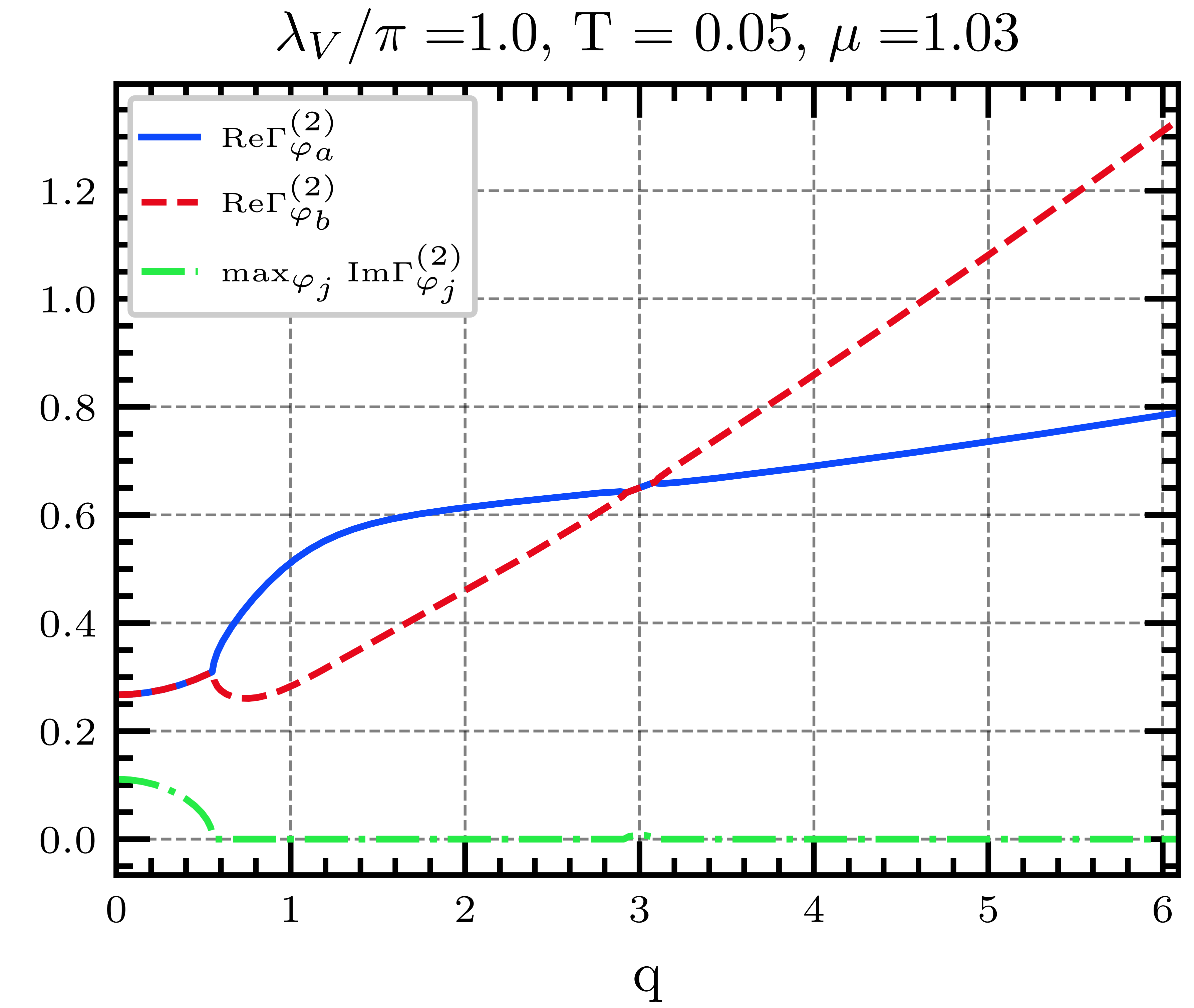
After a study of all models with $d \leq 4$, we find that the only ones with devil's flowers are dual to chiral spin models. All others have four homogeneous phases in their complex form. This is explained by Elitzur's theorem and the need for a scalar order parameter in the dual chiral form.

Schindler and Ogilvie, in preparation

Models and Mechanisms for QCD: Nambu-Jona Lasinio models at finite density

- Four-fermion models: non-renormalizable in d=4 and sensitive to regulator
- Inhomogeneous phases for $\mu \neq 0$ known; see Buballa (1406.1367; Prog. Part. Nucl. Phys. 81 (2015) for a review
- d=2 kinks, chiral spirals
- Lifshitz instability found in many models
- Sinusoidal modulation of correlation functions expected for $\mu \neq 0$ (relativistic Friedel oscillations: Kapusta and Tolmela, PRD 37 (1988)
- Mass mixing scenario occurs in a d=2+1 model: Winstel, PRD 110 (2024) 2403.07430

$$\mathscr{L}_E = \bar{\psi}\left(\gamma \cdot \partial + \mu\gamma_d\right)\psi + \sum_a \frac{\lambda_a}{2N}\left(\bar{\psi}\Gamma_a\psi\right)^2$$



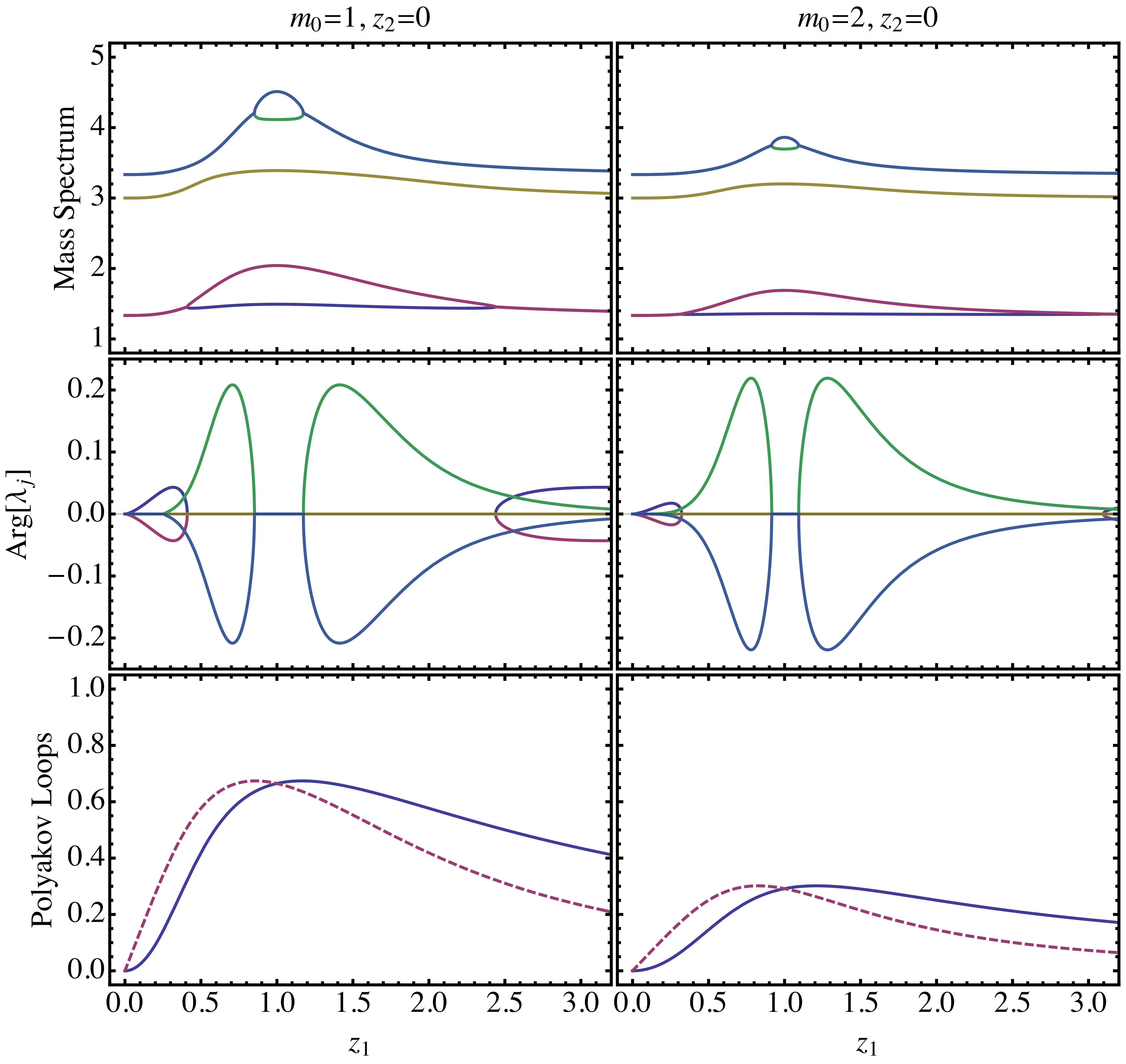
Mixing of $\sigma - \omega_3$ inverse propagators in a $d = 2+1$ model. Winstel, 2403.07430

Models and Mechanisms for QCD: Polyakov loop models

This class of models treats confinement effects but not chiral symmetry. They are often based on dimensional reduction at strong coupling to a 3d spin model of Polyakov loops $W(x)$

$$S_{eff} \simeq J \sum_{\langle x,y \rangle} \left[W(x)W^*(y) + W^*(x)W(y) \right] + \sum_x \left[zW(x) + z^{-1}W^*(x) \right]$$

A nonzero z leads to complex conjugate mass pairs in Polyakov loop correlation functions in both the $(3,\bar{3})$ and $(6,\bar{6})$ representations



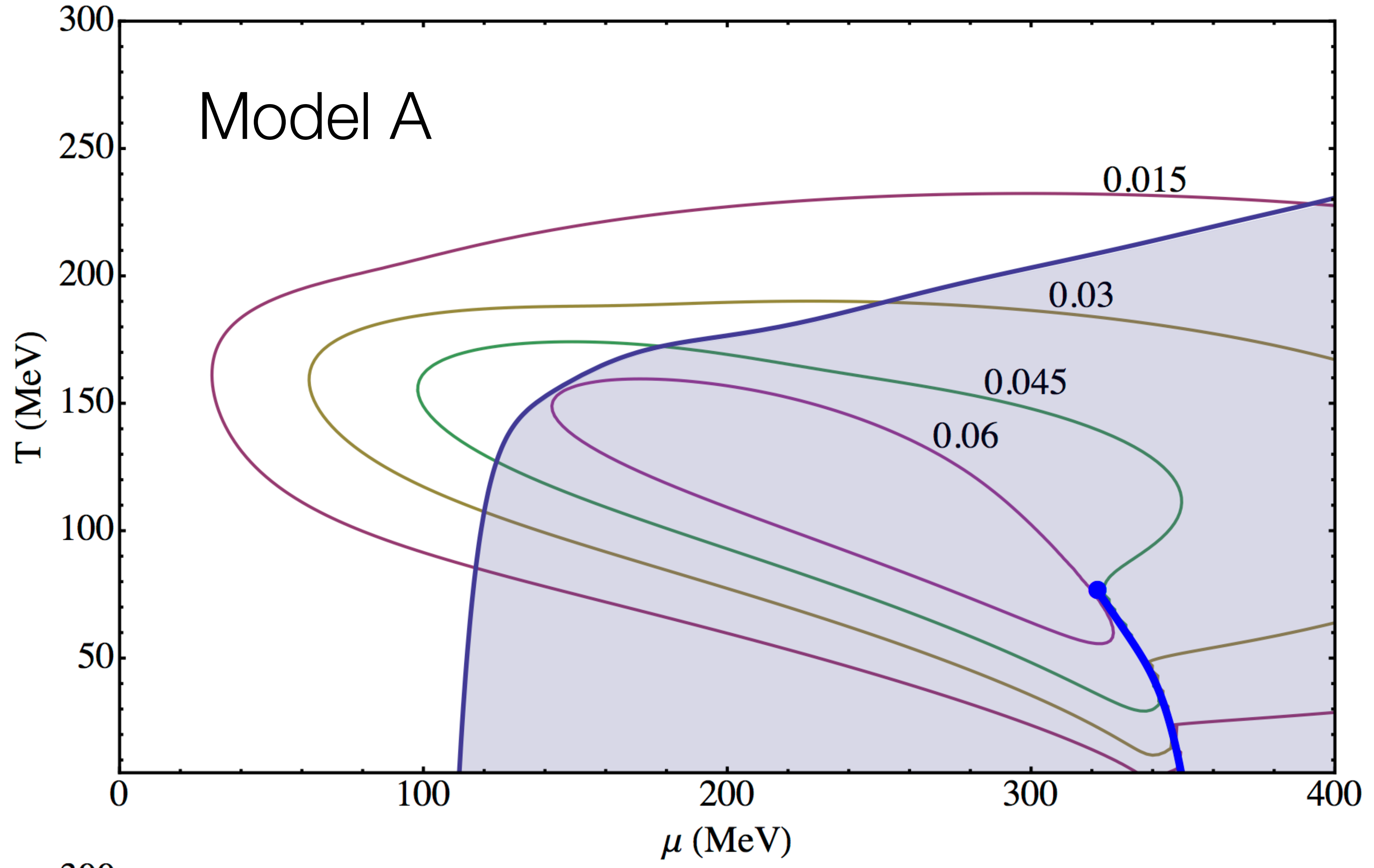
$\langle W \rangle < \langle W^* \rangle$ for $z < 1$ implying $F_Q > F_{\bar{Q}}$. This behavior reverses at $z = 1$ due to a particle-hole $z \to z^{-1}$ symmetry of this model.

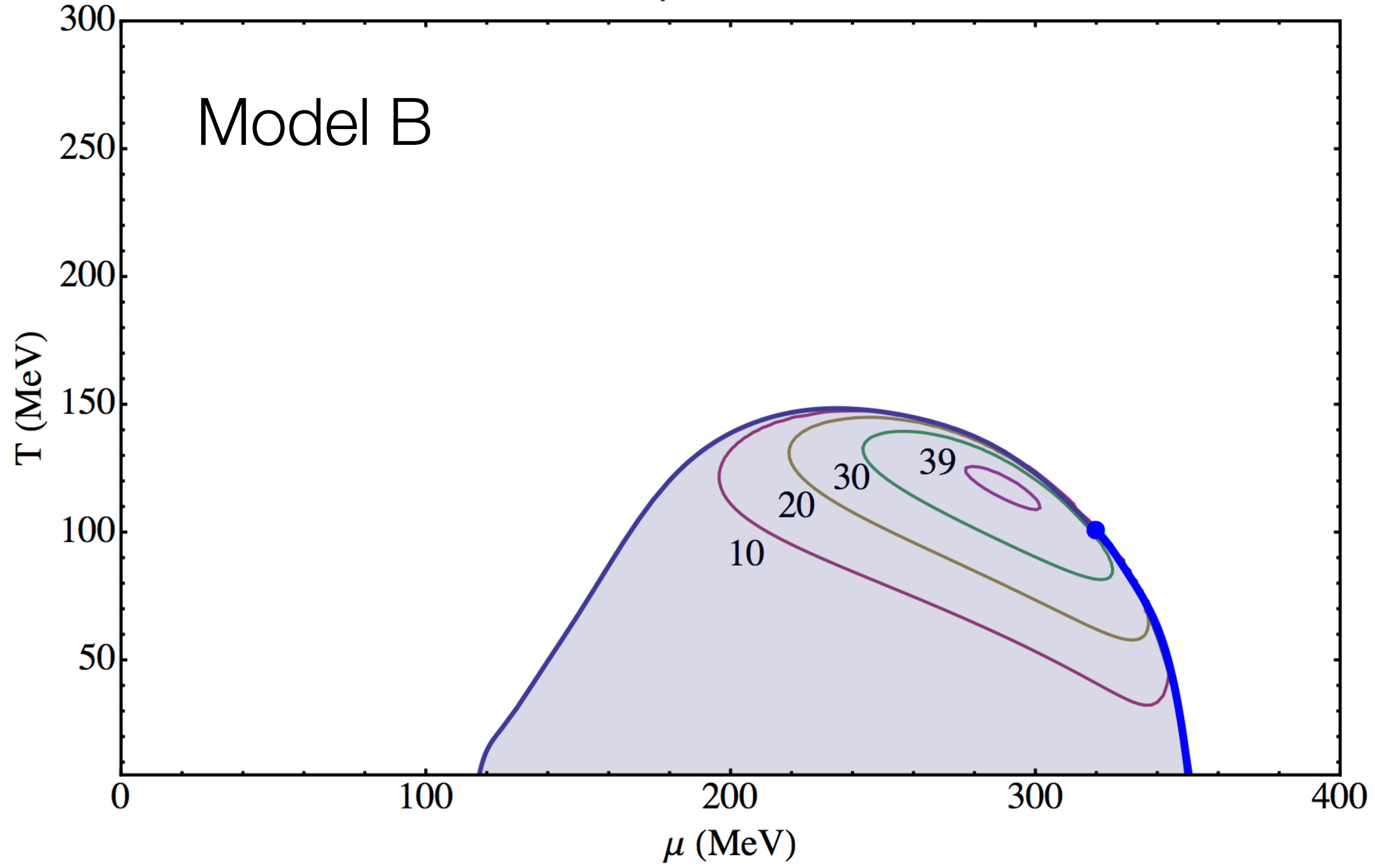
Nishimura, Ogilvie and Pangeni, PRD 93 (2016), 1512.09131; see Akerlund et al., JHEP 10 (2016) 1602.02925 for Z(3)

Models and Mechanisms for QCD: PNJL models

$$\Omega(\sigma,\omega_0,W,W^*) = V_\chi(\sigma) + U_{gauge}(W,W^*) - \frac{T}{\mathscr{V}}\ln\det\mathscr{D}\left(\sigma,W,W^*\right)$$

- PNJL models include both confinement and chiral symmetry effects. They have the same issues as NJL models plus sensitivity to confinement physics
- Free energy dominated by a complex saddle point with W and W^* real and unequal
- Sinusoidal modulation is seen but Lifshitz transition unclear





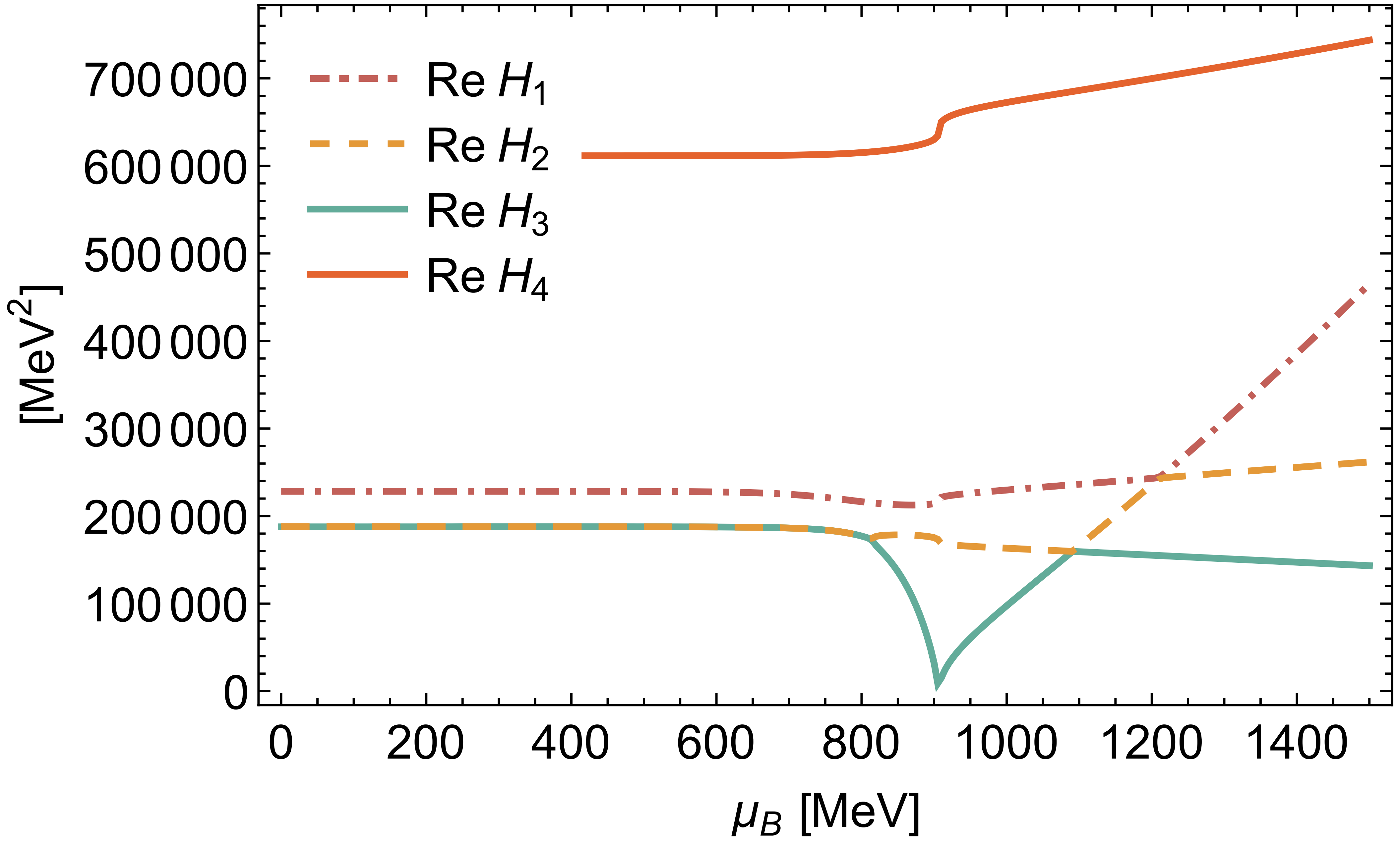
Figures show the critical line and disorder line in the $\mu - T$ plane for two different PNJL models; the contours show the imaginary part of the screening length. From Nishimura, Ogilvie, Pangeni PRD 91(2014) 1411.4959

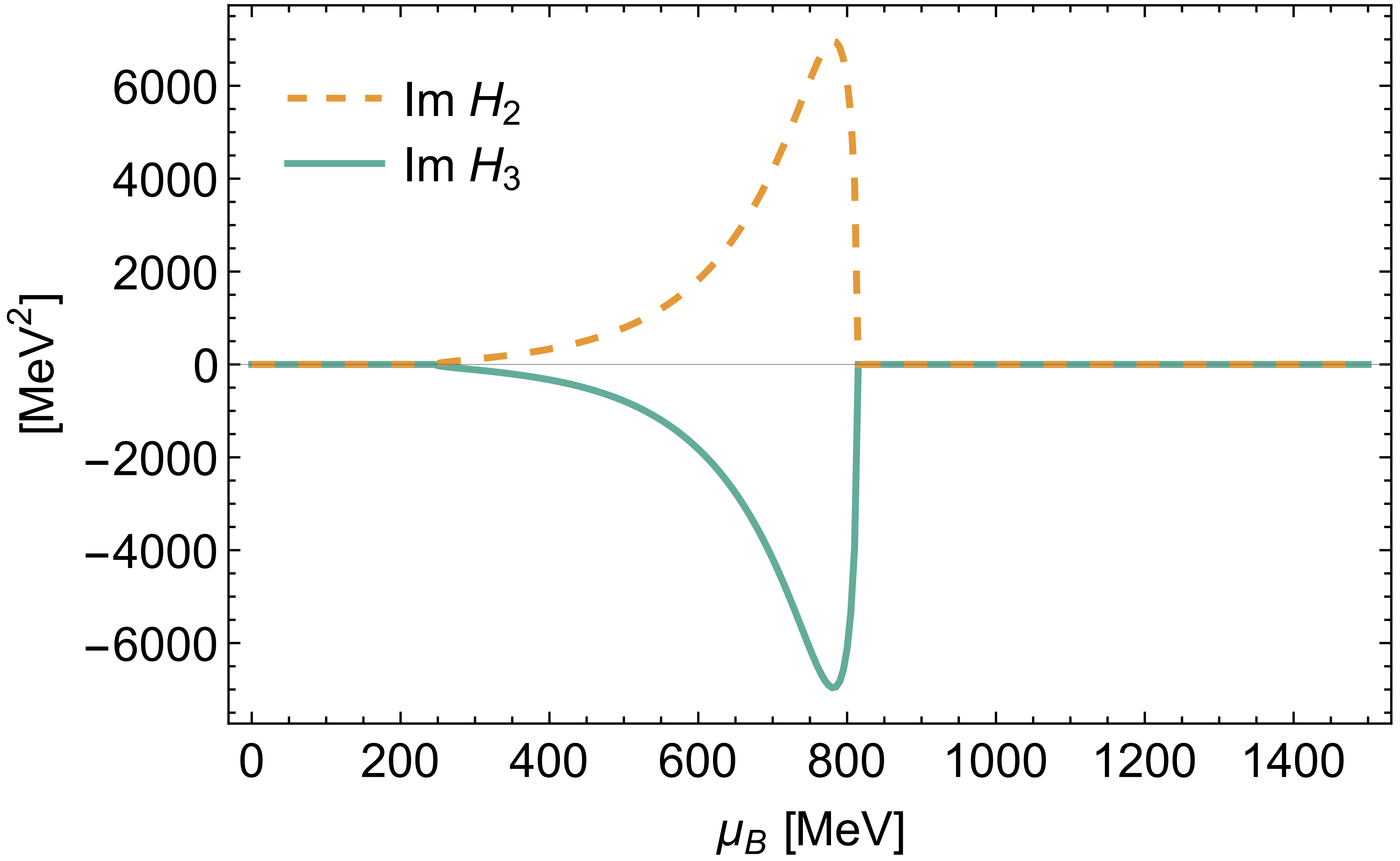
Models and Mechanisms for QCD: PQM_V model

Polyakov Quark model with vector (ω_0) repulsion

Haensch, Rennecke,von Smekal 2308.16244

$$\Omega(\sigma, \omega_0, W, W^*) = V_\chi(\sigma) + \frac{1}{2} m_\omega^2 \omega_0^2 + U_{gauge}(W, W^*)$$
$$- \frac{T}{\mathscr{V}} \Big[\ln \det \mathscr{D}\big(\sigma, \omega_0, W, W^*\big)$$
$$+ \ln \det \mathscr{D}_{\text{vac}}\big(\sigma\big) \Big] .$$

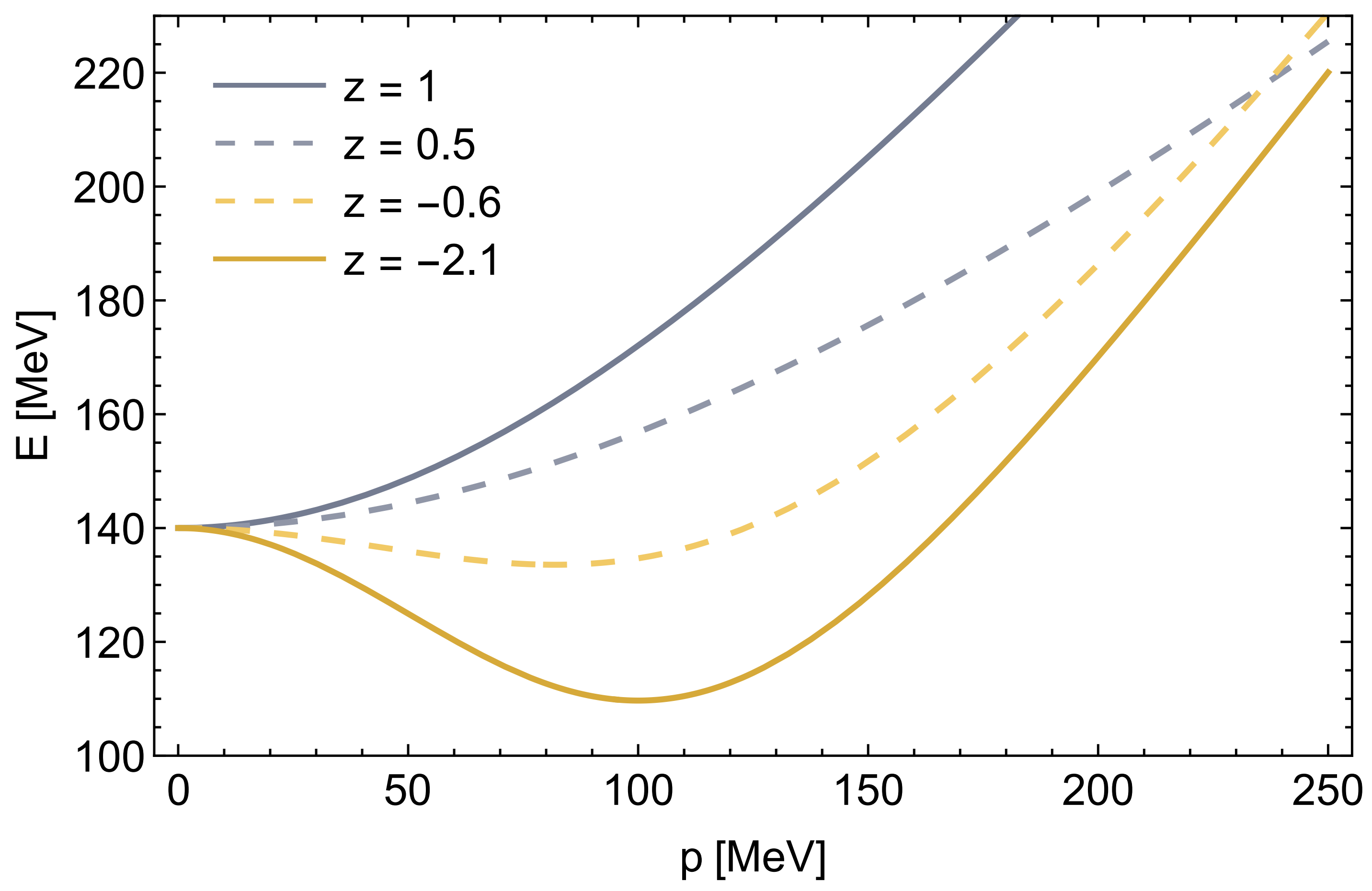




Experimental Signatures: HBT Interferometry of Moatons

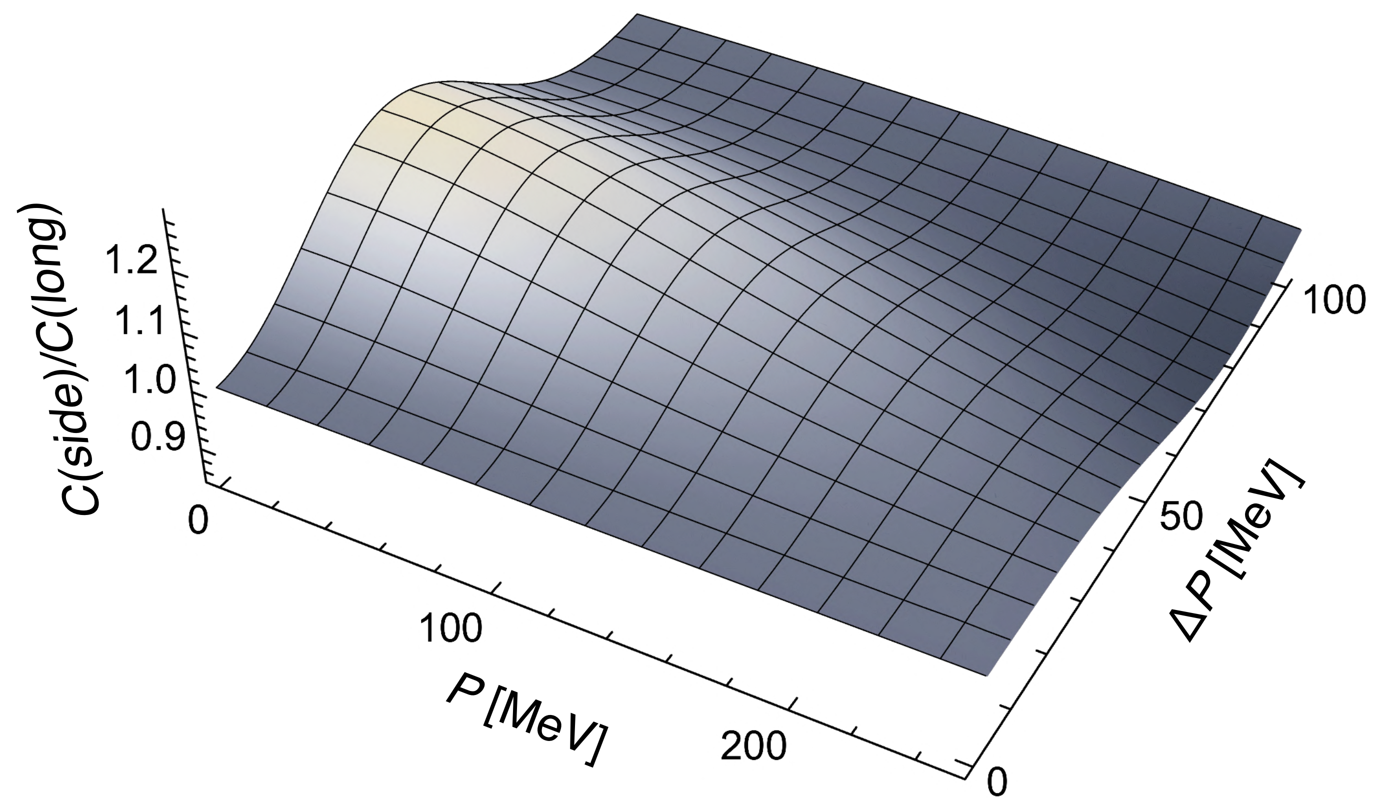
$$E^2 = Z(p)p^2 + m^2 = \left(1 - \frac{\lambda^2}{p^2 + M^2}\right)p^2 + m^2$$

$$E^2 \approx m^2 + \left(1 - \frac{\lambda^2}{M^2}\right)p^2 + \frac{\lambda^2}{M^4}p^4 + O(p^6)$$

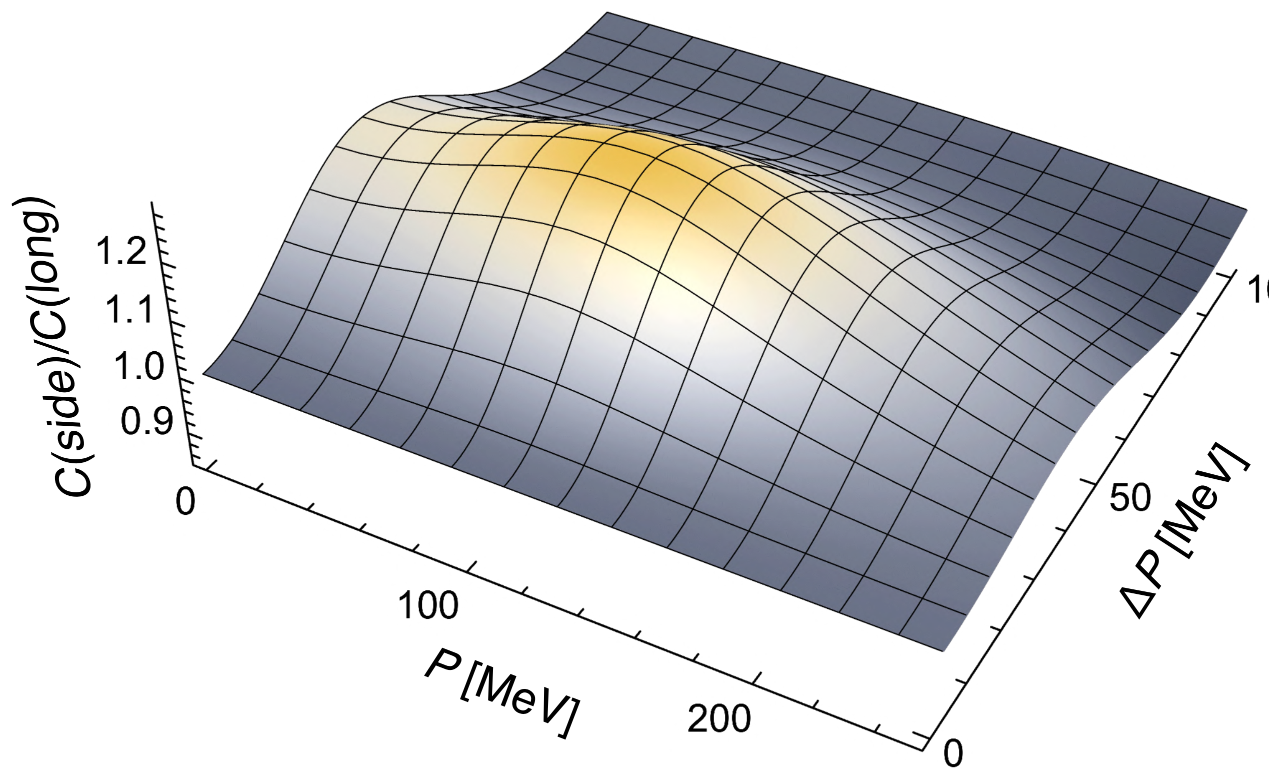


$$z = 1 - \frac{\lambda^2}{M^2}$$

Normal



Moatons



Rennecke and Pisarski PoS CPOD2021, 2110.02625; Rennecke, Pisarski and Rischke 2301.11484; PRD 107 (2023)

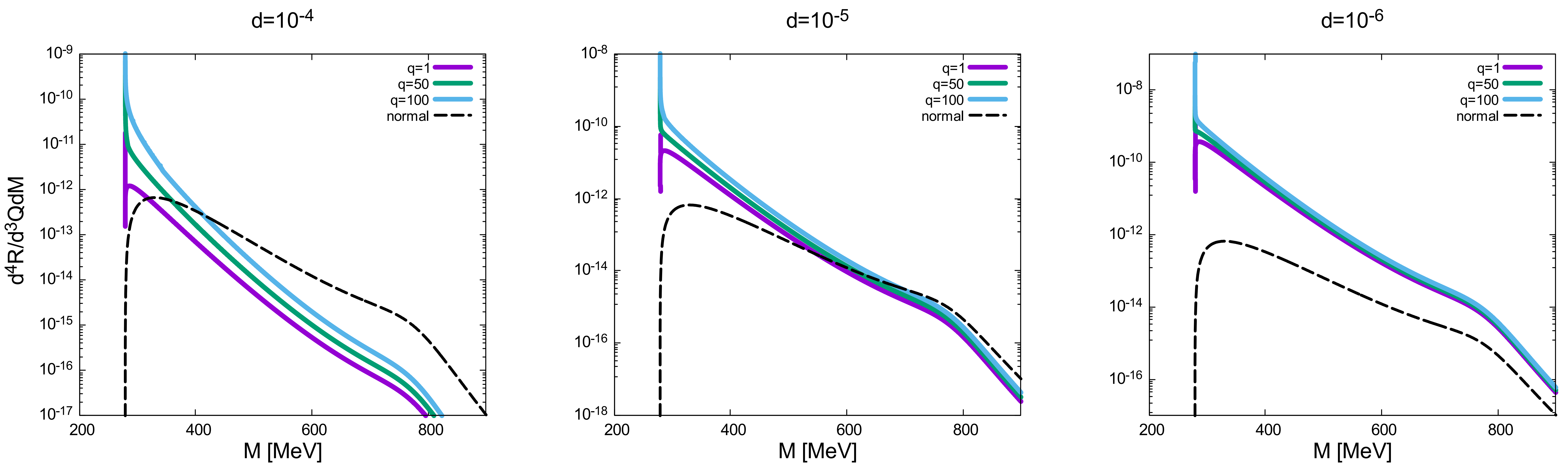
Experimental Signatures: Dilepton production

$$\pi^+ + \pi^- \rightarrow \gamma \rightarrow l^+ + l^-$$

Dilepton production in a background chiral spiral of wave number q:

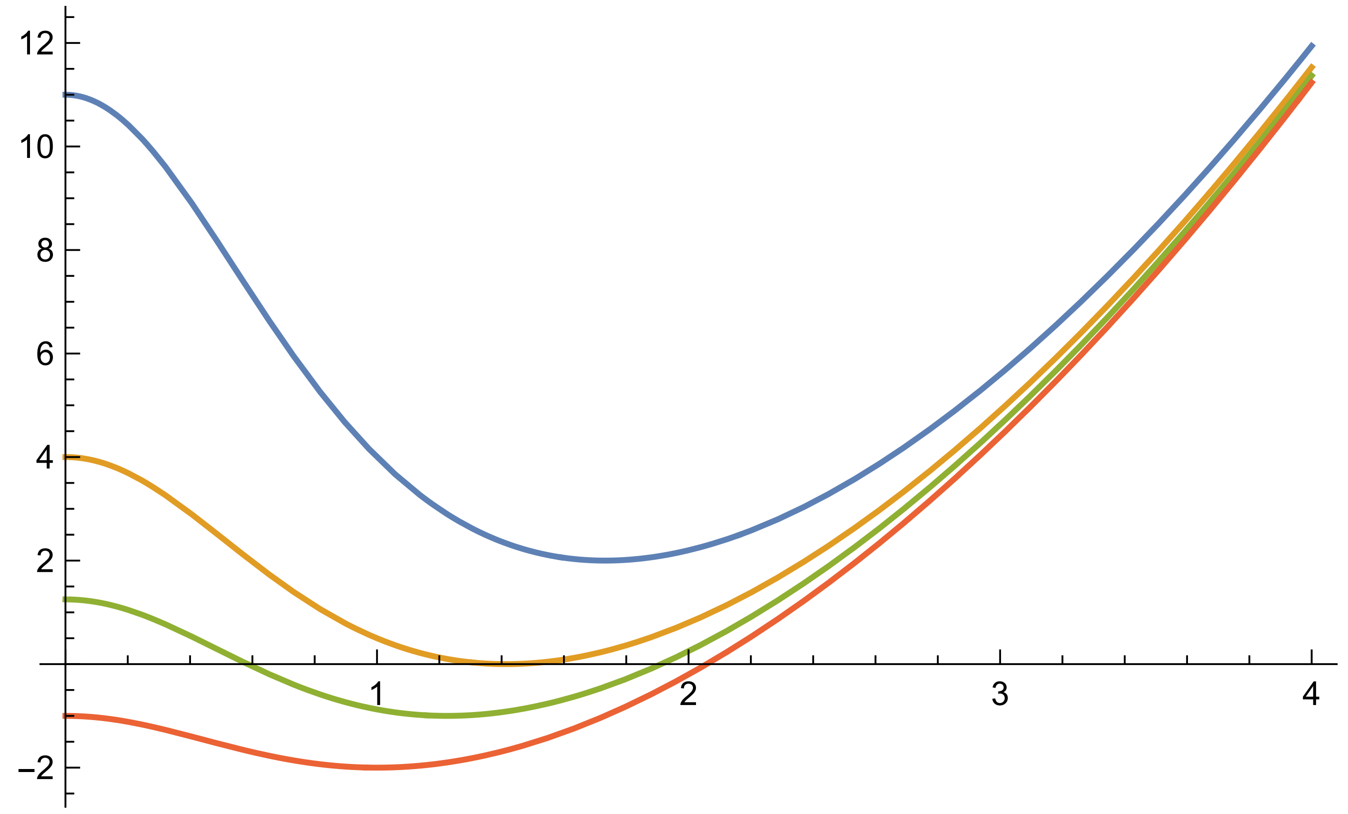
$$\langle \sigma + i\pi_3 \rangle = \sigma_0 \exp(i\vec{q} \cdot \vec{r})$$

Hayashi and Tsue, 2407.08523



Nussinov, Ogilvie, Pannullo, Pisarski, Rennecke, Schindler, Winstel, Valgushev, in preparation

- **Spike at threshold is a van Hove singularity: density of states diverges at a nontrivial extremum of the energy**
- **Direct consequence of exotic dispersion relation and non-Hermitian physics.**
- **Very general behavior: independent of any particular model or mechanism and an underlying inhomogeneous phase is not necessary**



Conclusions

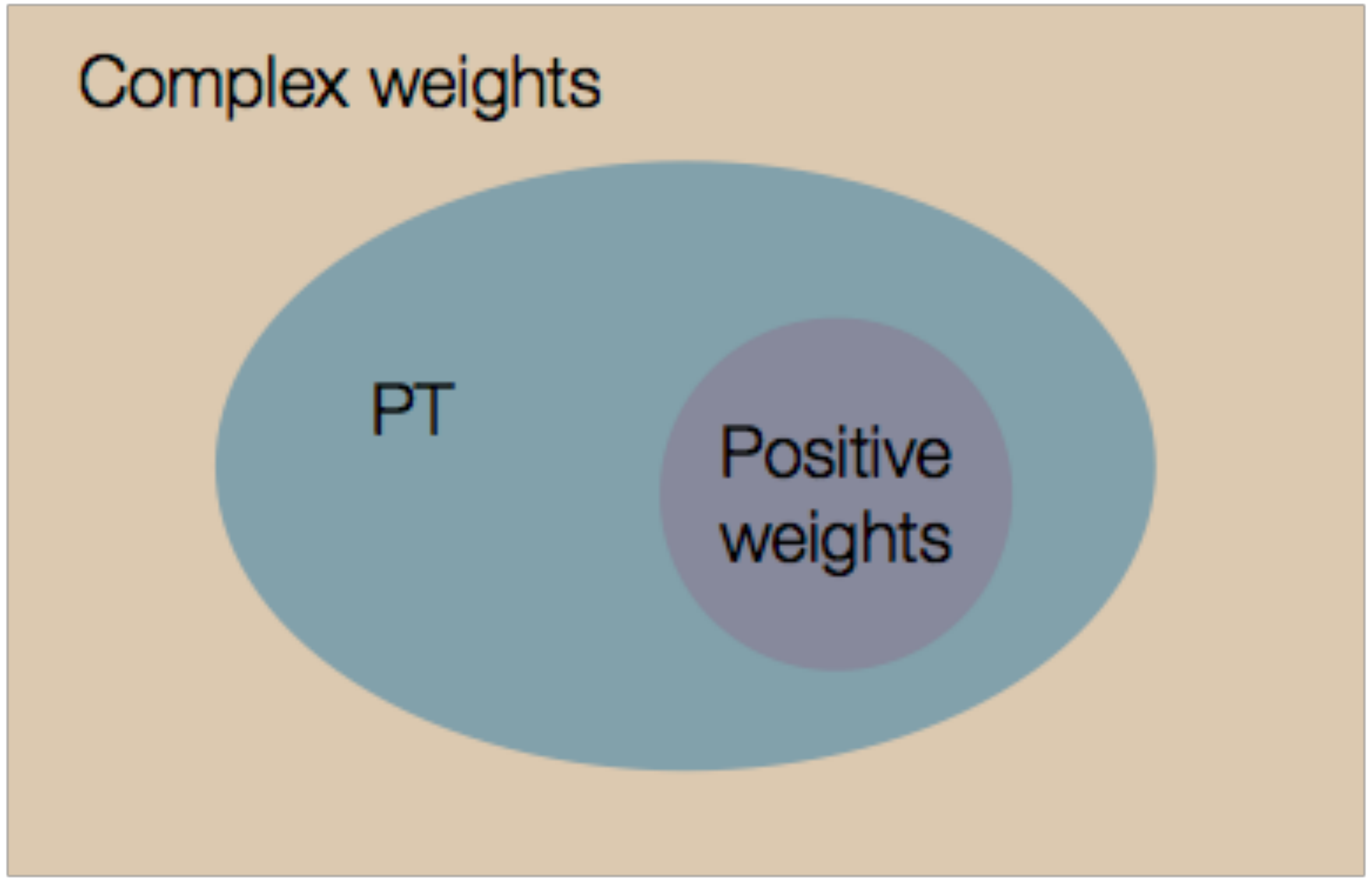
- High-density QCD is an important area of fundamental physics where non-Hermitian behavior occurs.
- Rich phase structure possible, with many models exhibiting exotic dispersion relations, and some have inhomogeneous phases due to a Lifshitz transition
- Potential experimental signals seem to be a generic feature of phases with exotic dispersion relations.
- Lessons are relevant for non-Hermitian field theories in other areas of particle physics.

Additional Slides

An algorithm for simulating some complex actions: Ogilvie and Medina, Lattice 2018, 1811.11112

$$S(\chi) = \sum_x \left[\frac{1}{2}(\partial_\mu \chi(x))^2 + V(\chi(x)) - ih(x)\chi(x)\right]$$

PT symmetry: $V(\chi)^* = V(-\chi) \Rightarrow \tilde{V}(\tilde{\chi}) \in \mathbb{R}$



$$\exp\left[-\frac{1}{2}\left(\partial_\mu \chi(x)\right)^2\right] = \int d\pi_\mu(x) \exp\left[\frac{1}{2}\pi_\mu(x)^2 + i\pi_\mu(x)\partial_\mu\chi(x)\right]$$

$$\exp\left[-V\left(\chi\left(x\right)\right)\right] = \int d\tilde{\chi}(x) \exp\left[-\tilde{V}\left(\tilde{\chi}\left(x\right)\right) + i\tilde{\chi}(x)\chi(x)\right]$$

If *dual weight positivity* holds

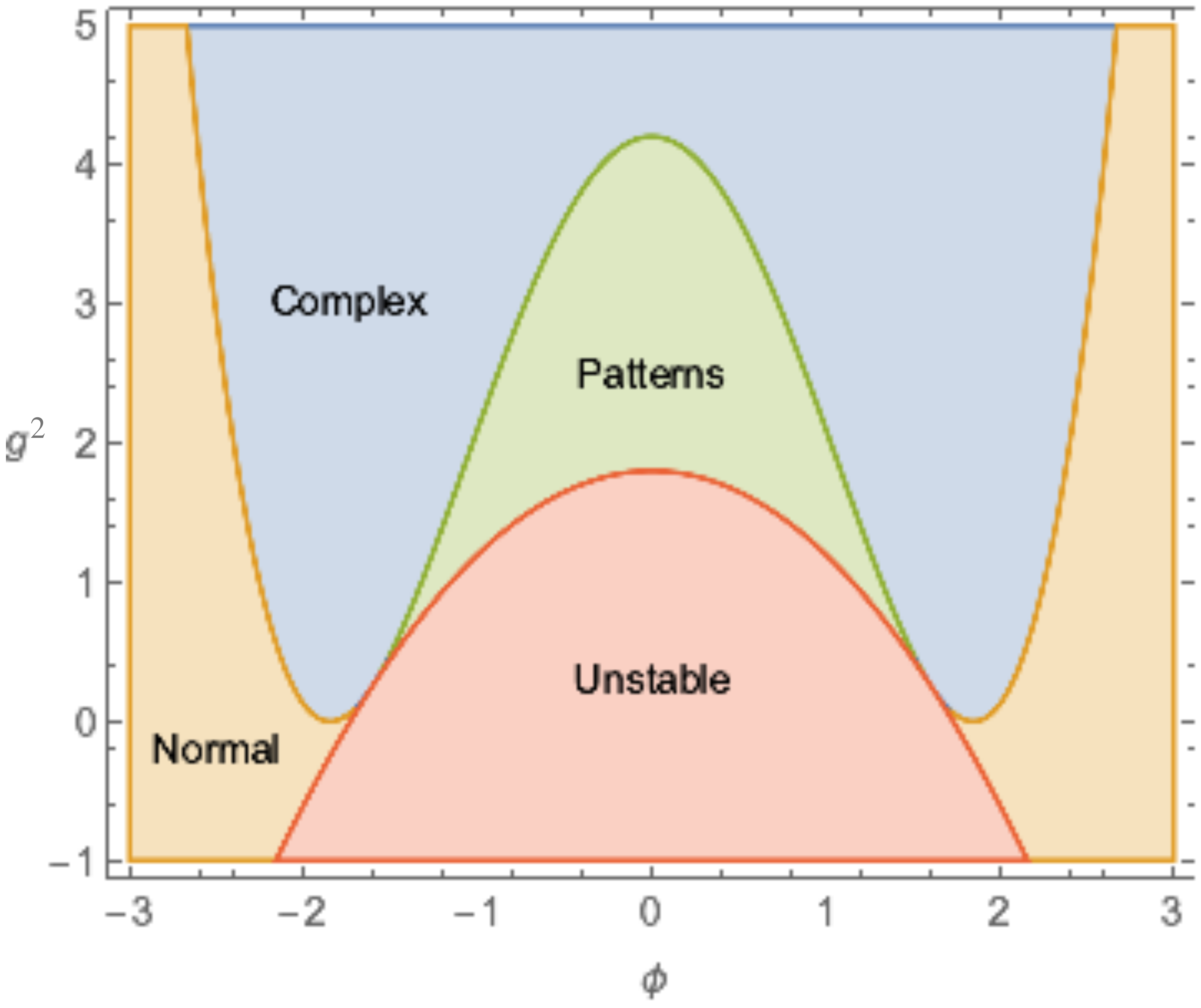
$$\tilde{w}[\tilde{\chi}(x)] \equiv \exp\left[-\tilde{V}(\tilde{\chi}(x))\right] \geq 0$$

the functional integral is manifestly positive and the dual action $\tilde{S}$ is simulatable by standard methods.

$$\tilde{S} = \sum_x \left[\frac{1}{2}\pi_\mu^2(x) + \tilde{V}(\partial \cdot \pi(x) - h(x))\right]$$

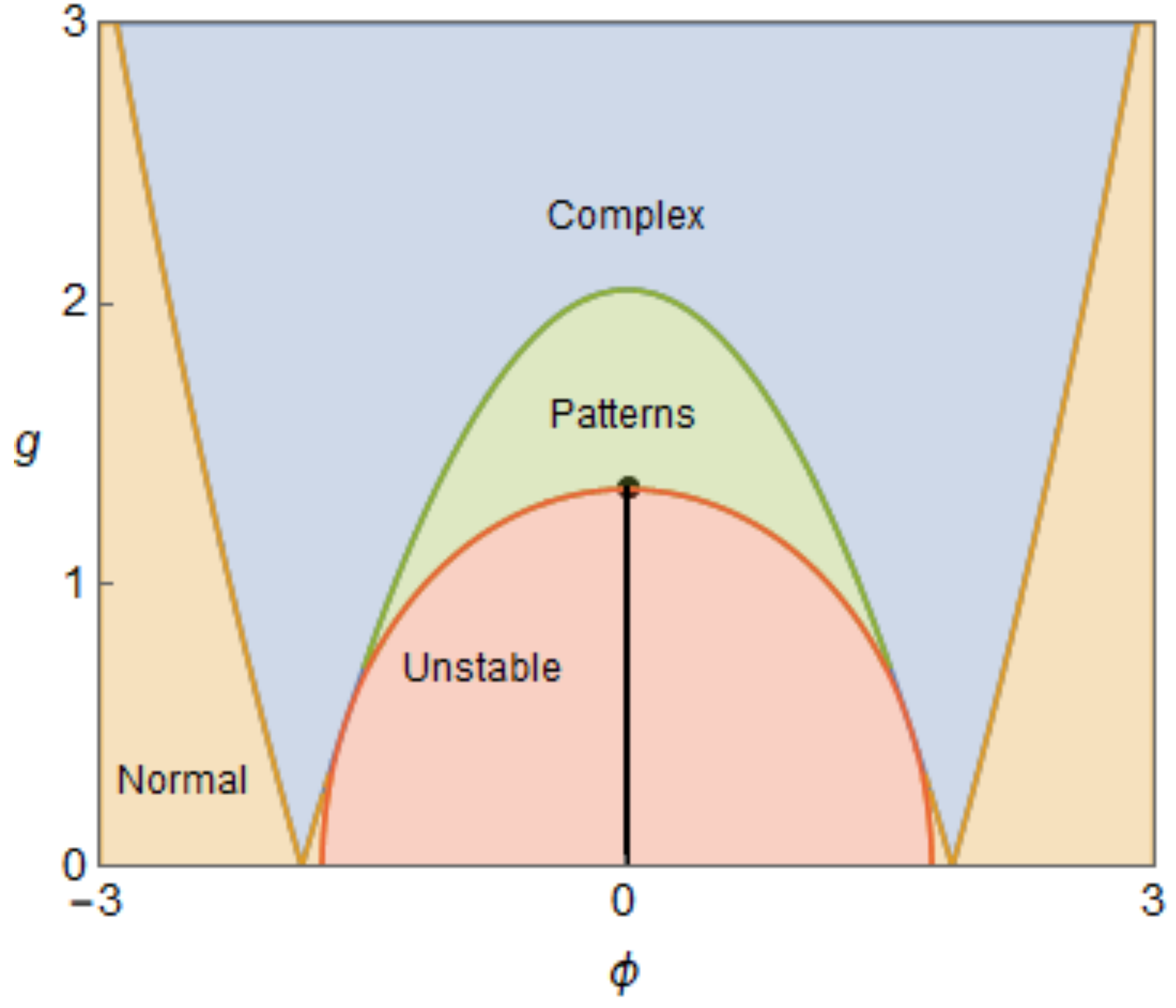
Dual weight positivity implies not only that standard lattice simulation methods can be be applied but also that mean field theory and other analytical methods can be used.

Mixing model: Phase diagram as a function of g^2

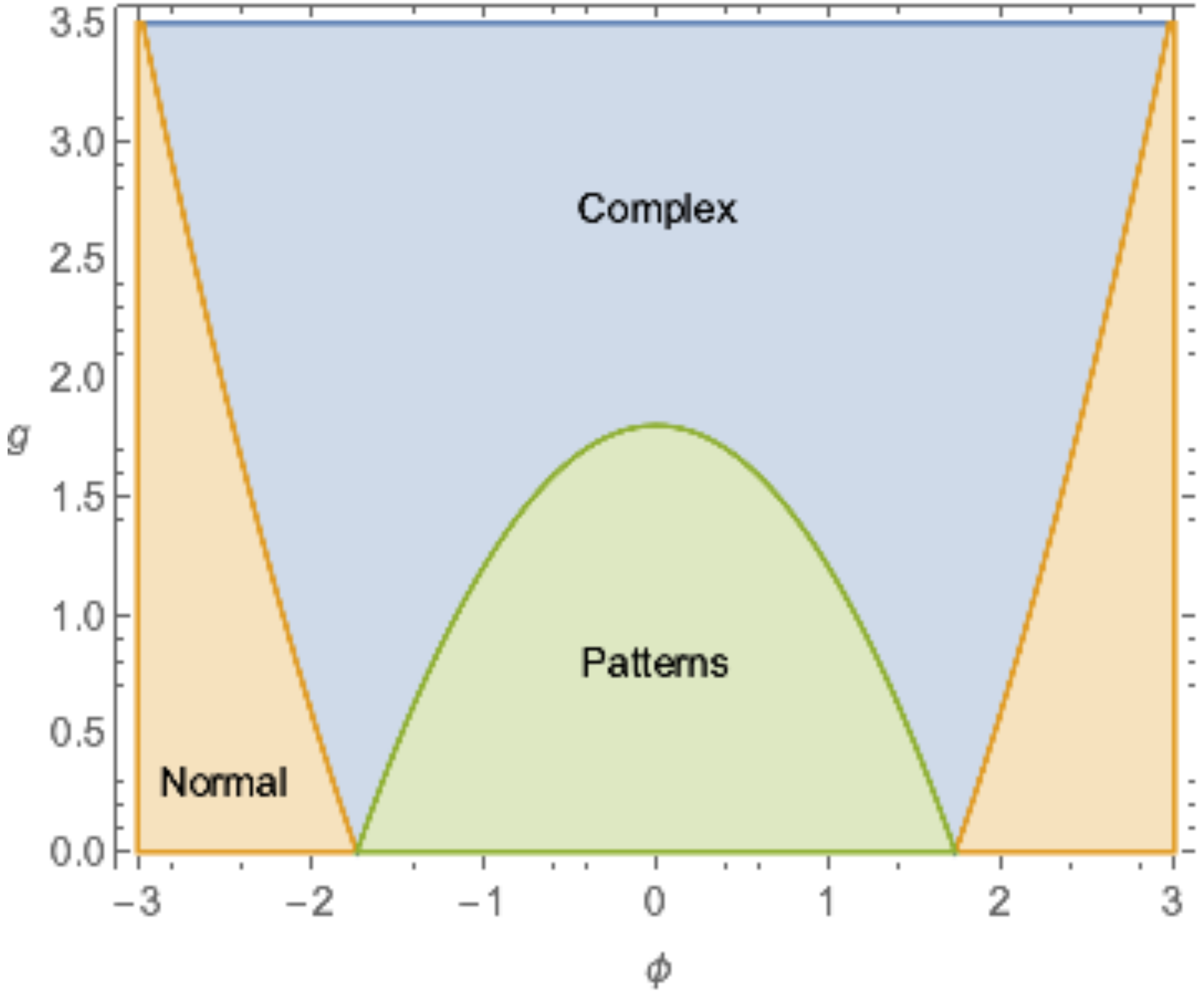


Mixing model: phase diagram in the Coulomb limit $m_\chi \to 0$

$m_\chi \neq 0$



$m_\chi = 0$



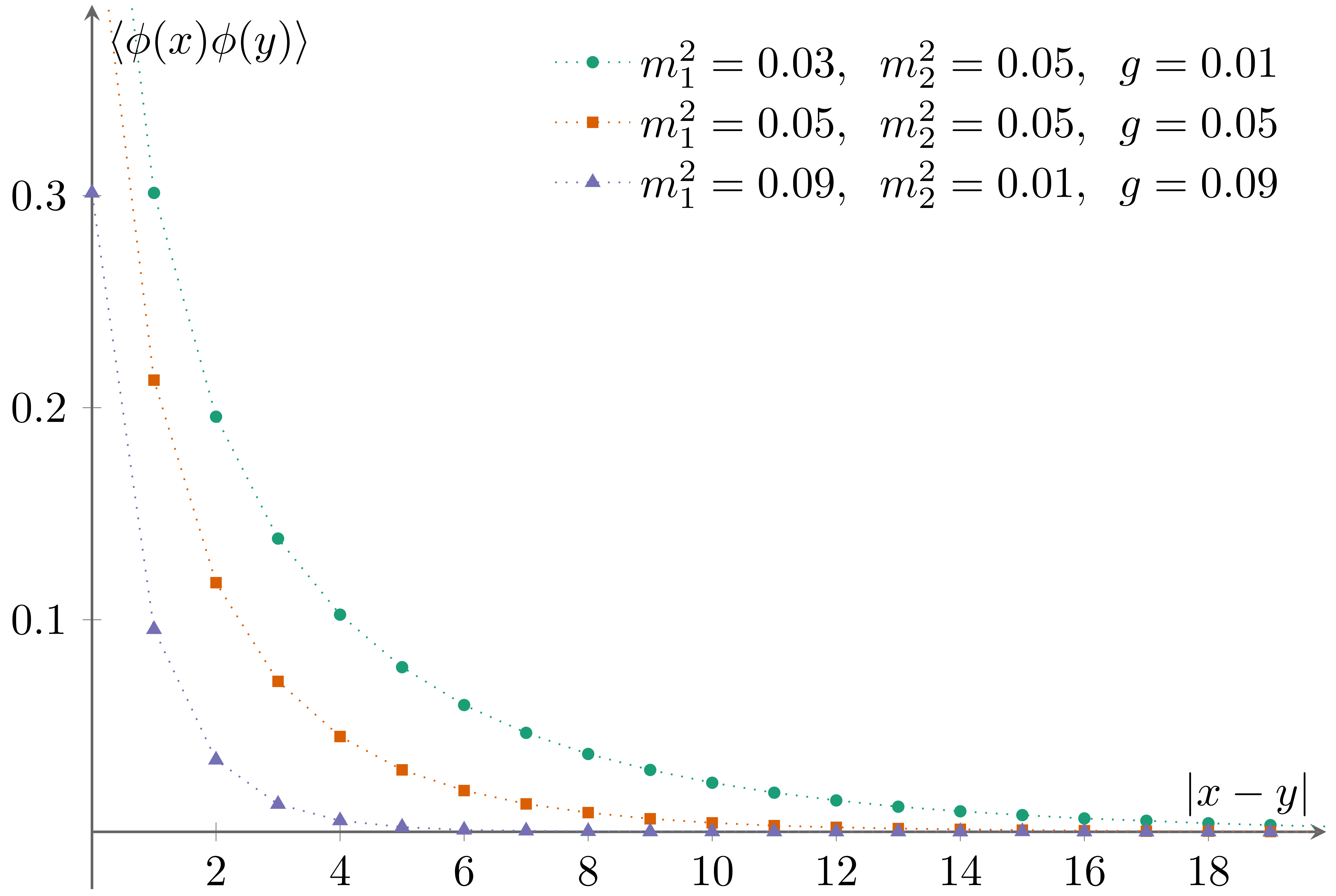
"avoided critical point"

Imaginary Yukawa coupling (ICY)

$$V(\phi,\chi) = m_\phi^2\phi^2/2 + m_\chi^2\chi^2/2 - ig\chi\phi^2$$

$$\tilde{V}(\phi,\pi_\mu) = m_\phi^2\phi^2/2 + (\partial\cdot\pi - g\phi^2)^2/2m_\chi^2$$

- No sign of any complex mass pairs in d=1,2 or 3.
- This model goes smoothly into a ϕ^4 model in a scaled limit where g and m_χ go to infinity.



Computational complexity

The well-known work of Troyer and Wiese (PRL 2005) shows that the sign problem of fermionic many-body systems is NP-hard by showing its equivalence to finding the ground state of a random-bond Ising model

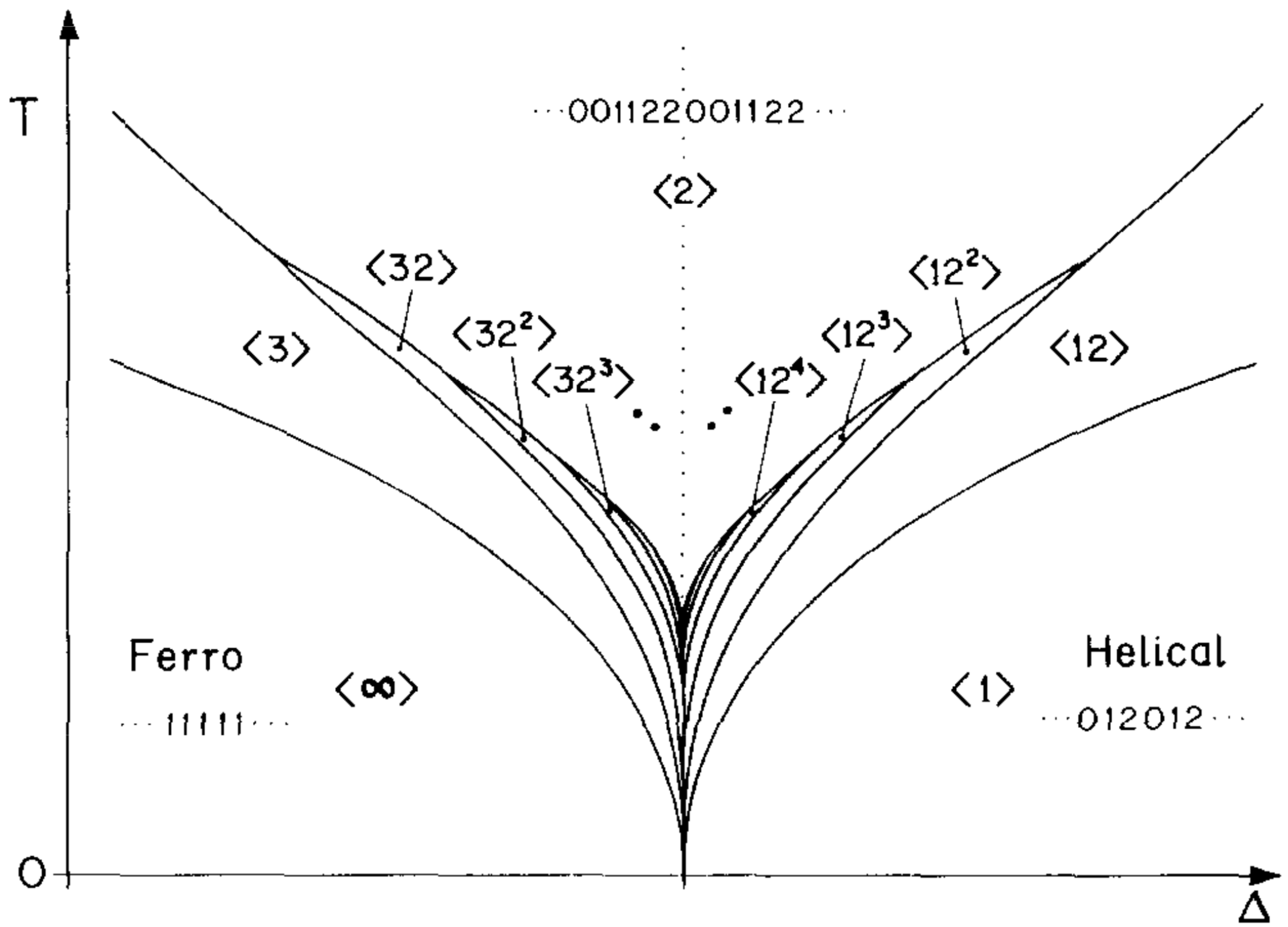
It has been proposed that scalar field theory models with long-range interactions (Schmalian and Wolynes, PRL 2001) and higher-derivative interactions (Westfahl et al, Chem. Phys. Lett 2002) can model glassy behavior, a prototypical NP-hard problem.

$$S_{\text{eff}} = \sum_x \left[\frac{1}{2}(\partial_\mu \phi(x))^2 + \lambda(\phi^2 - v^2)^2 + h\phi\right] + \frac{g^2}{2}\sum_{x,y}\phi(x)\Delta(x-y)\phi(y) \qquad \tilde{\Delta}(k) = \frac{1}{k^2}$$

Computational complexity in such systems has its origins in the complexity of the ground states and equilibrium states of the systems, in particular in spatial structure.

Z(N) spin models and pattern formation

Chiral $Z(3)$ Devil's Flower



Yeomans and Fisher, 1984

Basic model $$\mathscr{H} = -\frac{J}{2}\sum_{\langle j\nu\rangle}\left(z_j z^*_{j+\hat{\nu}} + z^*_j z_{j+\hat{\nu}}\right)$$

Chemical potential $$\Rightarrow e^{\mu} z_j z^*_{j+\hat{d}} + e^{-\mu} z^*_j z_{j+\hat{d}}$$

Chiral Z(N) model $$\Rightarrow e^{2\pi i\Delta/N} z_j z^*_{j+\hat{d}} + e^{-2\pi i\Delta/N} z^*_j z_{j+\hat{d}}$$

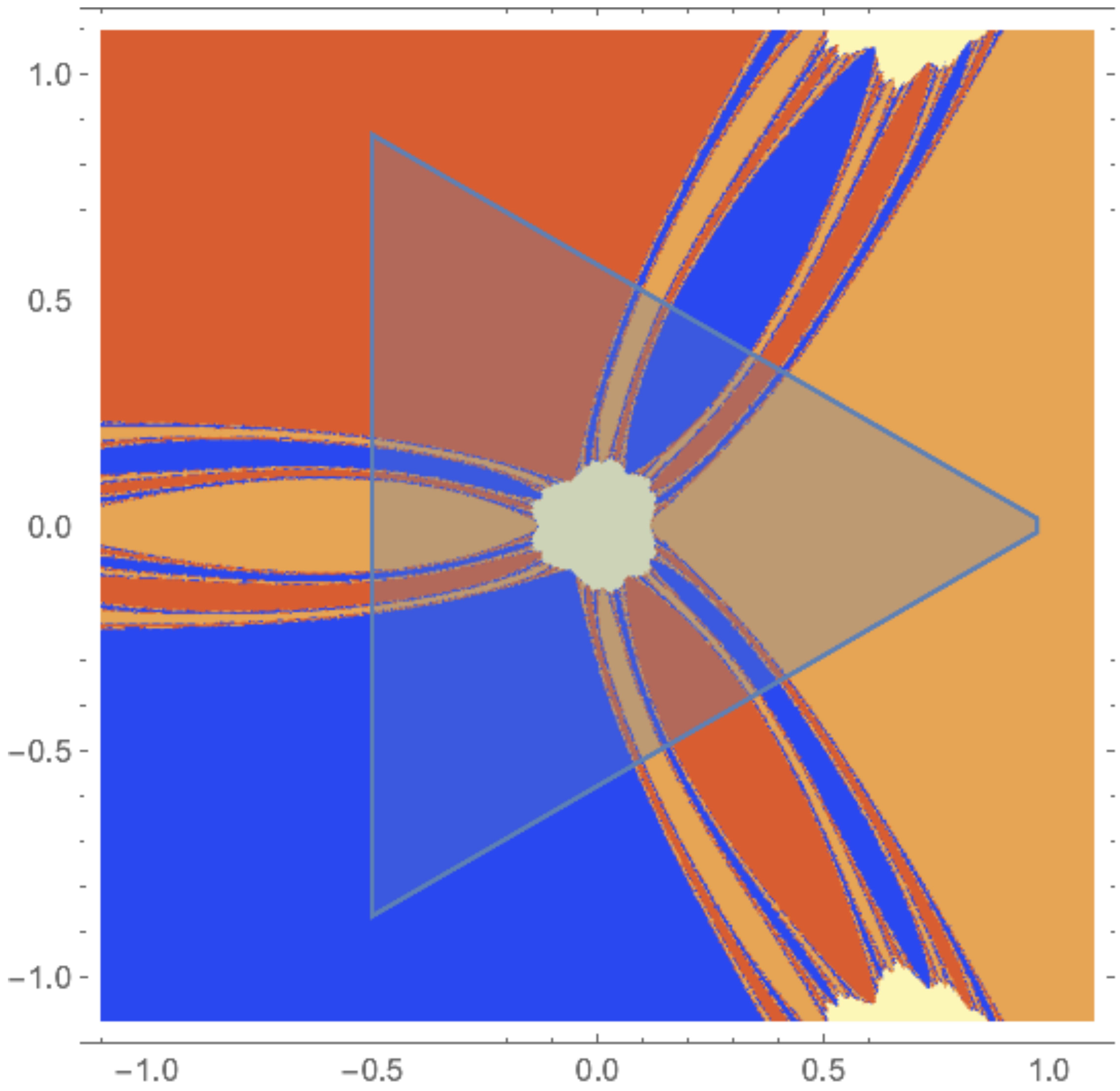
The Villain action Z(N) model has a simple dual form in all d.

$$J \to \tilde{J} = \frac{N^2}{4\pi^2 J} \qquad \mu \to \tilde{\mu} = -\frac{2\pi i J\mu}{N}$$

Meisinger and Ogilvie, 1306.1495, 1311.5515

The chiral model has an intricate low-temperature (large J) structure with patterned phases. These may be commensurate or incommensurate, depending on d. Lattice duality maps between classes of Hamiltonians, complex and real, with non-Hermitian transfer matrices. The 2d case is clear: we are looking at the universality class of 2d Z(N) parafermions and the patterned behavior in the chiral model corresponds to states with nonzero N-ality realized as kinks.

Complex $Z(N)$ models and the real-space renormalization group



Ogilvie, Schindler, Schindler, in preparation