

Gradient Properties of RG Flows

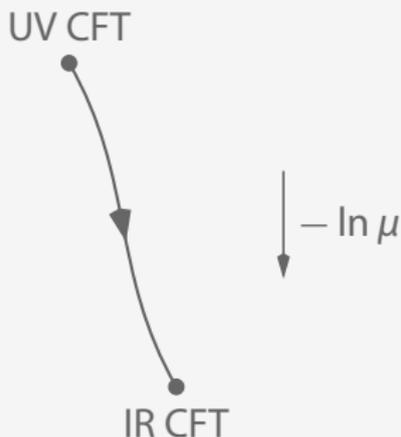
Andreas Stergiou



(Based on work with William Pannell, arXiv:2402.17817)

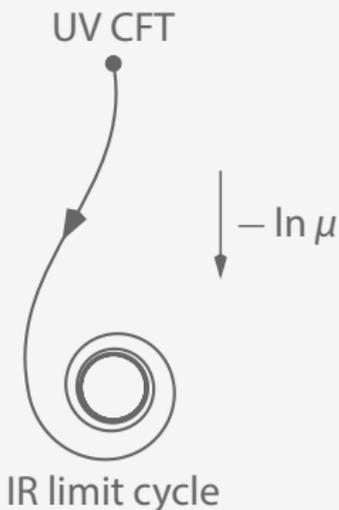
Renormalisation group flows

In relativistic QFT we typically think of RG flows as flows in theory space, with a particularly interesting case that of a CFT in the UV flowing to a CFT in the IR after a **relevant** deformation:



RG flow is described by **beta** functions. An essential constraint is **unitarity**.

Limit cycles



Limit cycles have been suggested as possible endpoints of RG flows in the early '70s by Wilson, but they have **never** been found in relativistic unitary QFT.

A certain type of limit cycle would arise for unitary theories that are scale-invariant but **not** conformal. (Fortin, Grinstein & AS, 2012)

a -theorem

Strongest version: Is there a Riemannian metric G_{IJ} in the space of couplings such that $\partial_I A = G_{IJ} \beta^J$ for some scalar A ?

Strong version: Is there a quantity that decreases monotonically in the flow from the UV to the IR?

Weak version: In the flow between a UV and an IR fixed point, is there a quantity a such that $a_{UV} > a_{IR}$?

A monotonically-decreasing quantity was found in $d = 2$ by Zamolodchikov in 1986.

At the RG flow endpoints it becomes the **central charge** of the corresponding CFT.

The RG flow in $d = 2$ is **gradient** in conformal perturbation theory.

a -theorem in $d = 4$

$$4d \text{ CFT in curved space: } T^\mu_\mu = aE_4 + cW^{\mu\nu\rho\sigma}W_{\mu\nu\rho\sigma}$$

It was suggested by Cardy in 1988 that the coefficient of the **Euler term** in the trace anomaly, called a , may be the quantity that satisfies a (weak) a -theorem in $d = 4$.

There have been lots of **successful** checks of Cardy's suggestion over the years.

A nice chain of arguments by Komargodski and Schwimmer proved the weak version of the a -theorem in 2011.

The relevant quantity is indeed a : $a_{UV} > a_{IR}$.

In perturbation theory, the **strong** version of the a -theorem was established by Jack and Osborn in 1990.

The quantity they considered also becomes a at fixed points.

Multiscalar models

We will be interested in the strongest version of the a -theorem.

We will first consider multiscalar theories in $d = 4$, defined by

$$\mathcal{L} = \frac{1}{2} \partial^\mu \varphi_i \partial_\mu \varphi_i + \frac{1}{2} \kappa_{ij} \varphi_i \varphi_j + \frac{1}{4!} \lambda_{ijkl} \varphi_i \varphi_j \varphi_k \varphi_l.$$

The beta function of the quartic coupling in these theories has been recently computed to **six** loops. (Bednyakov & Pikelner, 2021; using results of

Kompaniets & Panzer, 2017)

We want to see if

$$\beta^l = G^l \partial_l A, \quad \partial_l = \frac{\partial}{\partial \lambda^l}, \quad l = (ijkl).$$

We will also discuss this problem in $d = 4 - \varepsilon$.

The metric

We will demand that the metric G_{IJ} is Riemannian, i.e.

- symmetric, $G_{IJ} = G_{JI}$, and
- positive-definite, $G \succ 0$.

The symmetry property is necessary for one to be able to choose local coordinates such that, at a point,

$$G_{IJ} = \delta_{IJ} \quad \Rightarrow \quad \beta_I = \partial_I A,$$

and this implies that the eigenvalues of the matrix

$$\partial_I \beta_J = \partial_I \partial_J A,$$

are real at real fixed points and unitarity is maintained.

These eigenvalues determine scaling dimensions of quartic operators in φ at fixed points.

One loop

$${}^1\beta_{ijkl} = \lambda_{ijmn}\lambda_{mnkl} + \text{permutations} = \text{---} \circ \text{---}$$

Obviously

$${}^1A = \lambda_{ijkl}\lambda_{klmn}\lambda_{mnij} = \text{---} \circ \text{---}$$

$${}^1G_{ijkl;mnop} = \delta_{ijkl;mnop} = \text{---}$$

The solution is **not** unique as we can rescale A by any constant and the metric by the inverse of that constant.

Two loops

$${}^2\beta = {}^2b \text{ (diagram)} + {}^2c \text{ (diagram)}$$

$${}^2A = {}^2a_1 \text{ (diagram)} + {}^2a_2 \text{ (diagram)} + {}^2a_3 \text{ (diagram)}$$

$${}^2G = {}^2g \text{ (diagram)}$$

Demanding $\beta = G\partial A$ now gives

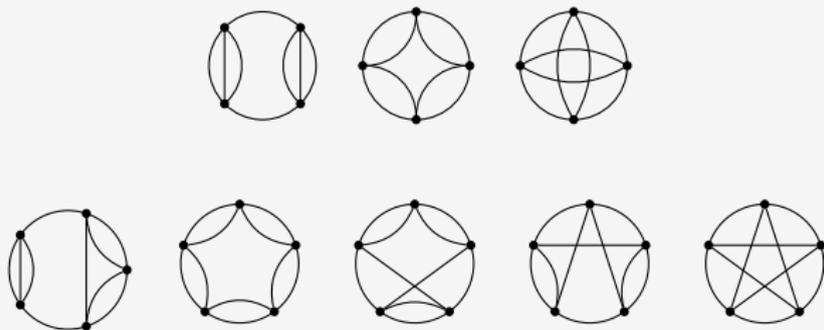
$$\begin{aligned} {}^2a_1 - {}^2c &= 0, \\ {}^2a_2 + \frac{1}{4} {}^1b {}^2g &= 0, \\ 2({}^2a_3) - 3({}^2b) + {}^1b {}^2g &= 0. \end{aligned}$$

These can be solved with ${}^2g = 0$.

Need for metric at three loops

At three loops it turns out that we need a **correction** to the metric to satisfy $\beta = G\partial A$. (Wallace & Zia, 1974)

Contrast diagrams needed at two vs. three loops:



Diagrams with inequivalent vertices give rise to **distinct** contributions to the beta function.

If those beta function contributions have **incompatible** coefficients, then only a metric can restore the gradient property.

Four loops and beyond

	1-loop	2-loop	3-loop	4-loop	5-loop	6-loop
A	1	3	5	17	42	177
G_{IJ}	1	1	7	18	97	453
Equations	1	3	10	36	164	819

Starting at four loops, it appears that we don't have enough **freedom** to satisfy all equations. At this order naive counting shows that we do, in fact, as some freedom remains from **undetermined** lower order coefficients.

Naive counting fails, however, as some of our degrees of freedom appear **only** in specific combinations.

At four loops $\beta = G\partial A$ cannot be satisfied unless the beta function coefficients satisfy **four** constraints. (Jack & Poole, 2018)

They **do**, so the flow is gradient at four loops.

Scheme dependence

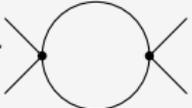
Coefficients in the beta function beyond leading order are **scheme-dependent**.

When a Feynman diagram has **subdivergences**, these need to be subtracted.

Of course one may decide to subtract subdivergences in different ways, i.e. one may choose to remove **finite** parts along with the divergences.

Such choices cannot possibly affect **physical** quantities or statements.

In the present context, scheme changes are described by coupling **redefinitions**:

$$\lambda \rightarrow \lambda + r \langle \text{circle diagram} \rangle + O(\lambda^3).$$


Relations like the ones satisfied by the four-loop beta function coefficients are **scheme-independent**.

Results at five loops

At **five** loops there are 110 beta function terms, 42 bubble diagrams and 97 contributions to the symmetric metric.

We found **37** constraints among beta function coefficients that need to be satisfied for the flow to be gradient.

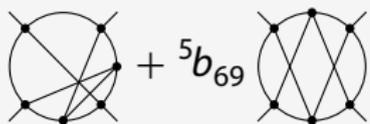
Some are simple, but most appear rather **complicated**, e.g.

$$\begin{aligned} & \beta_{116} + 2 \beta_{136} + 2 \beta_{137} + \beta_{13} \beta_3 + 2 \beta_{22} \beta_3 + 2 \beta_{37} + \frac{\beta_4^2}{4} + \beta_{40} \\ & + \beta_{10} \left(2 \beta_3^2 + \beta_4 + \frac{\beta_5}{3} \right) + \frac{\beta_3^2 \beta_5}{3} + \beta_{52} + \beta_{63} + \frac{\beta_{64}}{2} + \frac{\beta_{65}}{2} + 4 \beta_8^2 = \\ & 2 \beta_{10}^2 + 2 \beta_{113} + 4 \beta_{138} + 2 \beta_{140} + 2 \beta_{25} \beta_3 + 4 \beta_3 \beta_{30} + 2 \beta_3 \beta_{32} \\ & + \frac{\beta_3^2 \beta_4}{2} + 2 \beta_{42} + \beta_{44} + \frac{\beta_{47}}{2} + \frac{\beta_4 \beta_5}{6} + \frac{\beta_{51}}{2} + \beta_{54} + \beta_{56} + \beta_{68} + \frac{5 \beta_4 \beta_8}{2} \end{aligned}$$

They are all scheme-independent and satisfied!

A simple constraint at five loops

Consider the primitive diagrams

$${}^5\beta \supset {}^5b_{67} \text{ (diagram)} + {}^5b_{69} \text{ (diagram)} .$$


It turns out that these diagrams both come from

$${}^5A \supset {}^5a_{42} \text{ (diagram)}$$


and there is **no** other way to generate them in $G\partial A$.

This means that

$${}^5b_{67} = {}^5a_{42} \quad \text{and} \quad {}^5b_{69} = {}^5a_{42} \quad \Rightarrow \quad {}^5b_{67} = {}^5b_{69} .$$

This is satisfied since both coefficients are equal to $36\zeta_3^2$.

Results at six loops

At **six** loops there are 571 beta function terms, 177 bubble diagrams and 453 contributions to the symmetric metric.

We found **234** constraints among beta function coefficients that need to be satisfied for the flow to be gradient.

Again, few are simple, and most appear rather complicated.

They are **all** scheme-independent.

All **but 5** are satisfied.

This constitutes a direct **proof** that the beta function is **not** a gradient vector field.

But is the **RG flow** gradient or not?

Renormalisation subtleties

In dim-reg, beta functions arise from the **simple** poles of counterterms introduced via

$$\varphi_B = \mu^{-\varepsilon/2} Z^{1/2} \varphi_R, \quad \lambda_B = \mu^\varepsilon (\lambda_R + L).$$

But there is a **subtlety**: we never actually compute $Z^{1/2}$. Rather, we compute $(Z^{1/2})^T Z^{1/2}$, e.g. via



But this is **ambiguous**, because one can introduce an arbitrary orthogonal matrix between $(Z^{1/2})^T$ and $Z^{1/2}$. Then,

$$\gamma \rightarrow \gamma - \omega, \quad \beta_I \rightarrow \beta_I + (\omega\lambda)_I,$$

with ω in the Lie algebra of the orthogonal group.

Usually this subtlety is implicitly resolved by requiring that the anomalous dimension matrix be *symmetric*.

Nevertheless, physical statements should not depend on such choices.

It immediately follows that the beta function is *not* the vector field whose gradient properties we should be checking.

What is the *correct* vector field?

Jack & Osborn gave the answer in 1990.

The problematic equation turns out to be

$$T^{\mu}_{\mu} = \beta^I \mathcal{O}_I.$$

Trace of stress-energy tensor

This is true **only** for zero-momentum insertions of $T^\mu{}_\mu$. More generally,

$$T^\mu{}_\mu = \beta^l \mathcal{O}_l + \partial_\mu J^\mu.$$

If we have scalar fields, for example, $J^\mu = S_{ij} \varphi_i \partial^\mu \varphi_j$, with S_{ij} anti-symmetric.

Using the equations of motion we see then that

$$T^\mu{}_\mu = B^l \mathcal{O}_l, \quad B^l = \beta^l - (S\lambda)^l.$$

S can be computed in perturbation theory, e.g. using dim-reg.

B^l is **invariant** since $\beta^l \rightarrow \beta^l + (\omega\lambda)^l$ and $S \rightarrow S + \omega$.

The **lesson** is that the RG flow is governed by the B function, not the beta function.

How to compute S

This S is perhaps **unfamiliar**.

It appears due to the need to **renormalise** correlation functions involving both T^μ_μ and \mathcal{O}_I .

The renormalisation of such correlation functions can be **systematised** by considering **local** couplings, $\lambda \rightarrow \lambda(x)$, in addition to considering the theory in curved space.

New counterterms are then **necessary**, to cancel new divergences that arise in this background, e.g.

$$\mathcal{L}_{\text{c.t.}} \supset (N_I)_{ij} \partial^\mu \lambda^I \varphi_i \partial_\mu \varphi_j, \quad N_I = \sum_n \frac{N_I^{(n)}}{\epsilon^n}.$$

A careful treatment shows that

$$S_{ij} = (N_I^{(1)})_{ij} \lambda_I.$$

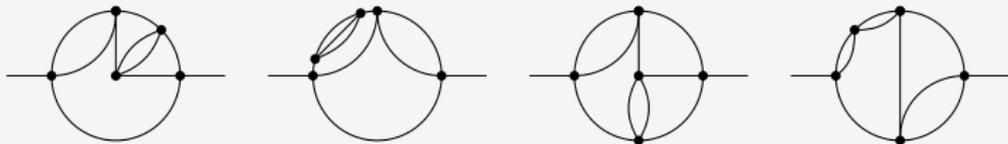
How to compute S

Diagrammatically, to determine N we have to look for diagrams that contribute to the anomalous dimension of the field, but we need to compute them allowing momentum to **flow out** of the coupling, e.g.



This specific diagram gives contributions to N , but not S . This is because it is **symmetric** under exchange of the external legs.

In multiscalar models, the first potential contribution to S is at **five** loops. There are **four** relevant asymmetric diagrams:



From β to B

At five loops we had found 37 constraints among beta function coefficients that need to be satisfied for the flow to be gradient.

It turns out that the four contributions to S at five-loop order do **not** affect these constraint equations.

At six loops there are 19 contributions to S .

We had found that the beta function was **not** a gradient vector field.

After including the five- and six-loop S , we find that with **four** conditions on S the B vector field is indeed gradient:

$${}^5s_2 = \frac{259}{4608},$$

$$192 {}^5s_3 - 384 {}^5s_4 = 31 - 36\zeta_3,$$

$${}^5s_1 - 12 {}^5s_2 - {}^5s_4 = 12 {}^6s_8 - 6 {}^6s_9 - 6 {}^6s_{16},$$

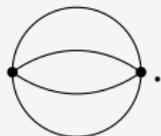
$$4 {}^5s_1 - 2 {}^5s_3 = -2 {}^6s_3 + {}^6s_6 + 4 {}^6s_{10} - 2 {}^6s_{15} + {}^6s_{17}.$$

4 – ϵ dimensions

In $d = 4 - \epsilon$ the beta function has **one** more term, linear in λ :

$$\beta_{ijkl} = -\epsilon \lambda_{ijkl} + O(\lambda^2).$$

This term is easy to accommodate via the bubble



However, such a contribution to A will then get multiplied with terms from the metric and will start appearing **all over the place** at higher orders.

These contributions can be absorbed in **order- ϵ** terms in A and G_{IJ} .

Both are needed, and then ϵ^2 terms need to **cancel**. This sets **further** constraints among beta function coefficients at five and six loops.

All of them (4 at five and 32 at six loops) are scheme-independent, **independent** of the constraints we found in $d = 4$ and **satisfied**!

Conclusion

Perturbative computations in a variety of contexts **support** a gradient property for the RG flow.

This is often highly **non-trivial** and follows from relations among beta function coefficients.

In the highest-loop case, the gradient property of the RG flow **rests** on relations among five- and six-loop contributions to S .

It is crucial to identify the **correct** vector field that describes the RG flow.

As a **future** direction we may consider the gradient question in theories with scalars, fermions and gauge fields in $d = 4$ and $d = 4 - \epsilon$. Some work already exists in the literature.