Integrability in Non-Hermitian Physics **and Field Theories**

Francisco Correa Universidad de Santiago de Chile

In collaboration with : M. Cárdenas, J. Cen, A. Fring, V. Jakubský, K. Lara, M. Pino, M. Plyushchay and T. Taira.

Applications of Field Theory to Hermitian and Non-Hermitian Systems

September 10-13, 2024

Integrable systems & Gravity

Inverse scattering method

Einstein-Maxwell equations

Gravity-Electromagnetism

-Cosmological models

-Kerr black hole

-Collision of exact gravity waves

-Schwarzschild black hole

And more...

-Taub-NUT

Universality in Binary Black Hole Dynamics: An Integrability Conjecture J.L. Jaramillo, B. Krishnan, C. F. Sopuerta, **arXiv:2305.08554**

Gravitational Solitons

VLADIMIR BELINSKI ENRIC VERDAGUER

ON MATHEMATICAL PHYSICS

C. Klein O. Richter

Ernst Equation and **Riemann Surfaces**

Analytical and Numerical Methods

quahlo cyctome & Eiold Thoovies **Integrable systems & Field Theories**

Guantum Chromodynamics & self-interacti els in 1+1 dimensions described by the Lagrangians [1, 2], ϵ equal the quantum-like ϵ *<u>Quantum Chromodynamics</u>* & Sen-mitera $\frac{1}{2}$ iermuons $\frac{1}{2}$ Quantum Chromodynamics & self-interacting fermions

$$
\mathcal{L}_{\rm GN} = \bar{\psi} i \partial \psi + \frac{g^2}{2} (\bar{\psi} \psi)^2
$$
 Gross-Neveu mo

, Gross-Neveu mode **Gross-Neveu model**

"2

$$
\mathcal{L}_{\rm NJL} = \bar{\psi} i \partial \psi + \frac{g^2}{2} \left[\left(\bar{\psi} \psi \right)^2 + \left(\bar{\psi} i \gamma_5 \psi \right)^2 \right] \text{ Nambu}
$$

where me omit the sum over the *N* flavours. These mod-

#!

 $Nambu-Jona-Lasinio model$ [∆] ⁼ [−]*^g*² 2 ona-Lasinio model **Nambu-Jona-Lasinio model**

q wake corresponding q . Cald the selfconsistency conditions. namical mass generation. Also represent the chiral sym-Integrable systems & Field Theories Recently, Dunne and Thies have found a generic class r_{min} - r_{min} and r_{min} and r_{min} and r_{min} and r_{min} we relate compute the superpotential from the superpotential fro **Integrable systems & Field Theories**

cosh

Guantum Chromodynamics & self-interacti els in 1+1 dimensions described by the Lagrangians [1, 2], ϵ equal the quantum-like ϵ *Quantum Chromodynamics & seif-interal* $\frac{1}{2}$ iermuons $\frac{1}{2}$ of time dependent solutions \mathbb{S}^3 , which generates \mathbb{S}^3 f interacting *W*+(*x*) = **Thromodynamics & self-interacting fern** ⁺ ^p3*^x* Quantum Chromodynamics & self-interacting fermions

$$
\mathcal{L}_{\text{GN}} = \bar{\psi} i \partial \psi + \frac{g^2}{2} (\bar{\psi} \psi)^2
$$
 Gross-Neveu model

and Nitta Editor and Nitta Editor Case, it is known in the time independent case, it is known in the time in

#!

where me omit the sum over the *N* flavours. These mod-

2

Gross-Neveu (GN) and Nambu-Jona-Lasinio (NJL) mod-

2

⇥

, Gross-Neveu mode α in the GN case, α in the GN case, α in the GN case, α **Gross-Neveu model**

"2

When shifting the overall energy by 3, this is an inverse cosmic the overall energy by 3, this is an inverse co
2-potential discussed in the overall discussed in the overall discussed in the overall except in the overall d

$$
\mathcal{L}_{\rm NJL} = \bar{\psi} i \partial \psi + \frac{g^2}{2} \left[\left(\bar{\psi} \psi \right)^2 + \left(\bar{\psi} i \gamma_5 \psi \right)^2 \right]
$$
Nambu-Jona-Lasinio model

Relativistic Hartree-Fock problem
$$
H\psi = E\psi
$$
 $H = \begin{pmatrix} -i\frac{d}{dx} & \Delta(x) \\ \Delta^*(x) & i\frac{d}{dx} \end{pmatrix}$

Consistency condition $\langle \psi \psi \rangle = 0$ Consistency condition

Consistency condition
$$
\langle \bar{\psi}\psi \rangle - i \langle \bar{\psi} i \gamma^5 \psi \rangle = -\Delta/g^2
$$

and Nitta Edition and that G. Dunne, G. Basar (2008) **The nonlinear Schro** [∆](*x*) = [−]*iNg*²tr*D,E* eralize the time-independent construction of Takahashi and Takahashi and Takahashi and Takahashi and Takahashi
Takahashi and Takahashi an **and Nitta and Nitta External The nonlinear Schrödinger equation** condensates \mathbf{B} de General (Bogoliubov- de Hamiltonian and insert- σ . Danne, σ . Dasar (2000) **THE HOTHER** find the exact analytical form of the resolvent operator is $n \nabla$ sk. Following the lines presenting presented in Γ nonimear schroainger equation the eigenfunction for the energy *^E* with potential *^U* ⁼ *U*⁰ sech2(↵*x*), ↵*, U*⁰ ² ^R as ¹ tanh2(↵*x*) 1 G. Dunne, G. Basar (2008) **The nonlinear Schrödinger equation**✏ *s, s* + ✏ + 1; ✏ + 1; **ility**

G. Dunne, G. Basar (2008)

Integrability & non-Hermitian physics PT-SYMMETRY

Classical and quantum many particle models

Solitons and non-linear integrable equations

Quantum spin chains

Integrability & non-Hermitian physics PT-SYMMETRY

Classical and quantum many particle models

Solitons and non-linear integrable equations

Quantum spin chains

A. Fring, Phil. Trans. R. Soc. A (2012)

Integrability & non-Hermitian physics

PRL 100, 030402 (2008)

PHYSICAL REVIEW LETTERS

week ending 25 JANUARY 2008

Optical Solitons in PT Periodic Potentials

Z. H. Musslimani

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K. G. Makris, R. El-Ganainy, and D. N. Christodoulides

College of Optics & Photonics-CREOL, University of Central Florida, Orlando, Florida 32816, USA (Received 1 September 2007; revised manuscript received 24 October 2007; published 23 January 2008)

We investigate the effect of nonlinearity on beam dynamics in parity-time (\mathcal{PT}) symmetric potentials. We show that a novel class of one- and two-dimensional nonlinear self-trapped modes can exist in optical PT synthetic lattices. These solitons are shown to be stable over a wide range of potential parameters. The transverse power flow within these complex solitons is also examined.

 $i\frac{\partial \psi}{\partial z} + \frac{\partial^2 \psi}{\partial x^2} + [V(x) + iW(x)]\psi + |\psi|^2 \psi = 0$

Integrability & non-Hermitian physics

PRL 110, 064105 (2013)

PHYSICAL REVIEW LETTERS

week ending 8 FEBRUARY 2013

Integrable Nonlocal Nonlinear Schrödinger Equation

Mark J. Ablowitz¹ and Ziad H. Musslimani²

¹Department of Applied Mathematics, University of Colorado, Campus Box 526, Boulder, Colorado 80309-0526 ²Department of Mathematics, Florida State University, Tallahassee, Florida 32306-4510 (Received 22 August 2012; published 7 February 2013)

A new integrable nonlocal nonlinear Schrödinger equation is introduced. It possesses a Lax pair and an infinite number of conservation laws and is PT symmetric. The inverse scattering transform and scattering data with suitable symmetries are discussed. A method to find pure soliton solutions is given. An explicit breathing one soliton solution is found. Key properties are discussed and contrasted with the classical nonlinear Schrödinger equation.

Outline

2 **Features of integrable non-Hermitian field theories**

Outline

1 **Integrability and non-Hermitian physics via a simple example**

Solitons and integrable equations Here we consider in detail one of the prototype nonlinear wave equations, the Korteweg**de Vries (20), sources in the complex field in the complex field** $\boldsymbol{\mu}$

$$
u_t + 6uu_x + u_{xxx} = 0
$$

Korteweg-de Vries equation

https://youtu.be/wEbYELtGZwI

Expected the interdisciplinaire CARNOT de Bourgogne, Équipe Solit symmetry is realized as *PT* : *x* → −*x*, *t* → −*t*, *i* → −*i*, *u* → *u*, leaving (1.1) invariant. As Laboratoire Interdisciplinaire CARNOT de Bourgogne, Équipe Solitons, Laser et Communications optiques

Solitons and integrable equations Here we consider in detail one of the prototype nonlinear wave equations, the Korteweg**de Vries (20), sources (Compact 19), sources in the complex field in the complex field of the complex field in the complex field** $\boldsymbol{\mu}$

$$
u_t + 6uu_x + u_{xxx} = 0
$$

Korteweg-de Vries equation

Two soliton

Solitons and integrable equations Here we consider in detail one of the prototype nonlinear wave equations, the Korteweg**de Vries (20), sources (Compact 19), sources in the complex field in the complex field of the complex field in the complex field** $\boldsymbol{\mu}$ $F - \frac{1}{2}$ $F - \frac{1}{2}$ ∂*u* = 6*u*² [∂]*^u* ∂*t* ∂*t* ∂*t*

• Conserved quantities are functions of the velocities itities are functions of the velocities

−∞ *H*[*u*(*x, t*)]*dx* = **• Total energy is the sum of individual solitons with different lenergies** • There is no a concept of a two-soliton with twice one soliton energy.... symmetry is realized as *PT* : *x* → −*x*, *t* → −*t*, *i* → −*i*, *u* → *u*, leaving (1.1) invariant. As **Mass** $m =$ nee $-\infty$ udx $^{\circ}$ f in \overline{v} u^2dx , H <u>wij</u> −∞ $\frac{du}{dx}$ $\frac{du}{dx}$ Mass Momentum Energy **gy** $p =$ 14fV $-\infty$ u^2dx , solitons w $\n *th*\n$ \int $H = E = \int dx \left(\frac{du}{2} \right) \frac{du}{dx} = u^3$ these quantities are real as they are meant to be physical, i.e. observable. **Energy** 1iffe −∞ $\frac{d\mathbf{r}}{dx}$ $\frac{1}{2}$ $\frac{du}{dx}$ $\hat{\mathbf{p}}$ $\overline{ }$!∂*u* ∂*x* \textbf{I} **itons with different lenergies** u^3 −∞ *u* different 1 2 !∂*u* ∂*x* $\dot{\mathbf{a}}$ ies $\frac{1}{2}$ = $\frac{1}{2}$ $\frac{1}{2}$ $-\infty$ $\frac{d}{dx}$ **t** le per $\frac{d}{dx}$ $\overrightarrow{\mathbf{g}}$ $\overline{1}$!∂*u* ∂*x* $\left(\frac{1}{2} \right)$ $H^{\text{c}} = E^{\text{c}} = \frac{1}{2}$ $v - q$ $\frac{d}{dx}$ **d** $\frac{d}{dx}$ $\frac{d}{dx}$ $\frac{d}{dx}$ **u** \overline{x} **a**iẽs $\left(\frac{1}{2} \right)$ $\mathop{{\bf Herer}}\limits^{*\!\bullet}d x\Big\vert \frac{1}{2}\Big\vert \frac{d\phi}{2}$ −∞ $\frac{u}{x}$ $\sum_{l=0}^{n}$

Solitons and integrable equations Here we consider in detail one of the prototype nonlinear wave equations, the Korteweg**de Vries (20), sources (Compact 19), sources in the complex field in the complex field of the complex field in the complex field** $\boldsymbol{\mu}$ **E** ! ∞ *H*[*u*(*x, t*)]*dx* =

$$
u_t + 6uu_x + u_{xxx} = 0
$$

depending on the *t* and space *x* and *t* and *x*. This equation is known to and *t* and *x*. This is known to and *t* and *x*. This is known to and *x*. This is known to and *x*. This is known to and *x*. This is known t Korteweg-de Vries equation

KdV equation remains invariant under

$$
\mathcal{PT}\colon\, x\to -x,\, t\to -t,\, i\to -i,\, u\to u
$$

Here we consider in detail one of the prototype nonlinear wave equations, the Korteweg**de Vries (Complex 501110115 and integrapie
KdV equation remain comple** \mathbf{x} cosh² ^η cos² ^θ*^a* [−] sinh² ^η sin² ^θ*^a* ⁺ ⁱ **(a)** integrable equations **Complex Solitons and integrable equations** \cdot \cdot \cdot *H*[*u*(*x, t*)]*dx* = *ations*

ux

$$
u_t + 6uu_x + u_{xxx} = 0
$$

depending on the *t* and space *x* and *t* and *x*. This equation is known to and *t* and *x*. This is known to and *t* and *x*. This is known to and *x*. This is known to and *x*. This is known to and *x*. This is known t Korteweg-de Vries equation $\frac{1}{2}$ biteweg-de vries equation

 $u = p -$

₂
2015 invariant 11nder
2016 in 2θ^a

$$
+ 6uux + uxxx = 0
$$

At v equation remains invariant under
 $\mathcal{PT}: x \to -x, t \to -t, i \to -i, u \to u$
between de Vries equation

^x (1.2)

p even function even function

 $u = p + iq$

q^{*x*} + *d*^{*x*} + 6 (*d*)*x* + 6 (*p*)*x* + 6 (*p u* odd function odd function $\overline{\Omega}$ and function that allows to implement degeneracies into Darboux-Crum transformations. We show that

" A. Khare, A. Saxena PLA (2016) *A*. Khare, A. Saxena PLA (2016)

1

 ι ²

2

HJ. Cen, A $\overline{ }$ *M. Knare, A. Saxena PLA (2016)*
J. Cen, A. Fring , JPhA (2016)

remains real despite the fact that the Hamiltonian density is complex [20]. The *PT p*_{*x*} HEP (2016) *q*_{*x*} + *x*^{*x*} + *x* F.C, A. Fring, JHEP (2016)

Rogue wave type

$$
u_t + 6uu_x + u_{xxx} = 0
$$

Korteweg-de Vries equation

$$
u = p + iq
$$

H.C. A. Fring, JHEP (2016) F.C, A. Fring, JHEP (2016)

1

u_x
 u^{*x*} **• Real conserved quantities (infinitely many)** *p^t* + *pxxx* + 6*pp^x* − 6*qq^x* = 0 (1.4)

remains real despite the fact that the Hamiltonian density is complex [20]. The *PT* - • Energies could be n times one soliton energy [©] © © ©

But if the KdV equation is so well known, how could this have gone unnoticed?

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V. Matveev, JMP (1994)

Singular solutions with infinite (divergent) energies

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Singular solutions with infinite (divergent) energies

But if the KdV equation is so well known, how could this have gone unnoticed?

V. Matveev, JMP (1994)

Singular solutions with infinite (divergent) energies

PT-SYMMETRY

Non-physical solitons are regularized: removing singularities and making charges finite!

An additional symmetry ensures the reality of the integrals of motion

All conserved charges are real, even though solitons are complex

F.C, A. Fring, JHEP (2016) J. Cen, F.C, A. Fring, Annals of Phys. (2017)

Why these solitons are relevant ???

Solitons + QM

Solitons + QM

M. Kac "Can one hear the shape of a drum ?"

Solitons + QM

M. Kac "Can one hear the eigenvalues of a potential ?"

Solitons + QM

Reflectionless potentials for all energies!

Why these potentials are interesting?

The Schrödinger equation

$$
i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x) \psi
$$

Formal coincidence

The Schrödinger equation

$$
\left(i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi\right)
$$

Formal coincidence

$$
i\frac{\partial E}{\partial z} = \frac{1}{2k}\frac{\partial^2 E}{\partial x^2} + k_0 n(x) E
$$

The paraxial equation of diffraction

The Schrödinger equation
\n
$$
\begin{aligned}\n\left(i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x) \psi \right) & \text{A complex potential} \\
\text{Formal coincidence} \\
\left(i\frac{\partial E}{\partial z} = \frac{1}{2k} \frac{\partial^2 E}{\partial x^2} + k_0 n(x) E \right) & \text{A refractive index with gain and loss}\n\end{aligned}
$$

The paraxial equation of diffraction

The Schrödinger equation

$$
i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x) \psi
$$

The probability density

Formal coincidence

$$
\biguparrow
$$

$$
i\frac{\partial E}{\partial z} = \frac{1}{2k}\frac{\partial^2 E}{\partial x^2} + k_0 n(x) E
$$

The power intensity

The paraxial equation of diffraction

 $|E(x, z)|$ The path of a beam of light through a material $|E(x, z)|^2$

F. C., V. Jakubsky, M. Plyushchay, PRA (2015)

 $|E(x, z)|$ The path of a beam of light through a material $|E(x, z)|^2$

Reflectionless but detectable, non trivial phase shift

F. C., $\ket{9}$ 5)

α *ditons* + Ω ^{*N*} the real (imaginary) part. On the right: intensity *[|]*ψ˜(*x, z*)*[|]* **Complex Solitons + QA + office**

the contract of the contract of

|*E*(*x*,*z*)| ⁰. The periodic system (d) has no bound states. In ^ψ˜(*x, z*), we fix ^σ = 2, *^x*⁰ ⁼ [−]50 and The path of a beam of light through a_{material}

(c) *E*⁰ = *k*²

20

2

discussed in section 5.1. The solid blue (dashed red) line represents the solid blue (dashed red) line represents

100

² of the light beam when propagating

(d) ψ⁰ = *eik*0*^x*

through the optical potential. There is a bound state of energy with (a) *^E*⁰ ⁼ [−]*k*²

Figure 2: Reflectionless optical potentials. On the left: *P T*-symmetric potential *V*˜ (*x*) =

as in (b) of Fig. 2. The two main reflections that appear about *x* = 0 correspond to those produced

Invisible potentials in both directions z $\frac{1}{2}$ visible potentials <u>in both directions</u>

 $F.$ C., (2015) by each potential well, see (b), $\mathbf{F}_{\mathbf{z}}$ (
Complex Solitons + QM + optics

the contract of the contract of

-

 \int_{0}^{0} $\left(\frac{50}{2}\right)$ 2 The path of a beam of light through a material

zv

Invisible optical crystal with a bound state in the continuum (BIC)

18

F. C., V. Jakubsky, M. Plyushchay, PRA (2015)

Outline

2 **Features of integrable non-Hermitian field theories**

Integrability & non-Hermitian physics

PRL 100, 030402 (2008)

PHYSICAL REVIEW LETTERS

week ending 25 JANUARY 2008

Optical Solitons in \mathcal{PT} Periodic Potentials

Z. H. Musslimani

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K. G. Makris, R. El-Ganainy, and D. N. Christodoulides

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 $i\frac{\partial \psi}{\partial z} + \frac{\partial^2 \psi}{\partial x^2} + [V(x) + iW(x)]\psi + |\psi|^2 \psi = 0$

The nonlinear Schrödinger equation

$$
iq_t + \alpha \left(q_{xx} - 2\kappa |q|^2 q \right) = 0
$$

The nonlinear Schrödinger equation

$$
iq_t + \alpha \left(q_{xx} - 2\kappa |q|^2 q \right) = 0
$$

Propagation of light in fiber optics

The nonlinear Schrödinger equation

$$
iq_t + \alpha \left(q_{xx} - 2\kappa |q|^2 q \right) = 0
$$

and many, many more applications...

- Water waves
- Plasma physics • Vater waves
• Plasma physics
- Bose-Einstein condensates and Bose-Einstein condensates
- Superconductivity For Superconductivity
• Gravity
	- Gravity
- Classical and quantum field theory
- Non-Hermitian physics equation (2.16) reduces to the complex modified Korteweg de-Vries with conjugate (2.17).

The nonline The nonlinear Schrödinger equation

$$
iq_t + \alpha \left(q_{xx} - 2\kappa |q|^2 q \right) = 0
$$

What is the origin of this equation?

Lax Pair formalism

Abstraction

$$
\iff
$$

$$
\Psi_t = V\Psi, \Psi_x = U\Psi \qquad \Longleftrightarrow \qquad \partial_t U - \partial_x V + [U, V] = 0
$$

$$
\Psi_{tx} = \Psi_{xt}
$$

the linear first order order order di Zakharov and Shabat (1972) zero curvature formalism *Lax Pair formalism* \iff *Zelo culvatule formalism* Z *Zakharov and Shabat (1972) ^t* = ↵ Eax Fan Formanship is the Known High Mac 2.16 is the Known and Shabat (1972)

$$
\partial_t U - \partial_x V + [U, V] = 0
$$

For a concrete model of the equations of the value of the value of the equations $U = \frac{1}{2}$ For a concrete model these equation have to hold up to the validity of the equation of $\Psi_{tx} = \Psi_{xt}$ Gauge field equations \qquad W is defined by $\mathcal{E}(\mathcal{E})$ reduces to the NLSE with conjugate (2.17) and (2.17) a There is a vast literature on quantum Calogero (or Calogero–Moser–Sutherland) models as a paradigm

Ax(*x, t*) = *q*(*x, t*)*C*(*x, t*) *r*(*x, t*)*B*(*x, t*)*,* (2.3) *Ax*(*x, t*) = *q*(*x, t*)*C*(*x, t*) *r*(*x, t*)*B*(*x, t*)*,* (2.3) *Ax*(*x, t*) = *q*(*x, t*)*C*(*x, t*) *r*(*x, t*)*B*(*x, t*)*,* (2.3) *C* = *i*↵*r^x* + 2↵*r* + ²*qr*² *^rxx* ²*ir^x* + 42*^r* The nonlinear Schrödinae The nonlinear Schrödinger equation Γ is not straightforward or obvious how to solve the eigenvalue equation for the potentials for the potentials of the potentials **The nonlinear Schrödinger equation**

The standard choice to achieve compatibility between (2.11) and (2.12) is to take *r*(*x, t*) = **q** the normical schroamger equation **The nonlinear Schrödinger equation**

Or What is the origin of this equation? *Interpretations*

 $\frac{1}{\sqrt{2\pi}}$ Lax Pair formalism

$$
\Leftrightarrow
$$

 \rightarrow 2ero *q i qxxx* 6 *|q|* **Example 1972 Zakharov and Shabat (1972)** zero curvature formalism \downarrow \downarrow

$$
\Psi_t = V \Psi, \Psi_x = U \Psi \qquad \Longleftrightarrow \qquad \partial_t U - \partial_x V + [U, V] = 0
$$

xx 2 *|q|* $\partial_t U - \partial_x V + [U, V] = 0$ \mathcal{C}_t and \mathcal{C}_t and \mathcal{C}_t is the value of the equation of $\mathcal{C}_$

$$
iq_t + \alpha \left(q_{xx} - 2\kappa |q|^2 q \right) = 0
$$

the Abiown2-reop-rewen-segor (ARTS) first ordiny *Integrable nonlocal Hirota equations* **The Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy**

Integral What is the origin of this equation?

Lax Pair formalism

$$
\iff
$$

$$
\Psi_t = V\Psi, \Psi_x = U\Psi \qquad \Longleftrightarrow \qquad \partial_t U - \partial_x V + [U, V] = 0
$$

$$
U = \begin{pmatrix} -i\lambda & q(x,t) \\ r(x,t) & i\lambda \end{pmatrix}
$$
 Exercise 16.1 Exercise 26.1 Exercise 36.1 Exercise 4.1 Exercise 4.

the linear function of the *Zakharov* and Shabat (1972) zero curvature formalism \downarrow \downarrow

$$
\Psi_t = V\Psi, \Psi_x = U\Psi \qquad \Longleftrightarrow \qquad \partial_t U - \partial_x V + [U, V] = 0
$$

- $Sine-Cordon$ For a concrete model these equation have to hold up to the validity of the equation of motion. When taking the matrix values of the matrix \sim Sine-Gordon \mathbf{e} • Sine-Gordon
	- *, V* \cdot X *A*(*x, t*) *B*(*x, t*) C ^{*x*}(*x*) *a*^{α} • Korteweg de Vries (KdV)
		- *U* = $\frac{1}{1}$ • mKdV
		- *r*(*x, t*) *i C*(*x*) Non-Imear Schrodinger
C(*X*) *A*irota • Non-linear Schrödinger
	- involving the constant spectral parameter \mathbf{r} parameter \mathbf{r} and \mathbf{r} \mathbf{r} arbitrary functions \mathbf{r} and \mathbf{r} arbitrary functions \mathbf{r} and \mathbf{r} arbitrary functions \mathbf{r} and \mathbf{r} arbi • Hirota

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$$
\Leftrightarrow
$$

 \rightarrow 2ero *q i qxxx* 6 *|q|* **Example 1972 Zakharov and Shabat (1972)** zero curvature formalism \downarrow \downarrow

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\Psi_t = V \Psi, \Psi_x = U \Psi \qquad \Longleftrightarrow \qquad \partial_t U - \partial_x V + [U, V] = 0
$$

xx 2 *|q|* $\partial_t U - \partial_x V + [U, V] = 0$ \mathcal{C}_t and \mathcal{C}_t and \mathcal{C}_t is the value of the equation of $\mathcal{C}_$

$$
iq_t + \alpha \left(q_{xx} - 2\kappa |q|^2 q \right) = 0
$$

The nonlocal nonlinear Schrödinger equation *rhe nonlocal nonlinear Schrödings ,* (2.6) ²*q*2*^r ^qxx* + 2*iq^x* + 42*^q ,* (2.7)

PRL 110, 064105 (2013)

 $PHYSICAL$ REVIEW LETTERS

week ending
8 FEBRUARY 2013

Integrable Nonlocal Nonlinear Schrödinger Equation

when *r*(*x*) and *r*(*x*) satisfy the two equations the two equations of two equations $\frac{1}{2}$

²Department of Mathematics, Florida State University, Tallahassee, Florida 32306-4510

(Received 22 August 2012; published 7 February 2013)
A new integrable nonlocal nonlinear Schrödinger equation is introduced. It possesses a Lax pair and an infinite number of conservation laws and is <i>PT symmetric. The data with suitable symmetries are discussed. A method to find pure soliton solutions is given. An explicit breathing one soliton solution is found. Key properties are discussed and contrasted with the classical nonlinear S

$$
iq_t + \alpha q_{xx} - 2\alpha q^2 r = 0,
$$

$$
ir_t - \alpha r_{xx} + 2\alpha qr^2 = 0
$$

equation (2.16) reduces to the complex modified Korteweg de-Vries with conjugate (2.17). The aforementioned *PT* -symmetry is preserved in these equations. we naplecal paplinger Schrädinger equation *P*(2.18)⇤ =(2.19). We also notice that a consequence of the introduction of the nonlocality *i* designed to linear Schrödinger equation **d** *±q*ˇ ⇤ <u>l</u> noi α 2⇤ nroainger equation **The nonlocal nonlinear Schrödinger equation**

- A parity transformed conjugate pair, $r(x,t) = \kappa q^*(-x,t)$ **i** parity transformed conjugate pair, $r(x,t) = \kappa q^*(-x)$ *±q* ˆ ⇤ *xx* 2*q*(ˆ*q*⇤) $r(x,t) = \kappa a^*(-x,t)$ *^x*)*.* (2.17) A parity transformed conjugate pair, $r(x,t) = \kappa q^*(-x,t)$
- A time-reversed pair, $r(x,t) = \kappa q^*(x, -t)$ A time-reversed pair, $r(x,t) = \kappa q^*(x,-t)$
- A real parity transformed conjugate pair, $r(x,t) = \pm q(-x,t)$ A real parity transformed conjugate pair, $r(x,t) = \pm q(-x,t)$
- $pair, r(x,t) = \pm q^{*}(-x,-t)$ DT gymmatric pair A \mathcal{PT} -symmetric pair, $r(x,t) = \pm q^*(-x,-t)$

The nonlocal nonlinear Schrödinger equation The aforementioned *PT* -symmetry is preserved in these equations. we naplecal paplinger Schrädinger equation *P*(2.18)⇤ =(2.19). We also notice that a consequence of the introduction of the nonlocality *i* designed to linear Schrödinger equation **d** *±q*ˇ ⇤ <u>l</u> noi α 2⇤ nroainger equation **The nonlocal nonlinear Schrödinger equation**

A parity transformed conjugate pair, $r(x,t) = \kappa q^*(-x,t)$ **i** parity transformed conjugate pair, $r(x,t) = \kappa q^*(-x)$ *±q* ˆ ⇤ *xx* 2*q*(ˆ*q*⇤) $r(x,t) = \kappa a^*(-x,t)$ *^x*)*.* (2.17) A parity transformed conjugate pair, $r(x,t) = \kappa q^*(-x,t)$

*q*ˆ

- $\bf me\text{-}reversed\,\, pair,\, \, \it r(x,t) =$ A time-reversed pair, $r(x,t) = \kappa q^*(x, -t)$ A time-reversed pair, $r(x,t) = \kappa q^*(x,-t)$
- mjugate pair, $r(x,t) = \pm q(-x,t)$
= $+a^*(-x,-t)$ *,* (2.8) A real parity transformed conjugate pair, $r(x,t) = \pm q(-x,t)$ A real parity transformed conjugate pair, $r(x,t) = \pm q(-x,t)$

<mark>. 42</mark>2个. 计文字文
一

formed into each other by means of a *PT* -symmetry transformation *PT* (2.19)⇤

Carparicy cransformed ed $pair, r(x,t) = \pm q^{*}(-x,-t)$ DT gymmatric pair A \mathcal{PT} -symmetric pair, $r(x,t) = \pm q^*(-x,-t)$

⇥

The nonlocal Hirota equation GMU J. C The nonlocal Hirota equation (3.10) *J. Cen, F.C, A. ^t* ⁼ *ⁱ*ˆ **The nonlocal Hirota equation**

↑ **1. Cen, F.C, A. Fring, JMP (2019)** *^x*)*.* (2.19)

xxx 6*q*ˆ

⇤*qq*ˆ

^x)*.* (2.21)

$$
q_t - i\alpha q_{xx} + 2i\alpha q^2 r + \beta [q_{xxx} - 6qrq_x] = 0,
$$

.
4번

+ (ˆ*q*⇤

 r New classes of integrable systems & solitons solutions A these of the gravic systems as sentons senation. *i*, we observe that (2.20) is the time-reversed of equations (2.21), i.e. *T* (2.21)=(2.20). classes of integrable systems & solitons solutions New classes of integrable systems & solitons solutions conjugation and a parity transformation (2.15), i.e. *P*(2.21)⇤

*iq*ˆ

The nonlocal nonlinear Schrödinger equation The aforementioned *PT* -symmetry is preserved in these equations. we naplecal paplinger Schrädinger equation *P*(2.18)⇤ =(2.19). We also notice that a consequence of the introduction of the nonlocality *i* designed to linear Schrödinger equation **d** *±q*ˇ ⇤ <u>l</u> noi α 2⇤ nroainger equation **The nonlocal nonlinear Schrödinger equation**

A parity transformed conjugate pair, $r(x,t) = \kappa q^*(-x,t)$ **i** parity transformed conjugate pair, $r(x,t) = \kappa q^*(-x)$ *±q* ˆ ⇤ *xx* 2*q*(ˆ*q*⇤) $r(x,t) = \kappa a^*(-x,t)$ *^x*)*.* (2.17) A parity transformed conjugate pair, $r(x,t) = \kappa q^*(-x,t)$

*q*ˆ

- $\bf me\text{-}reversed\,\, pair,\, \, \it r(x,t) =$ A time-reversed pair, $r(x,t) = \kappa q^*(x, -t)$ A time-reversed pair, $r(x,t) = \kappa q^*(x,-t)$
- mjugate pair, $r(x,t) = \pm q(-x,t)$
= $+a^*(-x,-t)$ *,* (2.8) A real parity transformed conjugate pair, $r(x,t) = \pm q(-x,t)$ A real parity transformed conjugate pair, $r(x,t) = \pm q(-x,t)$

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一

qxx 2*q*ˇ

^qxx ⌥ 2˜*qq*2⇤

xx 2*q*(ˇ*q*⇤)

Carparicy cransformed ed $pair, r(x,t) = \pm q^{*}(-x,-t)$ DT gymmatric pair A \mathcal{PT} -symmetric pair, $r(x,t) = \pm q^*(-x,-t)$

⇥

The nonlocal Hirota equation GMU J. C The nonlocal Hirota equation (3.10) *J. Cen, F.C, A. ^t* ⁼ *ⁱ*ˆ **The nonlocal Hirota equation**

↑ **1. Cen, F.C, A. Fring, JMP (2019)** *^x*)*.* (2.19)

xxx 6*q*ˆ

⇤*qq*ˆ

^x)*.* (2.21)

^x)*.* (2.23)

$$
q_t - i\alpha q_{xx} + 2i\alpha q^2 r + \beta [q_{xxx} - 6qrq_x] = 0,
$$

.
4번

 r New classes of integrable systems & solitons solutions $\sqrt{2\pi\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$ *i*, we observe that (2.20) is the time-reversed of equations (2.21), i.e. *T* (2.21)=(2.20). classes of integrable systems & solitons solutions New classes of integrable systems & solitons solutions

*iq*ˆ

$$
\mathbf{s}_{t} = -\alpha \mathbf{s} \times \mathbf{s}_{xx} - \frac{3}{2} \beta \left(\mathbf{s}_{x} \cdot \mathbf{s}_{x} \right) \mathbf{s}_{x} + \beta \mathbf{s} \times (\mathbf{s} \times \mathbf{s}_{xxx})
$$

+ (ˆ*q*⇤

Recalling here that the time-reversal map includes a conjugation, such that *T* : *q* ! *q*ˆ⇤*, i* !

+ (ˇ*q*⇤

 $transformations$

ansformations Extended Landau-Lifschitz equation \overline{O} \overline{O} \overline{O} **Gauge transformations** Extended Landau-Lifschitz equation

⇤*qq*ˇ

⇤

Next one needs to make sure that the sur \overline{a} 2⇤ **1. Cen, F.C, A. Fring, JPhA (2020)** *^x*)*.* (2.21) 2⇤ J. Cen, F.C, A. Fring, JPhA (2020) *J.* Cen

xxx 6*q*ˇ

New solutions and models....

....but something else ???

the Bullough-Dodd model (Tzitzeica) $\mathbf{r}_{\mathbf{a}}$ and $\mathbf{r}_{\mathbf{a$ described by the Lagrangian density of the form $22 - 1$ @*µ*'@*µ*' *^e*' ¹ The resulting classical nonlinear equation of motion **The Bullough-Dodd model (Tzitzéica)**

$$
\mathcal{L}_{\textrm{BD}}=\frac{1}{2}\partial_{\mu}\varphi\partial^{\mu}\varphi-e^{\varphi}-\frac{1}{2}e^{-2\varphi}+\frac{3}{2}
$$

$$
\ddot{\varphi} - \varphi'' + e^{\varphi} - e^{-2\varphi} = 0
$$

the Bullough-Dodd model (Tzitzeica) $\mathbf{r}_{\mathbf{a}}$ and $\mathbf{r}_{\mathbf{a$ described by the Lagrangian density of the form 2 2 (3.2) and subsequently reading o↵ the coecients of *ejkx*+*jlt* for *j* = 0*,...,* 8 leads to 9 @*µ*'@*µ*' *^e*' ¹ The resulting classical nonlinear equation of motion equations, that we do not report here, which may solved for the unknown constants in **The Bullough-Dodd model (Tzitzéica)**

$$
\mathcal{L}_{\text{BD}} = \frac{1}{2} \partial_{\mu} \varphi \partial^{\mu} \varphi - e^{\varphi} - \frac{1}{2} e^{-2\varphi} + \frac{3}{2} \qquad \ddot{\varphi} - \frac{3}{2} \qquad \ddot{\varphi
$$

$$
\ddot{\varphi} - \varphi'' + e^{\varphi} - e^{-2\varphi} = 0
$$

$$
\varphi_I^{\pm}(x,t) = \ln \left[\frac{\cosh \left(\beta + \sqrt{k^2 - 3t} + kx \right) \pm 2}{\cosh \left(\beta + \sqrt{k^2 - 3t} + kx \right) \mp 1} \right]
$$

the Bullough-Dodd model (Tzitzeica) $\mathbf{r}_{\mathbf{a}}$ and $\mathbf{r}_{\mathbf{a$ described by the Lagrangian density of the form $22 - 1$ @*µ*'@*µ*' *^e*' ¹ The resulting classical nonlinear equation of motion The Bullough-Dodd model (Tzitzéica) equations, that we do not report here, which may solved for the unknown constants in **The Bullough-Dodd model (Tzitzéica)**

$$
\mathcal{L}_{\text{BD}} = \frac{1}{2} \partial_{\mu} \varphi \partial^{\mu} \varphi - e^{\varphi} - \frac{1}{2} e^{-2\varphi} + \frac{3}{2} \qquad \ddot{\varphi} - \varphi'' + e^{\varphi} - e^{-2\varphi} = 0
$$

$$
\ddot{\varphi} - \varphi'' + e^{\varphi} - e^{-2\varphi} = 0
$$

$$
\varphi_I^{\pm}(x,t) = \ln \left[\frac{\cosh \left(\beta + \sqrt{k^2 - 3t} + kx \right) \pm 2}{\cosh \left(\beta + \sqrt{k^2 - 3t} + kx \right) \mp 1} \right]
$$
 When β is real the solutions are singular....

 $(t + kx) + 2$ $\frac{1}{2}$ $\left[\begin{array}{cc} 2 \\ 1 \end{array}\right]$ When β is real the solutions are singuian.... *^k*² ³*^t* ⁺ *kx*⌘ *When* β **is real the solutions** α **is real the solutions** are singular..... *are singular.....*

the Bullough-Dodd model (Tzitzeica) $\mathbf{r}_{\mathbf{a}}$ and $\mathbf{r}_{\mathbf{a$ described by the Lagrangian density of the form $22 - 1$ @*µ*'@*µ*' *^e*' ¹ The resulting classical nonlinear equation of motion The Bullough-Dodd model (Tzitzéica) equations, that we do not report here, which may solved for the unknown constants in **The Bullough-Dodd model (Tzitzéica)**

$$
\mathcal{L}_{\text{BD}} = \frac{1}{2} \partial_{\mu} \varphi \partial^{\mu} \varphi - e^{\varphi} - \frac{1}{2} e^{-2\varphi} + \frac{3}{2} \qquad \ddot{\varphi} - \varphi'' + e^{\varphi} - e^{-2\varphi} = 0
$$

$$
\ddot{\varphi} - \varphi'' + e^{\varphi} - e^{-2\varphi} = 0
$$

$$
\varphi_I^{\pm}(x,t) = \ln \left[\frac{\cosh \left(\beta + \sqrt{k^2 - 3t} + kx \right) \pm 2}{\cosh \left(\beta + \sqrt{k^2 - 3t} + kx \right) \mp 1} \right]
$$
 When β is real the solutions are singular....

 $(t + kx) + 2$ $\frac{1}{2}$ $\left[\begin{array}{cc} 2 \\ 1 \end{array}\right]$ When β is real the solutions are singuian.... *^k*² ³*^t* ⁺ *kx*⌘ *When* β **is real the solutions** α **is real the solutions** are singular..... *are singular.....*

the Bullough-Dodd model (Tzitzeica) $\mathbf{r}_{\mathbf{a}}$ and $\mathbf{r}_{\mathbf{a$ described by the Lagrangian density of the form $22 - 1$ @*µ*'@*µ*' *^e*' ¹ The resulting classical nonlinear equation of motion The Bullough-Dodd model (Tzitzéica) equations, that we do not report here, which may solved for the unknown constants in **The Bullough-Dodd model (Tzitzéica)**

3

$$
\mathcal{L}_{\text{BD}} = \frac{1}{2} \partial_{\mu} \varphi \partial^{\mu} \varphi - e^{\varphi} - \frac{1}{2} e^{-2\varphi} + \frac{3}{2} \qquad \ddot{\varphi} - \varphi'' + e^{\varphi} - e^{-2\varphi} = 0
$$

$$
\ddot{\varphi} - \varphi'' + e^{\varphi} - e^{-2\varphi} = 0
$$

$$
\varphi_I^{\pm}(x,t) = \ln \left[\frac{\cosh \left(\beta + \sqrt{k^2 - 3t} + kx \right) \pm 2}{\cosh \left(\beta + \sqrt{k^2 - 3t} + kx \right) \mp 1} \right]
$$
 When β is imaginary the solutions become regular....

 $(t + kx) + 2$ $\frac{1}{2}$ $\frac{1}{2}$ When β is imaginary the μ *D* μ μ μ μ μ $I\left(kx\right) \mp1$ **Solutions become regular....** When β is imaginary the **solutions become regular..... 1** *solutions become*

the Bullough-Dodd model (Tzitzeica) $\mathbf{r}_{\mathbf{a}}$ and $\mathbf{r}_{\mathbf{a$ described by the Lagrangian density of the form $22 - 1$ @*µ*'@*µ*' *^e*' ¹ The resulting classical nonlinear equation of motion The Bullough-Dodd model (Tzitzéica) equations, that we do not report here, which may solved for the unknown constants in *k*² 3*t*]*/k*, whereas ' *^I* is only real for *x<x^l* ⁼ [+ arccosh(2) + ^p The Bullough-Dodd model (Tzitzéica) negative in the complementary regime. Samples of these types of solutions in the di↵erent **The Bullough-Dodd model (Tzitzéica)**

3

$$
\mathcal{L}_{\text{BD}} = \frac{1}{2} \partial_{\mu} \varphi \partial^{\mu} \varphi - e^{\varphi} - \frac{1}{2} e^{-2\varphi} + \frac{3}{2} \qquad \ddot{\varphi} - \varphi'' + e^{\varphi} - e^{-2\varphi} = 0
$$

The first type of solutions we obtain a solutions we obtain a solutions we obtain a soliton soliton soliton so

PT : *x* ! *x, t* ! *t, i* ! *i,* ' ! '*.* (3.5)

The Bullough-Dodd Lagrangian (3.1) is trivially invariant under this symmetry and the

$$
\ddot{\varphi} - \varphi'' + e^{\varphi} - e^{-2\varphi} = 0
$$

identify various one-soliton soliton s
Ansatz in the general Ansatz in the general Ansatz in the general Ansatz in the general Ansatz in the general A

$$
\varphi_I^{\pm}(x,t) = \ln \left[\frac{\cosh \left(\beta + \sqrt{k^2 - 3t} + kx \right) \pm 2}{\cosh \left(\beta + \sqrt{k^2 - 3t} + kx \right) \mp 1} \right]
$$
 When β is imaginary the solutions become regular....

 $(t + kx) + 2$ $\frac{1}{2}$ $\frac{1}{2}$ When β is imaginary the sonutions become regular..... When β is imaginary the **solutions become regular.....**

$$
\mathcal{PT}: x \to -x, \quad t \to -t, \quad i \to -i, \quad \varphi \to \varphi.
$$

*k*² 3*t*]*/k* as the argument of the logarithm becomes

$$
\mathcal{PT}: \varphi_I^\pm \to \varphi_I^\pm
$$

the energies are real $E[\varphi_I^{\pm}] =$ *the energies are real* $E[\varphi^{\pm}_I] = -6|k|$ *k*² 3*t*]*/k* as the argument of the logarithm becomes the energies are real $E[\varphi_I^+] = -6|k|$ regimes are depicted in figure 1. In the case *[|]k[|] <* ^p3 both solutions '*[±]* complex, see figure 3. The type I solutions found here formally coincide with the solutions *^I* (*x, t*) evidently respect it for purely imaginary constants , i.e. *PT* : '*[±]* **The energies are real** $E[\varphi_I^+] = -6|k|$ $\mathbf{F}_{\mathbf{A}}$ see the suggestive are real E $rac{enc_{Bres}}{c_{Hres}}$ are real $D[\gamma]$ *E* $[\varphi_I^{\pm}] = -6|k|$

PT-SYMMETRY **regularizes solitons and explains the reality of conserved charges...**

......something else ???

Field theories and linear stability Field theories and linear stability

$$
\mathcal{L}=\frac{1}{2}\partial_{\mu}\varphi\partial^{\mu}\varphi-V(\varphi),
$$

Lagrangian density in 1+1 D Euler-Lagrange equations We use standard conventions and denote partial derivatives with respect to time *t* and space

$$
\partial_{\mu}\varphi\partial^{\mu}\varphi-V(\varphi),\hspace{1.5cm}\ddot{\varphi}-\varphi''+\frac{\partial V(\varphi)}{\partial\varphi}=0
$$

Field theories and linear stability Field theories and linear stability with the children and maintent bear may **Field theories and linear stability**

$$
\mathcal{L} = \frac{1}{2} \partial_{\mu} \varphi \partial^{\mu} \varphi - V(\varphi), \qquad \qquad \ddot{\varphi} - \varphi'' + \frac{\partial V(\varphi)}{\partial \varphi}
$$

Lagrangian density in 1+1 D Euler-Lagrange equations

$$
\mathcal{L} = \frac{1}{2} \partial_{\mu} \varphi \partial^{\mu} \varphi - V(\varphi), \qquad \qquad \ddot{\varphi} - \varphi'' + \frac{\partial V(\varphi)}{\partial \varphi} = 0
$$

@2*V* (')

Small perturbation $\varphi \rightarrow \varphi_s + \varepsilon \chi \qquad \varepsilon \ll 1$ We use standard conventions and denote partial derivatives with respect to time *t* and space *x* by overdots and dashes, respectively. The static solutions to these equations, i.e. the so-Small perturbation $\varphi \to \varphi_s + \varepsilon \chi \qquad \varepsilon \ll 1$ where '*^s* solves (2.2) and ' =: is a small perturbation. This converts the Euler-Lagrange

$$
\ddot{\varphi_s} - \varphi_s'' + \frac{\partial V(\varphi)}{\partial \varphi}\bigg|_{\varphi_s} + \varepsilon \left(\ddot{\chi} - \chi'' + \chi \frac{\partial^2 V(\varphi)}{\partial \varphi^2}\bigg|_{\varphi_s}\right) + \mathcal{O}(\varepsilon^2) = 0
$$

Field theories and linear stability Field theories and linear stability with regard to the main argument we have mainly **Field theories and linear stability**

$$
\mathcal{L} = \frac{1}{2} \partial_{\mu} \varphi \partial^{\mu} \varphi - V(\varphi), \qquad \qquad \ddot{\varphi} - \varphi'' + \frac{\partial V(\varphi)}{\partial \varphi}
$$

Lagrangian density in 1+1 D Euler-Lagrange equations T \cdot 1, \cdot (4.4)

$$
\mathcal{L} = \frac{1}{2} \partial_{\mu} \varphi \partial^{\mu} \varphi - V(\varphi), \qquad \qquad \ddot{\varphi} - \varphi'' + \frac{\partial V(\varphi)}{\partial \varphi} = 0
$$

@2*V* (')

Small perturbation $\varphi \rightarrow \varphi_s + \varepsilon \chi \qquad \varepsilon \ll 1$ We use standard conventions and denote partial derivatives with respect to time *t* and space *x* by overdots and dashes, respectively. The static solutions to these equations, i.e. the so-Small perturbation $\varphi \to \varphi_s + \varepsilon \chi \qquad \varepsilon \ll 1$ where $\frac{1}{2}$ is a substant perturbation. The Euler-Lagrange (2.2) and $\frac{1}{2}$

$$
\ddot{\varphi_s} - \varphi_s'' + \frac{\partial V(\varphi)}{\partial \varphi}\bigg|_{\varphi_s} + \varepsilon \left(\ddot{\chi} - \chi'' + \chi \frac{\partial^2 V(\varphi)}{\partial \varphi^2}\bigg|_{\varphi_s}\right) + \mathcal{O}(\varepsilon^2) = 0
$$

$$
\chi(x,t) = e^{i\lambda t} \Phi(x)
$$

Ansatz

Field theories and linear stability Field theories and linear stability riefly theories and intear stability $\frac{1}{2}$ and $\frac{1}{2}$ a **Field theories and linear stability**

$$
\mathcal{L} = \frac{1}{2} \partial_{\mu} \varphi \partial^{\mu} \varphi - V(\varphi), \qquad \qquad \ddot{\varphi} - \varphi'' + \frac{\partial V(\varphi)}{\partial \varphi} = 0
$$

Lagrangian density in 1+1 D Euler-Lagrange equations T \cdot 1, \cdot (4.4) \overline{a}

$$
\mathcal{L} = \frac{1}{2} \partial_{\mu} \varphi \partial^{\mu} \varphi - V(\varphi), \qquad \qquad \ddot{\varphi} - \varphi'' + \frac{\partial V(\varphi)}{\partial \varphi} = 0
$$

@'²

!

@2*V* (')

Small perturbation $\varphi \rightarrow \varphi_s + \varepsilon \chi \qquad \varepsilon \ll 1$ We use standard conventions and denote partial derivatives with respect to time *t* and space *x* by overdots and dashes, respectively. The static solutions to these equations, i.e. the so-Small perturbation $\varphi \to \varphi_s + \varepsilon \chi \qquad \varepsilon \ll 1$ where $\frac{1}{2}$ is a substant perturbation. The Euler-Lagrange (2.2) and $\frac{1}{2}$ nall perturbation $\varphi \to \varphi_s + \varepsilon \chi$ $\varepsilon \ll 1$ $\frac{1}{2}$ $\frac{1}{2}$ @' \overline{a} $\overline{1}$ $\frac{1}{\sqrt{2}}$ $\text{erturbation} \qquad \varphi \dashv$ $\sqrt{2}$ \overline{z} $\frac{1}{2}$ $\epsilon \ll 1$

$$
\ddot{\varphi_s} - \varphi_s'' + \left. \frac{\partial V(\varphi)}{\partial \varphi} \right|_{\varphi_s} + \varepsilon \left(\ddot{\chi} - \chi'' + \chi \frac{\partial^2 V(\varphi)}{\partial \varphi^2} \bigg|_{\varphi_s} \right) + \mathcal{O}(\varepsilon^2) = 0
$$

@'

$$
\chi(x,t) = e^{i\lambda t} \Phi(x) \qquad -\Phi_{xx} + V_1 \Phi = \lambda^2 \Phi
$$

Ansatz

$$
V_1(x) := \frac{\partial^2 V(\varphi)}{\partial \varphi^2} \bigg|_{\varphi_s}
$$

Sturm-Liouville eigenvalue problem/ Schrödinger equation with potential 1 φ_s te Sturm Liouville eigenvalue problem / Schrödinger equation with potential *x x* + *x z z <i>x iouville eigenvalue problem / Schrödinger equation w* tion $\ddot{\cdot}$ \mathbf{i} \overline{a} Sturm-Liouville eigenvalue problem/ Schrödinger equation with potential \mathbf{e} α . (2.6) simply introduce an oscillation in time around the solution '*s*, whereas when 2 C the sterme eigenvalue problem, bemoaniger equation with potential Sturm-Liouville eigenvalue problem/ Schrödinger equation with potential

Field theories and linear stability

Let's see some very well-known examples

e very well-known examples
\n
$$
-\Phi_{xx} + V_1 \Phi = \lambda^2 \Phi
$$
\n
$$
V_1(x) := \frac{\partial^2 V(\varphi)}{\partial \varphi^2} \bigg|_{\varphi_s}
$$

Field theories and linear stability

Let's see some very well-known examples

 $-\Phi_{xx} + V_1 \Phi = \lambda^2 \Phi$ $\partial \varphi^2 \big|_{\varphi_s}$ $V_1(x) := \frac{\partial^2 V(\varphi)}{\partial \varphi^2}$ $\partial \varphi^2$ $\overline{}$ $\begin{array}{c} \hline \end{array}$ $\begin{array}{c} \hline \end{array}$ $|_{\varphi_{s}}$

Let us next see how the energy of the perturbed solution behaves. Expanding up to

\mathbf{S} are the decay with time and are therefore unstable unit time are the are therefore unstable. Solution introduce an original intervalse and the solution in the solution of classical field theory^{*}

solutions will generally decay with the contract of the contra

 μ science and Department of Fhysics, Massachusetts Institute of Technology,
etts 02139 Let us next see the energy of the energy of the perturbation behaviors of the perturbed solution behaviors, combined by the perturbed solution behaviors of the perturbed solution behaviors of the perturbed solution behavio

^E['*^s* ⁺] = *^E*['*s*] +^Z *dx* " @*^V* (') investigations. We discuss how solutions to field equations, treated as classical, c-number nonlinear
differential equations, expose unexpected states in the quantal Hilbert space with novel quantum numbers $\frac{1}{2}$ Recent researches have shown that it is possible to obtain information about the physical content of nontrivial quantum field theories by semiclassical methods. This article reviews some of these *E*

and space-ti

provide non e from topological properties of the classical field configuration or from t
time symmetries. Also imaginary-time, c-number solutions are reviewed. m topological properties of the classical field configuration or from the m
symmetries. Also imaginary-time, *c*-number solutions are reviewed. It is s
turbative information about the vacuum sector of the quantum theory. $\frac{1}{t}$ the vacuum sea etor of the quantum the

 $n¹$ *v*. *iviou*. *i iiyo*. + 1, 681 (1977)
... Rev. Mod. Phys. 49, 681 (1977)

Field theories and linear stability '*^s* = tanh *x* **1** eories and $\ddot{ }$ **Field theories and linear stability**

Let's see some very well-known examples *x*
2 **x**
2 *x* exal

e very well-known examples
\n
$$
-\Phi_{xx} + V_1 \Phi = \lambda^2 \Phi
$$
\n
$$
V_1(x) := \frac{\partial^2 V(\varphi)}{\partial \varphi^2} \bigg|_{\varphi_s}
$$

Let us next see how the energy of the perturbed solution behaves. Expanding up to

second order in " and integrating two terms by parts, we easily derive

 \mathcal{L}^{ω} see how the perturbed solution behaves. Expanding up to the perturbed solution behaves. Expanding up to the perturbed solution behaves.

 $Sine-Gordon \quad V(\varphi) = -\cos \varphi$ $s - \cos \varphi$ introduce an oscillation in time around the solution in time around the solution φ solutions will grow or decay with time and are therefore unstable. s_{rel} introduce an orientation in time around the solution x

 static kink $\varphi_s = 4 \arctan e^x$ static kink $\varphi_s = \pm \text{al}(0, \theta)$

Field theories and linear stability '*^s* = tanh *x* **1** eories and $\ddot{ }$ **Field theories and linear stability**

Let's see some very well-known examples *x*
2 **x**
2 *x* exal

e very well-known examples
\n
$$
-\Phi_{xx} + V_1 \Phi = \lambda^2 \Phi
$$
\n
$$
V_1(x) := \frac{\partial^2 V(\varphi)}{\partial \varphi^2} \bigg|_{\varphi_s}
$$

 $Sine-Gordon \quad V(\varphi) = -\cos \varphi$ For $V(\varphi) = -\cos \varphi$ R $V(\varphi) = -\cos \varphi$ static kink $\varphi_s = 4 \arctan e^x$ $v_1(x) = -\frac{1}{\cosh^2 x} + 1$ static kink $\varphi_s = \pm \text{at}(0, \theta)$ and soliton reflection

$$
-\cos\varphi
$$
\n
$$
V_1(x) = -\frac{2}{\cosh^2 x} + 1
$$

For the eigenfunction solution solution reflectionless potential e^w and perturbed solution behaves. Expanding up to perturbed solution behaves. Expanding up to e^w second order in " and integrating two terms by parts, we easily derive Let us next see how the energy of the perturbed solution behaves. Expanding up to

Field theories and linear stability '*^s* = tanh *x* **1** eories and $\ddot{ }$ Eield theories and linear stability (*x*), the term of first order in " @*V* (') <u>ri</u> @2*V* (') Field theories and linear stability ⁺ *^O*("2)=0*.* (2.6)

Let's see some very well-known examples *x*
2 **x**
2 *x* exal Let s see some very well-known examples $\Omega_{\rm IZCD}$

Let's see some very well-known examples
$$
-\Phi_{xx}+V_1\Phi=\lambda^2\Phi
$$

$$
V_1(x):=\left.\frac{\partial^2 V(\varphi)}{\partial\varphi^2}\right|_{\varphi_s}
$$

Since-Gordon
$$
V(\varphi) = -\cos \varphi
$$

\nstatic kink $\varphi_s = 4 \arctan e^x$
\non-e-soliton reflections potential

For the eigenfunction solutions to this equation with 2 R the linear perturbation

Schr¨odinger equation

$$
-\cos\varphi
$$

\n
$$
V_1(x) = -\frac{2}{\cosh^2 x} + 1
$$

\n
$$
\coth^x
$$

actual extended to the expliton reflectionless potential and the linear perturbation one-soliton reflectionless potential second order in " and integrating two terms by parts, we easily derive Let us next see how the energy of the perturbed solution behaves. Expanding up to

will simply introduce an oscillation in time around the solution in time around the solution in the solution of
The solution in the solution of the solution in the solution of the solution of the solution of the solution o

$$
\varphi^4
$$
-theory $V(\varphi) = \frac{1}{2}(1 - \varphi^2)^2$
static kink $\varphi_s = \tanh x$

1

2

'*s*

Field theories and linear stability '*^s* = tanh *x* **1** eories and $\ddot{ }$ Eield theories and linear stability (*x*), the term of first order in " @*V* (') <u>ri</u> @2*V* (') Field theories and linear stability ⁺ *^O*("2)=0*.* (2.6)

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2 **x**
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\non-e-soliton reflections potential

For the eigenfunction solutions to this equation with 2 R the linear perturbation

Schr¨odinger equation

$$
\varphi^4\text{-theory} \quad V(\varphi) = \frac{1}{2}(1-\varphi^2)^2
$$
\ntwo-soliton reflections potential

\n
$$
V_1(x) = -\frac{6}{\cosh^2 x} + 4
$$
\nstatic kink

\n
$$
\varphi_s = \tanh x
$$

1

'*s*

$$
-\cos\varphi
$$

\n
$$
V_1(x) = -\frac{2}{\cosh^2 x} + 1
$$

\n
$$
\coth^x
$$

and *one-soliton reflectionless potential* one-so e^w and perturbed solution behaves. Expanding up to perturbed solution behaves. Expanding up to e^w second order in " and integrating two terms by parts, we easily derive Let us next see how the energy of the perturbed solution behaves. Expanding up to

two-soliton reflectionless potential
 $V_1(x) = -\frac{6}{x^2} + 4$ Ï

$$
V_1(x) = -\frac{6}{\cosh^2 x} + 4
$$

Field theories and linear stability '*^s* = tanh *x* **1** eories and $\ddot{ }$ Eield theories and linear stability (*x*), the term of first order in " @*V* (') <u>ri</u> @2*V* (') Field theories and linear stability ⁺ *^O*("2)=0*.* (2.6)

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\n
$$
V_1(x) = -\frac{6}{\cosh^2 x} + 4
$$
\nstatic kink

\n
$$
\varphi_s = \tanh x
$$

1

'*s*

$$
-\cos\varphi
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\n
$$
V_1(x) = -\frac{2}{\cosh^2 x} + 1
$$

\n
$$
\coth^x
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and *one-soliton reflectionless potential* one-so e^w and perturbed solution behaves. Expanding up to perturbed solution behaves. Expanding up to e^w second order in " and integrating two terms by parts, we easily derive Let us next see how the energy of the perturbed solution behaves. Expanding up to

two-soliton reflectionless potential
 $V_1(x) = -\frac{6}{x^2} + 4$ Ï

$$
V_1(x) = -\frac{6}{\cosh^2 x} + 4
$$

The Bullough-Dodd solutions and linear stability *[|]k[|] >* ^p3. Panel (a): Complex solution ' *^I* for *k* = 2 with = 3 and panel (b) *PT* regularized The Bullough-Dodd solutions and linear stability *[|]k[|] >* ^p3. Panel (a): Complex solution ' The Bullough-Dodd solutions and linear stability (*x*), the term of first order in "

$$
-\Phi_{xx} + V_1 \Phi = \lambda^2 \Phi
$$
 Bullough-Dodd static solution

$$
V_1(x) := \frac{\partial^2 V(\varphi)}{\partial \varphi^2}\Big|_{\varphi_s} \qquad \phi_I^{\pm}(x) = \ln \left[\frac{\cosh (\beta + \sqrt{3}x)}{\cosh (\beta + \sqrt{3}x)} \right]
$$

second order in " and integrating two terms by parts, we easily derive

 $-\Phi_{xx} + V_1 \Phi = \lambda^2 \Phi$ Bullough-Dodd static solutions n $\ddot{}$

$$
V_1(x) := \frac{\partial^2 V(\varphi)}{\partial \varphi^2} \bigg|_{\varphi_s} \qquad \phi_I^{\pm}(x) = \ln \left[\frac{\cosh (\beta + \sqrt{3}x) \pm 2}{\cosh (\beta + \sqrt{3}x) \mp 1} \right]
$$

 $\mathbb{R}^n \times \mathbb{R}^n$

⇤

$$
V_1^+(x) = 1 - \frac{3}{1 - \cosh(\beta + \sqrt{3}x)} + \frac{8\sinh^4\left[\frac{1}{2}(\beta + \sqrt{3}x)\right]}{\left[2 + \cosh(\beta + \sqrt{3}x)\right]^2},
$$

The Bullough-Dodd solutions and linear stability *^m* = 1*/*2, *^U*⁰ = 3*/*2, ↵ ⁼ ^p3*/*2. Since for these values *^s* = 1, there is only one bound state ugh-Dodd solutions: $\overline{ }$ *,* (3.8) shifting the overall energy by 3 we obtain *V*² = *U* for the parameter identifications ~ = 1, *m* The Bullough-Dodd solutions and linear state of the Bullough-Dodd solutions and linear state

 \mathbb{R} 3 costs

T

⇤

two-solit <u>ton</u> reflectionles $\overline{1}$ *two-soliton reflectionless potential* $\boldsymbol{\mathsf{R}}$ ⌘ 9 **two-soliton reflectionless potential**

$$
V_1^+(x) = 1 - \frac{3}{1 - \cosh(\beta + \sqrt{3}x)} + \frac{8\sinh^4\left[\frac{1}{2}(\beta + \sqrt{3}x)\right]}{\left[2 + \cosh(\beta + \sqrt{3}x)\right]^2},
$$

Ŧ

cosh

\n singularity
$$
x_0 = -\beta\sqrt{3}
$$
 when $\beta \in \mathbb{R}$ \n

\n\n no singularities **PTSYMMETRY** $\beta \in i\mathbb{R}$.\n

lates There exist linear stable perturbations! re exist linear st \mathbf{a} **b**le parturb *ear stable perturbations !* $\qquad \qquad$ There exist linear stable perturbations ! linearly perturbed by the shape mode (1) ¹ . Let us next consider the case when is taken

The round new complex solitons when shorten \blacksquare which are different We found new complex solitons with broken \mathbf{PT} which are unstable!

The scattering states for the inverse costs in the induction of Γ When adjusting the parameters therein to our equation (3.7) for *V*2, we obtain the two F.C, A. Fring, T. Taira, NPB (2022)
PT-SYMMETRY

It is not a mere artifact which measure reality energy conditions but also the physical sense of theories...

Non-Hermitian field integrable field theories

New classes of integrable systems & solitons solutions All conserved charges are real, even though solitons are complex An additional symmetry ensures the reality of the integrals of motion The linear stable pertubations display also **PT-symmetry** Are these features available only in this kind of integrable theories? Non-physical solitons are regularized (removing singularities!)

No, they can be applied everywhere.....

Applications of Field Theory to Hermitian and Non-Hermitian Systems

Non-Hermitian ideas provide new phenomena in (integrable) field theories

Outline

The Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy two linear first order diaerah first order diaerah first order diaerah diaerah diaerah diaerah diaerah diaerah
Tanggal egitimal egi

Lax Pair formalism

$$
\quad \Leftrightarrow \quad
$$

$$
\Psi_t = V\Psi, \Psi_x = U\Psi \qquad \Longleftrightarrow \qquad \partial_t U - \partial_x V + [U, V] = 0
$$

$$
U = \begin{pmatrix} -i\lambda & q(x,t) \\ r(x,t) & i\lambda \end{pmatrix}
$$
 Exercise 18.1 Exercise 21.1 Exercise 31.12 Exercise 41.13 Exercise 41.14 Exercise 41.14

the linear first order direct order order dividends for an auxiliary function \mathbb{Z} **akharov and Shabat (1972)** zero curvature formalism *Lax Pair formalism* → → *Zero curvature formalism*
Zakharov and Shabat (1972) \downarrow 2ero curvature formalism
air formalism
 \downarrow $\$

$$
\partial_t U - \partial_x V + [U, V] = 0
$$

- $r(f)$ concrete to the validity of the validity of the validity of the equation of the equatio $\overline{}$ • Sine-Gordon
- *i*, *Y* \cdot *Norteweg de Vries (KdV) A*(*x, t*) *B*(*x, t*) $\sum_{i=1}^{n}$ *,* (2.2) • Korteweg de Vries (KdV)
	- \mathbf{a} • mKdV
	- \cdot N_c *i q*(*x, t*) *r*(*x*) *i*^{*r*}(*x*) *i*^{*r*}(*x*) *i*^{*r*}(*x*) *i*^{*x*}(*x*) *i , V* = • Non-linear Schrödinger
	- *C*(*x, t*) *A*(*x, t*) \blacksquare **P**(*x*) \blacksquare • Hirota

Integrable systems and gravity @*tU* @*xV* + [*U, V*]=0 , *^t* = *V* , *^x* = *U .* (2.1) two linear first order diae first order differential equations for an auxiliary functions for an auxiliary func
The contract of an auxiliary functions for an auxiliary functions for an auxiliary functions for an auxiliary

 $U = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$ $U = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$ \Box Zakharov and Shabat (1972) Zero curvature formalism

$$
\partial_t U - \partial_x V + [U, V] = 0
$$

Zero curvature formalism
\n**Zakharov and Shabat (1972)**
\n
$$
\partial_t U - \partial_x V + [U, V] = 0
$$
\n
$$
V = \begin{pmatrix} -i\lambda & q(x, t) \\ r(x, t) & i\lambda \end{pmatrix}
$$
\n
$$
V = \begin{pmatrix} A(x, t) & B(x, t) \\ C(x, t) & -A(x, t) \end{pmatrix}
$$

Ax(*x, t*) = *q*(*x, t*)*C*(*x, t*) *r*(*x, t*)*B*(*x, t*)*,* (2.3) and *r*(*x, t*), the zero curvature condition holds when the matrix entries *A*, *B* and *C* satisfy **Coupled** into structure motion. When taking the matrix valued functions *U* and *V* to be of the general form **⁶⁰ This structure is intimately related with AdS3 gravity**

C*x*, *Cx*(*x*, *C*(*x*)₊₂*i*_{*C*(*x*)⁺²*i*_{*C*(*x*)⁺²^{*i*}C(*x*)⁺²^{*i*}C(*x*)⁺²^{*i*}C(*x*)⁺²^{*i*}C(*x*)⁺²^{*i*}C(*x*)⁺²^{*i*}C(*x*)⁺²^{*i*}C(*x*)⁺²^{*i*}C(*x*)⁺²^{*i*}C(*x*)⁺²^{*i*}C(*x*)⁺²^{*i*}C(}} *M. Cárdenas, F. C. K. Lara, M. Pino, PRL (2021) , V* = , (2.2)
, (2.2)
, (2.2) M. Cárdenas, F. C. K. Lara, M. Pino, PRL (2021)

Bx(*x, t*) = *qt*(*x, t*) 2*q*(*x, t*)*A*(*x, t*) 2*iB*(*x, t*)*,* (2.4)

$$
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{1}{\ell^2} g_{\mu\nu} = 0
$$

$$
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{1}{\ell^2} g_{\mu\nu} = 0
$$

BTZ black hole

VOLUME 69, NUMBER 13 PHYSICAL REVIEW LETTERS 28 SEPTEMBER 1992

Black Hole in Three-Dimensional Spacetime

Máximo Bañados, ^{(1),(a)} Claudio Teitelboim, ^{(1),(2),(a)} and Jorge Zanelli^{(1),(a)}
⁽¹⁾Centro de Estudios Científicos de Santiago, Casilla 16443, Santiago 9, Chile and Facultad de Ciencias, Universidad de Chile, Casilla 653, Santiago, Chile
⁽²⁾Institute for Advanced Study, Olden Lane, Princeton, New Jersey 08540
(Received 29 April 1992) As shown below, this geometrization of AKNS equa-

- Laboratory to understand the main features of black holes
- One of the best evidences for the AdS/CFT correspondence components near some surface. Although some surface in the some free of the some free of the some free of the
Although some free of the some free of the
- Very useful in several contexts beyond holography: string theory, QFT, the following and the superscript *±* is removed. Similar

the dynamics of AdS3 Einstein graduate and Advancement of Advancement and Advancement of Advancement and Advancement of Advancement and A **Integrable systems and gravity**

$$
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{1}{\ell^2} g_{\mu\nu} = 0
$$

Can be formulated as two independent Chern-Simons copies here the formulated as two independent Chern Simons copies can de formance as two maepenaem enem sinons copres Can be formulated as two independent Chern-Simons copies

Gauge connections $\frac{1}{2}$ Gauge connections **example to the dreibein connection** \mathbf{e} Gauge connections

 $A^{\pm} = \omega \pm e/\ell$ $\qquad \qquad \mathcal{F}^{\pm} = dA^{\pm} + A^{\pm} \wedge A^{\pm}$

spin connection & dreibein α dition and the constant curvature equation (see, for ex- α see, for ex-

 Ω (2 \mathbb{R}) boundary conditions for the gravitations for the gravitations for the gravitations for the gravitations of the
The gravitations for the gravitations for the gravitations for the gravitations of the gravitations of the gra $SL(Z, \mathbb{R})$ $SL(2,\mathbb{R})$ $\mathcal{L}(\mathcal{L}, \mathbf{u})$ *SL*(2,ℝ)

A Achiever and P K Townsend PI R (1986) F Witten NPR (1988) and the number of requirements $\frac{1}{2}$ is $\frac{1}{2}$ (i) it should require an $\frac{1}{2}$ (ii) it should render a $\frac{1}{2}$ A. Achucarro and P. K. Townsend, PLB (1986) E. Witten, NPB (1988) the dynamical evolution of the above geometry, accord-In Trendento and T. R. Townsend,

tors, The *n* 2 *x*, *n* relation *L*, *witten, NPD* (1988)

the dynamics of AdS3 Einstein graduate and Advancement of Advancement and Advancement of Advancement and Advancement of Advancement and A **Integrable systems and gravitation**

$$
\boxed{R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}-\frac{1}{\ell^2}g_{\mu\nu}=0}
$$

Can be formulated as two independent Chern-Simons copies \mathcal{L}_{h} to formulated as two independent Charp Simons series eart de formarie de two mae penaem enem omtond copied Can be formulated as two independent Chern-Simons copies

this approach, the theory is described by the difference

two independent copies of the well-known AKNS system

 $\frac{1}{2}$ Gauge connections **Exercise Server connections**

$$
\mathcal{A}^{\pm} = \omega \pm e/\ell \qquad \qquad \mathcal{F}^{\pm} = d\mathcal{A}^{\pm} + \mathcal{A}^{\pm} \wedge \mathcal{A}^{\pm}
$$
spin connection & dreibein

ample, Ref. [16]. These equations, in the contract of the cont

boundary conditions for the gravitations for the gravitation of the gravitation of the gravitation of the gravitation

dition and the constant constant constant constant constant α $\frac{2}{2}$ α Zero curvature condition

gµ⌫ =

altitude to the zero condition to the curvature conditions for the curvature $\frac{1}{2}$ This is done choosing specifi where h, i is the invariant bilinear form of the invariant bilinear form of the gauge α This is done choosing specific boundary conditions for the gravitational field. $\mathsf{h}\mathsf{I}$

A debucarre and P K Townsend-PI B (1986) F Witten NPB (1988) In Trendento and T. R. Townsend, and the number of requirements $\frac{1}{2}$ is $\frac{1}{2}$ (i) it should require an $\frac{1}{2}$ (ii) it should render a $\frac{1}{2}$ A. Achucarro and P. K. Townsend, PLB (1986) E. Witten, NPB (1988)

tors, The *n* 2 *x*, *n* relation *L*, *witten, NPD* (1988)

⌦*A*⁺

^µ A

nonvanishing components of the invariant bilinear formula of the invariant bilinear formula of the invariant b
The invariant bilinear formula of the invariant bilinear formula of the invariant bilinear formula of the inva **Little and Separate 1 and 1 a** are h*L*1*, L*¹i = 1 and h*L*0*, L*0i = 1*/*2. The boundary conditions computer with the conditions conditions and the gauge fields of the gauge fiel Consequently, the time evolution of the system also reis and gravity conditions, and \bm{s} i s possible when added when added i gravity, a reasonable choice should fulfill of two independent Chern-Simons actions with gauge Integrable systems and gravity

$$
\mathcal{A}^{\pm} = \omega \pm e/\ell \qquad \qquad \mathcal{F}^{\pm} = d\mathcal{A}^{\pm} + \mathcal{A}^{\pm} \wedge \mathcal{A}^{\pm} = 0
$$

\nZero curvature condition
\n
$$
\text{Coussaert, Henneaux, P. van Driel (1995)}
$$

Coussaert, Henneaux, P. van Driel (1995)

Boundary conditions
$$
A = b^{-1}(d+a)b
$$
 $b(\rho) = \exp \left[\log \left(\frac{\rho}{\ell}\right)L_0\right]$

$$
a=a_{\varphi}d\varphi+a_tdt
$$

$$
[L_n, L_m] = (n-m) L_{n+m} \qquad SL(2,\mathbb{R})
$$
 Generators

captures the radial dependence of the fields. For the fields. Ref. [18], the angular component reads as follows, As shown below, this geometrization of AKNS equatiliegi quie sysie *a[±]* ' The component along *L*⁰ is chosen to be a constant with-= ⌥2⇠*±L*⁰ *p±L±*¹ + *r±L*⌥¹*.* (9) out van die bysiems and gruvily integrate physical The gradie system α and the theory. In addition, the temporal computation, the temporal computation, the temporal comand gravity $A \rightarrow B$ and arovity **Integrable systems and gravity**

Unless stated only the + copy is treated on the + copy is treated in the + copy is treated in the + copy is tre
The + copy is treated in the + copy

$$
\boxed{a=a_\varphi d\varphi+a_t dt}
$$

$$
a_{\varphi} \sim 2i\lambda L_0 - q(\varphi, t)L_1 + r(\varphi, t)L_{-1} \qquad \qquad = \qquad U = \begin{pmatrix} -i\lambda & q(\varphi, t) \\ r(\varphi, t) & i\lambda \end{pmatrix}
$$

$$
a_t \sim -2A(\varphi, t)L_0 - B(\varphi, t)L_1 + C(\varphi, t)L_{-1} = V = \begin{pmatrix} A(\varphi, t) & B(\varphi, t) \\ C(\varphi, t) & -A(\varphi, t) \end{pmatrix}
$$

dynamics of the theory. In addition, the theory. In addition, the temporal com-temporal com-temporal com-tempo
The temporal com-temporal com-temporal com-temporal com-temporal com-temporal com-temporal com-temporal com-te

ponent of the gauge connection is given by the gauge connection is given by the gauge connection is given by t
The gauge connection is given by the gauge connection is given by the gauge connection is given by the gauge *a^t* ⇠ 2*AL*⁰ *BL*¹ + *CL*¹ (13) the following and the superscript **b** is superscript **z** is removed. Similar \mathbf{F} is \mathbf{F} is M Cárdenas E C V Lara M Pine PPI (2021) *a*^t. A baracino, **1***t*. C. **1***t***. Bara, 11***t*. **1110**, **1110** (2021) M. Cárdenas, F. C. K. Lara, M. Pino, PRL (2021)

 $\sqrt{2}$

(2*A±L*⁰ *± B±L±*¹ ⌥ *C±L*⌥¹)*.* (12)

captures the radial dependence of the fields. For the fields. Ref. [18], the angular component reads as follows, As shown below, this geometrization of AKNS equatiliegi quie sysie *a[±]* ' The component along *L*⁰ is chosen to be a constant with-= ⌥2⇠*±L*⁰ *p±L±*¹ + *r±L*⌥¹*.* (9) out van die bysiems and gruvily integrate physical The gradie system α and the theory. In addition, the temporal computation, the temporal computation, the temporal comand gravity Integrable systems and gravity two independent copies of the well-known AKNS system Ref. [18], the angular component reads as follows, $A \rightarrow B$ and arovity **Integrable systems and gravity**

$$
\boxed{a=a_\varphi d\varphi+a_t dt}
$$

$$
a_{\varphi} \sim 2i\lambda L_0 - q(\varphi, t)L_1 + r(\varphi, t)L_{-1} \qquad \qquad = \qquad U = \begin{pmatrix} -i\lambda & q(\varphi, t) \\ r(\varphi, t) & i\lambda \end{pmatrix}
$$

$$
a_t \sim -2A(\varphi, t)L_0 - B(\varphi, t)L_1 + C(\varphi, t)L_{-1} = V = \begin{pmatrix} A(\varphi, t) & B(\varphi, t) \\ C(\varphi, t) & -A(\varphi, t) \end{pmatrix}
$$

 ${\cal F}^\pm ~=~ d{\cal A}^\pm + {\cal A}^\pm \wedge {\cal A}^\pm ~\blacktriangleleft \hspace{-4mm} \longrightarrow ~ \partial_t U - \partial$ T^{\pm} and J^{\pm} is J^{\pm} in J^{\pm} in J^{\pm} $s = \omega \sqrt{1 + \omega \sqrt{2}}$ $d{\cal A}^{\pm} + {\cal A}^{\pm} \wedge {\cal A}^{\pm} \quad \Longleftrightarrow \quad \partial_t U - \partial_\varphi V + [U,V] = 0$ \overline{a} A^+ A^+ *C*('*, t*) *A*('*, t*) $\mathcal{F}^{\pm} = d\mathcal{A}^{\pm} + \mathcal{A}^{\pm} \wedge \mathcal{A}^{\pm}$ **a** $\longrightarrow \partial_t U - \partial_{\varphi} V + [U, V]$ Unless stated otherwise, only the + copy is treated in τ + τ i τ i $\mathcal{F}^{\pm} = d\mathcal{A}^{\pm} + \mathcal{A}^{\pm} \wedge \mathcal{A}^{\pm}$ is $\partial_t U - \partial_{\varphi} V + [U, V] = 0$ $\partial_t U - \partial_\varphi V + [U, V] = 0$

dynamics of the theory. In addition, the theory. In addition, the temporal com-temporal com-temporal com-tempo
The temporal com-temporal com-temporal com-temporal com-temporal com-temporal com-temporal com-temporal com-te

 α *General Relativity* $$ the dynamics of AdS³ Einstein gravity. General Relativity General Relativity α *General Relativity* dimensional gravity is captured by the torsionless con- $\overline{2}$ C_{cyc} under D_{e} Whichai Kelati General Relativity **AKNS** hierarchy

Unless stated only the + copy is treated on the + copy is treated in the + copy is treated in the + copy is tre
The + copy is treated in the + copy

*AAND DLL***_{***L***}***L***_{***L***}^{***L***}***<i>C</sub>* the following and the superscript \overline{a} *AKNS* hierarchy considerations can be applied to the copy.

↵ *,* (6)

nents *p[±]* and *r[±]* are the fields carrying the boundary

 $\sqrt{2}$

(2*A±L*⁰ *± B±L±*¹ ⌥ *C±L*⌥¹)*.* (12)

ponent of the gauge connection is given by the gauge connection is given by the gauge connection is given by t
The gauge connection is given by the gauge connection is given by the gauge connection is given by the gauge *a^t* ⇠ 2*AL*⁰ *BL*¹ + *CL*¹ (13) the following and the superscript **b** is superscript **z** is removed. Similar \mathbf{F} is \mathbf{F} is M Cárdenas E C V Lara M Pine PPI (2021) M. Cárdenas, F. C. K. Lara, M. Pino, PRL (2021)

We have developed a dictionary

zero curvature formulation

Integrable systems AdS_3 gravity

conserved quantities

M. Cárdenas, F. C. K. Lara, M. Pino, PRL (2021)

Integrable systems and gravity The temporal coordinate *t* range in R, (⇢*,*) are defined on the polar plane and ` is the

M. Cárdenas, F. C., M. Pino, *work in progress*

denas, F. C., M. Pino, work in progress \quad Soliton solutions in gravity

$$
ds^{2} = \left(\ell \frac{d\rho}{\rho} - \ell(\lambda^{+} + \lambda^{-})d\phi + (A^{-} - A^{+})dt\right)^{2} +
$$

$$
\left[\frac{\rho}{\ell}(p^{+}\ell d\phi - B^{+}dt) + \frac{\ell}{\rho}(r^{-}\ell d\phi + C^{-}dt)\right] \left[\frac{\rho}{\ell}(p^{-}\ell d\phi + B^{-}dt) + \frac{\ell}{\rho}(r^{+}\ell d\phi - C^{+}dt)\right]
$$

When we consider the Einstein's equations with a negative cosmological constant

AdS³ radius. All the functions *p±, r[±]* and *A±, B±, C[±]* depend on *t,* except from *[±]* that

Black hole horizon: Black flowers

Integrable systems and gravity The temporal coordinate *t* range in R, (⇢*,*) are defined on the polar plane and ` is the

M. Cárdenas, F. C., M. Pino, *work in progress*

denas, F. C., M. Pino, work in progress \quad Soliton solutions in gravity

$$
ds^{2} = \left(\ell \frac{d\rho}{\rho} - \ell(\lambda^{+} + \lambda^{-})d\phi + (A^{-} - A^{+})dt\right)^{2} + \left[\frac{\rho}{\ell}(p^{+}\ell d\phi - B^{+}dt) + \frac{\ell}{\rho}(r^{-}\ell d\phi + C^{-}dt)\right] \left[\frac{\rho}{\ell}(p^{-}\ell d\phi + B^{-}dt) + \frac{\ell}{\rho}(r^{+}\ell d\phi - C^{+}dt)\right]
$$

AdS³ radius. All the functions *p±, r[±]* and *A±, B±, C[±]* depend on *t,* except from *[±]* that

- Non-local PT inspired solutions
- Flat-limit of the AKNS boundary conditions
- Conformal symmetry AdS/CFT
- Interpretation of the quantum linear problem and black hole entropy
- Gravity analogue for integrable systems related by gauge transformations
- Higher dimensions and further hierarchies
- Different generating solution schemes
- How are these results connected with self-dual Yang-Mills description of integrable systems & Ward conjecture?

Outline

- Many open problems and new physics in solitons theory
- Meaning of complex and non-local solitons
- New ways to investigate gravity and more to be explored
- Electromagnetic solutions, self dual Yang-Mills and beyond
- Applications for AdS/CFT integrability?

4 **Discussion**

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