$\mathcal{PT}$ Phases and Dark Energy:

Road to Chern-Simons

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Overview

Talk based on: 1) Chern-Simons Gravity and $\mathcal{PT}$ Symmetry with N E Mavromatos, Phys. Rev. D **110** 045 2) Phases of scalar fields and $\mathcal{PT}$ symmetry with L. Chen, arXiv 2409.05439 [quant-ph]

1. Heuristics of $\mathcal{PT}$ symmetry
2. Emergent $\mathcal{PT}$ symmetry
3. String inspired gravity and $\mathcal{PT}$ symmetry
4. Consequences of nonperturbative renormalisation
5. Chern-Simons Gravity

Simple models with $\mathcal{PT}$ symmetry: global to local QFTs [1, 2]

- ϕ pseudo-scalar $(i\phi)^{\delta}\,\phi^2$, δ a real parameter, $\delta \to 2$ upside down potential
- $ie\overline{\psi}\gamma^{\mu}\psi A_{\mu}$, A_{μ} is an *axial* vector, e gauge charge
- Above changes sign of electric force: $\mathcal{PT}$ QED
- Axion electrodynamics: $ig\ \frac{\alpha}{4\pi f_a} a\,(x)\,F_{\alpha\beta}\,(x)\,\tilde{F}^{\alpha\beta}\,(x)$; $a\,(x)$ is a pseudoscalar, coupling g real; $\tilde{F}^{\alpha\beta}\,(x) = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}\,(x)$ [3]
- Axion term is $\mathcal{PT}$ symmetric since $F\tilde{F}$ and ia are both $\mathcal{PT}$ symmetric

Renormalisation and $\mathcal{PT}$ symmetry

Lee model [4, 5] : $\mathcal{PT}$ appears from nowhere

Toy model "fermionic" particles N and V, and bosonic particle θ; renormalisation exact
The interactions with coupling g in the model allow

$$\boxed{V \to N + \theta} \tag{1}$$

and so $g_0 \psi_V^\dagger \psi_N a$ and also the reverse process

$$\boxed{N + \theta \to V.} \tag{2}$$

No crossing symmetry:

$N \to V + \bar{\theta}$ is not allowed

Coupling constant renormalisation: $g^2 > M^2$, then the bare coupling can become imaginary: Hamiltonian is $\mathcal{PT}$-symmetric

$$g_0^2 = g^2 / \left(1 - \frac{g^2}{M^2}\right) \tag{3}$$

Renormalisation group flows

A non-toy relative of the Lee model:

Chiral Yukawa model [6, 7]

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2 + \overline{\psi}\left(i\not\partial - m\right)\psi - ig\overline{\psi}\gamma_5\psi\phi - \frac{u}{4!}\phi^4 \tag{4}$$

$u > 0$ Hermitian; $u < 0$ has a $\mathcal{PT}$ phase (conventionally unbounded); g, ig
Spacetime dimension $D = 4 - \epsilon$, $\epsilon > 0$

Unitarity in $\mathcal{PT}$

1. $u < 0$ is $\lim_{\delta\to 2} |u| (i\phi)^\delta \phi^2$ so not self-adjoint with DIrac inner product (i.p.); real eigenvalues
2. $\mathcal{PT}$ Hamiltonian H self-adjoint w.r.t. different i.p.; real eigenvalues
3. Formally Hilbert space has a new inner product η (hard to find)

Schwinger construction [9, 10]
$Z_1[j] = \int D\phi \exp\left(iS[\phi] - j(x)\phi(x)\right)$
$Z_2[j] = \langle 0|\eta T\left(\exp\left[-\int dx j(x)\phi(x)\right]\right)|0\rangle$
$Z[j] = Z_1[j] = Z_2[j]$
Schwinger-Dyson equations are the same using either Z_i
$Z_1[j] \to \Gamma[\phi]$ the effective action of Wetterich [11] and the functional RG

Renormalisation group [6, 8] I

Using a general purpose Mathematica program RGBeta [8] , in terms of $t = \log\mu$ and $h = g^2$ the renormalisation group beta functions are

$$\frac{dh}{dt} = \beta_h(h,u) \text{ and } \frac{du}{dt} = \beta_u(h,u) \tag{5}$$

where

$$\begin{aligned}\beta_h(h,u) = &-\epsilon h + \frac{1}{(4\pi)^2}10h^2 + \frac{1}{(4\pi)^4}\left(-\frac{57}{2}h^3 - 4h^2u + \frac{1}{6}hu^2\right)\\ &+ \frac{1}{(4\pi)^6}\left(\left[-\frac{339}{8} + 222\,\zeta(3)\right]h^4 + 72h^3u + \frac{61}{24}h^2u^2 - \frac{1}{8}hu^3\right)\end{aligned} \tag{6}$$

and

$$\beta_u(h,u) = -\epsilon u + \frac{1}{(4\pi)^2}\left(-48h^2 + 8hu + 3u^2\right) + \frac{1}{(4\pi)^4}\left(384h^3 + 28h^2u - 12hu^2 - \frac{17}{3}u^3\right). \tag{7}$$

Renormalisation group [6, 8] II

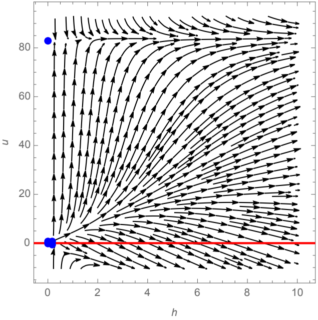

Figure: Global flow for $\epsilon = 0.01$. There are a group of four fixed points that are close to the origin, and one high-u fixed point that we ignore from concerns over its validity in perturbation theory. 4 nontrivial fixed points, 1 trivial fixed point. Flow from infrared Hermitian fixed point to nonHermitian UV fixed point

Renormalisation group [6, 8] III

Finite RG flows

1)No flows from positive to negative h and vice versa
2) Flows from positive u to negative u, i.e. Hermitian to $\mathcal{PT}$-symmetric .

Fixed points are $O(\epsilon)$ Flows where pertubation theory valid.
One-loop β function of g as $\epsilon \to 0^+$

$$\frac{d}{dt}(g^2) = \frac{5}{8\pi^2}(g^2)^2 - \epsilon g^2, \tag{8}$$

where $d/dt \equiv \mu\, d/d\mu$, and μ is a transmutation mass scale. The solution with $\epsilon t \ll 0$ is

$$g^2 \approx -\frac{1}{C + \frac{5}{8\pi^2}t}, \tag{9}$$

where C is an integration constant.
Hermitian couplings, $C < 0$, g^2 increases, g^2 pole at finite $t = t_p = 8\pi^2|C|/5$ [13]

$t > t_p$ $g^2 < 0$

$\mathcal{PT}$ symmetric, asymptotically free

Renormalisation group [6, 8] IV

Conjecture to avoid pole [14, 15, 16]

Conjecture in higher D

$$\ln Z_{PT}(g) = \mathrm{Re}\left[\ln Z\left(\lambda = -g + i0^{+}\right)\right]$$

based on weak coupling, zero temperature, quartic scalar interactions.

In other situations, the conjecture, as expressed in the above form, will not be expected to hold [17].

Something similar may be valid though.

Gravitational multiplet in string theory I

Compactification to 4 space-time dimensions:
the bosonic ground state closed-string sector [18]
consists of *massless* fields in the so-called *gravitational multiplet*, which contains

- spin-0 (scalar) dilaton $\Phi(x)$
- $g_{\mu\nu}(x)$, (3+1)-dimensional graviton
- spin-1 antisymmetric tensor *gauge* Kalb-Ramond field $B_{\mu\nu}(x) = -B_{\nu\mu}(x)$
- field strength $\mathcal{H}_{\mu\nu\rho}(x) = \partial_{[\mu} B_{\nu\rho]}(x) + \Delta_{\mu\nu\rho}$ from Green-Schwarz mechanism (cancels gauge and gravity anomalies)
- $\Delta_{\mu\nu\rho}$ leads to Chern-Simons gravity [19], since Bianchi identity

$$\boxed{\varepsilon^{\mu\nu\rho\sigma}\,\mathcal{H}_{[\nu\rho\sigma\,;\,\mu]}} = \varepsilon_{abc}{}^{\mu}\,\mathcal{H}^{abc}{}_{;\mu} = \boxed{\frac{\alpha'}{32\,\kappa}\,\sqrt{-g}\,\Big(R_{\mu\nu\rho\sigma}\,\widetilde{R}^{\mu\nu\rho\sigma} - \mathbf{F}_{\mu\nu}\,\widetilde{\mathbf{F}}^{\mu\nu}\Big)}\,, \tag{10}$$

Gravitational multiplet in string theory II

- In $3+1$ dimensions KR axion field b is the dual of the KR field strength $\mathcal{H}_{\mu\nu\rho}$:

$$\boxed{\partial_\sigma b \propto \varepsilon_{\sigma\mu\nu\rho}\mathcal{H}^{\mu\nu\rho}}\,. \tag{11}$$

- Euclidean effective string action:

$$S_{\rm B}^{\rm eff(I)} = -\int d^4x\sqrt{-g}\Big[\frac{1}{2\kappa^2}\,R + \frac{1}{6}\,\mathcal{H}_{\lambda\mu\nu}\,\mathcal{H}^{\lambda\mu\nu} - \boxed{\sqrt{\frac{2}{3}}\,\frac{\alpha'}{96\,\kappa}\,b(x)\,\Big(R_{\mu\nu\rho\sigma}\,\widetilde{R}^{\mu\nu\rho\sigma} - \mathbf{F}_{\mu\nu}\,\widetilde{\mathbf{F}}^{\mu\nu}\Big)} + \dots\Big], \tag{12}$$

Gravitational multiplet in string theory III

Ambiguity: emergence of $\mathcal{PT}$ symmetry [20]

From (11) we need to evaluate $\varepsilon_{\mu\nu\rho\lambda}\,\varepsilon^{\mu\nu\rho\sigma}$ which can be done before or after continuing to Minkowski space from Euclidean space:

$$\varepsilon^{(\mathrm{E})}_{\mu\nu\rho\lambda}\,\varepsilon^{\mu\nu\rho\sigma\,(\mathrm{E})} = +6\,\delta^{\sigma}_{\lambda}\,, \tag{13}$$

and

$$\varepsilon_{\mu\nu\rho\lambda}\,\varepsilon^{\mu\nu\rho\sigma} = -6\delta^{\sigma}_{\lambda}\ , \tag{14}$$

So $\frac{1}{6}\,\mathcal{H}_{\lambda\mu\nu}\,\mathcal{H}^{\lambda\mu\nu} = \pm\frac{1}{2}\partial_\mu b\partial^\mu b$
Minus choice leads to ghost b field: $b\,(x) \to ib\,(x)$ canonical kinetic term

BUT

$$\boxed{i\sqrt{\frac{2}{3}}\,\frac{\alpha'}{96\,\kappa}\,b(x)\,\Big(R_{\mu\nu\rho\sigma}\,\widetilde{R}^{\mu\nu\rho\sigma} - \mathbf{F}_{\mu\nu}\,\widetilde{\mathbf{F}}^{\mu\nu}\Big)}$$

which is $\mathcal{PT}$ symmetric.

Simplification:

Gravitational multiplet in string theory IV

- Reduce gauge sector to $U(1)$ in flat space
- Yang-Mills contribution topological and so not dependendet on metric
- leads to Hermitian or $\mathcal{PT}$ symmetric axion electrodynamics
- axion gravitodynamics part ignored: comes from anomaly cancellation (which arises through a one loop calculation); maybe not affected by renormalisation

Above needs further justification: gravitons could affect axion self energy

Axion electrodynamics [3]

RG flows axion electrodynamics

Euclidean Lagrangian $\mathcal{L}_E$

$$\mathcal{L}_E = \frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{1}{2}\partial_\mu b \partial_\mu b + \frac{1}{2}m_R^2 b^2 + \frac{1}{4}i\mathfrak{g}_{\mathrm{R}}\, b\, F_{\mu\nu}\tilde{F}_{\mu\nu}\,, \tag{15}$$

$[\mathfrak{g}_R] = -1$ is -1

Theory is not perturbatively renormalisable.

Functional RG $\longrightarrow$ $\quad \partial_t g^2 = (2 + 2\gamma_F + \gamma_a)\, g^2, \quad \partial_t m^2 = (\gamma_a - 2)\, m^2$

where (*scheme dependent*,g, m dimensionless,$m^2 \equiv \frac{m_R^2}{k^2}$, $\mathfrak{g}_R^2 k^2$)

$$\gamma_a = \frac{g^2}{6\,(4\pi)^2}\left(2 - \frac{\gamma_F}{4}\right), \tag{16}$$

$$\gamma_F = \frac{g^2}{6(4\pi)^2}\left(\frac{(2-\frac{\gamma_a}{4})}{(1+m^2)^2} + \frac{(2-\frac{\gamma_F}{4})}{1+m^2}\right) \tag{17}$$

Wetterich equation [11]

Derived from Wetterich equation (which can treat nonrenormalisable theories)
Effective action functional Γ_k with *smoothly suppressed* infrared modes

$$\boxed{\partial_t \Gamma_k [\phi] = \frac{1}{2} Tr \left[\frac{1}{\Gamma_k^{(2)} + R_k} \partial_t R_k \right]} \tag{18}$$

where $\Gamma_k^{(2)}$ is a second functional derivative and R_k is a smoothing function.
Ansatz for Γ_k:

$$\Gamma_k = \int d^4x \left[\begin{array}{c} \frac{Z_F}{4} (F_{\mu\nu}(x))^2 + \frac{Z_a}{2} (\partial_\mu b(x))^2 + \frac{\bar{m}_k^2}{2} b(x)^2 \\ + i \frac{\bar{g}_k}{4} b(x) F_{\mu\nu}(x) \tilde{F}_{\mu\nu}(x) \end{array} \right] + Z_F \int d^4x \; L_{GaugeFix}$$

Fixed points and singularities

Massless case (relevant for Kalb-Ramond axion and Chern-Simons gravity):

$$\boxed{\partial_t g^2 = 2g^2 \frac{13g^4 \; - \; 8064\, \pi^2 g^2 - 147456\pi^4}{g^4 - 384\pi^2 g^2 - 147456\pi^4}} \tag{19}$$

$$\text{Singularity of } \beta_{\mathrm{g}} \text{ occurs for } g_{\mathrm{sing}} : (\pm\, 48.3728 i, \quad \pm 78.2689)\,. \tag{20}$$

$\mathcal{PT}$ symmetric singularity, Hermitian singularity $g^2_{sing} = (-2339.93, 6126.02)$

Nontrivial fixed points denoted by $g^\star$ are:

$$g^\star = (\pm 13.2389\; i, \quad \pm 79.3174) \Rightarrow g^2_* = (-175.268, 6291.25)\,. \tag{21}$$

$\mathcal{PT}$ symmetric fixed point, Hermitian fixed point

Beta function 1a

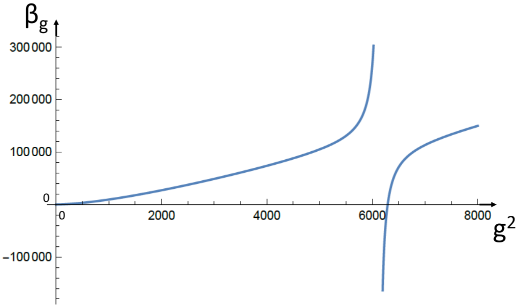

Figure: The beta function $\beta_g \equiv \partial_t g^2$ of the Hermitian theory

Exhibits the proximity of a singular behavior at a finite running coupling

Beta function 1b

Fixed point behavior $(\beta_g(g)\big|_{g=g_\star}=0)$, characterised by the existence of a trivial infra-red fixed point at zero coupling, and a non-trivial infra-red fixed point at strong coupling, $g_\star^2 \gg 1$.

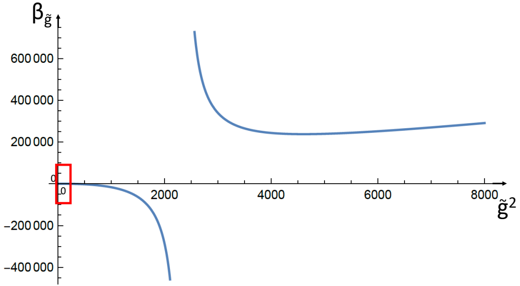

Figure: Nonperturbative RG beta function for non-Hermitian case as a function of $\tilde{g}^2$, is qualitatively different from the Hermitian beta function. Non-pole singular behaviour at finite coupling.

Scale $k = k_{\rm sing}$ corresponding to singularity by solving RG equation. $g(k)$ has infinte sheeted Riemann sheet structure near singularity (not a pole)

Beta function 1c

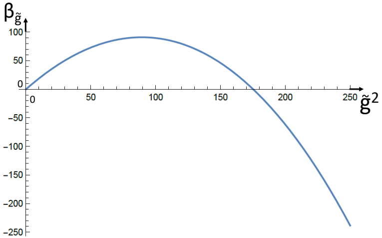

Figure: Zoomed-in Nonperturbative RG beta function for non-Hermitian case

Connection to running gravity

Axion coupling and Newton's constant

$$g^2 \equiv (\mathfrak{g}_R(k)k)^2 = \frac{1}{864}\,\frac{M^2_{\mathrm{Pl,R}}(k)}{M_s^4}\,k^2 = \frac{1}{864}\,\frac{1}{\widetilde{\mathrm{G}}_{\mathrm{N}}}\,\frac{k^2}{M_s^2}\,. \tag{22}$$

M_s: UV cutoff in the effective low-energy theory, $t \equiv \log\left(\frac{k}{M_s}\right)$

G_{N} is (3+1)-dimensional gravitational (Newton's) coupling,

$\widetilde{\mathrm{G}}_{\mathrm{N}} \equiv \mathrm{G}_{\mathrm{N}}\, M_s^2$ its dimensionless counterpart.

$$\boxed{\frac{d}{dt}\log\left(g^2\,\widetilde{\mathrm{G}}_{\mathrm{N}}\right) = 2 \quad \Rightarrow \quad g^2\,\widetilde{\mathrm{G}}_{\mathrm{N}} = \mathfrak{D}\exp\left(2t\right)}\,, \tag{23}$$

where $\mathfrak{D}$ is a positive integration constant to be fixed by the boundary conditions

UV regime, $t \to 0$

Connection to running gravity 2

Infrared (IR) region corresponds to $k \to 0$, or, some times, for practical purposes $k \to k_0 = m_{\rm IR} \ll M_s$, with $m_{\rm IR}$ an infrared mass cutoff.

$$\widetilde{\rm G}_{\rm N}(t \to -\infty) = \lim_{t \to -\infty} \frac{g^2(t=0)}{g^{\star\,2}}\, \widetilde{\rm G}_{\rm N}(t=0)\, e^{2t} = 0\,, \tag{24}$$

indicating that the gravitational constant at the singularity goes to zero (using $\mathcal{PT}$ axion electrodynamics with $g^{\star\,2}$ the non-trivial UV fixed point).

The finite value $\widetilde{\rm G}_{\rm N}(t=0)$ can be identified with the Newton's constant in the UV regime (short distances)

Conclusions

The inclusion of axion graviton interactions is one next step to see the robustness of the findings, but $\mathcal{PT}$ and axion physics is an interesting area in terms of repulsive gravity.

References I

- Bender, C.M. Making sense of non-Hermitian Hamiltonians. *Rept. Prog. Phys.*. **70** pp. 947 (2007)
- Bender, C. & Boettcher, S. Real spectra in nonHermitian Hamiltonians having PT symmetry. *Phys. Rev. Lett.*. **80** pp. 5243-5246 (1998)
- Eichhorn, A., Gies, H. & Roscher, D. Renormalization Flow of Axion Electrodynamics. *Phys. Rev. D*. **86** pp. 125014 (2012)
- Bender, C., Brandt, S., Chen, J. & Wang, Q. Ghost busting: PT-symmetric interpretation of the Lee model. *Phys. Rev. D*. **71** pp. 025014 (2005)
- Lee, T. Some Special Examples in Renormalizable Field Theory. *Phys. Rev.*. **95** pp. 1329-1334 (1954)

References II

- Croney, L. & Sarkar, S. Renormalization group flows connecting a $4-\epsilon$ dimensional Hermitian field theory to a PT-symmetric theory for a fermion coupled to an axion. *Phys. Rev. D*. **108** pp. 085024 (2023)
- C. Schubert, Nucl. Phys. B **323** (1989), 478-492 doi:10.1016/0550-3213(89)90153-3
- Thomsen, A. Introducing RGBeta. *Eur. Phys. J. C*. **81**, 408 (2021)
- Jones, H. & Rivers, R. Which Green Functions Does the Path Integral for Quasi-Hermitian Hamiltonians Represent?. *Phys. Lett. A*. **373** pp. 3304-3308 (2009)
- Rivers, R. PATH INTEGRAL METHODS IN QUANTUM FIELD THEORY. (Cambridge University Press,1988,10)
- Wetterich, C. Exact evolution equation for the effective potential. *Physics Letters B*. **301**, 90-94 (1993)

References III

- Mavromatos, N. & Sarkar, S. Chern-Simons gravity and PT symmetry. *Phys. Rev. D*. **110**, 045004 (2024)
- Romatschke, P. Life at the Landau pole. *AppliedMath*. **4** pp. 55-69 (2024)
- Ai, W., Bender, C. & Sarkar, S. PT-symmetric $-g\phi^4$ theory. *Phys. Rev. D*. **106**, 125016 (2022)
- Chen, L. & Sarkar, S. Phases of scalar fields and $\mathcal{PT}$ symmetry. arXiv:2409.05439 [quant-ph] (2024,9)
- Collins, J. & Soper, D. Large Order Expansion in Perturbation Theory. *Annals Phys.*. **112** pp. 209-234 (1978)
- Lawrence, S., Weller, R., Peterson, C. & Romatschke, P. Instantons, analytic continuation, and PT-symmetric field theory. *Phys. Rev. D*. **108**, 085013 (2023)

References IV

- Green, M., Schwarz, J. & Witten, E. Superstring Theory Vol. 1: 25th Anniversary Edition. (Cambridge University Press,2012,11)
- Duncan, M., Kaloper, N. & Olive, K. Axion hair and dynamical torsion from anomalies. *Nucl. Phys. B*. **387** pp. 215-235 (1992)
- Mavromatos, N. Non-Hermitian Yukawa interactions of fermions with axions: potential microscopic origin and dynamical mass generation. *J. Phys. Conf. Ser.*. **2038** pp. 012019 (2020,10)
- Cesare, M., Mavromatos, N. & Sarkar, S. On the possibility of tree-level leptogenesis from Kalb–Ramond torsion background. *Eur. Phys. J. C*. **75**, 514 (2015)