Dynamics of thermal bubble nucleation

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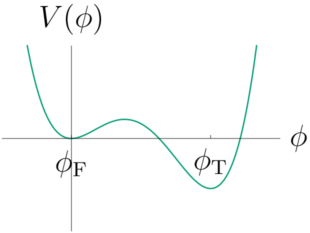
Tunneling in QFT — online seminars

Upcoming: indico.cern.ch/e/tunnelingqft

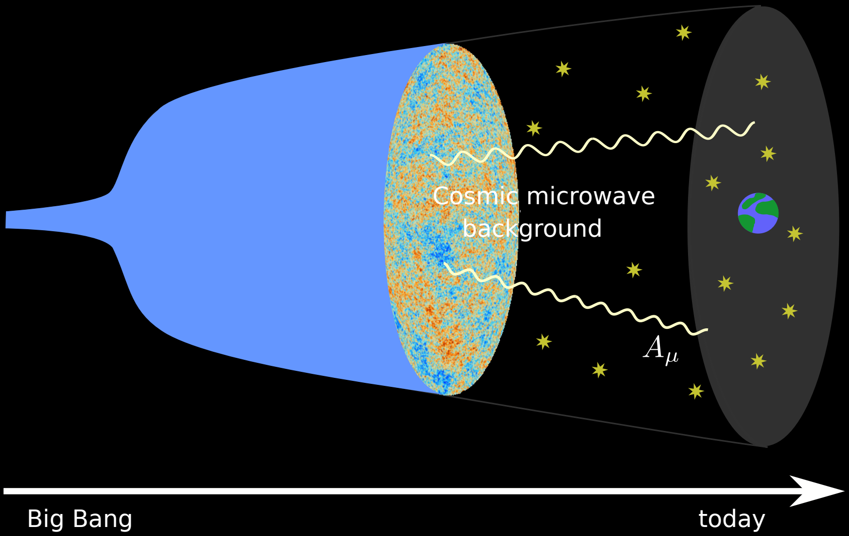
- 19 September: Rosemary Zielinski, *Particle escape in QFT*
- 17 October: Simone Blasi, *Nucleation on domain wall seeds*
- 14 November: Mark Hindmarsh, *Testing tunneling theory in ^{3}He*

Previous: youtube.com/@tunnelingqft

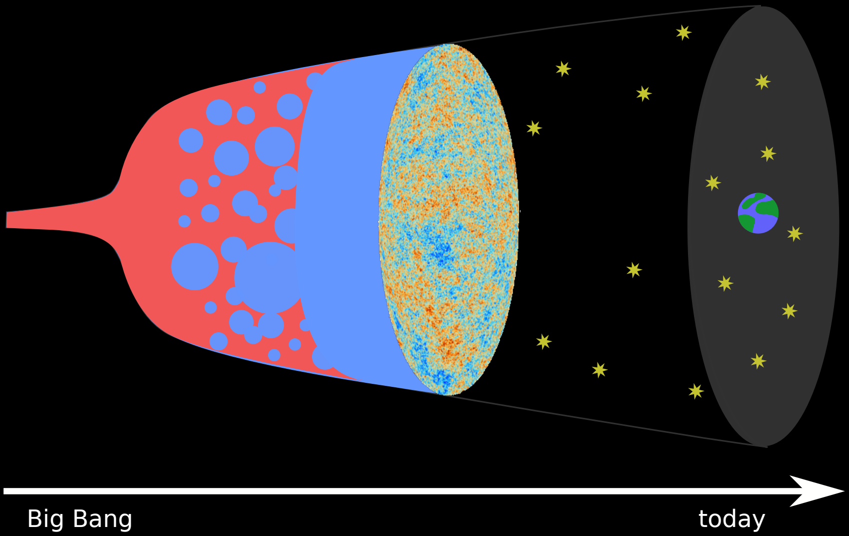
- Laura Batini
- Wen-yuan Ai
- Lorenzo Ubaldi
- Patrick Draper
- Yutaro Shoji
- Silvia Pla Garcia
- Ian Moss



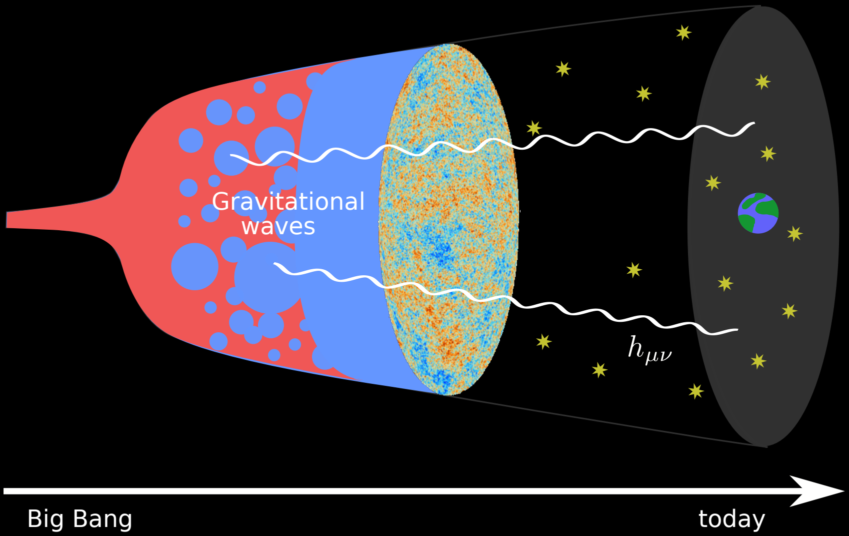
Andreas Ekstedt, OG & Miha Nemevšek



Cosmic microwave background
A_μ
Big Bang
today



Big Bang
today



Gravitational waves
$h_{\mu\nu}$
Big Bang
today

GW frequencies and Hubble radius

- Red-shifting signal produced on frequencies $f_* \geq H_*$:

$$f_{\text{today}} = \left(\frac{a_*}{a_0}\right) f_* \propto g_*^{1/6} T_*$$

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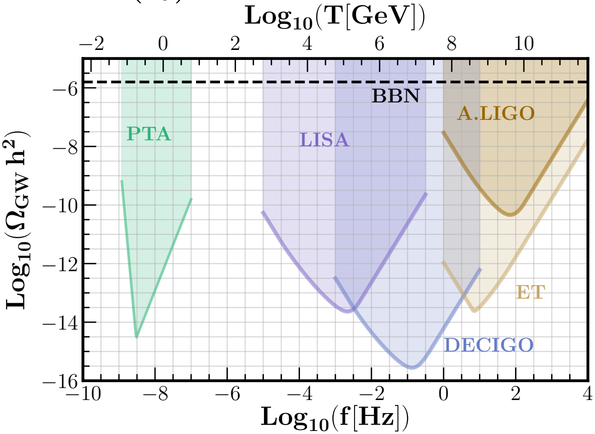


- PTA $\sim$ GeV
- LISA $\sim$ TeV
- LIGO $\sim$ EeV

Domenech 2109.01398

Cosmological 1st-order phase transitions

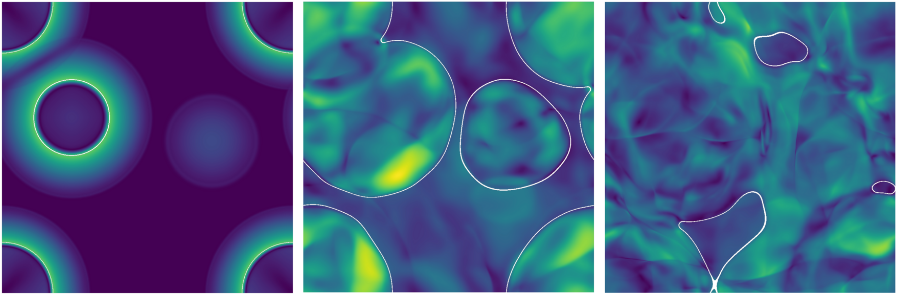


Figure: Cutting et al. arXiv:1906.00480.

- Universe supercools
- Bubbles nucleate, expand and collide
- This creates long-lived fluid flows
- And creates gravitational waves: $\Box h_{ij}^{(\mathrm{TT})} \sim T_{ij}^{(\mathrm{TT})}$

Gravitational wave spectrum

GW signal depends strongly on 4 phase transition quantities,

$$\Omega_{\mathrm{GW}} = F(T_*, R_*, \alpha_*, v_{\mathrm{w}}),$$

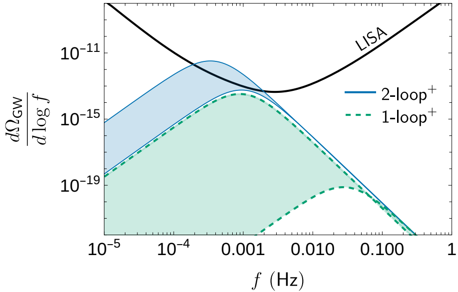
T_* : percolation temperature,

R_* : bubble radius,

α_* : transition strength,

v_{w} : bubble wall speed.

Each depends on the bubble nucleation rate.

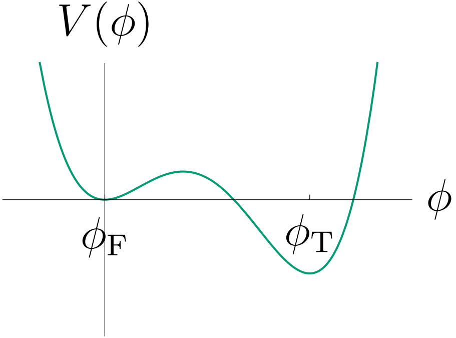


Large uncertainties linked to predictions of nucleation rate.

OG & Tenkanen '21

Zeroing in on vacuum decay

Fate of the false vacuum



Vacuum decay rate per unit volume,

$$\Gamma \approx \left(\frac{S_{\rm E}[\phi_{\rm B}]}{2\pi}\right)^2 \left|\frac{\det' S''_{\rm E}[\phi_{\rm B}]}{\det S''_{\rm E}[\phi_{\rm F}]}\right| \times e^{-S_{\rm E}[\phi_{\rm B}]}.$$

Coleman '77

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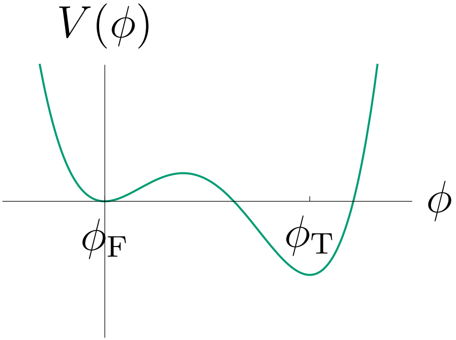

Vacuum decay rate per unit volume,

$$\Gamma \approx \underbrace{\left(\frac{S_E[\phi_B]}{2\pi}\right)^2 \left|\frac{\det' S_E''[\phi_B]}{\det S_E''[\phi_F]}\right|^{-1/2}}_{\text{our focus next, often ignored}} \times \underbrace{e^{-S_E[\phi_B]}}_{\text{standard stuff}} .$$

Coleman '77

Functional determinants for vacuum decay

We encounter objects like

$$\det S_{\mathsf{E}}''[\phi_{\mathsf{B}}] = \det \left[-\nabla^2 + W(\rho)\right].$$

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where for each orbital number l, the radial determinant

$$\det\Bigg[\underbrace{-\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - \frac{3}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} + \frac{l(l+2)}{\rho^2}}_{-\nabla_l^2} + W(\rho)\Bigg]$$

can be computed with the Gelfand-Yaglom theorem.

Gelfand & Yaglom '60

Renormalising functional determinants

Adding one-loop counterterms

$$S[\phi] = S_0[\phi] + \hbar S_{\mathrm{c.t.}}[\phi] + O(\hbar^2),$$

Where are the UV divergences?

$$\Gamma \approx \underbrace{\left(\frac{S_0[\phi_{\mathrm{B}}]}{2\pi}\right)^2 \left|\frac{\det' S_0''[\phi_{\mathrm{B}}]}{\det S_0''[\phi_{\mathrm{F}}]}\right| e^{-S_{\mathrm{c.t.}}[\phi_{\mathrm{B}}]}}_{\text{UV finite}} \times \underbrace{e^{-\frac{1}{\hbar}S_0[\phi_{\mathrm{B}}]}}_{\text{UV finite}} \times \left[1 + O(\hbar^2)\right],$$

Konoplich '87

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Konoplich '87

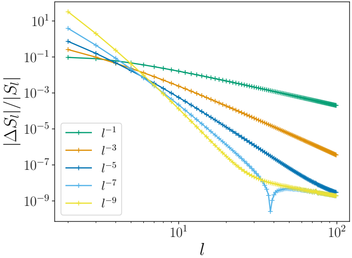
Reveals determinant is exponential, not just $\sim m_\phi^4$

Functional determinant numerics

How to renormalise numerically?

- Each orbital number l is UV finite
  $\Rightarrow$ UV divergences from $l \to \infty$.

- WKB approx. $= l^{-1}$ expansion.



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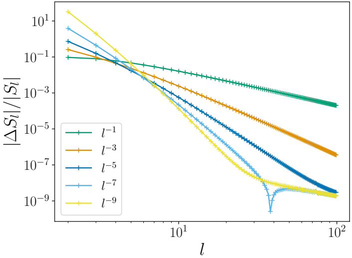


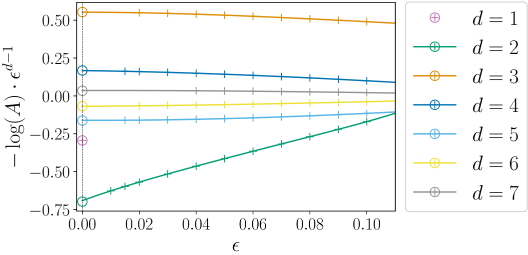
- Add $0 = -S^{\text{WKB}} + S^{\text{WKB}}$ and collect terms:

$$\sum_{l=2}^{\infty} S_l + S_{\text{c.t.}} = \underbrace{\sum_{l=2}^{\infty}(S_l - S_l^{\text{WKB}})}_{\text{finite and converges faster}} + \underbrace{\sum_{l=2}^{\infty} S_l^{\text{WKB}} + S_{\text{ct}}}_{\text{finite and known}} .$$

Dunne et al. '04, Ekstedt, OG, Hirvonen '23

BubbleDet

First public code for computing these bubble determinants.



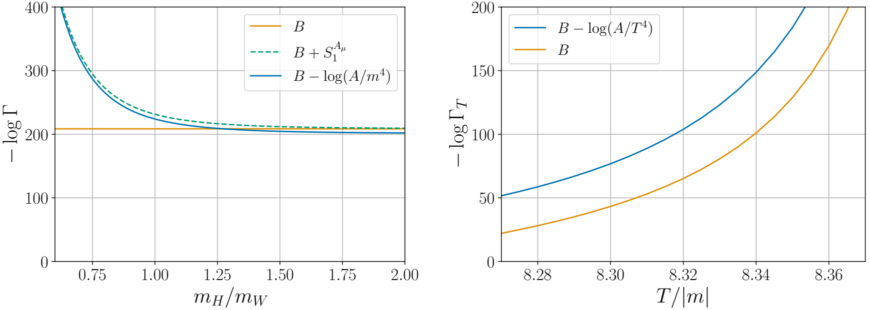
Ekstedt, OG, Hirvonen '23

An example, for $\lambda\phi^4$ in the thin wall limit for $d = 3$,

$$\Gamma \sim \underbrace{m^3 \exp\left(-\frac{1}{\epsilon^2}\left[\frac{10}{27}+\frac{\log 3}{6}\right]\right)}_{\text{one-loop}}\underbrace{\exp\left(-\frac{1}{\epsilon^2\lambda}\left[\frac{32\pi}{81}\right]\right)}_{\text{tree-level}}$$

Munster & Rotsch '00, Matteini et al. 24

Decay rates at one loop



(a) Vacuum decay in U(1) Higgs

(b) Thermal decay in Yukawa model

$$\Gamma = A\, e^{-B}$$

Simple scaling $A \sim m^4$ or T^4 is not a great approximation.

Ekstedt, OG, Hirvonen '23

Thermal bubble nucleation

Quantum, stochastic, thermal

Stochastic FT in 4+1D

Wick rotation

Quantum FT in 3+1D

Zero temperature

Energies << T

Stochastic FT in 3+1D

Wick rotation

Quantum FT in 2+1D

High temperature

Feynman & Vernon '63, Parisi & Wu '80, Morikawa '86, Greiner & Müller '96

Stochastic quantization

Euclidean QFT

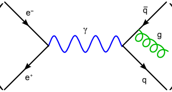
$$\langle \phi(x)\phi(y)\phi(z)\rangle = \frac{1}{Z}\int \mathcal{D}\phi\ \phi(x)\phi(y)\phi(z)e^{-S_{\mathrm{E}}[\phi]}$$

Stochastic evolution reproduces all correlation functions

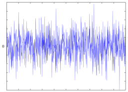
$$\frac{\partial \phi}{\partial t_5} = -\frac{\delta S_{\mathrm{E}}}{\delta \phi} + \underbrace{\eta}_{\text{noise}},$$

so that in Euclidean signature

$\Rightarrow \Bigg($



$\Bigg)_{4\mathrm{D}} = \Bigg($



$\Bigg)_{4+1\mathrm{D}}$

Parisi & Wu '80

Time$_5$less decay

Different stochastic equations are possible

$$\frac{\partial \phi}{\partial t_5} = -\frac{\delta S_{\mathrm{E}}}{\delta \phi} + \eta, \qquad \frac{\partial^2 \phi}{\partial t_5^2} + \gamma \frac{\partial \phi}{\partial t_5} = -\frac{\delta S_{\mathrm{E}}}{\delta \phi} + \eta, \qquad \ldots$$

as long as the distribution has the late-time attractor solution

$$P(\phi) \propto e^{-S_{\mathrm{E}}[\phi]},$$

the specifics of the t_5 time evolution are irrelevant.

Ryang, Saito & Shigemoto '85, Horowitz '85

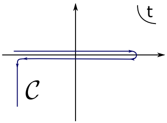
$\Rightarrow$vacuum $\Gamma_{4\mathrm{D}}$ is pure equilibrum from 4+1D perspective

$S_{\mathrm{E}}[\phi_{\mathrm{B}}]$ is *energy* of bubble in 4+1D, then one-loop determinant is *entropy*.

Quantum thermal nucleation rate

What about real thermal stochasticity?

In real-time at $T \neq 0$, correlation functions look like



$$\begin{aligned}\langle \mathcal{O}(t)\mathcal{O}(0)\rangle &= \frac{1}{Z}\mathrm{Tr}\left[e^{-\hat{H}/T}\left(e^{i\hat{H}t}\mathcal{O}(0)e^{-i\hat{H}t}\right)\mathcal{O}(0)\right], \\ &= \int_{\mathcal{C}} \mathcal{D}\phi\ \mathcal{O}(t)\mathcal{O}(0)\ e^{iS[\phi]}, \\ &= \int \mathcal{D}\phi\mathcal{D}\phi'\ \mathcal{O}(t)\mathcal{O}(0)\ e^{iS_T[\phi]-iS_T[\phi']}.\end{aligned}$$

Schwinger '61, Keldysh '65

How can we arrive at thermal stochastic equations?

Influence functional and open EFT

Split the field based on momentum

$$\phi(t,\mathbf{p}) = \underbrace{\theta(\Lambda - |\mathbf{p}|)\phi(t,\mathbf{p})}_{\phi_{\text{IR}}} + \underbrace{\theta(|\mathbf{p}| - \Lambda)\phi(t,\mathbf{p})}_{\phi_{\text{UV}}},$$

Bödeker, McLerran & Smilga '95, Lombardo & Mazzitelli '95

and integrate over the UV modes, assuming $\rho = \rho_{\text{UV}} \times \rho_{\text{IR}}$,

$$\int \mathcal{D}\phi\mathcal{D}\phi' \rho e^{i(S[\phi]-S[\phi'])}$$
$$= \int \mathcal{D}\phi_{\text{IR}}\mathcal{D}\phi'_{\text{IR}}\rho_{\text{IR}} e^{i(S[\phi_{\text{IR}}]-S[\phi'_{\text{IR}}]+S_{\text{IF}}[\phi_{\text{IR}},\phi'_{\text{IR}}])}.$$

The influence functional S_{IF} gives the effect of the UV modes, in the in-in formalism.

Feynman & Vernon '63

Complex influence functionals

In general the action for the IR modes is **complex**

$$e^{i(S[\phi_{\rm IR}]-S[\phi'_{\rm IR}]+{\rm Re}S_{\rm IF}[\phi_{\rm IR},\phi'_{\rm IR}])}\times e^{-{\rm Im}S_{\rm IF}[\phi_{\rm IR},\phi'_{\rm IR}]},$$

but not arbitrarily so,

$$\text{unitarity} \Rightarrow S_{\rm IF}[\phi_{\rm IR},\phi'_{\rm IR}] = S_{\rm IF}[\phi'_{\rm IR},\phi_{\rm IR}]^*, \qquad S_{\rm IF}[\phi_{\rm IR},\phi_{\rm IR}]=0,$$
$$\text{thermality} \Rightarrow S_{\rm IF}[\phi_{\rm IR}(t),\phi'_{\rm IR}(t')] = S_{\rm IF}[\phi_{\rm IR}(-t),\phi'_{\rm IR}(-t'-i/T)].$$

see Crossley, Glorioso & Liu '17; Haehl, Loganayagam & Rangamani '17

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see Crossley, Glorioso & Liu '17; Haehl, Loganayagam & Rangamani '17

Rephrasing with a Hubbard-Stratonovich trasform,

$$\underbrace{e^{-\frac{1}{2}\phi_{\rm IR}\cdot{\rm Im}\Gamma^{(2)}\cdot\phi_{\rm IR}}}_{\text{for example}} = \frac{1}{\sqrt{\det {\rm Im}\Gamma^{(2)}}}\int \mathcal{D}\chi e^{-\frac{1}{2}\chi\cdot({\rm Im}\Gamma^{(2)})^{-1}\cdot\chi}e^{i\chi\cdot\phi_{\rm IR}},$$

we get a **unitary** $\phi_{\rm IR}$ evolution coupled to a **stochastic** χ evolution.

Effective stochastic $\lambda\phi^4$

The semiclassical limit is Langevin evolution, e.g. for $\lambda\phi^4$,

$$-\Box\phi_{\mathrm{IR}}(x) + m^2\phi_{\mathrm{IR}}(x) + \lambda\phi_{\mathrm{IR}}(x)^3 + \int_{t_i}^{t} d^4y\,\mathrm{Re}\Gamma^{(2)}(x-y)\phi_{\mathrm{IR}}(y) = \chi(x) + \dots$$

where the stochastic variable satisfies

$$\langle\chi(x)\chi(y)\rangle = \mathrm{Im}\Gamma^{(2)}(x-y),$$

and where $\Gamma^{(2)}$ is the UV contribution to the IR self-energy.

Gleiser & Ramos '93, Greiner & Müller '96

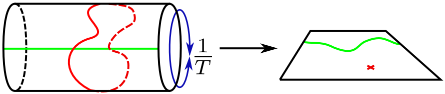
High-temperature dimensional reduction

Thermal Langevin equations are stochastic quantisation of a *2+1D vacuum QFT* (Wick rotated), leading to

$$\underbrace{\Gamma_T^{(3+1\mathrm{D})}}_{\text{thermal nucleation}} = \underbrace{(\text{dynamical factor})}_{\equiv \kappa_{\mathrm{dyn}}/(2\pi)} \times \underbrace{\Gamma_0^{(2+1\mathrm{D})}}_{\text{vacuum decay}}$$

OG & Hirvonen '21

where the 2+1D theory is that of high-T dimensional reduction.



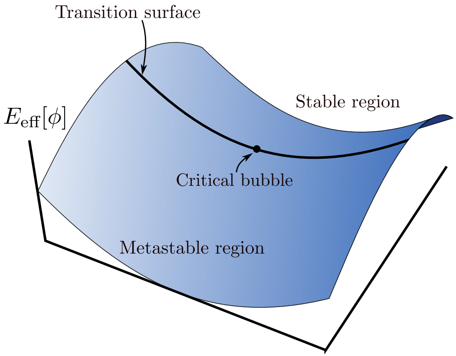
Appelquist & Pisarski '81, Kajantie et al. '95, Braaten & Neito '95

Out-of-equilibrium effects on nucleation

So thermal nucleation is not just energy and entropy,

$$\Gamma_T \approx \frac{\kappa_{\text{dyn}}}{2\pi} m^3 e^{-(E_\text{b} - S_\text{b} T)/T}$$

Langer '69



and κ_{dyn} depends on dynamics, and out-of-equilibrium backreaction of particles Δf:

$$\kappa_{\text{dyn}}^2 \Delta\phi = \underbrace{(\nabla^2 - V''[\phi_b])}_{\text{Affleck '81}} \Delta\phi - \underbrace{\sum_a \frac{\mathrm{d}m_a^2}{\mathrm{d}\phi} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E} \Delta f}_{\text{Hirvonen '24}},$$

Do we really understand bubble nucleation?

- At least five different expressions for κ_{dyn}.

  Langer '69, Affleck '81, Linde '81

  Arnold & McLerran '87, Hirvonen '24

- Infrared divergences in κ_{dyn} for weak damping.

  Hangi et al. '90, Ekstedt '22

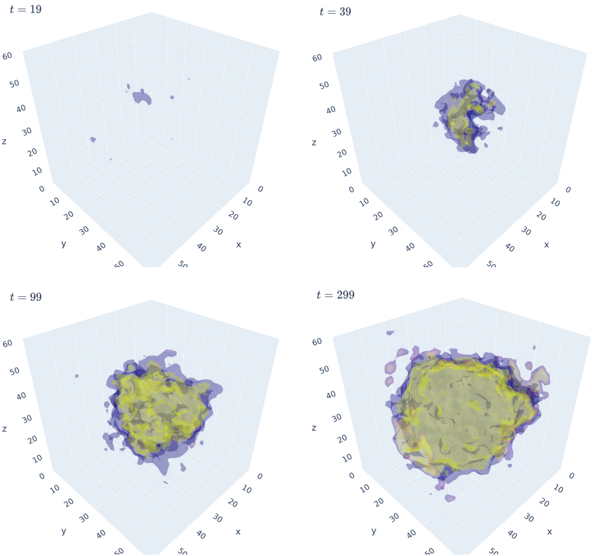
- Radical proposals:
  - additional saddlepoints
  - nucleation via intermediate solitons/oscillons

  Tye & Wong '11, Pîrvu, Johnson & Sibiryakov '23

- How could we tell if we understand bubble nucleation?
  - analogue experiments
  - lattice simulations

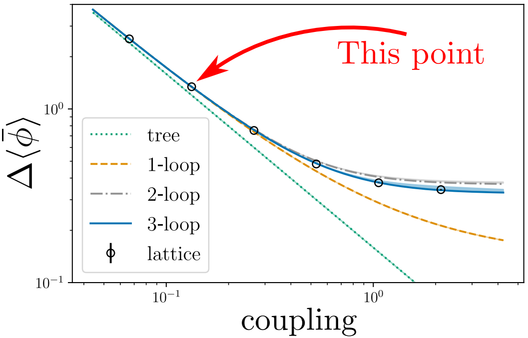
Skeletons in the closet

Stochastic lattice simulations



OG, Güyer & Rummukainen '22

A super perturbative benchmark point



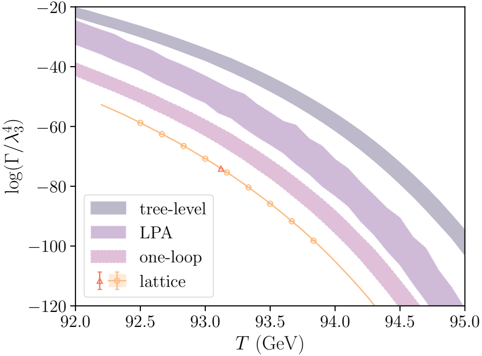
Perturbation theory converging *very* quickly for $\Delta\langle\bar{\phi}\rangle$ in $g\phi^3 + \lambda\phi^4$,

$$\underbrace{1.341(2)}_{\text{lattice}} \stackrel{?}{=} \underbrace{1.2}_{\text{tree}} + \underbrace{0.1378}_{\text{1-loop}} + \underbrace{0.0054}_{\text{2-loop}} - \underbrace{0.0016}_{\text{3-loop}} + \dots$$

$$\stackrel{\checkmark}{=} 1.34170(4)$$

OG '21

Benchmarking against the lattice



Qualitative agreement for log rate, but *way* worse than $\Delta\langle\bar{\phi}\rangle$,

$$\underbrace{-74.09(5)}_{\text{lattice}} \overset{?}{=} \underbrace{-38.02}_{\text{tree}} - \underbrace{25.32}_{\text{1-loop}} + \dots$$

$$\overset{x}{=} -63(3)$$

OG, Kormu & Weir '24, see also Pîrvu, Shkerin & Sibiryakov '24

Conclusions

- **Nucleation rate** $\to \Omega_{\mathrm{GW}}$ predictions.
- **Vacuum decay** $=$ energy $+$ entropy.
- **Thermal nucleation** $=$ energy $+$ entropy $+$ dynamics.
- **Dynamics** $\to$ lattice rate *slower* than expected.
- **What are we missing?**

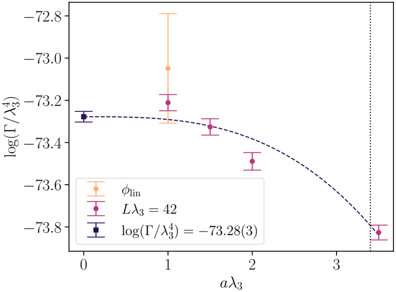
Conclusions

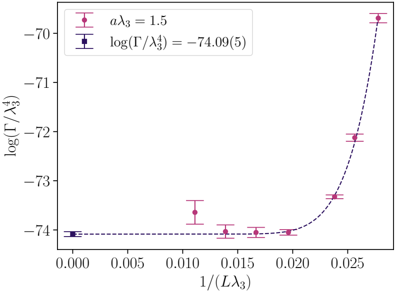
- **Nucleation rate** → Ω_{GW} predictions.
- **Vacuum decay** = energy + entropy.
- **Thermal nucleation** = energy + entropy + dynamics.
- **Dynamics** → lattice rate *slower* than expected.
- **What are we missing?**

Thanks for listening!

Backup slides

Continuum limits





Expected approach to continuum is

$$\log \Gamma(a, L) = \log \Gamma(0, \infty) \left[1 + c_a (ma)^3 + c_L e^{-mL} + \dots \right],$$

where $c_a = O(1)$ and $c_L = O(e^{2mR})$.

OG, Kormu & Weir '24