New Hermitian/non-Hermitian Toda field theories and Calogero models from infinite symmetries

Andreas Fring

Applications of Field Theory to Hermitian and Non-Hermitian Systems

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Outline

(1) Motivation
(2) n-Extended Lorentzian Kac-Moody algebras
(3) Toda field theories
(4) Calogero models
(5) Conclusions

mainly based on:

- AF, S. Whittington, n-Extended Lorentzian Kac-Moody algebras, *Lett. in Math. Phys.* 110 (2020) 1689.
- AF, S. Whittington, Lorentzian Toda field theories, *Rev. in Math. Phys.* 33 (2021) 2150017.
- F. Correa, AF, O. Quintana, J. Phys. A: Math and Theor. 57 (2024) 055203

Motivation

Symmetry algebras

- finite dimensional Lie algebras:
  electromagnetic force u(1), weak force su(2), strong force su(3)
- infinite dimensional Kac-Moody algebras:
  string theory, conformal field theories
- Lorentzian Kac-Moody algebras:
  type II string theory, M-theory

Model building for extensions of subsectors of the standard model or description of other physical systems?

AffineToda scalar field theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{m^2}{\beta^2} \sum_{k=\mathbf{a}}^{\ell} n_k \exp(\beta \alpha_k \cdot \phi)$$

$a = 1 \equiv$ conformal field theory (finite dimensional Lie algebras)
$a = 0 \equiv$ massive field theory (Kac-Moody algebras)
$\beta \in \mathbb{R} \equiv$ no backscattering (bootstrap equations)
$\beta \in i\mathbb{R} \equiv$ backscattering (Yang-Baxter, quantum groups)

[T. Hollowood, *Nucl. Phys.* B384 (1992) 523] :
- naively taking $\beta \to i\beta$ gives more poles in the S-matrix
- change of sign in the residues $\Rightarrow$ non-unitary theory
- kink solutions present at classical level
- are the energies real?

Answer: yes, reason modified $\mathcal{CPT}$ (see Francisco's talk)

Here: can we have $a = -1, -2, -3, -4, \ldots$?

Calogero(-Moser-Sutherland) models:

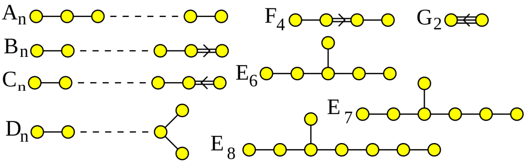
$$H=\frac{1}{2}p^2+\sum_{\alpha\in\Delta_{\mathbf{g}}\rightarrow\textbf{infinite dimensional g}}\frac{c_\alpha}{(\alpha\cdot q)^2}=\frac{1}{2}p^2+\sum_{i=1}^{r}\sum_{n=1}^{h\rightarrow\infty}\frac{c_{in}}{[\sigma^n(\gamma_i)\cdot q]^2}$$

What properties do these theories have?
In particular: Are they integrable?

Assemble mathematical tools:

Dynkin diagrams (finite dimensional case)

simple roots: $\alpha_i \in \mathbb{R}^n$ with $i = 1, 2, \ldots, \ell$
Cartan matrix: $K_{ij} = 2\alpha_i \cdot \alpha_j / \alpha_j^2$ $(\cdot) \equiv$ Euclidean inner product
K is positive definite and encoded in Dynkin diagrams:



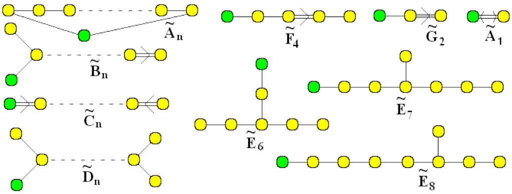
Associated Chevalley generators satisfy Serre relations:

$$[H_i, H_j] = 0, \;\; [H_i, E_j] = K_{ij} E_j, \;\; [H_i, F_j] = -K_{ij} F_j, \;\; [E_i, F_j] = \delta_{ij} H_i,$$

$$[E_i, \ldots, [E_i, E_j] \ldots] = 0, \qquad [F_i, \ldots, [F_i, F_j] \ldots] = 0$$

Dynkin diagrams (affine extended case)

simple roots: $\alpha_i \in \mathbb{R}^n$ for $i = 0, 1, 2, \ldots, \ell$, $\alpha_0 = -\theta = \sum_{i=1}^{\ell} n_i \alpha_i$
Kac labels: $n_i \in \mathbb{N}$
Cartan matrix: $K_{ij} = 2\alpha_i \cdot \alpha_j / \alpha_j^2$ $(\cdot) \equiv$ Euclidean inner product
K is positive semi-definite and encoded in Dynkin diagrams:



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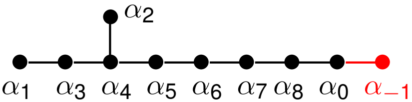
$$[E_i, \ldots, [E_i, E_j] \ldots] = 0, \qquad [F_i, \ldots, [F_i, F_j] \ldots] = 0$$

Dynkin diagrams (n-extended case)

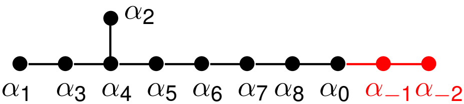
Over-extended:

$$\mathbf{E}_8^{(1)} \equiv \mathbf{E}_{10} :$$



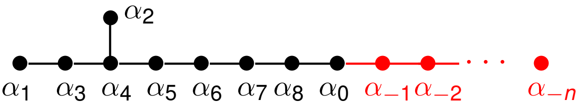
Very-extended:

$$\mathbf{E}_8^{(2)} \equiv \mathbf{E}_{11} :$$



n-extended:

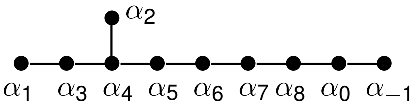
$$\mathbf{E}_8^{(n)} \equiv \mathbf{E}_{n+9} :$$



Definition: hyperbolic Kac-Moody algebra

The deletion of **any one node** leaves a possibly disconnected set of connected Dynkin diagrams each of which is of finite type, except for at most one affine type.

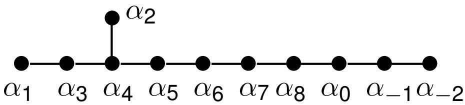
$$\mathbf{E}_8^{(1)} \equiv \mathbf{E}_{10} :$$



Definition: Lorentzian Kac-Moody algebra

The deletion of **at least one node** leaves a possibly disconnected set of connected Dynkin diagrams each of which is of finite type, except for at most one affine type.

$$\mathbf{E}_8^{(2)} \equiv \mathbf{E}_{11} :$$



N-extended root lattices, simple construction scheme

Self-dual Lorentzian lattice $\Pi^{1,1}$:
For $z, w \in \Pi^{1,1}$, $z = (z^+, z^-)$, $w = (w^+, w^-)$ define:

$$z \cdot w = -z^+ w^- - z^- w^+$$

- null vectors: $k = (1,0)$, $\bar{k} = (0,-1)$, $k^2 = \bar{k}^2 = 0$, $k \cdot \bar{k} = 1$
- length two vectors: $\pm\left(k + \bar{k}\right)$

[M. Gaberdiel, D. Olive, P. West, *Nucl. Phys.* B 645 (2002) 403]

algebra	root lattice	added root	Dynkin diagram	expl.
$\mathbf{g}_0$	$\Lambda_{\mathbf{g}_0} = \Lambda_{\mathbf{g}} \oplus \Pi^{1,1}$	$\alpha_0 = k - \theta$	$\cdots \underset{\alpha_i}{\circ} - \underset{\alpha_0}{\bullet}$	$E_8^{(0)}$
$\mathbf{g}_{-1}$	$\Lambda_{\mathbf{g}_{-1}} = \Lambda_{\mathbf{g}} \oplus \Pi^{1,1}$	$\alpha_{-1} = -\left(k + \bar{k}\right)$	$\cdots \underset{\alpha_i}{\circ} - \underset{\alpha_0}{\circ} - \underset{\alpha_{-1}}{\bullet}$	$E_8^{(1)} \equiv E_{10}$
$\mathbf{g}_{-2}$	$\Lambda_{\mathbf{g}_{-2}} = \Lambda_{\mathbf{g}_{-1}} \oplus \Pi^{1,1}$	$\alpha_{-2} = k - \left(\ell + \bar{\ell}\right)$	$\cdots \underset{\alpha_i}{\circ} - \underset{\alpha_0}{\circ} - \underset{\alpha_{-1}}{\circ} - \underset{\alpha_{-2}}{\bullet}$	$E_8^{(2)} \equiv E_{11}$

$$\Lambda_{\mathbf{g}_{-n}} = \Lambda_{\mathbf{g}} \oplus \Pi_1^{1,1} \oplus \ldots \oplus \Pi_n^{1,1}$$

$$\alpha^{(n)} := \left\{\alpha_1, \ldots, \alpha_r, \alpha_0 = k_1 - \theta, \ldots, \alpha_{-j} = k_{j-1} - \left(k_j + \bar{k}_j\right)\right\}$$

N-extended weight lattices

fundamental weights $\lambda_i^{(n)}$: $\lambda_i^{(n)} \cdot \alpha_j^{(n)} = \delta_{ij}, i, j = -n, 0, 1, \ldots, r$

With $\lambda_i^{(n)} = \sum_{j=-n}^{r} K_{ij}^{-1} \alpha_j^{(n)}$, $K_{ij}^{-1} = \lambda_i^{(n)} \cdot \lambda_j^{(n)}$:

$$
\begin{aligned}
\lambda_i^{(n)} &= \lambda_i^f + n_i \lambda_0^{(n)}, \quad i = 1, \ldots, r, \\
\lambda_0^{(n)} &= \bar{k}_1 - k_1 + \frac{1}{n} \sum_{i=2}^{n} \left[k_i + (n+1-i)\bar{k}_i \right], \\
\lambda_{-1}^{(n)} &= -k_1, \\
\lambda_{-2}^{(n)} &= -\frac{1}{n} \sum_{i=2}^{n} \left[k_i + (n+1-i)\bar{k}_i \right], \\
&\ldots \\
\lambda_{-j}^{(n)} &= \frac{(1-j)}{n} \sum_{i=2}^{n} \left[k_i + (1-i)\bar{k}_i \right] + \sum_{i=2}^{j-1} \left[k_i + (1-i)\bar{k}_i \right] + (1-j) \sum_{i=j}^{n} \bar{k}_i
\end{aligned}
$$

N-extended Weyl vector

$$\begin{aligned}\rho^{(n)} &= \sum_{j=-n}^{r} \lambda_j \\ &= \rho^f + h\bar{k}_1 - (1+h)k_1 + \sum_{i=2}^{n} \left[\left(\frac{h}{n} + \frac{n+1-2i}{2} \right) k_i \right. \\ &\quad \left. + \frac{(n+1-i)(2h+n(1-i))}{2n} \bar{k}_i \right]\end{aligned}$$

$h \equiv$ Coxeter number

Freudenthal-de Vries strange formula

$$\rho^{(n)} \cdot \rho^{(n)} = \frac{h(h+1)r + n(n^2-1)}{12} - \frac{h(h+n)(1+n)}{n}$$

for $n = -1$ this reduces to the well-known formula

$$\left(\rho^f\right)^2 = \frac{h}{12}\dim\mathbf{g} = \frac{h(h+1)r}{12}.$$

SO(1,2) and SO(3) principal subalgebras

KM: principal $SO(3)$-generators $[J_+, J_-] = J_3$, $[J_3, J_\pm] = \pm J_\pm$

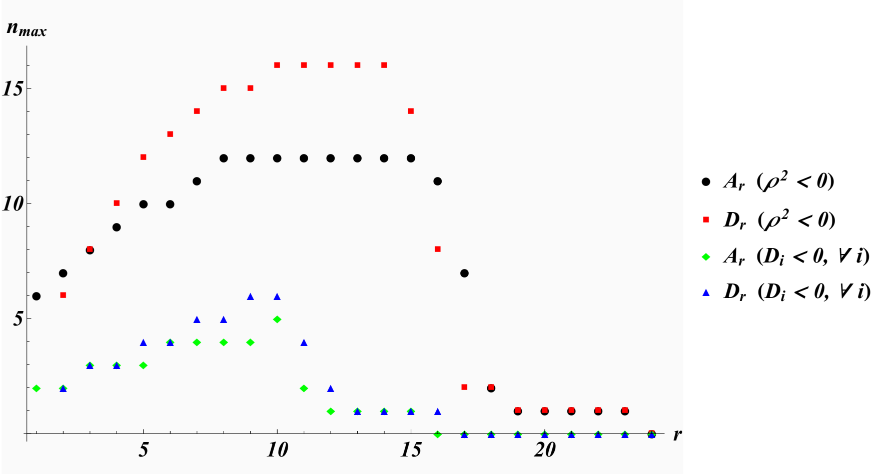
$$J_3 = \sum_{i=1}^{r} D_i H_i, \;\; J_+ = \sum_{i=1}^{r} \sqrt{D_i} E_i, \;\; J_- = \sum_{i=1}^{r} \sqrt{D_i} F_i, \quad D_i := \sum_{j=1}^{r} K_{ji}^{-1}$$

LKM: principal $SO(1,2)$-generators $\left[\hat{J}_+, \hat{J}_-\right] = -\hat{J}_3$, $\left[\hat{J}_3, \hat{J}_\pm\right] = \pm \hat{J}_\pm$

$$\hat{J}_3 = -\sum_{i=1}^{r} \hat{D}_i H_i, \quad \hat{J}_+ = \sum_{i=1}^{r} p_i E_i, \quad \hat{J}_- = \sum_{i=1}^{r} q_i F_i, \quad \hat{D}_i := \sum_{j=1}^{r} K_{ji}^{-1}$$

A necessary and sufficient condition for the existence of a $SO(3)$-principal subalgebra or a $SO(1,2)$-principal subalgebra is $D_i > 0$ or $\hat{D}_i < 0$ for all i, respectively.

A necessary condition for the existence of a $SO(1,2)$ principle subalgebra is $\rho^2 < 0$:



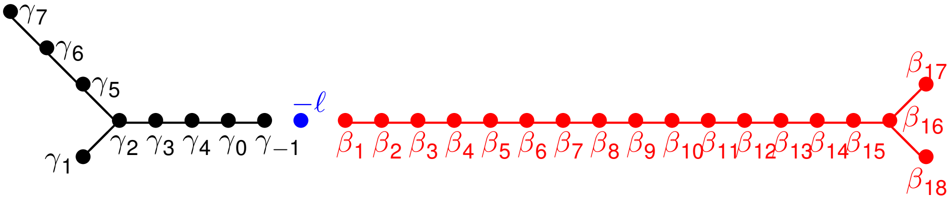
$n_{\max} \equiv$ maximum value n for $\mathbf{g}_{-n}$ to posses a $SO(1,2)$ PSA

Decomposition of $\mathbf{g}_{-n}$

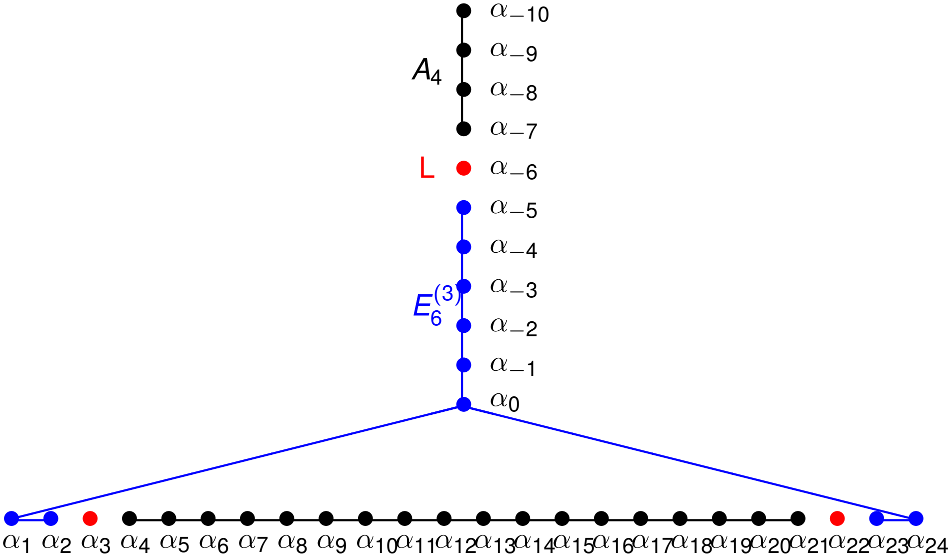
Compute

$$D_k^{(n)} = \sum_{j=-n}^{r} K_{kj}^{-1} = \rho^{(n)} \cdot \lambda_k^{(n)} = 0, \qquad k = -n, \ldots, -1, 0, 1, \ldots, r,$$

Reduced Dynkin diagram of $D_{25}^{(2)} = E_7^{(1)} \diamond L \diamond D_{18}$:



Reduced Dynkin diagram of $A_{24}^{(10)} = E_6^{(3)} \diamond L \diamond A_4 \diamond L^2 \diamond A_{18}$:



$D_{17}^{(1)} = E_8^{(5)} \diamond L \diamond D_4$	$D_{18}^{(1)} = E_8^{(4)} \diamond L \diamond D_6$	$D_{20}^{(1)} = E_8^{(3)} \diamond L \diamond D_9$
$D_{39}^{(1)} = E_8^{(1)} \diamond L \diamond D_{30}$	$D_{13}^{(2)} = E_7^{(4)} \diamond L \diamond A_3$	$D_{14}^{(2)} = E_7^{(3)} \diamond L \diamond D_5$
$D_{25}^{(2)} = E_7^{(1)} \diamond L \diamond D_{18}$	$D_{13}^{(3)} = \hat{A}_{10}^{(1,5)} \diamond L \diamond D_5$	$D_{16}^{(3)} = \hat{A}_9^{(1,5)} \diamond L \diamond D_9$
$D_{11}^{(4)} = \hat{A}_{13}^{(1,6)} \diamond L^2$	$D_{12}^{(4)} = \hat{A}_{11}^{(1,6)} \diamond L \diamond D_4$	$D_{14}^{(4)} = \hat{A}_{10}^{(1,6)} \diamond L \diamond D_7$
$D_{21}^{(4)} = E_7^{(2)} \diamond L \diamond D_{15}$	$D_{11}^{(5)} = \hat{A}_{13}^{(1,7)} \diamond L \diamond A_1^2$	$D_{81}^{(5)} = E_8^{(1)} \diamond L \diamond D_{76}$
$D_{13}^{(6)} = \hat{A}_{12}^{(1,5)} \diamond L \diamond D_6$	$D_{52}^{(6)} = E_8^{(2)} \diamond L \diamond D_{47}$	$D_{43}^{(7)} = E_8^{(3)} \diamond L \diamond D_{38}$
$D_{11}^{(8)} = \mathbf{A}_{\mathbf{16}}^{(\mathbf{1,7})} \diamond L \diamond A_1^2$	$D_{13}^{(8)} = \mathbf{A}_{\mathbf{14}}^{(\mathbf{1,5})} \diamond L \diamond D_6$	$D_{17}^{(8)} = E_7^{(6)} \diamond L \diamond D_{11}$
$D_{39}^{(8)} = E_8^{(4)} \diamond L \diamond D_{34}$	$D_{37}^{(9)} = E_8^{(5)} \diamond L \diamond D_{32}$	$D_{11}^{(10)} = \mathbf{A}_{\mathbf{19}}^{(\mathbf{1,8})} \diamond L^2$
$D_{36}^{(10)} = E_8^{(6)} \diamond L \diamond D_{31}$	$D_{12}^{(11)} = \mathbf{A}_{\mathbf{18}}^{(\mathbf{1,6})} \diamond L \diamond D_4$	$D_{14}^{(13)} = \mathbf{A}_{\mathbf{19}}^{(\mathbf{1,5})} \diamond L \diamond D_7$
$D_{36}^{(14)} = \mathbf{E}_{\mathbf{8}}^{(\mathbf{10})} \diamond L \diamond D_{31}$	$D_{13}^{(16)} = \mathbf{A}_{\mathbf{23}}^{(\mathbf{1,6})} \diamond L \diamond D_5$	$D_{37}^{(16)} = \mathbf{E}_{\mathbf{8}}^{(\mathbf{12})} \diamond L \diamond D_{32}$
$D_{39}^{(19)} = \mathbf{E}_{\mathbf{8}}^{(\mathbf{15})} \diamond L \diamond D_{34}$	$D_{16}^{(20)} = \mathbf{A}_{\mathbf{26}}^{(\mathbf{1,5})} \diamond L \diamond D_9$	$D_{21}^{(20)} = \mathbf{E}_{\mathbf{7}}^{(\mathbf{18})} \diamond L \diamond D_{15}$
$D_{13}^{(24)} = \mathbf{A}_{\mathbf{33}}^{(\mathbf{1,8})} \diamond L \diamond A_3$	$D_{43}^{(24)} = \mathbf{E}_{\mathbf{8}}^{(\mathbf{20})} \diamond L \diamond D_{38}$	$D_{14}^{(26)} = \mathbf{A}_{\mathbf{34}}^{(\mathbf{1,7})} \diamond L \diamond D_5$
$D_{52}^{(34)} = \mathbf{E}_{\mathbf{8}}^{(\mathbf{30})} \diamond L \diamond D_{47}$	$D_{25}^{(48)} = \mathbf{A}_{\mathbf{54}}^{(\mathbf{1,5})} \diamond L \diamond D_{18}$	$D_{17}^{(64)} = \mathbf{A}_{\mathbf{76}}^{(\mathbf{1,11})} \diamond L \diamond D_4$
$D_{81}^{(64)} = \mathbf{E}_{\mathbf{8}}^{(\mathbf{60})} \diamond L \diamond D_{76}$	$D_{18}^{(68)} = \mathbf{A}_{\mathbf{79}}^{(\mathbf{1,10})} \diamond L \diamond D_6$	$D_{20}^{(76)} = \mathbf{A}_{\mathbf{86}}^{(\mathbf{1,9})} \diamond L \diamond D_9$
$D_{39}^{(152)} = \mathbf{A}_{\mathbf{160}}^{(\mathbf{1,7})} \diamond L \diamond D_{30}$		

Decomposition of the n-extended algebras $D_r^{(n)}$

Lorentzian Toda field theories

$$\mathcal{L}_{\mathbf{g}_{-n}} = \frac{1}{2}\partial_\mu\phi \cdot \partial^\mu\phi - \frac{g}{\beta^2}\sum_{i=-n}^{r} e^{\beta\alpha_i\cdot\phi}$$

- coupling constants $g, \beta \in \mathbb{R}$
- simple roots α_i of dimension $\ell + 2m$
- scalar field $\phi(x,t)$ with $\ell + 2m$ components $\phi^a(x,t)$
- Lorentzian inner product:
  for $x = (x_1, \ldots, x_{\ell+2m})$, $y = (y_1, \ldots, y_{\ell+2m})$ define

$$x \cdot y := \sum_{\beta=1}^{\ell} x_\beta y_\beta - \sum_{\beta=1}^{m} \left(x_{\ell+2\beta-1} y_{\ell+2\beta} + x_{\ell+2\beta} y_{\ell+2\beta-1}\right)$$

corresponding to

$$\Lambda_{\mathbf{g}_{-n}} = \Lambda_{\mathbf{g}} \oplus \Pi_1^{1,1} \oplus \ldots \oplus \Pi_n^{1,1}$$

$\mathcal{L}_{\mathbf{g}_{-n}}$-extended Lorentzian Toda field theory

Construction:

$$\mathcal{L}_{\mathbf{g}_1} \overset{\alpha_0}{\rightarrow} \mathcal{L}_{\mathbf{g}_0} \overset{\alpha_{-1}}{\rightarrow} \mathcal{L}_{\mathbf{g}_{-1}} \overset{\alpha_{-2}}{\rightarrow} \mathcal{L}_{\mathbf{g}_{-2}}$$

$\mathcal{L}_{\mathbf{g}_1} \equiv$ conformal Toda field theory

add modified affine root $\alpha_0 = k - \sum_{i=1}^{r} n_i \alpha_i \Rightarrow$

$\mathcal{L}_{\mathbf{g}_0} \equiv$ affine Toda field theory

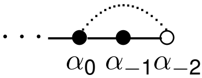
add Lorentzian roots $\alpha_{-1} = -\left(k + \bar{k}\right) \Rightarrow$

$\mathcal{L}_{\mathbf{g}_{-1}} \equiv$ conformal affine Toda field theory ($\Theta^{\mu}_{\ \mu} = 0$)

add Lorentzian roots $\alpha_{-2} = \bar{k} \Rightarrow$

$\mathcal{L}_{\mathbf{g}_{-2}} \equiv$ massive Lorentzian Toda field theory

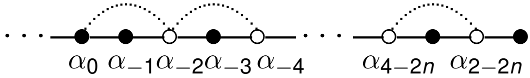
$\mathbf{g}_{-2}$:



continue:

$$\alpha_i \equiv \text{simple roots of } \mathbf{g} \qquad \text{for } j = 1, \ldots, r$$
$$\alpha_{-(2i-2)} = k_i - \sum\nolimits_{j=-(2i-3)}^{r} n_j \alpha_j \qquad \text{for } i = 1, \ldots, n$$
$$\alpha_{-(2i-1)} = -(k_i + \bar{k}_i) \qquad \text{for } i = 1, \ldots, n$$

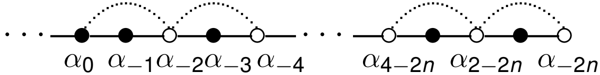
massless models:

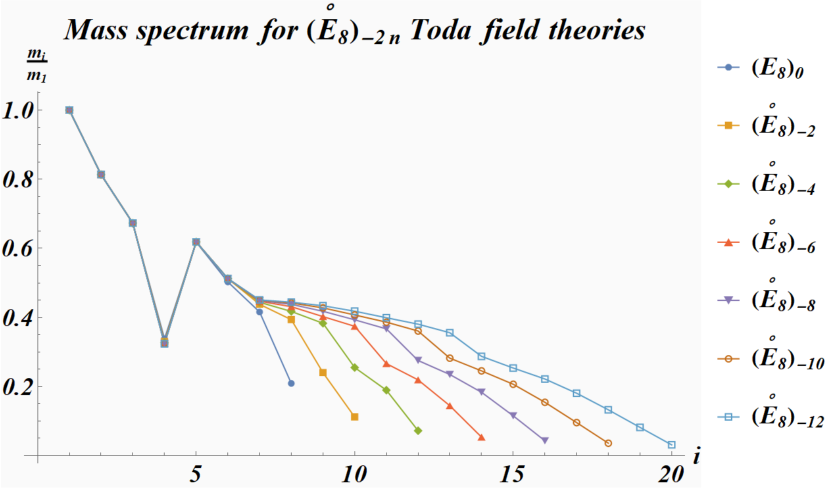
$\mathbf{g}_{1-2n}$:


add

$$\alpha_{-(2n)} = -\sum\nolimits_{j=-(2n-1)}^{r} n_j \alpha_j$$

massive models:

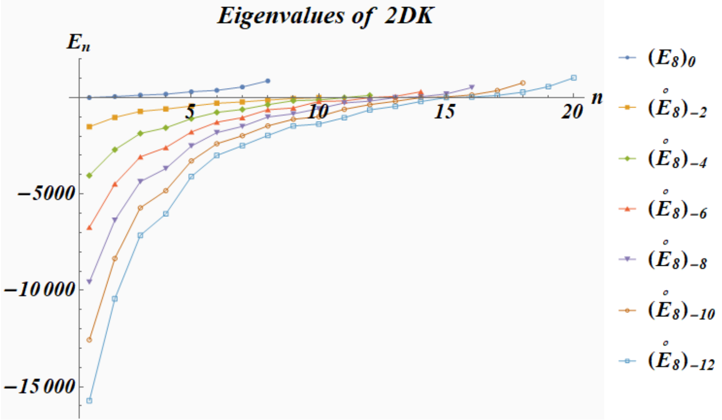
$\mathbf{g}_{-2n}$:




Notice the almost stable noncrystallographic H_4 compound

Integrability - Painlevé test

For a Lorentzian Toda field theory to be integrable all eigenvalues of the matrix $P = 2DK$ must be positive integers.



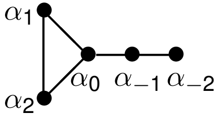
Thus, apart from the affine E_8 Toda theory none of the Lorentzian Toda theories is integrable.

Calogero models

$$H = \frac{1}{2}p^2 + \sum_{\alpha \in \Delta_{\mathbf{g}}} \frac{c_\alpha}{(\alpha \cdot q)^2}$$

Example:

$(\mathbf{A}_2)_{-2}$:



$$K_{ij} = 2\frac{\alpha_i \cdot \alpha_j}{\alpha_j \cdot \alpha_j} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

Representation of the roots:

$\alpha_1 = (1, -1, 0; 0, 0, 0, 0)\,,\ \alpha_2 = (0, 1, -1; 0, 0, 0, 0)\,,$

$\alpha_0 = (-1, 0, 1; 1, 0, 0, 0)\,,\ \alpha_{-1} = (0, 0, 0; -1, 1, 0, 0)\,,\ \alpha_{-2} = (0, 0, 0; 1, 0, -1, 1)$

Lorentzian inner product:

$$x \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_5 - x_5 y_4 - x_6 y_7 - x_7 y_6$$

Demanding the length of an arbitrary root

$$\alpha = p\alpha_{-2} + q\alpha_{-1} + l\alpha_0 + m\alpha_1 + n\alpha_2 \in \Delta_{(\mathbf{A}_2)_{-2}}, \qquad p, q, q, m, n \in \mathbb{Z},$$

to be 2, corresponds to the Diophantine equation

$$l^2 - l(m+n+q) + m^2 - mn + n^2 + p^2 - pq + q^2 = 1 \qquad \Leftrightarrow \qquad \alpha \cdot \alpha = 2.$$

For every root α_i we define a Weyl reflection

$$\sigma_i(x) := x - (\alpha_i \cdot x)\alpha_i$$

symmetries generated by the Weyl reflections

$$\sigma_{-2}(\alpha) : p \to q - p, \;\; \sigma_{-1}(\alpha) : q \to l + p - q, \;\; \sigma_0(\alpha) : l \to q + m + n - l,$$
$$\sigma_1(\alpha) : m \to l + n - m, \;\; \sigma_2(\alpha) : n \to l + m - n$$

The invariant potential may then be written as

$$V(q) = \sum_{\substack{p,q,l,m,n=0 \\ \text{Diophantine equation}}}^{\infty} \frac{g_{pqlmn}}{[(p\alpha_{-2} + q\alpha_{-1} + l\alpha_0 + m\alpha_1 + n\alpha_2) \cdot q]^2}$$

Alternative representation of the potential:

$$V(q) = \sum_{i=1}^{\ell} \sum_{n=-\infty}^{\infty} \frac{g_{in}}{[\sigma^n(\gamma_i) \cdot q]^2}$$

with Coxeter element $\sigma := \sigma_{-2}\sigma_{-1}\sigma_0\sigma_1\sigma_2$.
Now affine Weyl group with $\sigma_a := \sigma_0\sigma_1\sigma_2$,
Closed formula for arbitrary powers:

$$\begin{aligned}
\sigma_a^k(\alpha) \quad = \quad & p\alpha_{-2} + q\alpha_{-1} \\
& + \left[\frac{1}{8}\left(6k^2+1\right) q + \frac{1}{4}(6k+3)l - \frac{1}{4}(6k+1)n + \frac{m}{2} + (-1)^k \frac{2l-4m+2n-q}{8} \right] \alpha_0 \\
& + \left[\frac{1}{8}\left(6k^2-4k-1\right) q + \frac{1}{4}(6k+1)l - \frac{1}{4}(6k-1)n + \frac{m}{2} - (-1)^k \frac{2l-4m+2n-q}{8} \right] \alpha_1 \\
& + \left[\frac{1}{8}\left(6k^2-8k+1\right) q + \frac{1}{4}(6k-1)l - \frac{1}{4}(6k-3)n + \frac{m}{2} + (-1)^k \frac{2l-4m+2n-q}{8} \right] \alpha_2
\end{aligned}$$

Then for one representative of the orbit α_2

$$V_2(q) = \sum_{n=-\infty}^{\infty} \frac{g}{[\sigma_a^n(\alpha_2) \cdot q]^2}$$

This infinite sum can be computed

$$V_2(q) = \sum_{n=-\infty}^{\infty} \frac{16g}{\{2\left[(-1)^n - 1\right] q_1 - 2\left[(-1)^n + 1\right] q_2 + \left[-6n + (-1)^n - 1\right] q_5 + 4q_3\}^2}$$

$$= \frac{\pi^2}{9q_5^2} \left\{ \frac{g}{\sin^2\left[\frac{\pi}{3q_5}(q_2 - q_3)\right]} + \frac{g}{\sin^2\left[\frac{\pi}{3q_5}(q_1 - q_3 - q_5)\right]} \right\}$$

Next seek representatives for other orbits such that $V(q)$ becomes invariant

$$V(q) = \sum_{n=-\infty}^{\infty} \frac{g}{[\sigma^n(\alpha_2)\cdot q]^2} + \frac{g}{[\sigma^n(\alpha_2)\cdot \sigma_0(q)]^2} + \frac{g}{[\sigma^n(\alpha_2)\cdot \sigma_1(q)]^2}$$
$$+ \frac{g}{[\sigma^n(\alpha_2)\cdot \sigma_1\sigma_0(q)]^2} + \frac{g}{[\sigma^n(\alpha_2)\cdot \sigma_0\sigma_1(q)]^2} + \frac{g}{[\sigma^n(\alpha_2)\cdot \sigma_0\sigma_1\sigma_0(q)]^2}$$
$$+ \frac{g}{[\sigma^n(\alpha_2)\cdot \sigma_2\sigma_0(q)]^2} + \frac{g}{[\sigma^n(\alpha_2)\cdot \sigma_2\sigma_1(q)]^2} + \frac{g}{[\sigma^n(\alpha_2)\cdot \sigma_2\sigma_0\sigma_2(q)]^2}$$

Computing each term gives

$$V(q) = \frac{2\pi^2 g}{9q_5^2}\left(V_{12} + V_{13} + V_{23} + V_{125}^{+} + V_{125}^{-} + V_{135}^{+} + V_{135}^{-} + V_{235}^{+} + V_{235}^{-}\right)$$

where

$$V_{ij} := \frac{1}{\sin^2\left[\frac{\pi}{3q_5}(q_i - q_j)\right]}, \quad V_{ijk}^{\pm} := \frac{1}{\sin^2\left[\frac{\pi}{3q_5}(q_i - q_j \pm q_k)\right]}, \quad i,j,k = 1,2,3,4,5$$

$V(q)$ is invariant under the entire affine Weyl group!

$$\begin{array}{llllll}
\sigma_0 : & V_{12} \leftrightarrow V_{235}^{+} & V_{13} \leftrightarrow V_{135}^{-} & V_{23} \leftrightarrow V_{125}^{+} & V_{125}^{-} \leftrightarrow V_{235}^{-} & V_{135}^{+} \circlearrowleft \\
\sigma_1 : & V_{13} \leftrightarrow V_{23} & V_{125}^{+} \leftrightarrow V_{125}^{-} & V_{135}^{-} \leftrightarrow V_{235}^{-} & V_{135}^{+} \leftrightarrow V_{235}^{+} & V_{12} \circlearrowleft \\
\sigma_2 : & V_{12} \leftrightarrow V_{13} & V_{125}^{+} \leftrightarrow V_{135}^{+} & V_{235}^{+} \leftrightarrow V_{235}^{-} & V_{125}^{-} \leftrightarrow V_{135}^{-} & V_{23} \circlearrowleft
\end{array}$$

Similarly for the hyperbolic and Lorentzian Kac-Moody algebra.

$\mathcal{PT}$-extensions

$$\mathcal{H}_\mu = \frac{1}{2}p^2 + \frac{1}{2}\sum_{\alpha\in\Delta} g_\alpha^2 V(\alpha\cdot q) + i\mu\cdot p$$

$\cdot\ \mu = 1/2\sum_{\alpha\in\Delta}\tilde{g}_\alpha f(\alpha\cdot q)\alpha$, $f(x) = 1/x\ \ V(x) = f^2(x)$

- Not so obvious that one can re-write

$$\mathcal{H}_\mu = \frac{1}{2}(p+i\mu)^2 + \frac{1}{2}\sum_{\alpha\in\Delta}\hat{g}_\alpha^2 V(\alpha\cdot q), \quad \hat{g}_\alpha^2 = \begin{cases} g_s^2 + \alpha_s^2\tilde{g}_s^2 & \alpha\in\Delta_s \\ g_l^2 + \alpha_l^2\tilde{g}_l^2 & \alpha\in\Delta_l \end{cases}$$

$\Rightarrow \mathcal{H}_\mu = \eta^{-1}h_{\text{Cal}}\eta \qquad$ with $\quad \eta = e^{-q\cdot\mu}$

- integrability follows trivially $\dot{L} = [L, M]$: $L(p)\to L(p+i\mu)$
- computing backwards for any CMS-potential

$$\mathcal{H}_\mu = \frac{1}{2}p^2 + \frac{1}{2}\sum_{\alpha\in\Delta}\hat{g}_\alpha^2 V(\alpha\cdot q) + i\mu\cdot p - \frac{1}{2}\mu^2$$

- $\mu^2 = \alpha_s^2\tilde{g}_s^2\sum_{\alpha\in\Delta_s}V(\alpha\cdot q) + \alpha_l^2\tilde{g}_l^2\sum_{\alpha\in\Delta_l}V(\alpha\cdot q)$ only for V rational

[AF, Mod. Phys. Lett. A21 (2006) 691, Acta P. 47 (2007) 44]

$\mathcal{PT}$-deformations

$$\mathcal{H}_{adC}(p,q) = \frac{p^2}{2} + \frac{\omega^2}{4} \sum_{\tilde{\alpha}\in\tilde{\Delta}^+} (\tilde{\alpha}\cdot q)^2 + \sum_{\tilde{\alpha}\in\Delta^+} \frac{g_{\tilde{\alpha}}}{(\tilde{\alpha}\cdot q)^2}$$

Example A_3 :

$$\tilde{\alpha}_1 = \cosh\varepsilon\alpha_1 + (\cosh\varepsilon - 1)\alpha_3 - \imath\sqrt{2}\sqrt{\cosh\varepsilon}\sinh\left(\frac{\varepsilon}{2}\right)(\alpha_1 + 2\alpha_2 + \alpha_3),$$

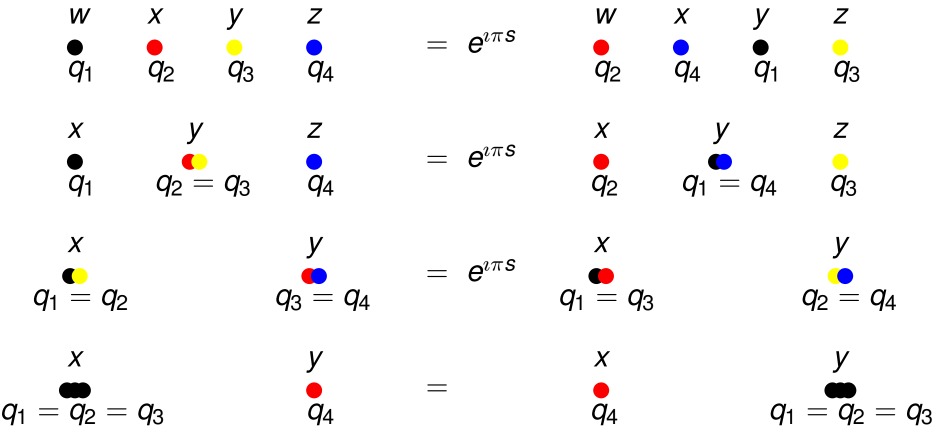
$$\tilde{\alpha}_2 = (2\cosh\varepsilon - 1)\alpha_2 + 2\imath\sqrt{2}\sqrt{\cosh\varepsilon}\sinh\left(\frac{\varepsilon}{2}\right)(\alpha_1 + \alpha_2 + \alpha_3),$$

$$\tilde{\alpha}_3 = \cosh\varepsilon\alpha_3 + (\cosh\varepsilon - 1)\alpha_1 - \imath\sqrt{2}\sqrt{\cosh\varepsilon}\sinh\left(\frac{\varepsilon}{2}\right)(\alpha_1 + 2\alpha_2 + \alpha_3)$$

$\mathcal{PT}$-symmetric potentials ($q_{ij} = q_i - q_j$):

$$\begin{aligned}
\tilde{\alpha}_1\cdot q &= q_{43} + \cosh\varepsilon(q_{12}+q_{34}) - \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}(q_{13}+q_{24}) \\
\tilde{\alpha}_2\cdot q &= q_{23}(2\cosh\varepsilon - 1) + \imath 2\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}q_{14} \\
\tilde{\alpha}_3\cdot q &= q_{21} + \cosh\varepsilon(q_{12}+q_{34}) - \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}(q_{13}+q_{24})
\end{aligned}$$

Anyonic exchange factors in the 4-particle scattering process



[AF, M. Smith, J Phys. A 43 (2010): 325201, 45 (2012): 085203.]

Conclusions

- We defined and investigated a new class of Kac-Moody algebras.
- Q: Constructions of algebras besides the root spaces?
- Lorentzian Toda field theories can be seen as a systematic framework of perturbed integrable systems
- Q: quantum corrections to masses, couplings, scattering matrices, form factors, correlation functions,...?
- We found a systematic way to generate the root spaces of Lorentzian Weyl groups from orbits of the associated Coxeter elements
- Calogero models invariant under the infinite affine, hyperbolic and Lorentzian Kac-Moody algebras have been constructed
- Q: quantum versions, other algebras, $\mathcal{PT}$-versions....