R-defects
Topological defects in non-Abelian Chern-Simons

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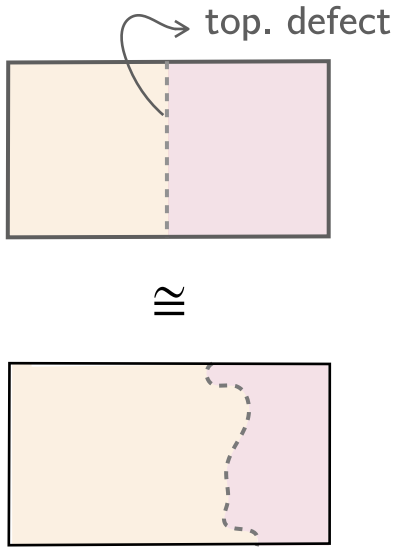
Ben Gurion University

Work in progress with Alex Arvanitakis, Lewis Cole, Daniel Thompson

genHET meeting @CERN, Wednesday 30 April, 2024

Topological defects



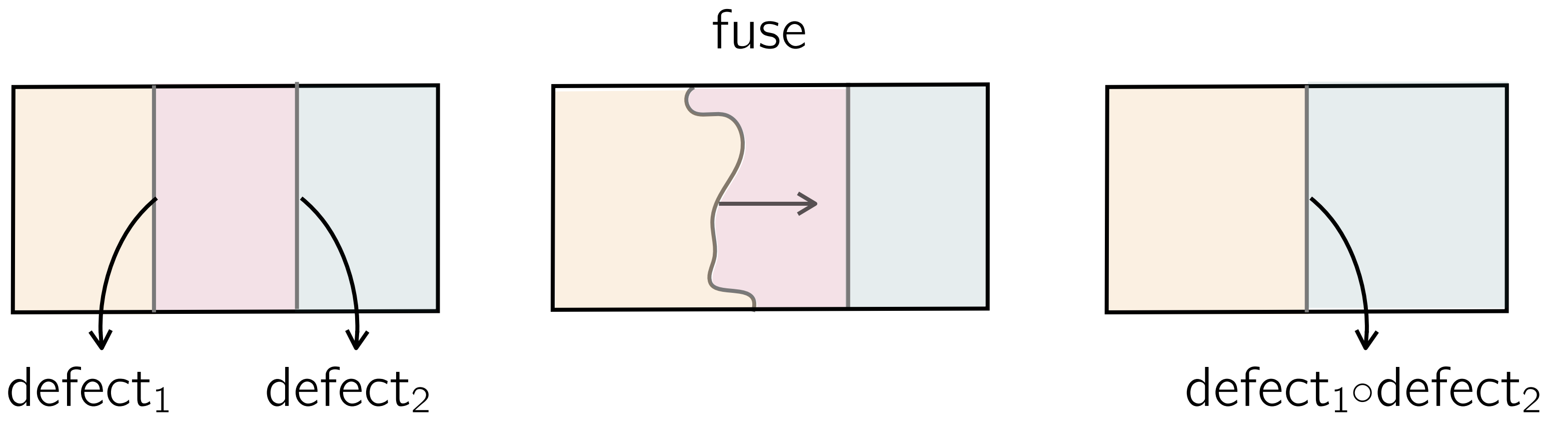
A defect is called topological when the energy momentum tensor

satisfies $T_L = T_R$ & $\bar{T}_L = \bar{T}_R$ at the defect locus

→ "Topological" = can be deformed and moved at no cost

→ Encode dualities and symmetries [Bachas, de Boer, Dijkgraaf, Ooguri, Kapustin,Tikhonov, Fröhlich, Fuchs, Gaberdiel, Runkel, Schweigert, Brunner, Roggenkamp, Carqueville,...]

→ "Fusion" = move and compose topological defects



Motivation

T-duality is an example of **a topology defect** [Fuchs, Gaberdiel, Runkel, Schweigert] [Kapustin, Saulina][Niro, Roumpedakis, Sela]

Talks of Camille and Marcia

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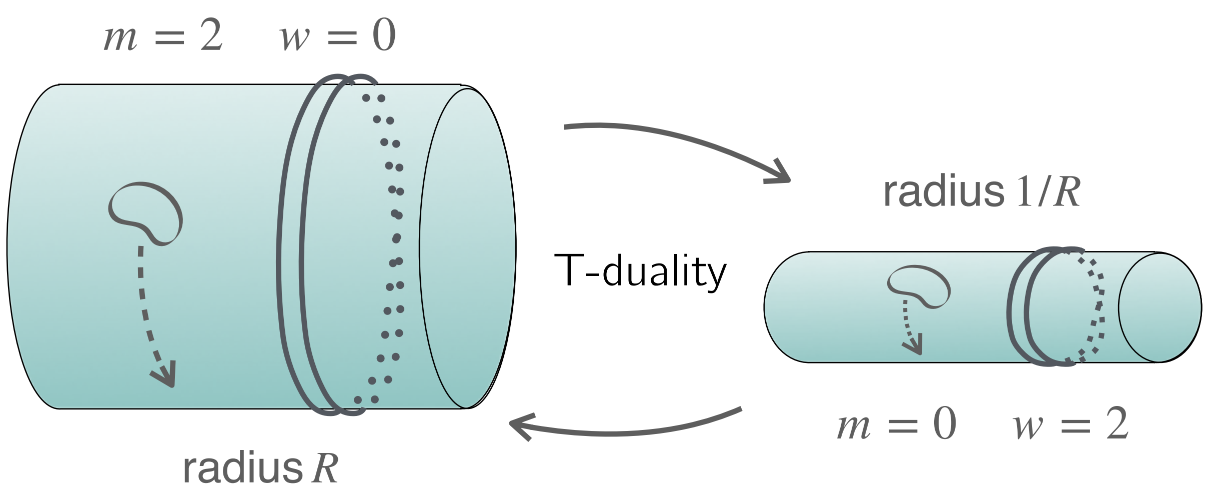

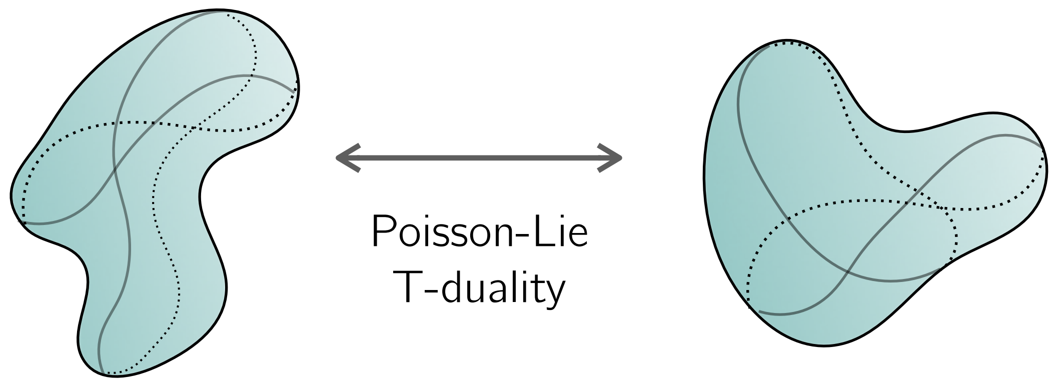

Exact symmetry of string theory

is a topological defect !

SUGRA solution generating technique

.... but is also a topological defect ! [SD, Raml]

Goal: Use the technology of defects and their fusion to understand generalised T-duality

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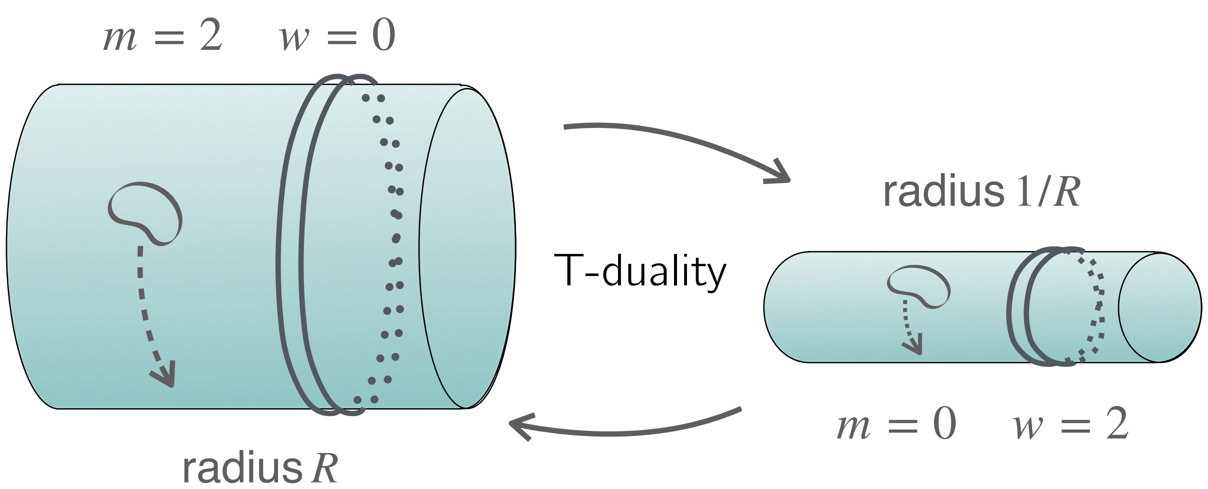

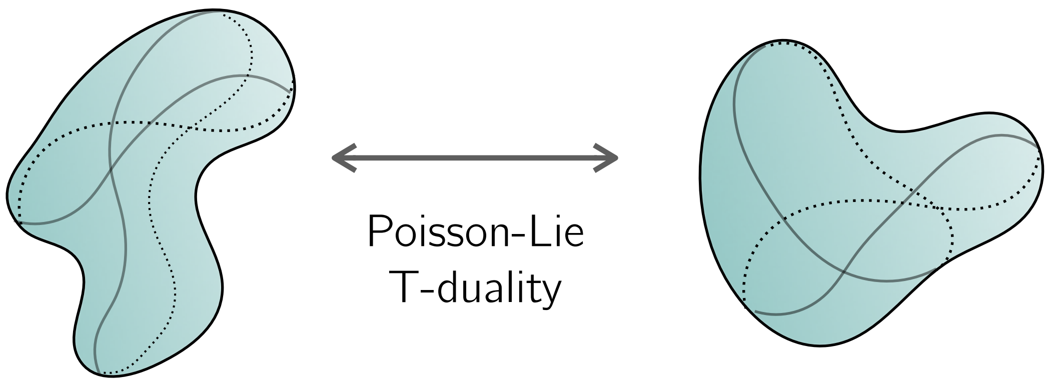

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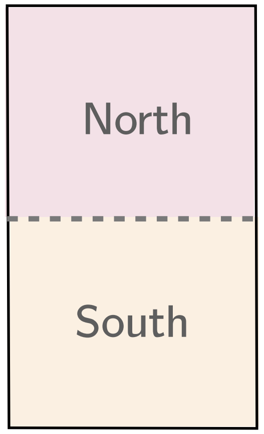
[SD, Raml]

Goal: Use the technology of defects and their fusion to understand **generalised T-duality**

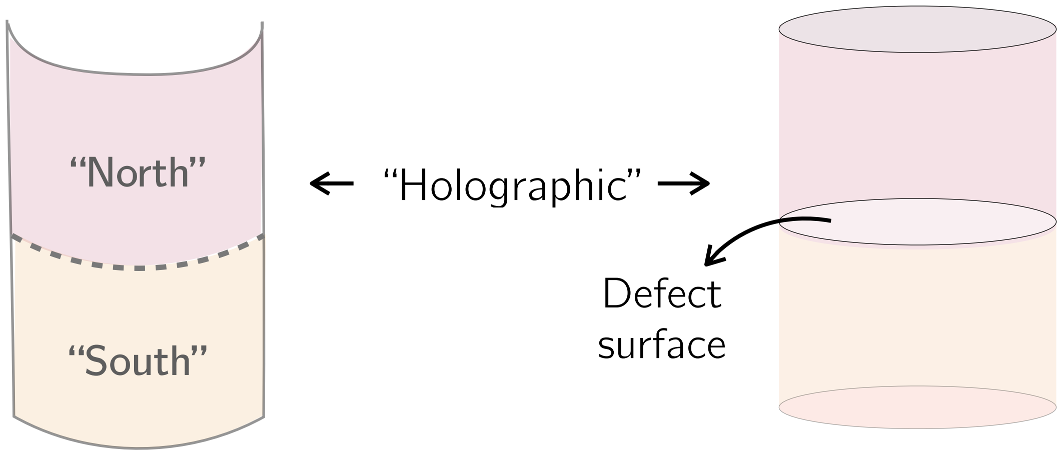
→ **problem:** fusion remained a difficult question

→ needed a way to **construct topological defects for non-Abelian Chern-Simons**...

“Holographic picture” and the folding trick



“Holographic picture” and the folding trick



$$S_{\text{unfolded}}[A_N, A_S] = S_{\text{CS}}[A_N] + S_{\text{CS}}[A_S] + S_D[\mathbb{A}]$$

with $\quad A_S \in \mathfrak{u}(1)^d_S\,, \qquad A_N \in \mathfrak{u}(1)^d_N$

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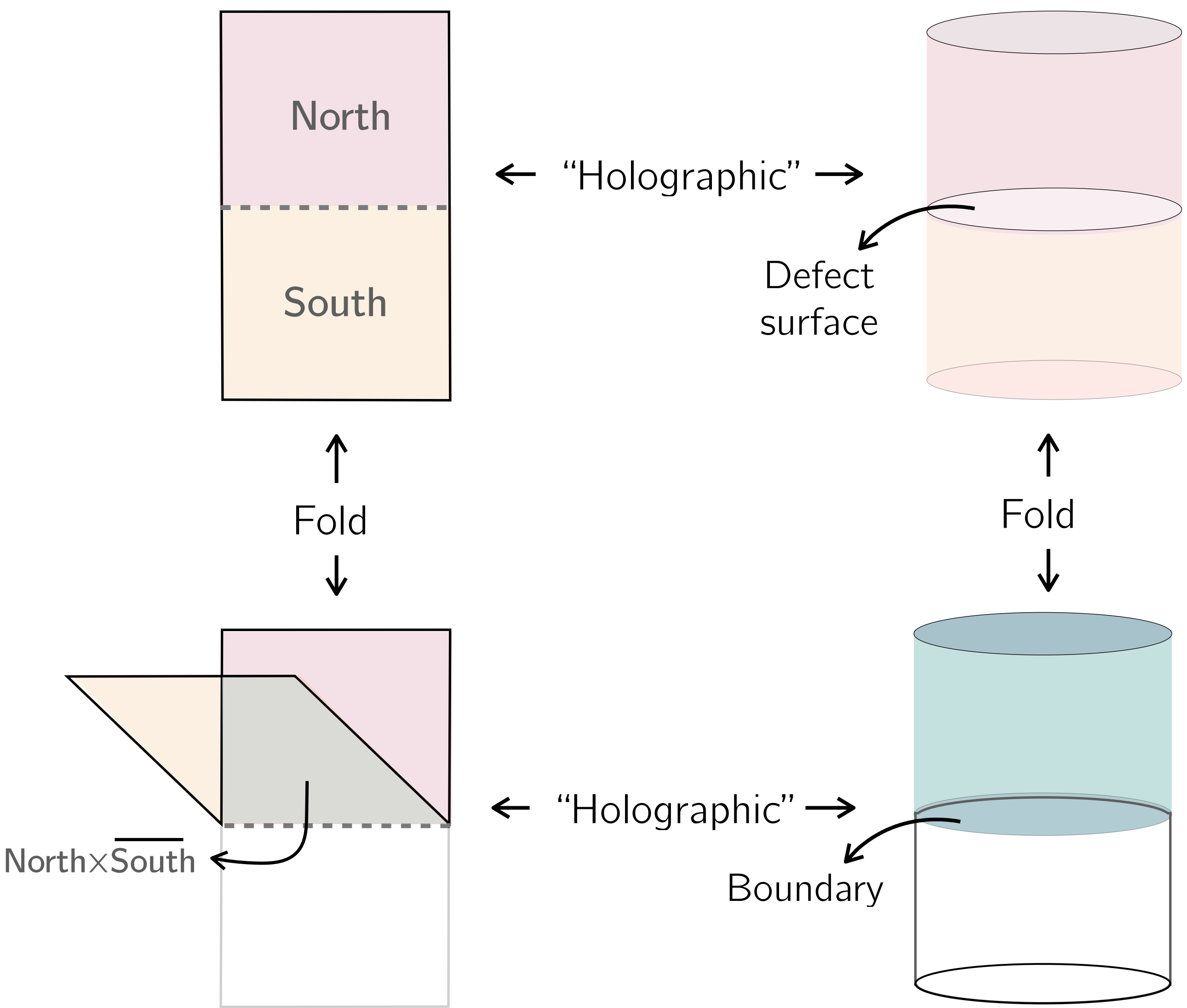

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↑
=
↓

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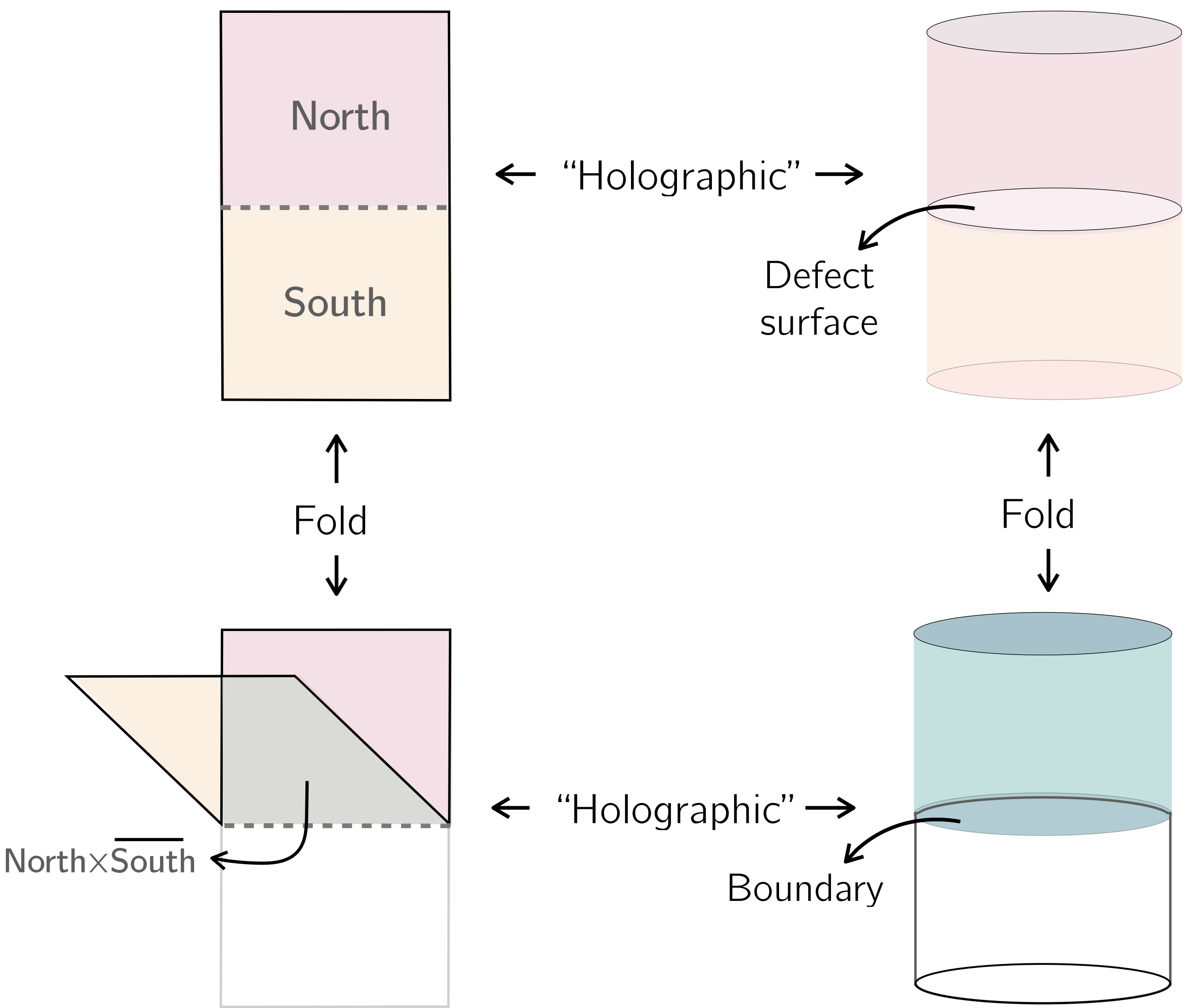

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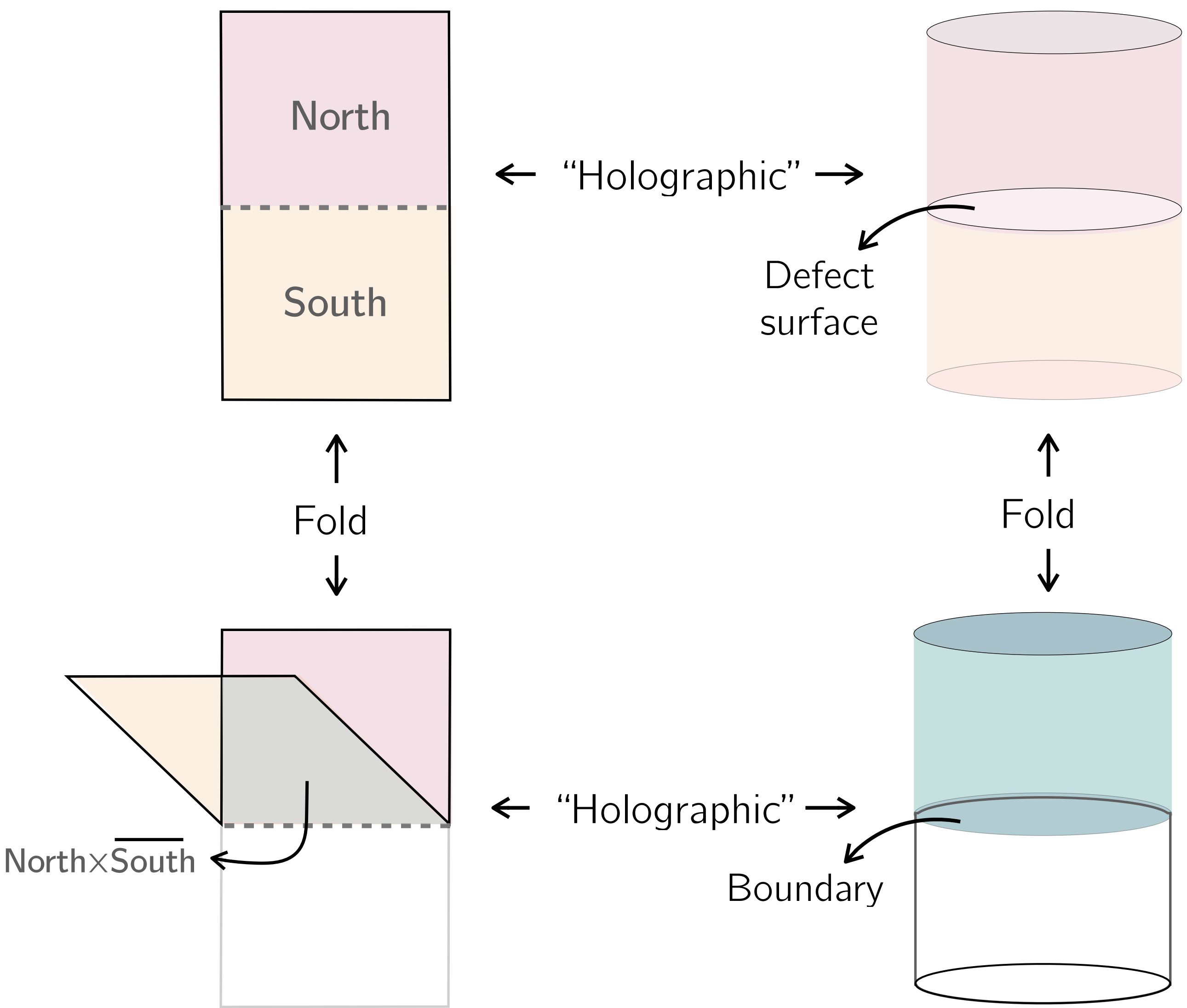

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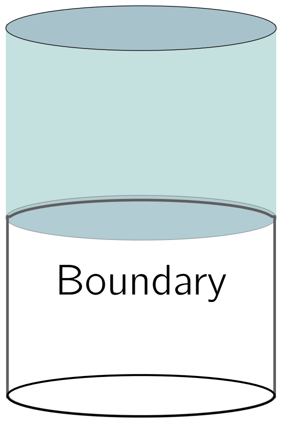
In the 3d Abelian Chern-Simons:

when does Chern-Simons theory admit a topological surface ?

Elegant algebraic answer: [Kapustin, Saulina] Look for Lagrangian subalgebras !

Lagrangians as topological boundary conditions



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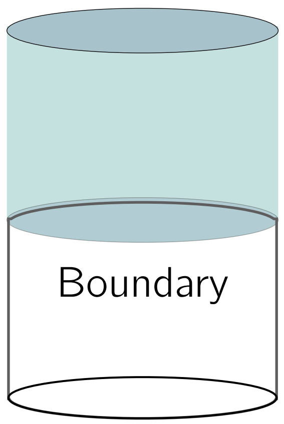

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Varying the action

$$\delta S_{\mathrm{CS}} = 2\,k \int_{M_N} \underbrace{\langle\langle \delta\mathbb{A} \,, \mathrm{d}\mathbb{A} \rangle\rangle} + k \int_D \underbrace{\langle\langle \delta\mathbb{A} \,, \mathbb{A} \rangle\rangle}$$

Vanishes by the e.o.m. $F[\mathbb{A}] \equiv \mathrm{d}\mathbb{A} = 0$

requires a boundary cond. that we will take to be topological

Demand that $\mathbb{A}|_D \in$ "Lagrangian subspace S" of $\mathfrak{u}(1)^{2d} = \mathfrak{u}(1)^d_N \oplus \mathfrak{u}(1)^d_S$

$$\langle\langle \mathbb{X} \,, \mathbb{Y} \rangle\rangle = 0 \qquad \forall\, \mathbb{X}, \mathbb{Y} \in S$$

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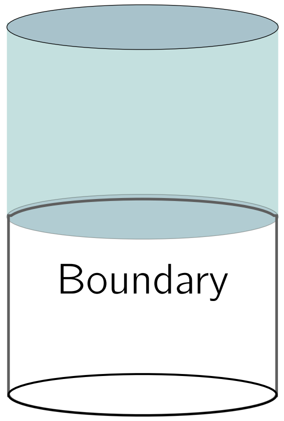

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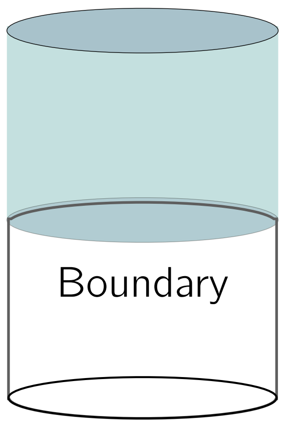
Instead of boundary condition: include the boundary term

projector

$$S = k \int_{M_N} \langle\!\langle \mathbb{A} \,, {\rm d}\mathbb{A} \rangle\!\rangle + k \int_D \langle\!\langle \mathbb{A} \,, P_S\, \mathbb{A} \rangle\!\rangle$$

Lagrangians as topological boundary conditions



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For Abelian Chern-Simons, the gauge group is

$$\mathfrak{u}(1)^{2d} = \mathfrak{u}(1)^d_N \oplus \mathfrak{u}(1)^d_S$$

And there are two Lagrangians

$$\begin{aligned} \text{diagonal} \quad &: \quad \mathfrak{u}(1)^d_+ = \{(X_N, X_S) \in \mathfrak{u}(1)^{2d} \mid X_N = X_S\} \ , \\ \text{anti-diagonal} \quad &: \quad \mathfrak{u}(1)^d_- = \{(X_N, X_S) \in \mathfrak{u}(1)^{2d} \mid X_N = -X_S\} \end{aligned}$$

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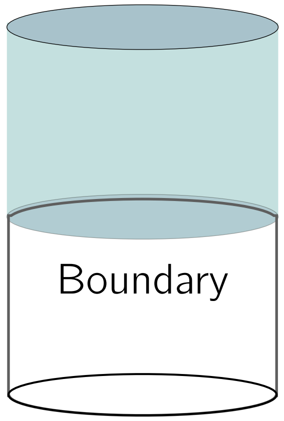

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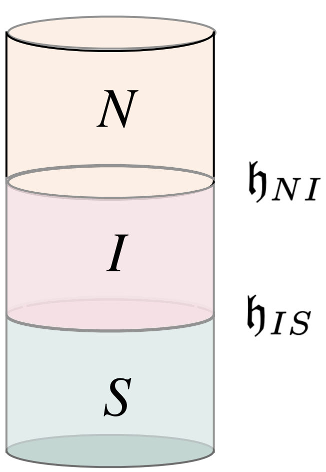

Yields the fusion algebra

$$\mathfrak{u}(1)^d_- \circ \mathfrak{u}(1)^d_- = \mathfrak{u}(1)^d_+ \qquad \mathfrak{u}(1)^d_+ \circ \mathfrak{h} = \mathfrak{h} \ , \qquad \mathfrak{h} \circ \mathfrak{u}(1)^d_+ = \mathfrak{h}$$

That is $\left(\{\mathfrak{u}(1)^d_+ \, , \mathfrak{u}(1)^d_-\} \, , \circ\right) \cong \mathbb{Z}_2$

identity idempotent

Topological surfaces in non-Abelian Chern-Simons

In the 3d non-Abelian Chern-Simons:

when does Chern-Simons theory admit a topological surface ?

[Fuch, Schweigert, Valentino]
[Kapustin, Saulina]

$$S_{\rm CS} = k \int_{M_N} \left(\langle\!\langle \mathbb{A} \,, \mathrm{d}\mathbb{A} \rangle\!\rangle + \frac{1}{3} \langle\!\langle \mathbb{A} \,, [\![\mathbb{A} \,, \mathbb{A}]\!] \rangle\!\rangle \right)$$

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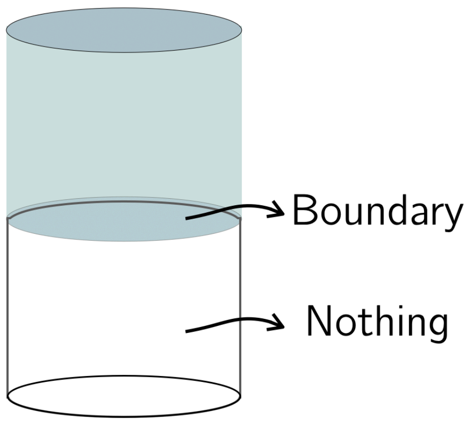

Problem

find Lagrangian subalgebras and corresponding boundary term
for the gauge group of "doubled" Chern-Simons

When gauge group is non-Abelian: hard problem

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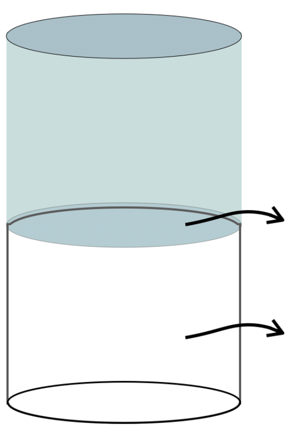

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When gauge group is non-Abelian: hard problem

We showed: [Arvanitakis, Cole, SD, Thompson *WIP*]

→ Canonical way to constructed topological defects

→ Crucial tool: (modified) Yang-Baxter equation

→ Defined and studied their fusion

Lagrangians via the mCYBE: $\mathcal{R}$-defects

Strategy: simplify one's life a little by looking for a subclass of defects

Solve the modified classical Yang-Baxter equation

$$[\mathcal{R}X\,,\mathcal{R}Y] - \mathcal{R}\big([\mathcal{R}X\,,Y] + [X\,,\mathcal{R}Y]\big) + [X\,,Y] = 0$$

Yields a Lagrangian subalgebra $\quad \mathfrak{g}_{\mathcal{R}} = \{\big((\mathcal{R}+1)X, (\mathcal{R}-1)X\big) \in \mathfrak{d}\}$

With Lie-bracket $\quad [X\,,Y]_{\mathcal{R}} = [\mathcal{R}X\,,Y] + [X\,,\mathcal{R}Y]$

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Called a “bi-algebra” or “Manin triple” $\mathfrak{d} = \mathfrak{g}_{\Delta} \oplus \mathfrak{g}_{\mathcal{R}}$

→ Directly generalises the Abelian case $\mathfrak{u}(1)^{2d} = \mathfrak{u}(1)^d_{\Delta} \oplus \mathfrak{u}(1)^d_{-}$

→ Technical requirement $\mathfrak{d} = \mathfrak{g}_{\Delta} \oplus \mathfrak{g}_{\mathcal{R}} \cong \mathfrak{g} \oplus \mathfrak{g}$

Effectively specialising to non-compact algebras

Lagrangians via the mCYBE: $\mathcal{R}$-defects

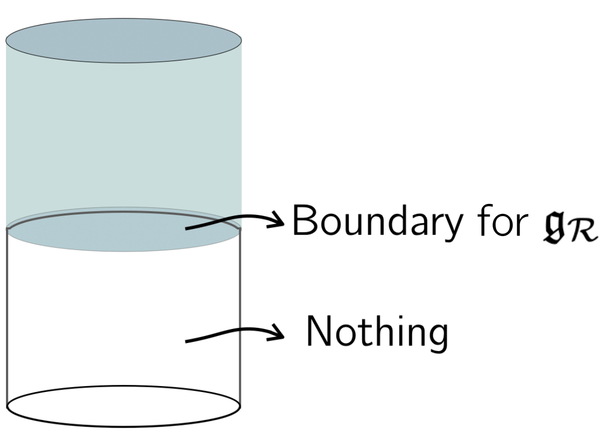
We have a topological boundary condition for the Lagrangian subalgebra

$$\mathfrak{g}_{\mathcal{R}} = \{\big((\mathcal{R}+1)X, (\mathcal{R}-1)X\big) \in \mathfrak{d}\}$$

Since with the R-matrix we can construct a projector

$$\langle\!\langle \mathbb{A} \,, \mathcal{P}_{\mathcal{R}}\mathbb{A} \rangle\!\rangle = \langle A_S \,, A_N \rangle + \tfrac{1}{2}\langle A_N - A_S \,, \mathcal{R}(A_N - A_S)\rangle$$

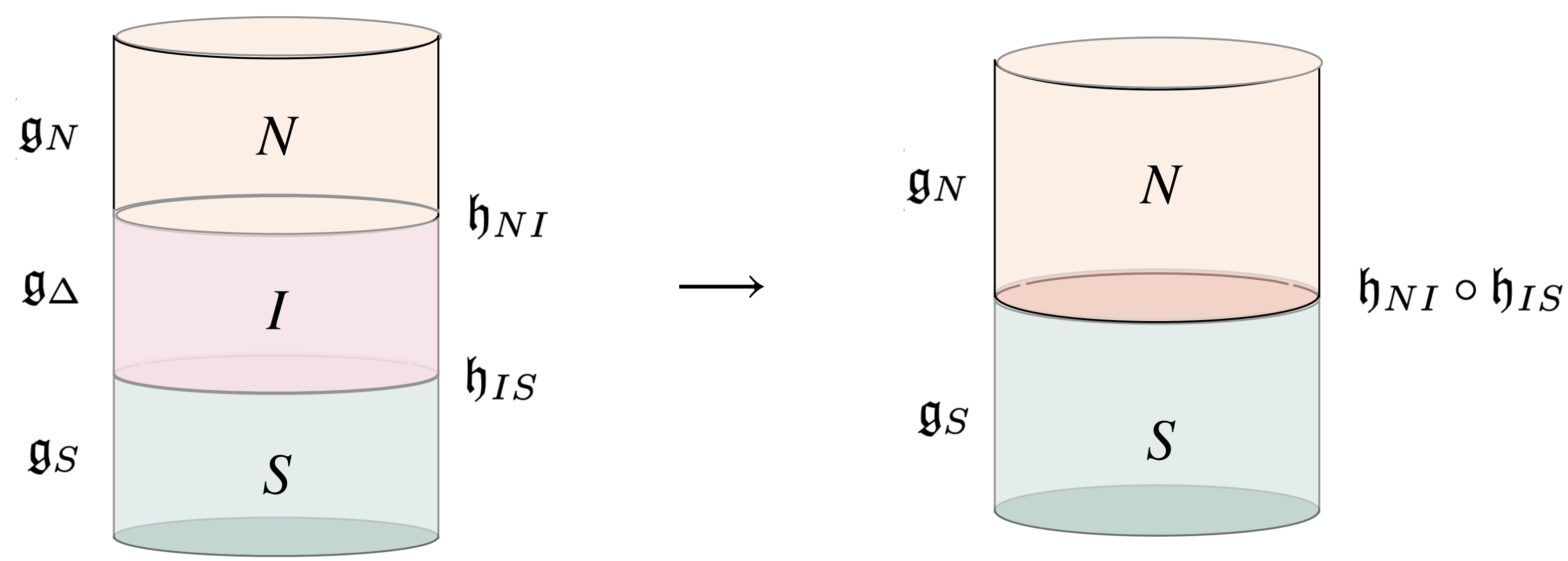
Yielding the (folded) Chern-Simon action



$$S_{\text{folded}} = \int \mathrm{CS}[\mathbb{A}] + \int_D \langle\!\langle \mathbb{A} \,, \mathcal{P}_{\mathcal{R}}\mathbb{A} \rangle\!\rangle$$

Fusion at the level of the algebra

[Arvanitakis] [Fuchs, Schweigert, Waldorf] [Burzstyn]



$$\mathfrak{h}_{NI} \circ \mathfrak{h}_{IS} = \Pi_{NS}\Big((\mathfrak{h}_{NI} \oplus \mathfrak{h}_{IS}) \cap (\mathfrak{g}_N \oplus \mathfrak{g}_\Delta \oplus \mathfrak{g}_S) \Big)$$

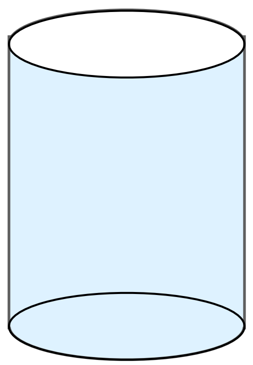
$$\mathfrak{g}_\mathcal{R} \circ \mathfrak{g}_\mathcal{R} = \{ (X_N, X_S) \in \mathfrak{d} \mid X_N^{\mathfrak{t}} = X_S^{\mathfrak{t}} \,, X_N^- = 0 \,, X_S^+ = 0 \}$$

$\longrightarrow$ Proved that $\mathfrak{h}_{NI} \circ \mathfrak{h}_{IS}$ is

- ✓ Lagrangian
- ✓ a subalgebra
- with identity element $\mathfrak{g}_\Delta \circ \mathfrak{h} = \mathfrak{h}$, $\mathfrak{h} \circ \mathfrak{g}_\Delta = \mathfrak{h}$
- reduces to Lagrangian fusion in the Abelian case

AdS$_3$ gravity

$$S_{\text{grav}} = \underbrace{\frac{1}{16\pi G}\int_M \sqrt{g}(R+2)}_{\text{Einstein-Hilbert action}} + \overbrace{\frac{1}{8\pi G}\int_{\partial M} \mathrm{d}^2x\sqrt{h}K}^{\text{Gibbons-Hawking-York}} - \underbrace{\frac{1}{8\pi G}\int_{\partial M}\sqrt{h}}_{\text{Boundary area counter-term}}$$



Admits a re-formulation in terms of Chern-Simons theories

$$S_{\text{grav}} = S_{\text{CS}}[A] - S_{\text{CS}}[\bar{A}] + S_{\text{bdy}}$$

"folded" Chern-Simons theory for the gauge group $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$

Where the gauge connections are related to the metric vielbein and the soldering form

$$A^a = \omega^a + \frac{1}{\ell}e^a \,, \qquad \tilde{A}^a = \omega^a - \frac{1}{\ell}e^a$$

With respect to the $\mathfrak{sl}(2,\mathbb{R})$ -generators $[L_a, L_b] = (a-b)L_{a+b}$

AdS$_3$ gravity

Gibbons-Hawking-York

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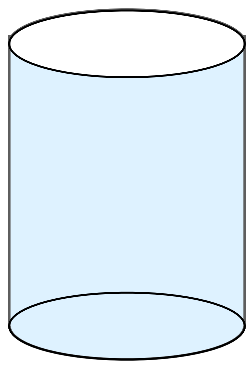
Einstein-Hilbert action

Boundary area counter-term



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In the Feffer-Graham gauge the boundary term is [Llabres; Apolo; Ebert, Hijano, Kraus, Monten, Myers]

$$S_{\text{bdy}} = \frac{k}{4\pi}\int_{\partial M} \text{tr}\, A \wedge \bar{A} - \frac{k}{2\pi}\int_{\partial M} \text{tr}\left(L_0(A-\bar{A}) \wedge (A-\bar{A})\right)$$

Cartan generator of $\mathfrak{sl}(2,\mathbb{R})$

AdS$_3$ gravity

The **boundary term corresponds precisely** the R-boundary term [Arvanitakis, Cole, SD, Thompson]

$$S_{\text{bdy}} = c \int_{\partial M} \langle A, \bar{A} \rangle - \tfrac{1}{2} \langle \bar{A} - A, \mathcal{R}(\bar{A} - A) \rangle$$

where the R-matrix is the **Drinfel'd-Jimbo R-matrix** for

$$\mathcal{R} L_0 = 0 \,, \qquad \mathcal{R} L_\pm = \pm L_\pm$$

AdS$_3$ gravity

The boundary term corresponds precisely the R-boundary term [Arvanitakis, Cole, SD, Thompson]

$$S_{\rm bdy} = c\int_{\partial M}\langle A,\bar{A}\rangle - \tfrac{1}{2}\langle \bar{A}-A,\mathcal{R}(\bar{A}-A)\rangle$$

where the R-matrix is the Drinfel'd-Jimbo R-matrix for

$$\mathcal{R}L_0 = 0\,, \qquad \mathcal{R}L_\pm = \pm L_\pm$$

Higher spin AdS$_3$ gravity also admits Chern-Simons formulation

$$S_{\rm grav} = S_{\rm CS}[A] - S_{\rm CS}[\bar{A}] + S_{\rm bdy}$$

with connections for the algebra $\mathfrak{sl}(N,\mathbb{R})$

with spin fields $\phi_{\mu_1\ldots\mu_{s-1}\mu_s} \sim {\rm Tr}\left(e_{(\mu}\ldots e_{\mu_{s-1}}e_{\mu_s)}\right)$

→ what type of BC does the R-boundary term give rise to?

$$S_{\rm bdy} = c\int_{\partial M}\langle A,\bar{A}\rangle - \tfrac{1}{2}\langle \bar{A}-A,\mathcal{R}(\bar{A}-A)\rangle$$

with the Drinfel'd-Jimbo matrix $\mathcal{R}H_i = 0\ ,\quad \mathcal{R}E_\alpha = +c\,E_\alpha\ ,\quad \mathcal{R}E_{-\alpha} = -c\,E_{-\alpha}$

to be continued....

→ Summary

Constructed **topological boundary conditions in non-Abelian Chern-Simons**

- Led to a subclass: R-defects
- Looked into their fusion
- Application to AdS_3 -gravity

→ What's next ?

- Can we identify the surface defect for Poisson-Lie T-duality ?
- Does the R-defect boundary make sense in higher spin 3d gravity
- SymTFT description ?

Thank you for you attention !