Cosmological complexity

Shajid Haque
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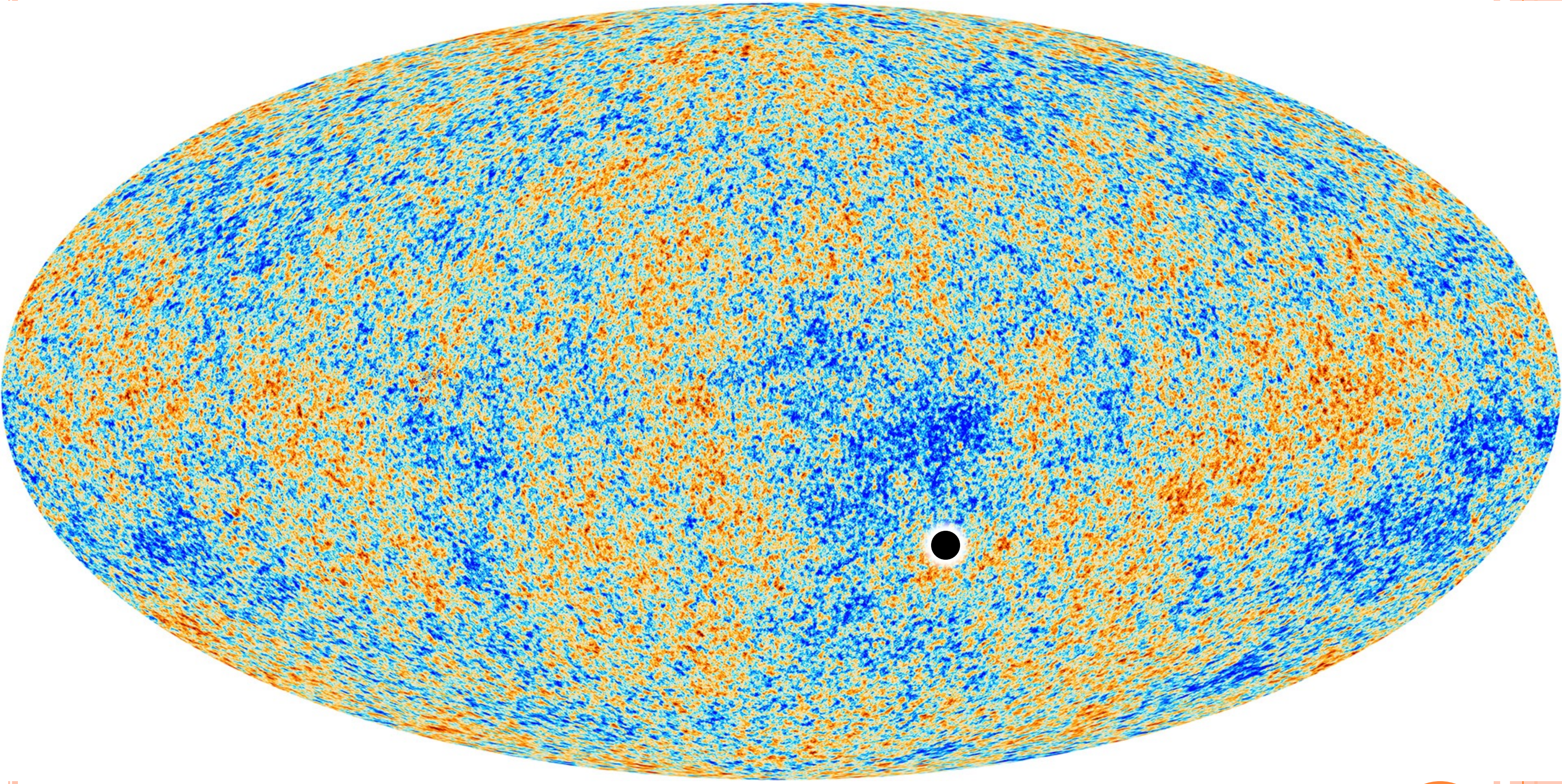
Based on

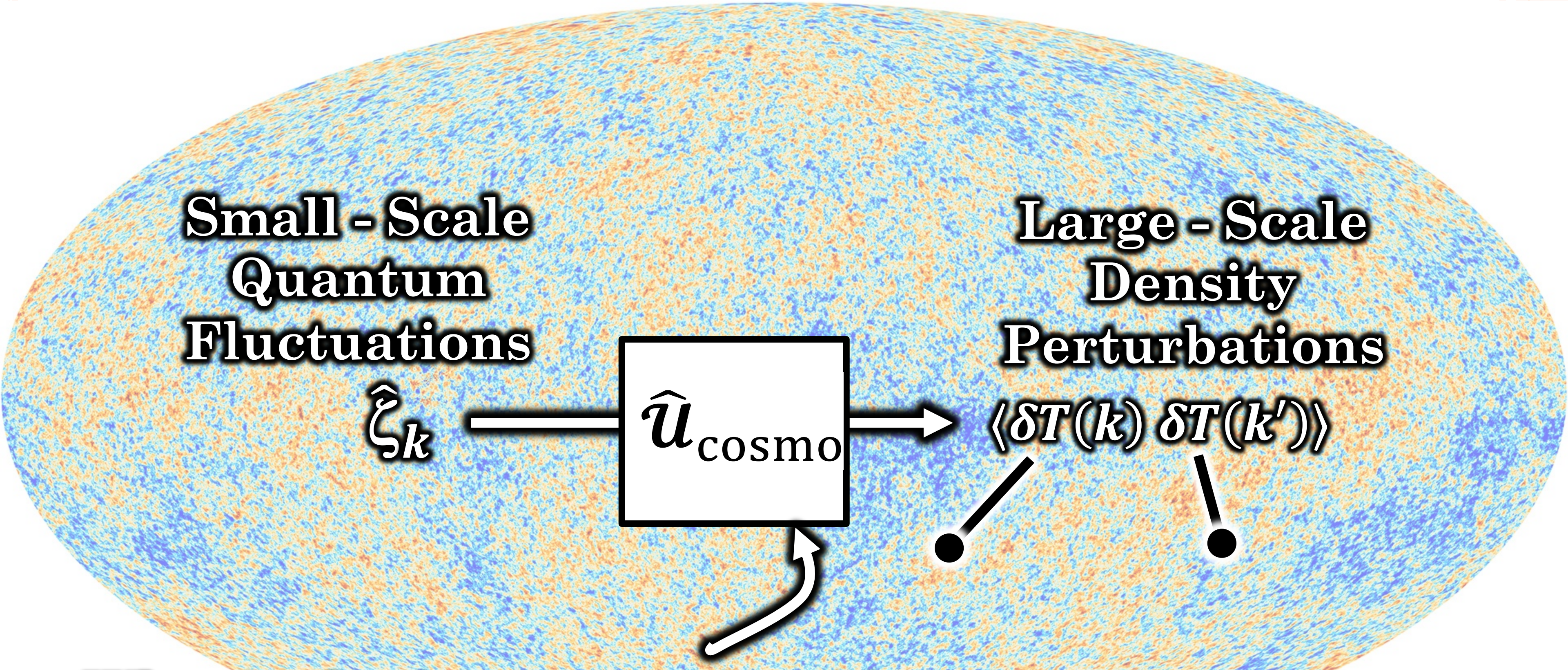
Bhattacharyya, Das, **SH**, Underwood, 2001.08664
Bhattacharyya, Das, **SH**, Underwood, 2005.10854
SH, Jana, Underwood, 2107.08969
SH, Jana, Underwood, 2110.08356

Cosmic Microwave Background



ESA and Planck Collaboration

Cosmic Microwave Background



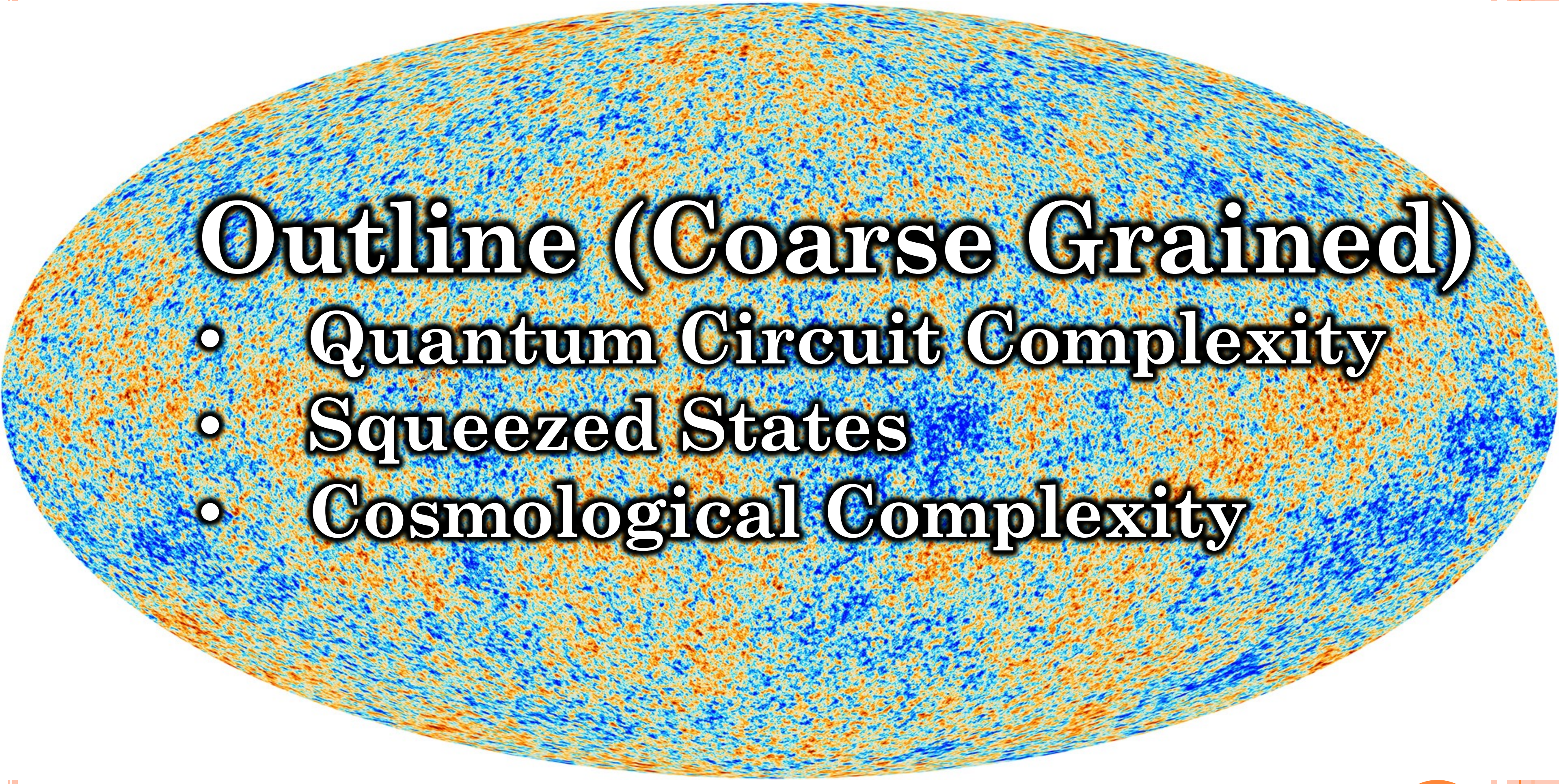
What is the (quantum circuit) complexity of this process?

ESA and Planck Collaboration

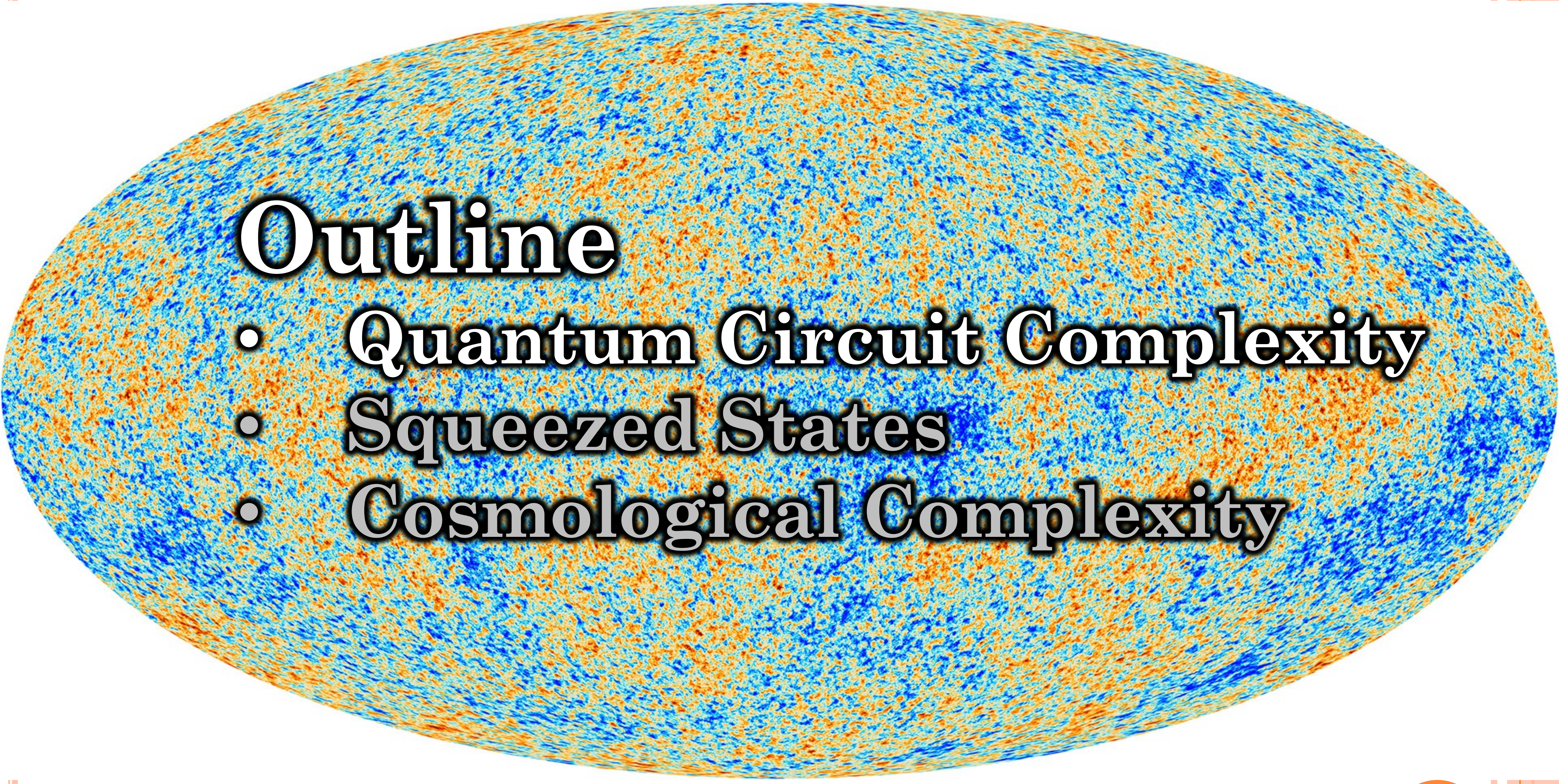
- Growth of complexity with time?
- Bounds on the growth of complexity?
- Complexity for decohered state?

Outline (Coarse Grained)

- Quantum Circuit Complexity
- Squeezed States
- Cosmological Complexity



ESA and Planck Collaboration



Outline

- Quantum Circuit Complexity
- Squeezed States
- Cosmological Complexity

ESA and Planck Collaboration

Background and Motivation

Question: How do we probe the interior of a Black Hole?

Eternal Black Hole

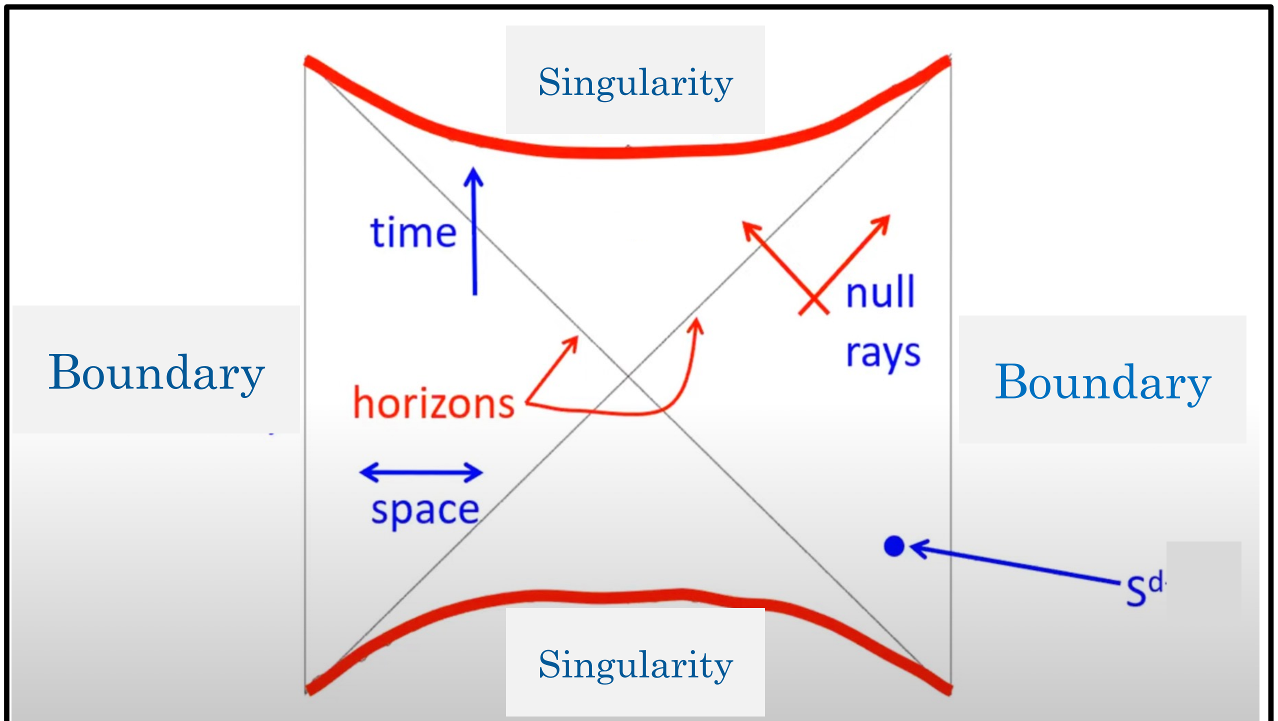


Figure courtesy: Rob Myers

Background and Motivation

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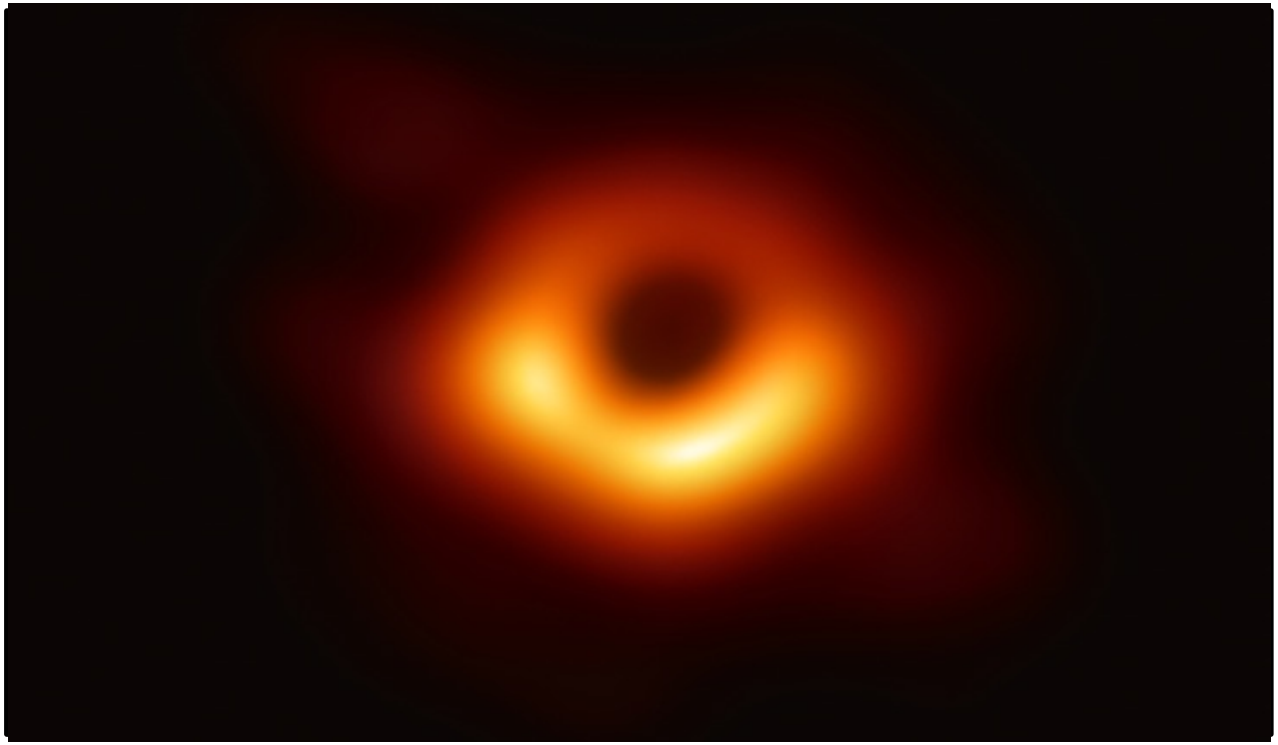

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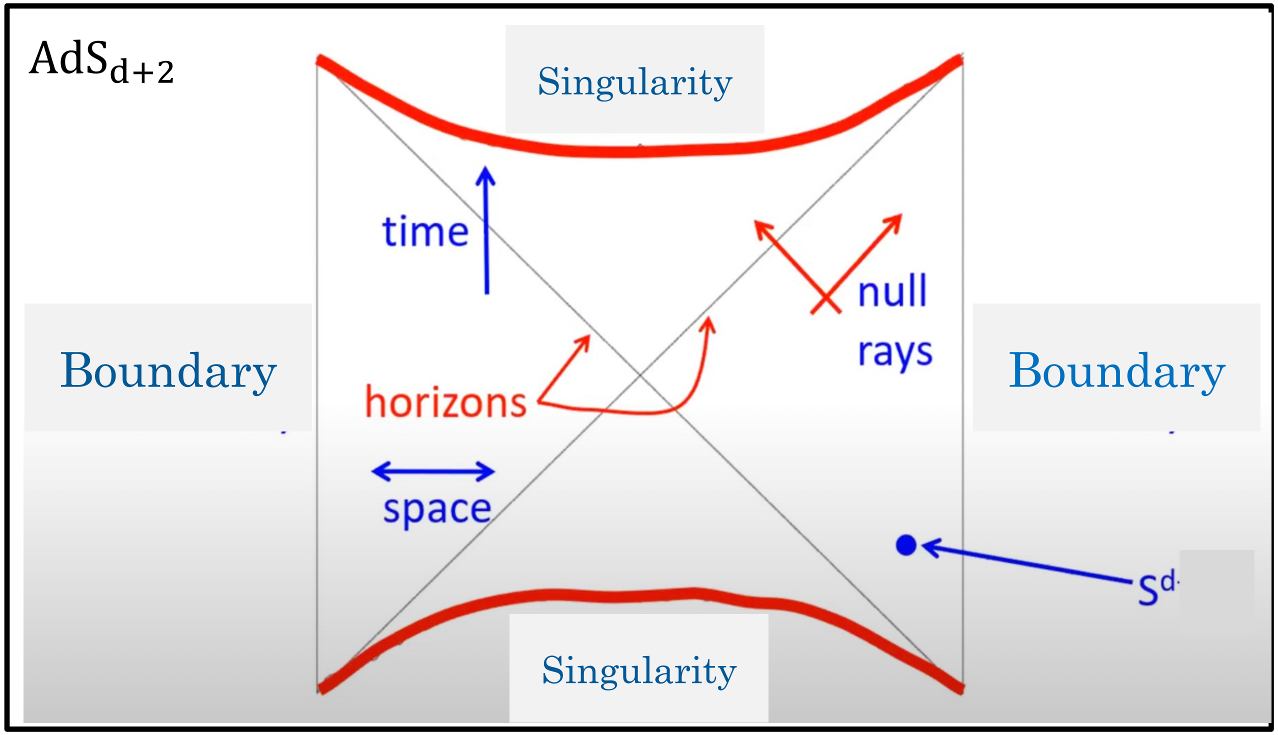


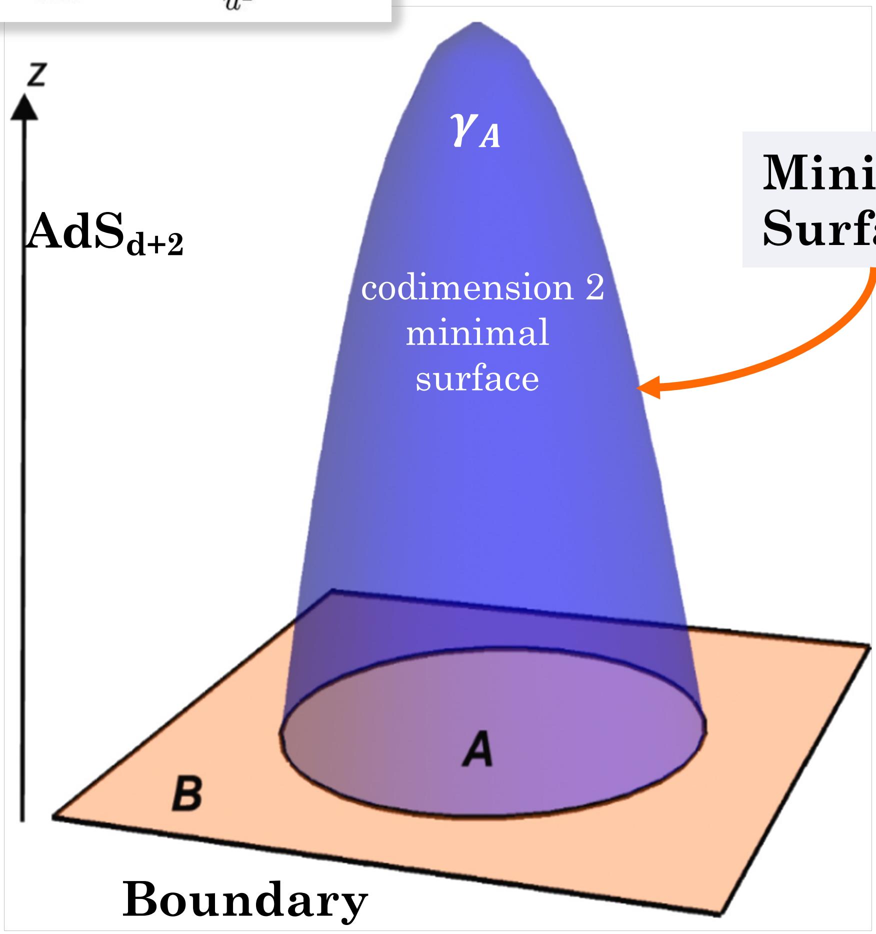
Figure courtesy: Rob Myers

Background and Motivation

EE in (d + 1) dim. CFT can be obtained from the area of d dim. minimal surfaces in AdS_{d+2}

Ryu, Takayanagi 2006

$$ds^2_{\mathrm{AdS}} = R^2_{\mathrm{AdS}} \frac{-dt^2 + dx^i dx^i + du^2}{u^2}$$



Boundary
$z = 0$

Minimal Surface ⟺ (AdS/CFT Duality) **Entanglement Entropy**

$$S_A = \frac{(\mathrm{Area})_{\min}}{4G} = \frac{\mathrm{Area}\,(\gamma_A)}{4G}$$

Background and Motivation

Eternal Black Hole ↔ (AdS/CFT duality) ↔ 2 Copies of CFT in Thermofield Double

↓ ↓

Ryu-Takayanagi Surface ↔ (AdS/CFT duality) ↔ Entanglement Entropy

Ryu, Takayanagi 2006

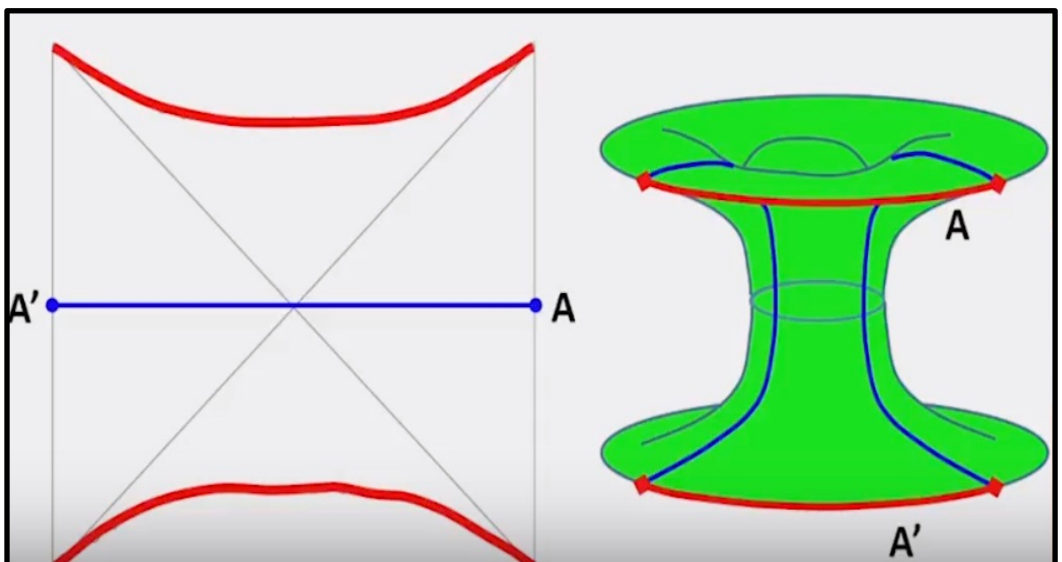


Figure courtesy: Rob Myers

Background and Motivation

Eternal Black Hole ⟷ (AdS/CFT duality) ⟷ 2 Copies of CFT in Thermofield Double

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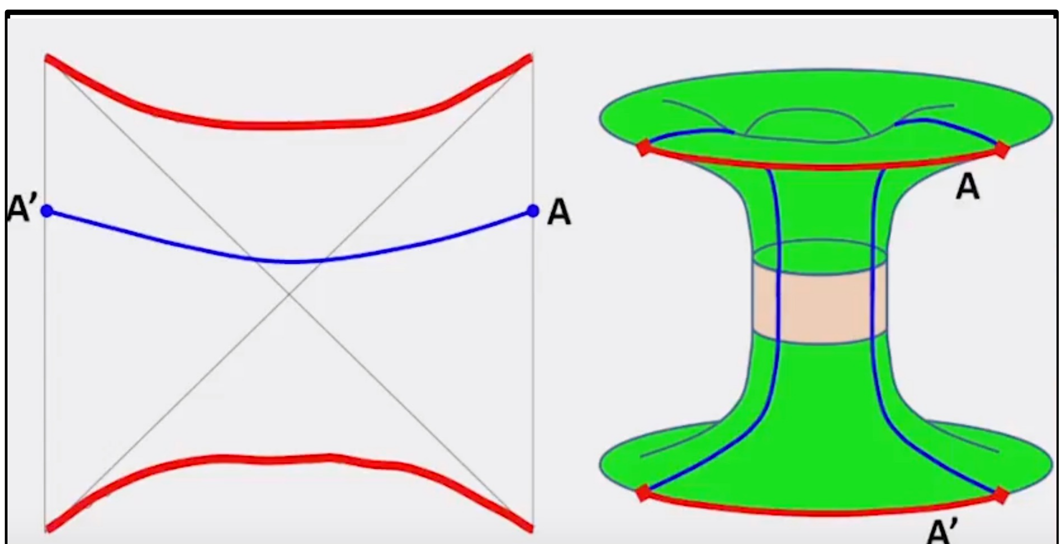


Figure courtesy: Rob Myers

Background and Motivation

Entanglement Entropy is NOT enough! – Susskind, 2014

Eternal Black Hole ⟷ (AdS/CFT duality) ⟷ 2 Copies of CFT in Thermofield Double

↓ ↓

RT Surface is Disconnected ⟷ (AdS/CFT duality) ⟷ Entanglement Entropy Saturates

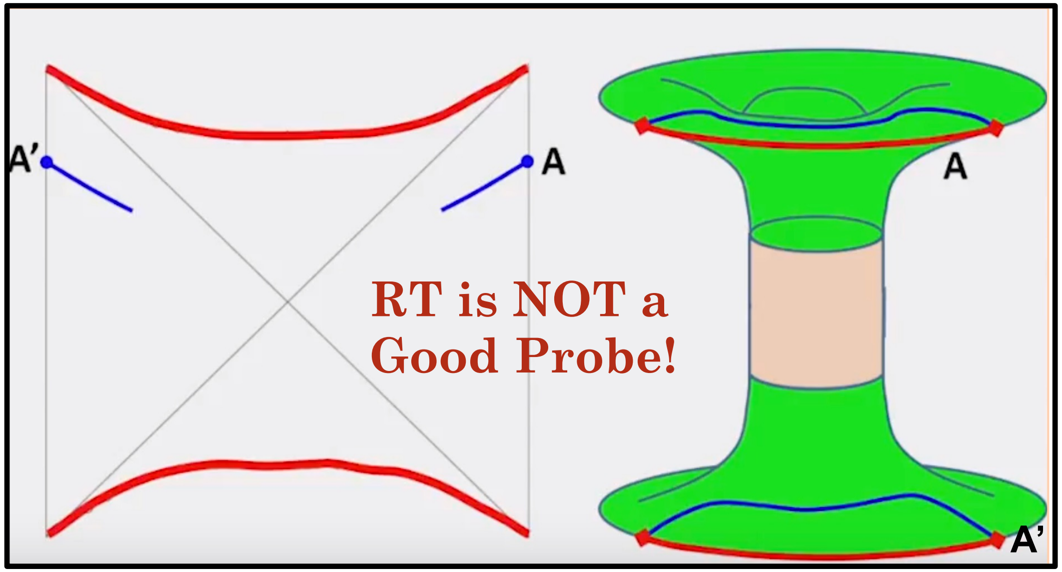
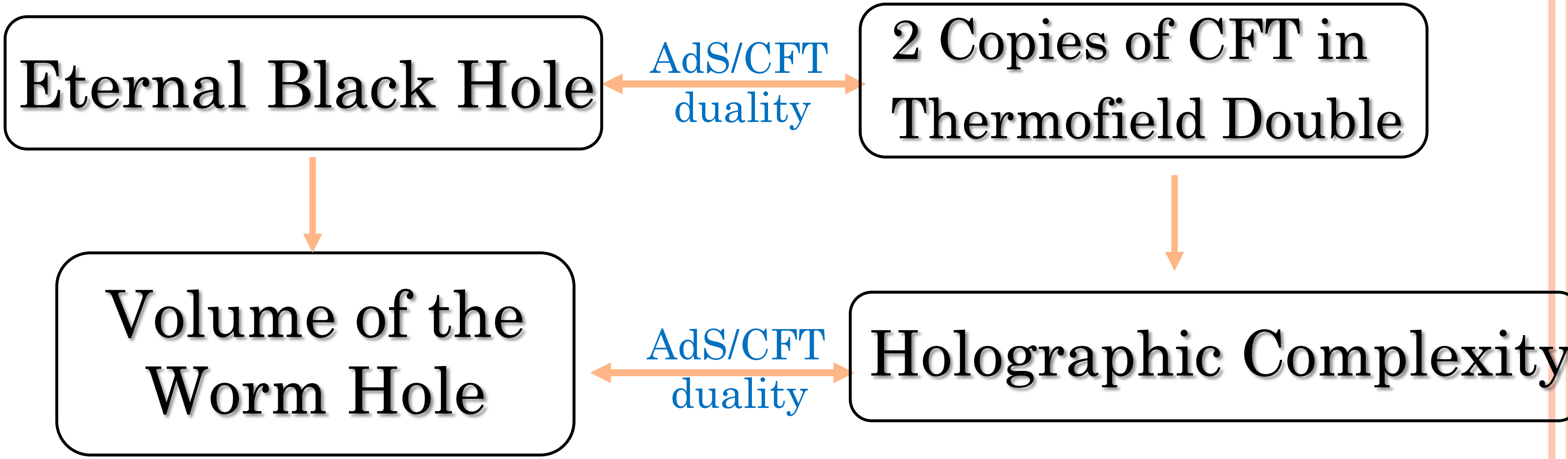


Figure courtesy: Rob Myers

Background and Motivation

Entanglement Entropy is NOT enough! – Susskind, 2014



Complexity is a well-defined Quantity

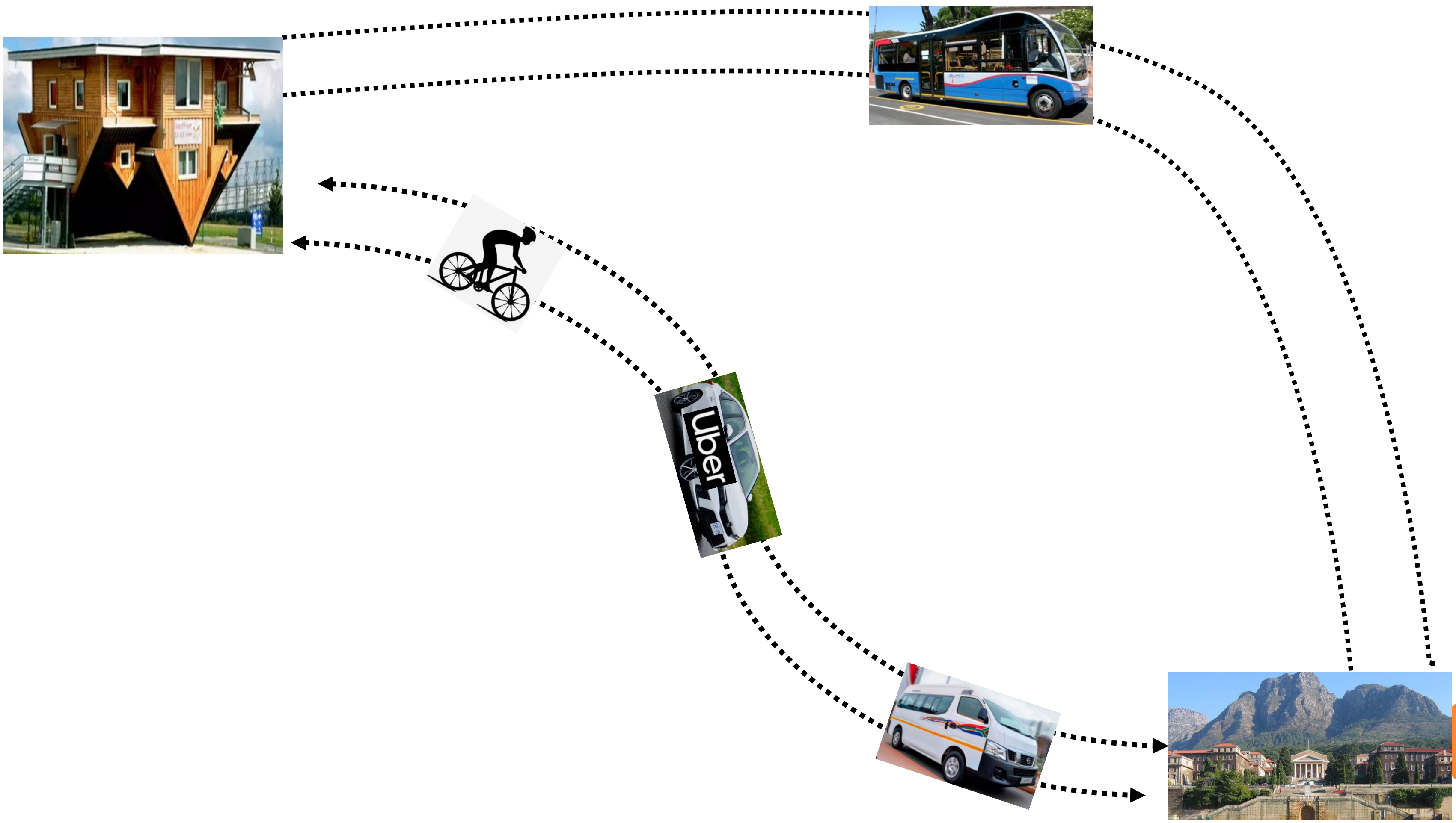
in <u>Quantum Information Theory</u>!

Incorporating an idea from a different Field

Computational Complexity

Generically: A measure of difficulty to implement a task.

Example: How difficult it is to come to work/school everyday?



Computational Complexity

Generically: A measure of difficulty to implement a task.

Example: How difficult it is to come to work everyday?

Physics Question:

How difficult it is to prepare a particular State in a Quantum Theory?

We will use a particular Model – Quantum Circuit Model

$$|\psi\rangle = U\,|\psi_0\rangle$$

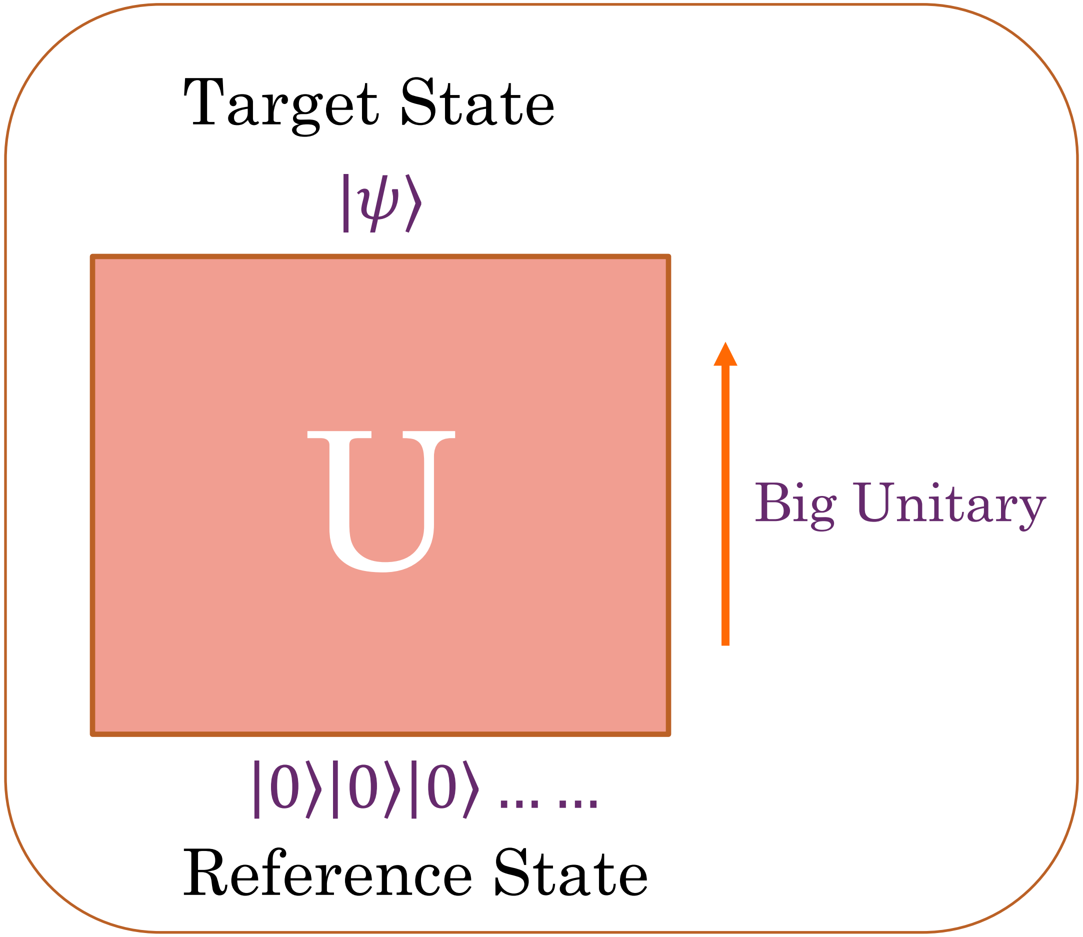
Target State

Unitary Operator

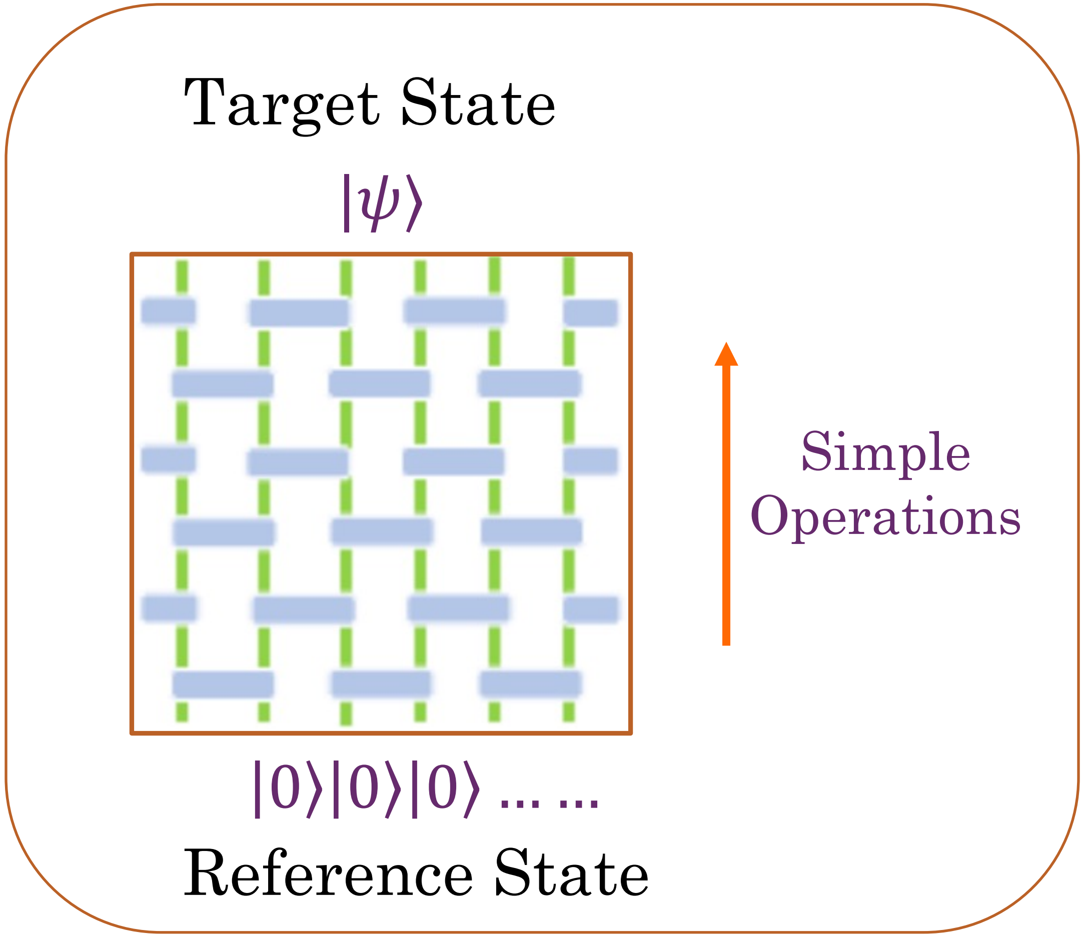
Analog of Classical Logic Gate

Reference State: $|0\rangle|0\rangle|0\rangle \ldots \ldots$

Quantum Circuit



Quantum Circuit



How to **Minimize** the number of operations/Quantum Gates?

How to find the **Optimal Quantum Circuit?**

Complexity = Minimum no of Gates required to Prepare the Target State

QUANTUM CIRCUIT

More Precisely
Approximate Target state with Unitary operations built from these Quantum Gates

$$|\psi\rangle = \underbrace{g_n \cdots \cdots \cdots g_2 g_1}_{\textbf{Quantum Gates}} |\psi_0\rangle$$

with some tolerance

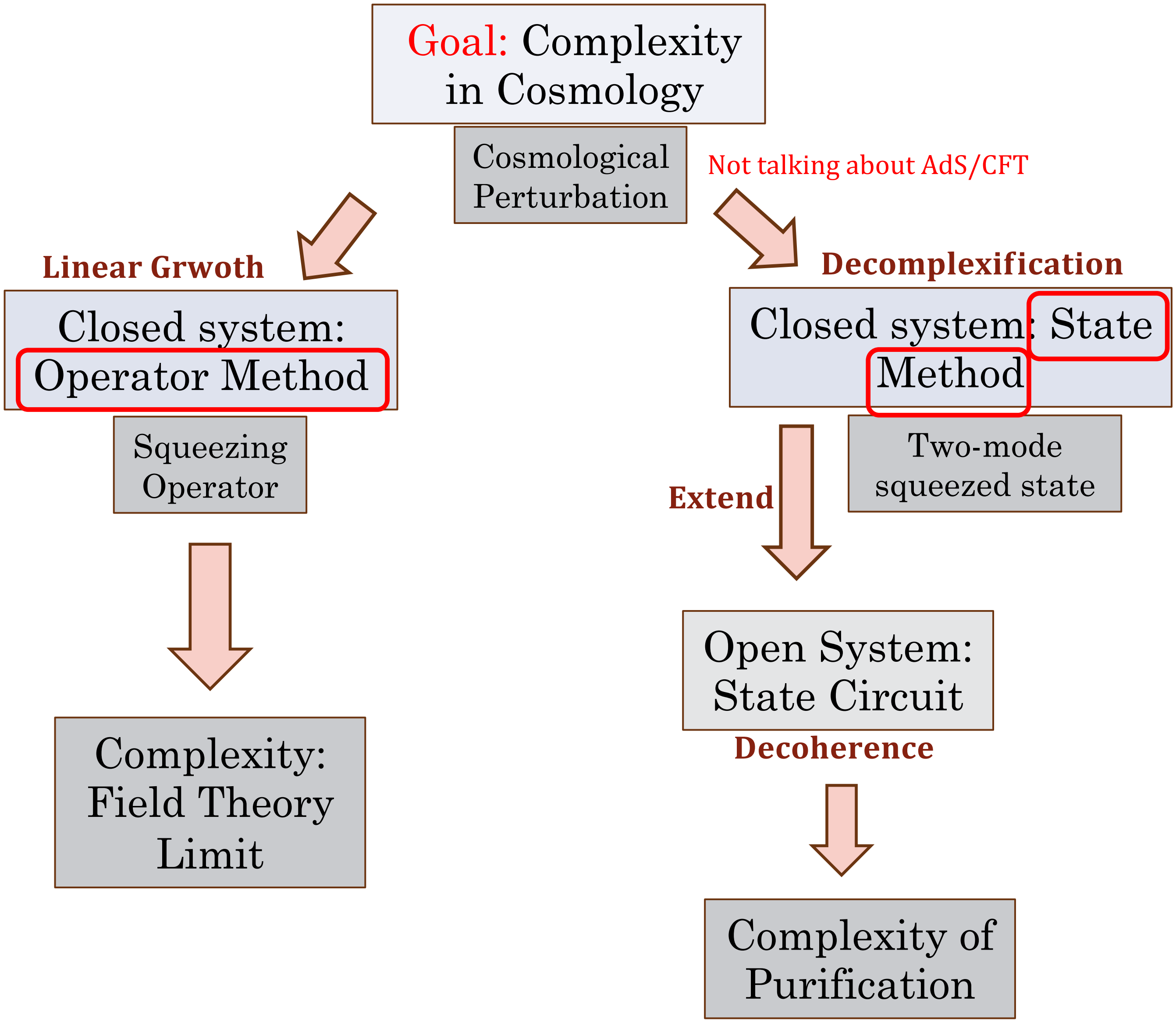
$$\langle \psi_T | \psi_T^{(\epsilon)} \rangle \geq 1 - \epsilon$$

There are some Universal Gates

There could be many choices for Universal Gates

Any Universal (one or two qubits) Gate sets is good as any other provided we only care about some tolerance

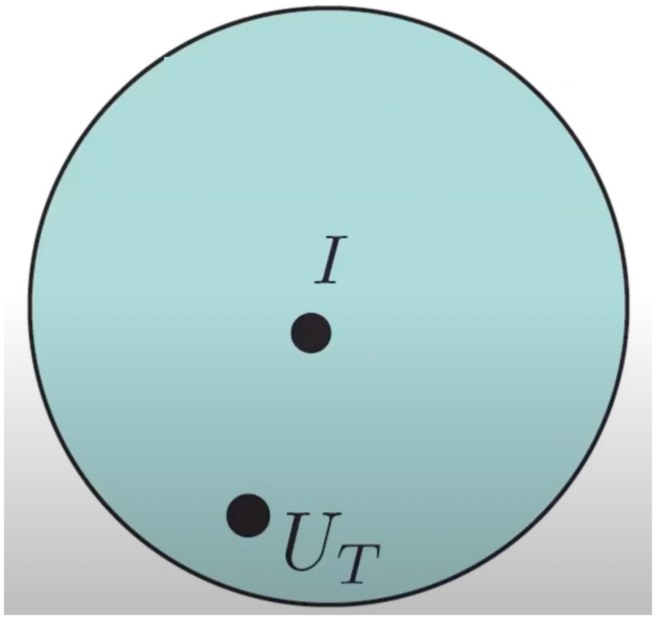
Different Circuits



Operator Complexity

Quantum circuit is a **unitary operator** that transforms a given reference state to a specified target state.

$$|\psi\rangle_{\mathrm{T}} = \hat{\mathcal{U}}_{\mathrm{target}}\ |\psi\rangle_{\mathrm{R}}$$



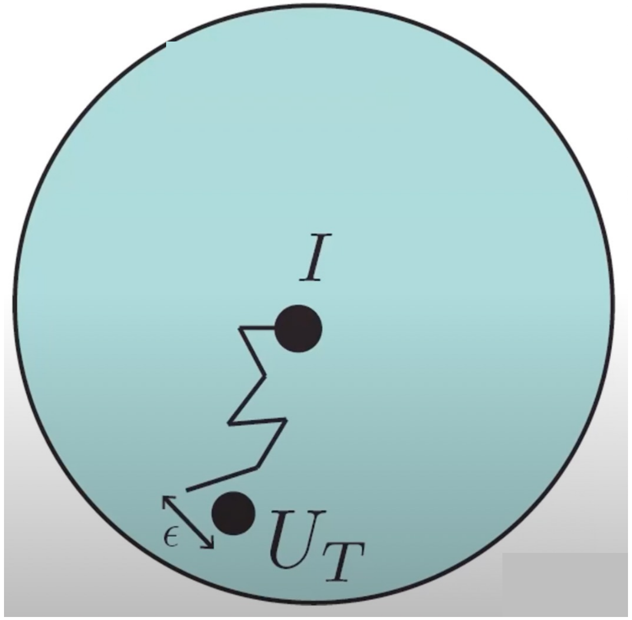
$|\psi_T\rangle \longleftarrow |\psi_R\rangle$ State

$U_{target} \longleftarrow I$ Operator

Operator Complexity

Quantum circuit is a **unitary operator** that transforms a given reference state to a specified target state.

$$|\psi\rangle_{\mathrm{T}} = \hat{\mathcal{U}}_{\mathrm{target}} \; |\psi\rangle_{\mathrm{R}}$$



$$U_{target} = g_n \cdots \cdots . \, g_2 g_1$$

Drawbacks of the discrete Gates

- Sensitive to arbitrary tolerance
- Discontinuous (overshooting problem)

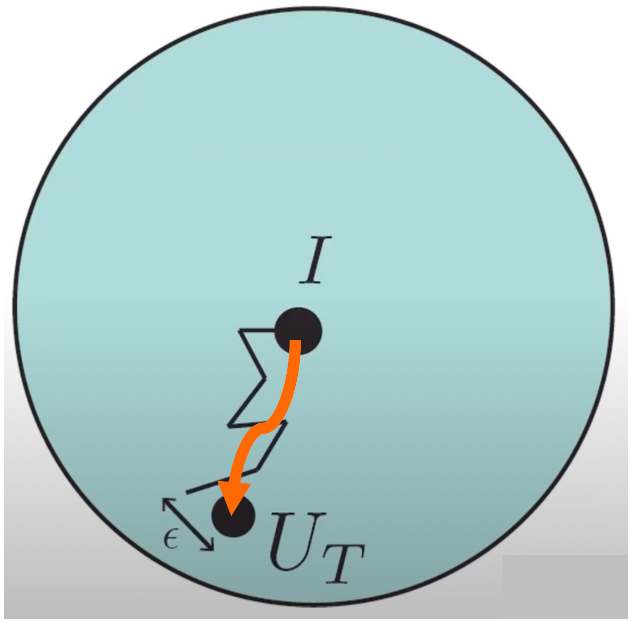
Operator Complexity

Quantum circuit is a **unitary operator** that transforms a given reference state to a specified target state.

$$|\psi\rangle_{\mathrm{T}} = \hat{\mathcal{U}}_{\mathrm{target}}\ |\psi\rangle_{\mathrm{R}}$$

Need a continuous description of complexity! **Nielsen 2005**

Finding optimal circuit ➔ Geometric problem of studying geodesic on a group manifold



Group Manifold Approach

Target unitary is generated from a set of fundamental operators, which form a Lie algebra.

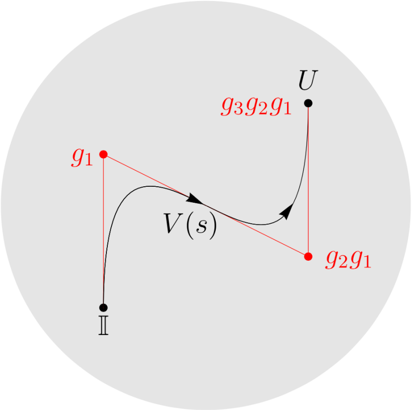
Advantage?

- Geometry is determined by the generators of the Lie algebra
- Manifestly independent of the states.
- Continuous trajectories than discrete ones

Geometry suggests new approach to Quantum Algorithms!

Operator Complexity

Target Unitary consists of a continuum of operators parametrized by a parameter s.



$$\hat{\mathcal{U}}_{\text{target}} = \mathcal{P} \exp\left[-i\int_0^1 V^I(s)\hat{\mathcal{O}}_I \, ds\right]$$

Vectors that specify the path of the sequence of operators

Universal Operators.

$$\left[\hat{\mathcal{O}}_I, \hat{\mathcal{O}}_J\right] = i f_{IJ}^P \, \hat{\mathcal{O}}_P$$

Path ordering ensures that the operators are applied sequentially from $s = 0$ to 1

It is useful to introduce a s-dependent unitary

$$\hat{U}(s) = \mathcal{P} \exp\left[-i\int_0^s V^I(s')\hat{\mathcal{O}}_I \, ds'\right]$$

Solution to Equation

$$\frac{d\hat{U}(s)}{ds} = -iV^I(s)\,\hat{\mathcal{O}}_I\,\hat{U}(s)$$

Subject to the BC: $\hat{U}(0) = \mathbb{1}$ and $\hat{U}(1) = \hat{\mathcal{U}}_{\text{target}}$

Operator Complexity

From the Unitary we define a Cost Function $F\left(U(t), \dot{U}(t)\right)$

Define a Distance Functional

$$\mathcal{D}\left[V^I\right] = \int_0^1 \sqrt{G_{IJ} V^I V^J}\, ds$$

The optimal quantum circuit is the one with Minimal length

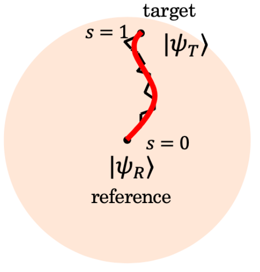
$$\mathcal{C}_{\text{target}} = \min_{\{V^I\}} \mathcal{D}\left[V^I\right] = \min_{\{V^I\}} \int_0^1 \sqrt{G_{IJ} V^I V^J}\, ds$$

The minimal path $V^I(s)$ is a geodesic on the space, which solves

Euler-Arnold equation

$$G_{IJ} \frac{dV^J}{ds} = f_{IJ}^{P}\, V^J G_{PL} V^L$$

Summary



Unitary evolution from reference state $|\psi_R\rangle$ to target state $|\psi_T\rangle$

- First, we identify the target unitary U_{target} and select a set of basis operators, with associated Lie group.

$$\hat{\mathcal{U}}_{\text{target}} = \overleftarrow{P} \exp\left[\int_0^1 V^I(s)\, \hat{\mathcal{O}}_I\, ds\right]$$

$\{\hat{\mathcal{O}}_I\}$: basis of gates
$V^I(s)$: tangent vectors

- We use these operators to construct the unitary $U(s)$.

- By solving E.A. eq. we get paths that define a set of geodesics $V^I(s)$ on this space.

- We then restrict this set of geodesics to those that realize the target unitary through the D.E. of $U(s)$ and the boundary conditions:

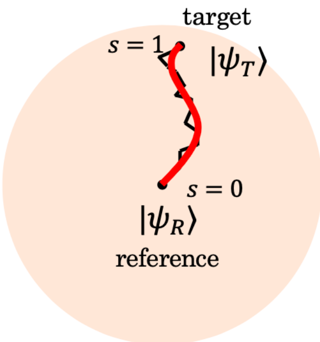
$$\frac{d\hat{U}(s)}{ds} = -iV^I(s)\, \hat{\mathcal{O}}_I\, \hat{U}(s) \qquad \hat{U}(0) = \mathbb{1} \quad \text{and} \quad \hat{U}(1) = \hat{\mathcal{U}}_{\text{target}}$$

- Assign a circuit depth $\mathcal{D}[V^I] = \int_0^1 \sqrt{G_{IJ}\, V^I V^J}\, ds$ with $G_{IJ} = \delta_{IJ}$ "gate cost"

- Finally, we use the resulting optimal construction of the unitary to calculate the complexity since Circuit Complexity is depth minimized over paths

$$\mathcal{C} = \min_{\{V^I\}} \mathcal{D}[V^I]$$

COMMENTS



Unitary evolution from reference state $|\psi_R\rangle$ to target state $|\psi_T\rangle$

$$\mathcal{C}_{\text{target}} = \min_{\{V^I\}} \mathcal{D}\left[V^I\right] = \min_{\{V^I\}} \int_0^1 \sqrt{G_{IJ} V^I V^J}\, ds\,,$$

- Metric G_{ij} is the operational cost to build the path with any particular operator.
- A natural choice is the Cartan Killing Form.
- This is Not generically possible, so we will choose δ_{ij}

Next: We will review the <u>state circuit complexity</u>

FURTHER COMMENTS ON THE METRIC

Consider the space of arbitrary group manifold G

Let's U be an element and choose a basis on the tangent space (i.e. the lie algebra) by O_I .

Then a path is given by $U(s)$. A simple metric would be

$$d\ell^2 = \mathrm{tr}(\Omega^\dagger \mathbb{J} \Omega) ds^2$$

The matrix quantity Ω is the velocity on the group manifold and is related to group elements along the path by

$$\Omega = i\dot{U}U^{-1}$$

and J is some fixed matrix in the definition of the inner product

If we expand it in the basis O_I as $\Omega = V^I(s) O_I$

Then the metric elements in this basis will be given by

$$\mathrm{tr}(\mathcal{O}_I^\dagger \mathbb{J}\, \mathcal{O}_J) = G_{IJ}.$$

Further Comments on the Metric

The metric elements:

$$\mathrm{tr}(\mathcal{O}_I^\dagger \mathbb{J}\, \mathcal{O}_J) = G_{IJ}.$$

- In the case of an orthonormal basis, and if we choose J to be the identity matrix then the metric will be

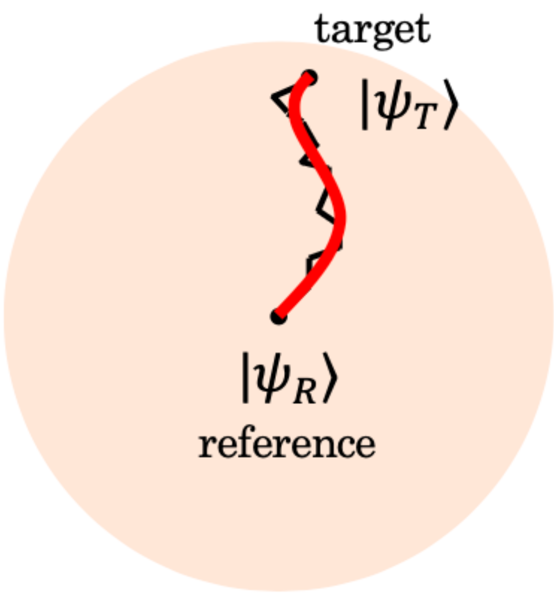
$$G_{IJ} = \delta_{IJ}$$

- If for a given group, we use the adjoint representation for the basis matrices O_I, then the metric is given by Kartan-Killing form

$$G_{IJ} = K_{IJ} = f_{IM}^{L}\, f_{LJ}^{M}$$

Now we will review the state circuit complexity

Quantum Circuit Complexity: State Computation



$$|\psi_R\rangle \rightarrow |\psi_T\rangle$$

Jefferson, Myers

$$|\psi_T\rangle = \hat{U}\,|\psi_R\rangle$$

Unitary transformation from "reference" state to "target" state

$$\hat{U} = \overleftarrow{P}\exp\left[\int_0^1 ds \sum_I Y^I(s)\ \widehat{M_I}\right]$$

defines a path through space of operators (gates)

$H(s)$

Operators

Tangent vector to a trajectory in the space of Unitary

$\hat{M}_I \sim i\epsilon\ \hat{q}_a\ \hat{p}_b$

Gates--preserve the general Gaussian form

example: Scaling and Entangling

$$\hat{U} = \overleftarrow{P}\exp\left[\int_0^s ds\, H(s)\right]$$

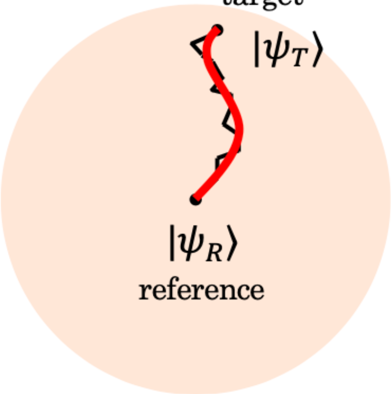
In this general space the paths satisfy the boundary condition: $U(s=0) = 1\,, U(s=1) = U$

Next, we need to define a cost, such as

$$F_2(Y) = \sqrt{\sum_I (Y^I)^2}$$

Provide Riemannian Geometry

Quantum Circuit Complexity: State Computation



Other choices possible:

$$F_1(U,Y) = \sum_I |Y^I| \,, \qquad F_p(U,Y) = \sum_I p_I \, |Y^I| \,,$$

$$F_2(U,Y) = \sqrt{\sum_I (Y^I)^2} \,, \qquad F_q(U,Y) = \sqrt{\sum_I q_I \, (Y^I)^2}$$

By using this cost function, we define a distance functional

$$\mathcal{D}(U(t)) = \int_0^1 \mathrm{d}t \; F\Big(U(t), \dot{U}(t)\Big)$$

Complexity is defined by the minimization of the distance functional over paths

Complexity= Minimization of "length" along path

$$C = \min \, \mathcal{D}(U(t))$$

Quantum Circuit Complexity: State Computation

Jefferson, Myers

Change the problem to the wavefunctions language

$$\psi \simeq \exp\left[-\frac{1}{2} x_a\, A_{ab}\, x_b\right] \quad \text{Gaussian}$$

with $\psi_T = U\, \psi_R$

Boundary conditions

$$U(s=0) = 1\,, U(s=1) = U_{target} \quad \Rightarrow \quad A_{\rm R} = \omega_0 \mathbb{1}\,, \qquad A_{\rm T} = \begin{pmatrix} \omega_1 & \beta \\ \beta & \omega_2 \end{pmatrix}$$

Then we need to translate the scaling and entangling gates to this matrix representation.

That is, we build a representation of these operators as 2×2 matrices which act on the symmetric matrices A.

The gate matrices act as

$$A' = Q_{ab}\, A\, Q_{ab}^T$$

$$Q_{ab} = \exp[\epsilon\, M_{ab}] \qquad \text{with} \quad [M_{ab}]_{\,cd} = \delta_{ac}\, \delta_{bd}\,.$$

The Basis of generators M_I

$$M_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad M_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$M_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad M_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Quantum Circuit Complexity: State Computation

In wavefunction language

Circuits form a representation of GL(2,R)	⇒	U(s) are trajectories in the space of GL(2, R) transformations.

In this matrix formulation, the path-ordered exponentials are replaced by

$$U(s) = \overleftarrow{\mathcal{P}} \exp \int_0^s \mathrm{d}\tilde{s}\, Y^I(\tilde{s})\, M_I \ , \qquad \text{with} \ \ A_{\mathrm{T}} = U(s=1)\, A_{\mathrm{R}}\, U^T(s=1)$$

Then we get

Simple expression of the velocity vector.

$$Y^I(s)\, M_I = \partial_s U(s)\, U^{-1}(s) \quad \Longrightarrow \quad Y^I(s) = \mathrm{tr}\left(\partial_s U(s)\, U^{-1}(s) M_I^T\right)$$

⇓

$$\mathrm{d}s^2 = \delta_{IJ}\, \mathrm{tr}\left(\mathrm{d}U\, U^{-1}\, M_I^T\right)\, \mathrm{tr}\left(\mathrm{d}U\, U^{-1}\, M_J^T\right)$$

Now we explicitly construct a parametrization of U(s) to construct the desired geodesics.

$$\mathcal{D}(U) = \int_0^1 \mathrm{d}s \sqrt{g_{ij}\, \dot{x}^i\, \dot{x}^j} \equiv k \, .$$

The minimum value of k is then the depth of the optimal circuit, and by extension, the complexity of the target state.

NOTE

Wavefunction language

Jefferson, Myers

- Our task is to find the shortest geodesic on GL(2, R) that connects the initial and final states, A_R and A_T.
- There is a continuous family of geodesics connecting the desired states. This non-uniqueness arises because our space of circuits is 4d (since dim GL(2, R) = 4) whereas our space of states is only three-dimensional (since the 2×2 matrices A_{ij} are symmetric).
- The complexity is defined as the cost of the optimal circuit. Hence this one-parameter family of solutions is merely the set of all possible circuits. To find the optimal circuit, we simply need to find the geodesic within this family with the shortest length.

Quantum Circuit Complexity

- Model as continuous application of operators Nielsen et al

$$\hat{\mathcal{U}}_{\text{target}} = \overleftarrow{P} \exp\left[\int_0^1 V^I(s)\, \hat{\mathcal{O}}_I\, ds\right]$$

$\{\hat{\mathcal{O}}_I\}$: basis of gates
$V^I(s)$: tangent vectors

Operator Circuit Complexity

- Characterize gates by structure constants

$$[\hat{\mathcal{O}}_I, \hat{\mathcal{O}}_J] = i\, f_{IJ}^K \hat{\mathcal{O}}_K$$

- Minimization:
  $\Rightarrow$ Euler-Arnold eq on group manifold

$$G_{IJ} \frac{dV^J}{ds} = f_{IJ}^K\, G_{KL}\, V^J\, V^L$$

❑ Advantage: Focus is on target unitary
❑ Disadvantage: Euler-Arnold eq can be difficult to solve

Balasubramanian, Decross, Kar, Parrikar
Basteiro, Erdmenger, Fries, Goth, Matthaiakakis, Meyer

(Gaussian) State Circuit Complexity

- Characterize target operator by its action on Gaussian states

$$\langle x|\psi_R\rangle \sim e^{-\frac{1}{2}\,\omega_0 \Sigma_k x_k^2}$$

$$\hat{\mathcal{O}}_k \sim e^{-i\hat{x}_k\hat{p}_k} \longrightarrow \langle x|\psi_T\rangle \sim e^{-\frac{1}{2}\,\Sigma_k \Omega_k x_k^2}$$

$\{\hat{\mathcal{O}}_I\}$: basis for GL(2, R)

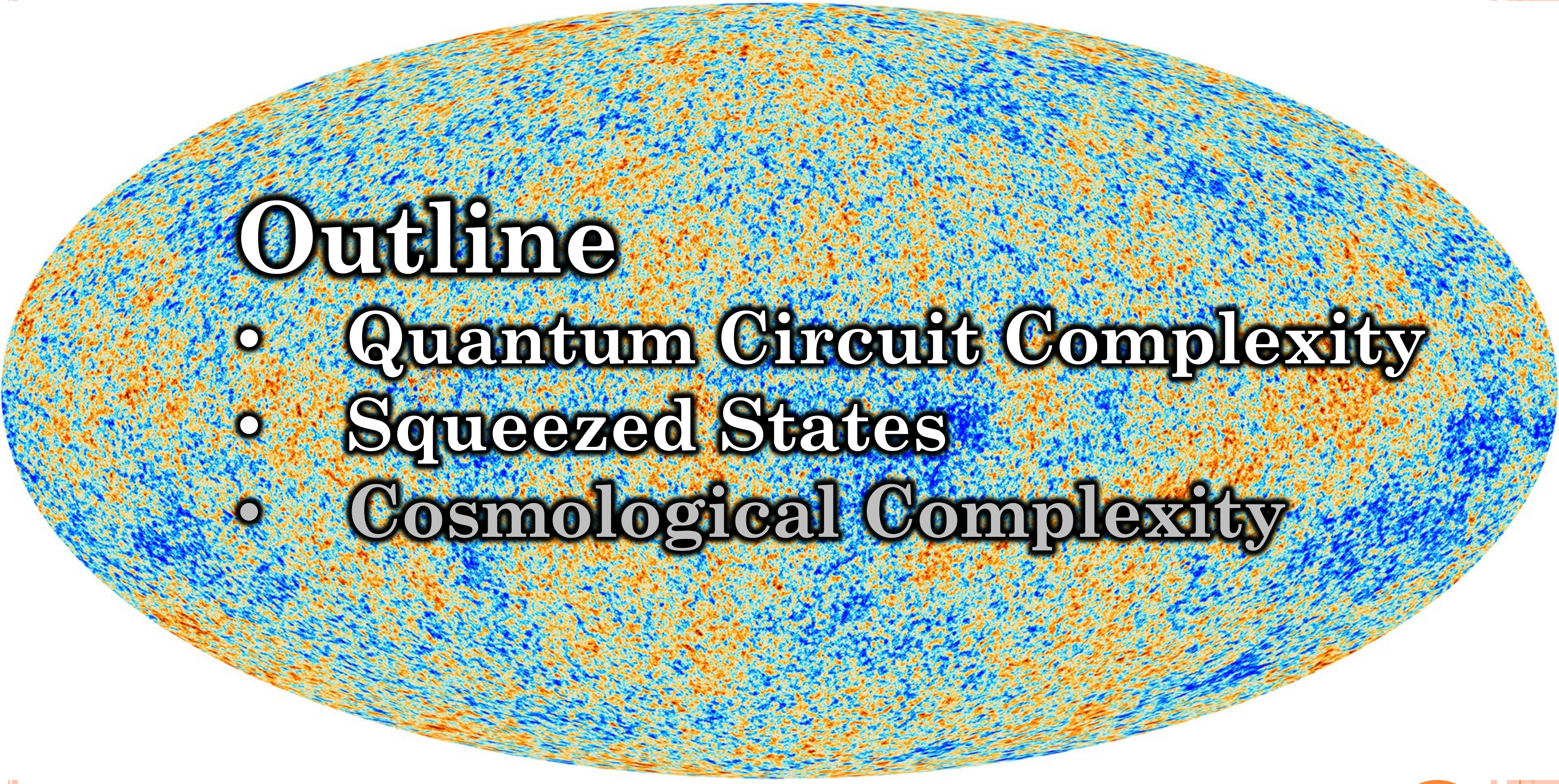
❑ Advantage: Simple to set up and find optimal path
❑ Disadvantage: Restricted to Gaussian states

Jefferson, Myers
Ali, Bhattacharyya, Haque, Kim, Moynihan, Murugan

End of Lecture 1

Thank You

Lecture 2



Outline

- Quantum Circuit Complexity
- Squeezed States
- Cosmological Complexity

ESA and Planck Collaboration

Lecture 2
Motivation
(not Holographic)

- Quantum complexity is versatile, is a proxy for various physical quantities. Useful for understanding Quantum Chaos, Quantum Phase transition etc.
- It gives an additional label to states $\Rightarrow$ additional information about quantum evolution.
- Complexity applied to coherent and squeezed states, that are essential building blocks of quantum optics and quantum computation.
- Can it say anything about cosmology?
- Can we understand decoherence?

Lecture 2

Last time/Lecture 1

- Background and Motivation for studying quantum complexity
- What is Circuit Complexity?
- Operator and State circuit complexities

Lecture 2

- Examples: Displacement operator, Harmonic Oscillators, Free field Theory (I will apply both state and operator circuits)
- What is Squeezed States?
- Complexity of Purification

Quantum Circuit Complexity: State Computation

In this construction we are interested in the Gaussian States.

The position space wave function for two coupled Harmonic Oscillators

$$\psi_R = \exp\left[-\left(\frac{1}{2}\right)\omega_R \sum_{a=1}^{2} x_a^2 \right] \Longrightarrow \psi_T = \exp\left[-\left(\frac{1}{2}\right)\omega \sum_{a,b=1}^{2} \mathrm{x_a}\, \Omega_{ab}\, x_b \right]$$

Ground state wavefunction

Time evolved wave function

The reference and the target states are simultaneously diagonalized,

$$A_R = \omega_R \mathbb{1}, \qquad A_T = \omega \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix}$$

The unitary operator acts on the reference matrix as

$$A(s) = \hat{U}(s) A_R \hat{U}^T(s).$$

Since Ω's can be complex, we will use the diagonal elements of GL(2,C) as our set of gate operators

$$\hat{U}(s) = \exp\left[y^a(s) M_a^{\mathrm{diag}}\right] = \exp\left[\begin{pmatrix} y^1(s) & 0 \\ 0 & y^2(s) \end{pmatrix}\right]$$

U(s) are trajectories in the space of GL(2, C) transformations.

$y^a(s) = \alpha^a(s) + i\beta^a(s)$ are complex.

Note: The off-diagonal components will increase the distance between states; the shortest distance corresponds to them being set to zero.

Two coupled Harmonic Oscillators

The resulting metric on the reduced space of operators becomes

$$ds^2 = G_{IJ} dY^I dY^J = \sum_{a=1}^{2} |y^a|^2 = \sum_{a=1}^{2} \left[(\alpha^a)^2 + (\beta^a)^2 \right]$$

The resulting circuit depth

$$\mathcal{C} = \int_0^1 \sqrt{G_{IJ} dY^I dY^J}\; ds = \int_0^1 \sqrt{\sum_{a=1}^{2} \left[(\alpha^a)^2 + (\beta^a)^2 \right]}\; ds,$$

is **minimized,** subject to the boundary conditions, by the straight-line geodesic

$$\alpha^a(s) = \ln \left| \Omega_a \frac{\omega}{\omega_R} \right| \; s, \qquad \beta^a(s) = \arctan \left[\frac{\mathrm{Im}(\Omega_a)}{\mathrm{Re}(\Omega_a)} \right] \; s$$

Choosing the reference frequency to be the ground state frequency of the oscillator

$$\omega_R = \omega$$

leads to the **complexity**

$$\mathcal{C} = \frac{1}{2} \sqrt{\sum_{a=1}^{2} \left[\left(\ln |\Omega_a| \right)^2 + \left(\arctan \left[\frac{\mathrm{Im}(\Omega_a)}{\mathrm{Re}(\Omega_a)} \right] \right)^2 \right]}\,.$$

Quantum Circuit Complexity

- Model as continuous application of operators — Nielsen et al

$$\hat{\mathcal{U}}_{\text{target}} = \overleftarrow{P} \exp\left[\int_0^1 V^I(s)\, \hat{\mathcal{O}}_I\, ds\right]$$

$\{\hat{\mathcal{O}}_I\}$: basis of gates

$V^I(s)$: tangent vectors

Operator Circuit Complexity

- Characterize gates by structure constants

$$[\hat{\mathcal{O}}_I, \hat{\mathcal{O}}_J] = i\, f_{IJ}^K \hat{\mathcal{O}}_K$$

- Minimization:

  ⇒ Euler-Arnold eq on group manifold

$$G_{IJ} \frac{dV^J}{ds} = f_{IJ}^K\, G_{KL}\, V^J\, V^L$$

❑ Advantage: Focus is on target unitary

❑ Disadvantage: Euler-Arnold eq can be difficult to solve

Balasubramanian, Decross, Kar, Parrikar
Basteiro, Erdmenger, Fries, Goth, Matthaiakakis, Meyer

(Gaussian) State Circuit Complexity

- Characterize target operator by its action on Gaussian states

$$\langle x|\psi_R\rangle \sim e^{-\frac{1}{2}\,\omega_0 \Sigma_k x_k^2}$$

$$\hat{\mathcal{O}}_k \sim e^{-i\hat{x}_k\hat{p}_k} \longrightarrow \langle x|\psi_T\rangle \sim e^{-\frac{1}{2}\,\Sigma_k\, \Omega_k\, x_k^2}$$

$\{\hat{\mathcal{O}}_I\}$: basis for $GL(N, \mathbb{C})$

❑ Advantage: Simple to set up and find optimal path

❑ Disadvantage: Restricted to Gaussian states

Jefferson, Myers
Ali, Bhattacharyya, Haque, Kim, Moynihan, Murugan

Let's do an example on Operator Circuit Complexity

Operator Complexity for Displacement operator

Operator Circuit Complexity

Displacement Operator is important ==> can generate Coherent States

$$\hat{\mathcal{U}}_{\text{target}}(t) = \hat{\mathcal{D}}(\alpha, t) = e^{i\hat{H}_0 t}\hat{\mathcal{D}}(\alpha)e^{-i\hat{H}_0 t} \qquad \hat{H}_0 = \omega \hat{a}^\dagger \hat{a}.$$

$$\hat{\mathcal{D}}(\alpha) = \exp\left[\alpha \hat{a}^\dagger - \alpha^* \hat{a}\right]$$

$$\hat{e}_1 = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{a}^\dagger\right), \quad \hat{e}_2 = \frac{i}{\sqrt{2}}\left(\hat{a} - \hat{a}^\dagger\right), \quad \hat{e}_3 = \hat{\mathbb{1}},$$

$$\hat{\mathcal{U}}_{\text{target}}(t) = \exp\left[\sqrt{2}i \ \text{Im}[\alpha(t)] \ \hat{e}_1 + \sqrt{2}i \ \text{Re}[\alpha(t)] \ \hat{e}_2\right].$$

Heisenberg Lie Algebra

$$[\hat{e}_1, \hat{e}_2] = -i\hat{e}_3,$$

Euler Arnold Equations

$$\frac{dV^1}{ds} = -V^2V^3; \quad \frac{dV^2}{ds} = V^1V^3; \quad \frac{dV^3}{ds} = 0.$$

Solutions

$$V^1(s) = v_1\cos(v_3 s) + v_2\sin(v_3 s); \quad V^2(s) = v_1\sin(v_3 s) - v_2\cos(v_3 s); \quad V^3(s) = v_3;$$

This is the path that minimizes the circuit depth

Resulting circuit complexity along this minimal path

$$\mathcal{C}_{\text{target}} = \int_0^1 \sqrt{G_{IJ}V^I(s)V^J(s)} \ ds = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

Operator Circuit Complexity

Since the operator e_3 is the central element, it just gives an overall phase

$$\exp\left[-i\int_0^1 V^I \hat{e}_I \, ds\right] |\psi\rangle_{\rm R} = \exp\left[-i\int_0^1 \left(V^1(s)\hat{e}_1 + V^2(s)\hat{e}_2\right) ds\right] e^{-iv_3\hat{\mathbb{1}}} |\psi\rangle_{\rm R}$$
$$= \exp\left[-i\int_0^1 \left(V^1(s)\hat{e}_1 + V^2(s)\hat{e}_2\right) ds\right] e^{-iv_3} |\psi\rangle_{\rm R} .$$

we can set $v_3 = 0$

$$\mathcal{C}_{\rm target} = \sqrt{v_1^2 + v_2^2} .$$

Now let's construct the $U(s)$

Target operator is the $s = 1$ boundary condition of the s-dependent unitary operator

$$\hat{U}(s) = \mathcal{P}\exp\left[-i\int_0^s V^I(s')\hat{e}_I \, ds'\right]$$

$$\hat{e}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{e}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{e}_3 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\hat{U}(s) = \begin{pmatrix} 1 & ia(s) & c(s) \\ 0 & 1 & ib(s) \\ 0 & 0 & 1 \end{pmatrix}$$

General Element of Heisenberg Group

Operator Circuit Complexity

with this group element and the sols of the E A equations in

$$\frac{d\hat{U}(s)}{ds} = -iV^I(s)\ \hat{e}_I\ \hat{U}(s)$$

solved by the parametrization

$$\begin{cases} a(s) &= a_0 - v_1 s\,; \\ b(s) &= b_0 - v_2 s\,; \\ c(s) &= c_0 + b_0 v_1 s - \frac{1}{2} v_1 v_2 s^2 \end{cases}$$

Then

$$\hat{U}(s) = \begin{pmatrix} 1 & ia(s) & c(s) \\ 0 & 1 & ib(s) \\ 0 & 0 & 1 \end{pmatrix} \quad \xrightarrow{s=0,\ \hat{U}(s=0)=\hat{\mathbb{1}}} \quad \hat{U}(s) = \begin{pmatrix} 1 & -iv_1 s & -\frac{1}{2}v_1 v_2 s^2 \\ 0 & 1 & -iv_2 s \\ 0 & 0 & 1 \end{pmatrix}$$

We will determine the constants v_1 and v_2 by applying the boundary conditions:

$$s = 1,\ \hat{U}(s=1) = \hat{\mathcal{U}}_{\text{target}}$$

$$v_1 = -\sqrt{2}\ \text{Im}[\alpha(t)], v_2 = -\sqrt{2}\ \text{Re}[\alpha(t)]$$

Complexity

$$\mathcal{C}_{\text{Heis}} = \sqrt{2}\,|\alpha|\,,$$

Operator Circuit Complexity

Complexity $\mathcal{C}_{\text{Heis}} = \sqrt{2}\,|\alpha|$

Significance?

Note: the average number density – or equivalently, the average energy – of a vacuum coherent state

$$\langle E\rangle \sim \bar{N}_\alpha = \langle\alpha|\hat{a}^\dagger \hat{a}|\alpha\rangle = |\alpha|^2\,.$$

$$\mathcal{C}_{\text{displacement}} \sim \sqrt{\langle E\rangle}\,.$$

In QI protocols, the energy needed to prepare a state or set of gates can be an important resource.

The energy required to build a coherent state with some fixed complexity grows quadratically with that complexity $\langle E\rangle \sim C^2$.

These scaling might have general lessons for building QI protocols?

Next: Operator circuit complexity for free scalar field

Complexity: Free Harmonic Oscillator

Operator Circuit Complexity

Example: Free Harmonic Oscillator

$$|\psi_T\rangle = \hat{\mathcal{U}}_{\text{target}}\, |\psi_R\rangle$$

$$|\psi_T\rangle = e^{-i\hat{H}_0 t}\ \ |\psi_R\rangle$$

$$\hat{H}_0 = \frac{\omega}{2}\left(\hat{a}^\dagger \hat{a} + \hat{a}\,\hat{a}^\dagger\right)$$

SH, Jana, Underwood

Model as continuous application of operators

$$\hat{\mathcal{U}}_{\text{target}} = \overleftarrow{P} \exp\left[\int_0^1 V^I(s)\, \hat{\mathcal{O}}_I\, ds\right]$$

$\{\hat{\mathcal{O}}_I\}$: basis of gates
$V^I(s)$: tangent vectors

$$\left\{ \hat{\mathcal{O}}_1 = \frac{\hat{a}^2 + \hat{a}^{\dagger 2}}{4} \quad \hat{\mathcal{O}}_2 = i\,\frac{\hat{a}^2 - \hat{a}^{\dagger 2}}{4} \quad \hat{\mathcal{O}}_3 = \frac{\hat{a}^\dagger \hat{a} + \hat{a}\,\hat{a}^\dagger}{4} \right\}$$

Characterize gates by structure constants $\left[\hat{\mathcal{O}}_I, \hat{\mathcal{O}}_J\right] = i\, f_{IJ}^{K} \hat{\mathcal{O}}_K$

$$\left[\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2\right] = -i\hat{\mathcal{O}}_3, \qquad \left[\hat{\mathcal{O}}_3, \hat{\mathcal{O}}_1\right] = i\hat{\mathcal{O}}_2, \qquad \left[\hat{\mathcal{O}}_2, \hat{\mathcal{O}}_3\right] = i\hat{\mathcal{O}}_1$$

su(1,1)

Minimization ⇒ Euler-Arnold eq on group manifold ($G_{IJ} = \delta_{IJ}$)

$$G_{IJ}\frac{dV^J}{ds} = f_{IJ}^{K}\, G_{KL}\, V^J\, V^L$$

Complexity: Free Harmonic Oscillator

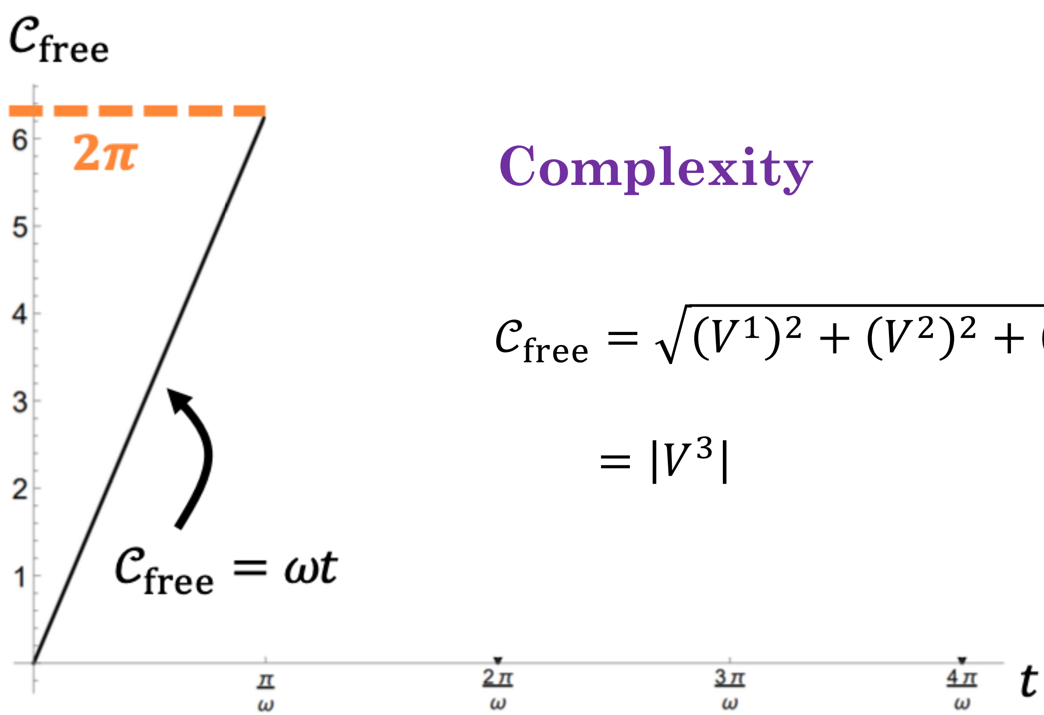
Operator Circuit Complexity

Minimization $\Rightarrow$ Euler-Arnold eq on group manifold ($G_{IJ} = \delta_{IJ}$)

$$G_{IJ}\frac{dV^J}{ds} = f_{IJ}^{K}\, G_{KL}\, V^J\, V^L \longrightarrow$$

$$V^1 = 0,\quad V^2 = 0,\quad V^3 = \min\begin{cases}2(\omega t - 2\pi n)\\ 2(2\pi n - \omega t)\end{cases}$$

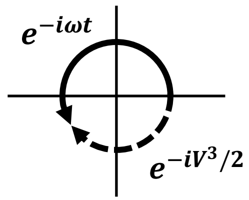
V^3 is a compact direction



Complexity

$$\mathcal{C}_{\text{free}} = \sqrt{(V^1)^2 + (V^2)^2 + (V^3)^2}$$

$$= |V^3|$$



Complexity: Free Scalar Field

Operator Circuit Complexity

SH, Jana, Underwood

Free scalar field ϕ in $(d+1)$-dimensions, mass m, box L with periodic boundary conditions

$\Lambda = N_{\max}\pi/L$ UV cutoff

$$\hat{\phi} = \sum_{\vec{n}}^{N_{\max}} \frac{1}{\sqrt{2\,E_{\vec{n}}}} \left(\hat{a}_{\vec{n}}\, e^{i\vec{p}_{\vec{n}}\cdot\vec{x}} + \hat{a}_{\vec{n}}^{\dagger} e^{-i\vec{p}_{\vec{n}}\cdot\vec{x}} \right) \quad \text{Mode expansion: } \begin{cases} \vec{p}_{\vec{n}} = \vec{n}\pi/L \\ E_{\vec{n}} = \sqrt{p_{\vec{n}}^2 + m^2} \end{cases}$$

We **cutoff** the infinite sums of modes at the UV scale $\Lambda = N_{max}\frac{\pi}{L}$ for $N_{max} \gg 1$

The Hamiltonian becomes a sum over modes

$$\hat{H} = \frac{1}{2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\dots\sum_{n_d=1}^{\infty} E_{\vec{n}}\left(\hat{a}_{\vec{n}}^{\dagger}\hat{a}_{\vec{n}} + \hat{a}_{\vec{n}}\hat{a}_{\vec{n}}^{\dagger}\right)$$

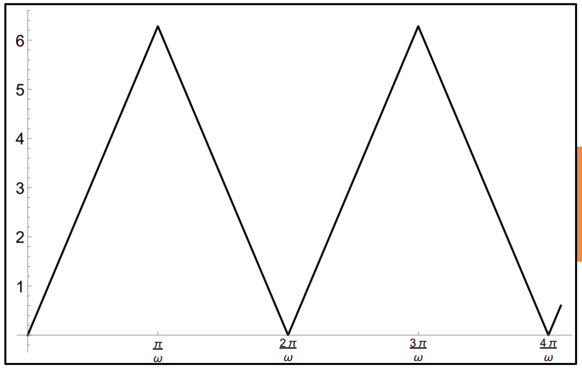
Target Unitary

$$\mathcal{U}_{\text{target}} = \prod_{\vec{n}}^{N_{\max}} e^{-i\frac{1}{2}E_{\vec{n}}\left(\hat{a}_{\vec{n}}^{\dagger}\hat{a}_{\vec{n}}+\hat{a}_{\vec{n}}\hat{a}_{\vec{n}}^{\dagger}\right)}$$

Complexity for a single mode $\vec{n}$

$$\mathcal{C}_{\text{free}}^{\vec{n}} = |v_3^{\vec{n}}| = \begin{cases} 2(E_{\vec{n}}t - 2\pi m) & \text{for } 2\pi m < E_{\vec{n}}t < \pi(2m+1) \\ 2(2\pi m - E_{\vec{n}}t) & \text{for } \pi(2m-1) < E_{\vec{n}}t < 2\pi m \end{cases} \quad \text{for some integer } m$$

copies of free oscillator for each mode



Complexity of the free scalar field

$$\mathcal{C}_\phi = \sqrt{\sum_{\vec{n}}^{N_{\max}} \left(V_{\vec{n}}^3\right)^2} \sim L^{d/2}\sqrt{\int^{\Lambda}\left(V^3(p)\right)^2\, d^d p}$$

COMPLEXITY: FREE SCALAR FIELD

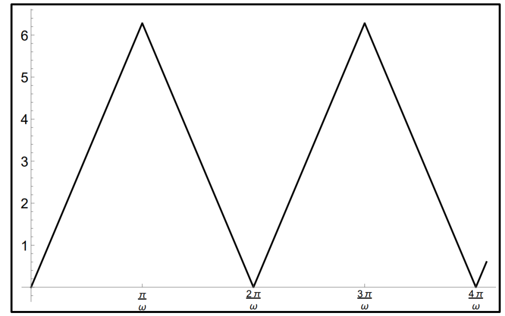
Operator Circuit Complexity

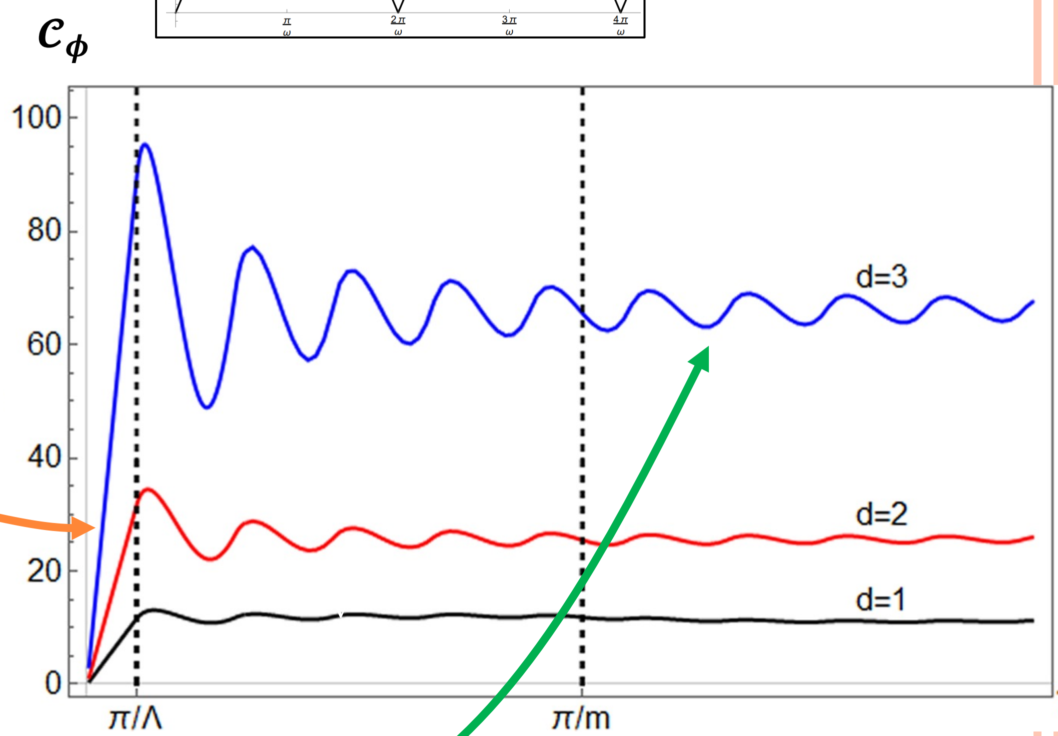
Complexity of free scalar field

$$\mathcal{C}_\phi = \sqrt{\sum_{\vec{n}}^{N_{\max}} \left(V^3_{\vec{n}}\right)^2} \sim L^{d/2} \sqrt{\int^{\Lambda} \left(V^3(p)\right)^2 \, d^d p}$$

continuum limit

$$\sim \begin{cases} L^{d/2}\Lambda^{d/2}\,(\Lambda t) & \text{early times } t \ll \pi/\Lambda \\ L^{d/2}\Lambda^{d/2} & \text{late times } t \gg \pi/\Lambda \end{cases}$$

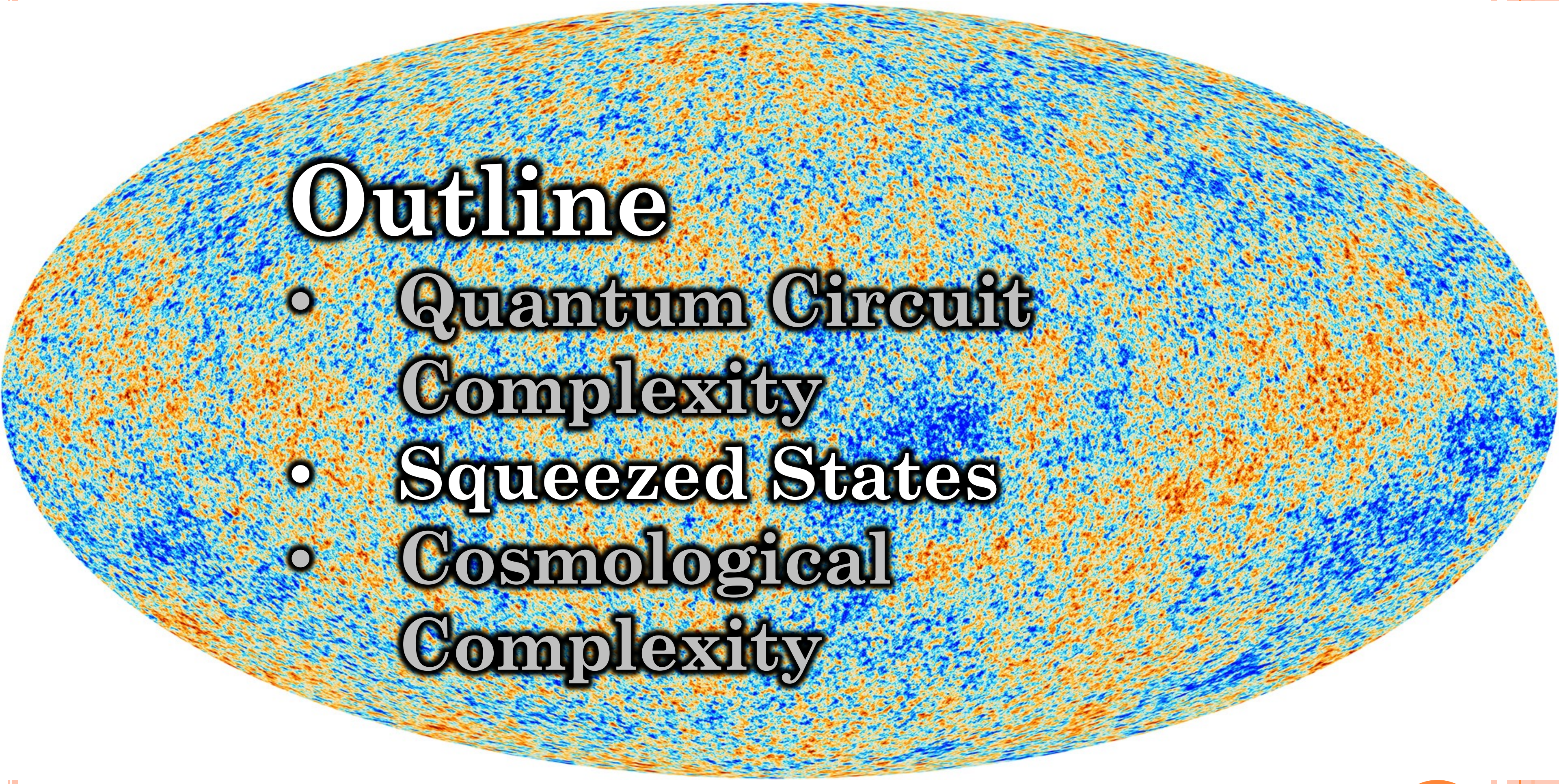




Linear Growth:
complexity of only one mode growing

Saturation:
complexity of all modes oscillating, average out

Let's change the gear…



Outline

- Quantum Circuit Complexity
- Squeezed States
- Cosmological Complexity

ESA and Planck Collaboration

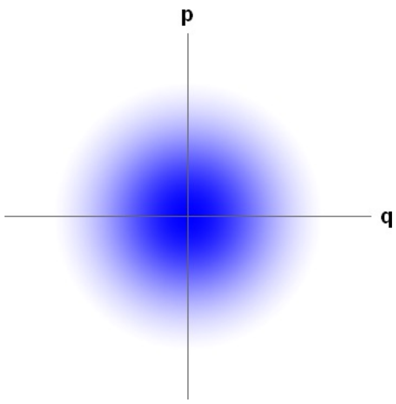
Squeezed States

Vacuum States

$$\psi(q) \sim e^{-\frac{q^2}{2}}, \qquad \varphi(p) \sim e^{-\frac{p^2}{2}}$$

$$\langle \Delta\hat{q}^2 \rangle = \frac{1}{2} \qquad \langle \Delta\hat{p}^2 \rangle = \frac{1}{2}$$

$$\langle \Delta\hat{q}^2 \rangle \langle \Delta\hat{p}^2 \rangle = \frac{1}{4}$$



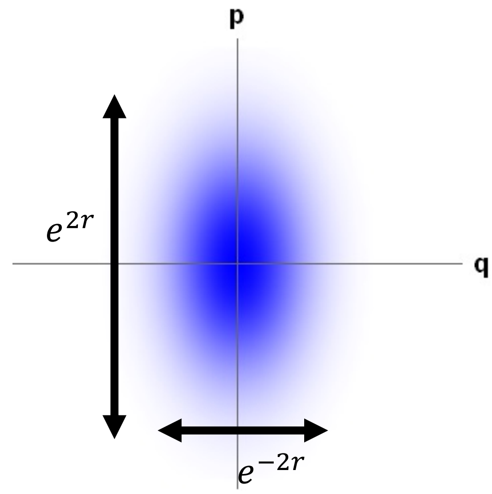
$r = 0$

Squeezed Vacuum State

$$\psi(q) \sim e^{-\frac{q^2}{2}e^{2r}}, \qquad \varphi(p) \sim e^{-\frac{p^2}{2}e^{-2r}}$$

$$\langle \Delta\hat{q}^2 \rangle = \frac{1}{2}e^{-2r} \qquad \langle \Delta\hat{p}^2 \rangle = \frac{1}{2}e^{2r}$$

$$\langle \Delta\hat{q}^2 \rangle \langle \Delta\hat{p}^2 \rangle = \frac{1}{4}$$

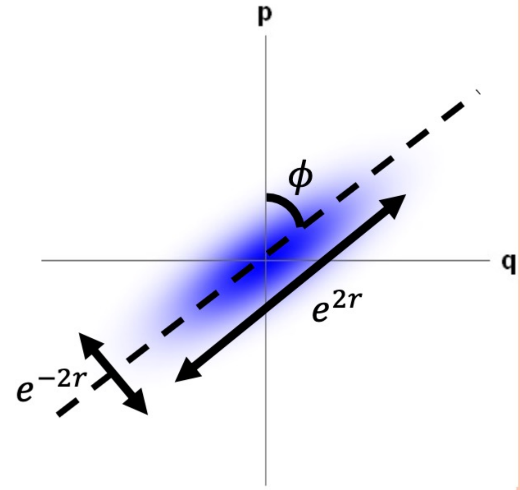


$r = 0.5$
$\phi = 0$

Squeezed, rotated Vacuum State

$$\hat{q}_+ = \hat{p}\sin\phi + \hat{x}\cos\phi$$
$$\hat{q}_- = \hat{p}\cos\phi - \hat{x}\sin\phi$$

$$\langle \Delta\hat{q}_+{}^2 \rangle \langle \Delta\hat{q}_-{}^2 \rangle = \frac{1}{4}$$



$r = 0.5$
$\phi = \pi/3$

Squeezed States

Described by **squeezing parameter r, squeezing angle ϕ**, and **rotation angle θ**

$$|r,\phi,\theta\rangle = \hat{\mathcal{S}}(r,\phi)\,\hat{\mathcal{R}}(\theta)\,|0\rangle$$

$$|\psi_T\rangle = \hat{\mathcal{U}}_{\text{target}}\,|\psi_R\rangle \qquad \hat{\mathcal{U}}_{\text{target}} = \hat{S}\,(r,\phi)\,\hat{R}(\theta)$$

where $\hat{\mathcal{S}}(r,\phi) \equiv \exp\left[\frac{r(t)}{2}\left(e^{-2i\phi}\,\hat{a}^2 - e^{2i\phi}\,\hat{a}^{\dagger 2}\right)\right]$ **squeezing operator**

$\hat{\mathcal{R}}(\theta) \equiv \exp\left[-i\theta\,\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right)\right]$ **rotation operator**

Time evolution by generic quadratic Hamiltonian $\hat{H}_2 = \Omega\,\hat{a}^\dagger\hat{a} + \frac{1}{2}\left(\Delta\,\hat{a}^2 + \Delta^*\,\hat{a}^{\dagger 2}\right)$

We can write $\hat{\mathcal{U}} = \hat{S}\,(r,\phi)\,\hat{R}(\theta)$

Example, for IHO we get the Squeezed, rotated Vacuum State (single mode)

$$|\Psi(t)\rangle = \hat{\mathcal{U}}|0\rangle = \frac{e^{i\theta}}{\sqrt{\cosh r}}\sum_{n=0}^{\infty}(-1)^n\,e^{-2in\phi}\tanh^n r\,\frac{\sqrt{(2n)!}}{2^n n!}\,|2n\rangle$$

"World record" laboratory squeezing $r \approx 1.7$

Vahlbruch, et al, 2016

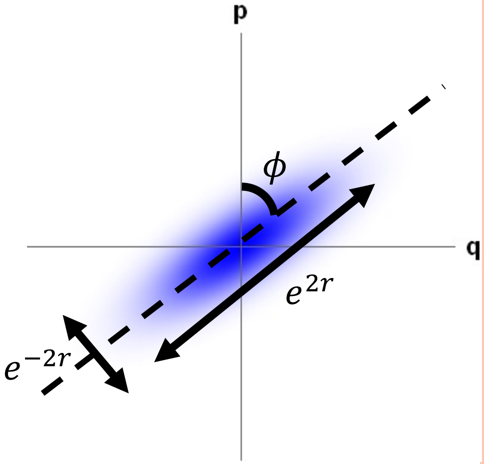
Squeezed States found in:

- Quantum Optics
- Gravitational Wave Detection
- Cosmological Perturbations

Squeezed, ***Rotated*** Vacuum State

$\hat{q}_+ = \hat{p}\sin\phi + \hat{x}\cos\phi$
$\hat{q}_- = \hat{p}\cos\phi - \hat{x}\sin\phi$

$\langle\Delta\hat{q}_+{}^2\rangle\langle\Delta\hat{q}_-{}^2\rangle = \frac{1}{4}$



$r = 0.5$
$\phi = \pi/3$

WHY SQUEEZED STATES?

- Squeezed states appear naturally in Cosmological Scalar Perturbation Model.
- The time evolution can be written as a product of Squeezing and Rotation Operator.
- We can apply both Operator approach and State approach (by using wave function).
- These squeezed states can be realized as a **TFD.**
- We can get mixed (thermal) state by tracing out degrees of freedom, hence study **Decoherence.**
- Natural setup to study open quantum system and hence **Complexity of Purification**.
- Perform a comparison: Open vs closed system complexity.
- Applications in quantum optics and quantum computations

Not today

Squeezed States: Inverted Harmonic oscillator

'The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction.'

Sidney Coleman

SQUEEZED STATES: INVERTED HARMONIC OSCILLATOR

State Circuit Complexity

Why inverted harmonic oscillator (IHO)?
Similar situation happens in cosmological perturbation model

IHO is defined by a Hamiltonian with a "wrong sign" of the restoring force

$$\hat{H} = \frac{1}{2}\hat{p}^2 - \frac{1}{2}k^2\hat{x}^2$$

Using raising and lowering operators based on the non-inverted harmonic oscillator

$$\hat{x} = \frac{1}{\sqrt{2k}}\left(\hat{a}^\dagger + \hat{a}\right), \qquad \hat{p} = i\sqrt{\frac{k}{2}}\left(\hat{a}^\dagger - \hat{a}\right),$$

Hamiltonian $$\hat{H} = -\frac{k}{2}\left(\hat{a}^2 + \hat{a}^{\dagger 2}\right)$$

If a system starts in the "vacuum state" annihilated by the lowering operator

$$\hat{a}|0\rangle = 0$$

then it will naturally evolve into a **squeezed state** at later times.

unitary evolution can be parameterized as: $$\hat{\mathcal{U}} = \hat{\mathcal{S}}(r,\phi)\hat{\mathcal{R}}(\theta)$$

Squeezed States: Inverted Harmonic oscillator

State Circuit Complexity

Why inverted harmonic oscillator?
Similar situation happens in cosmological perturbation model

Squeezed, Rotated Vacuum State (single mode)

$$|\psi_T\rangle = \hat{\mathcal{U}}_{\text{target}}\,|\psi_R\rangle \qquad \hat{\mathcal{S}}(r,\phi) \equiv \exp\left[\frac{r(t)}{2}\left(e^{-2i\phi}\,\hat{a}^2 - e^{2i\phi}\,\hat{a}^{\dagger 2}\right)\right]$$

$$\hat{u}_{\text{target}} = \hat{S}(r,\phi)\;\;\hat{R}(\theta) \qquad \hat{\mathcal{R}}(\theta) \equiv \exp\left[-i\theta\,\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right)\right]$$

$$|\Psi(t)\rangle = \hat{\mathcal{U}}|0\rangle = \frac{e^{i\theta}}{\sqrt{\cosh r}}\sum_{n=0}^{\infty}(-1)^n\,e^{-2in\phi}\tanh^n r\,\frac{\sqrt{(2n)!}}{2^n n!}\,|2n\rangle$$

Schrodinger equation

$$i\frac{d}{dt}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle$$

Squeezing equations of motion

$$\dot{r} = k\sin(2\phi)\,;$$

$$\dot{\phi} = k\coth(2r)\cos(2\phi)$$

Solution: $r(t) = kt, \qquad \phi(t) = \pi/4\,.$

Squeezed States: Inverted Harmonic oscillator

State Circuit Complexity

Why inverted harmonic oscillator?
Similar situation happens in cosmological perturbation model

Reference state: unsqueezed vacuum $\langle x|0\rangle$

Circuit?

Target state: squeezed state $\langle x|\psi\rangle$

Complexity

$$\mathcal{C}_2 = \frac{1}{2}\sqrt{\left(\ln\left|\frac{\Omega(t)}{k}\right|\right)^2 + \left(\tan^{-1}\left(\frac{\mathrm{Im}\,\Omega}{\mathrm{Re}\Omega}\right)\right)^2}$$

$$\Omega(t) = \frac{k}{e^{2r}\sin^2\phi + e^{-2r}\cos^2\phi}\left(1 - i\sin(2\phi)\sinh(2r)\right)$$

For small amounts of squeezing $r \ll 1$ $\qquad \mathcal{C}_2 \approx 0$

For large squeezing $r \gg 1, \phi \sim \frac{\pi}{4}$ $\qquad \mathcal{C}_2 \approx \frac{1}{2}\sqrt{\left(\tan^{-1} e^{2r}\right)^2} \approx \frac{\pi}{4}$

Complexity of a single mode vacuum squeezed state saturates at late times

Note: If we do the same computation for operator circuit, we get $\boldsymbol{C_{op} = 2\,r}$

It is insensitive to the squeezing angle.

What Next?

- Examples:

  Displacement operator : Operator complexity
  Harmonic Oscillator : State and Operator complexity
  Free field Theory: Operator complexity

- What is Squeezed States?

  Inverted Harmonic Oscillator

- Complexity of Purification

  Open Quantum System

Squeezed States, Thermal Density Matrix and TFD

Consider
Thermal state

$$\hat{\rho}_{\text{th}} = \frac{1}{Z}\sum_{n=0}^{\infty} e^{-\beta E_n}|n\rangle\langle n|$$

A straightforward purification of this generic thermal state is the TFD state:

$$|\text{TFD}\rangle = \frac{1}{\sqrt{Z}}\sum_{n=0}^{\infty} e^{-\beta E_n/2}|n\rangle \otimes |n\rangle_{\text{anc}}$$

This is not a unique purification, and it is possible **to include an additional phase**

$$|\Psi\rangle_\phi = |\text{TFD}\rangle_\phi = \frac{1}{\sqrt{Z}}\sum_{n=0}^{\infty}(-1)^n e^{-2in\phi} e^{-n\beta\omega/2}|n\rangle \otimes |n\rangle_{\text{anc}}$$

We recognize this as a two-mode squeezed vacuum state

$$|\Psi_{sq}\rangle = \frac{1}{\cosh r}\sum_{n=0}^{\infty}(-1)^n e^{-2in\phi}\tanh^n r|n\rangle \otimes |n\rangle_{\text{anc}} \equiv \hat{S}_{\text{sq}}(r,\phi)|0\rangle \otimes |0\rangle_{\text{anc}}$$

$$\boxed{\beta\omega = -\ln\tanh^2 r}$$

$$\boxed{\hat{\rho}_{\text{pure}} = |\Psi_{sq}\rangle\langle\Psi_{sq}| = \frac{1}{\cosh^2 r_k}\sum_{n,m=0}^{\infty}(-1)^{n+m}e^{-2i(n-m)\varphi_k}\tanh^{n+m} r_k\,|n, n_{anc}\rangle\langle m, m_{anc}|}$$

Squeezed States, Thermal Density Matrix and TFD

SH, Jana, Underwood

Now consider the circuit $|\psi_T\rangle = \hat{\mathcal{U}}\ |\psi_R\rangle$

with (purified) ground state as **Reference State** $|\psi_R\rangle = |0\rangle \otimes |0\rangle_{\text{anc}}$

Position space wavefunction

$$\langle q, q_{\text{anc}}|\psi_R\rangle = \mathcal{N}_R \exp\left[-\frac{1}{2}\omega(q^2 + q_{\text{anc}}^2)\right]$$

For Target state $|\psi_T\rangle = |\Psi\rangle_\phi$

position space wavefunction

$$\Psi_{\text{sq}}(q, q_{\text{anc}}) = \langle q, q_{\text{anc}}|\Psi\rangle_\phi = \mathcal{N} \exp\left\{-\frac{\omega}{2} A(q^2 + q_{\text{anc}}^2) - \omega B\, q\, q_{\text{anc}}\right\}$$

$$A = \frac{1 + e^{-4i\phi}\tanh^2 r}{1 - e^{-4i\phi}\tanh^2 r}, \qquad B = \frac{2\tanh r\, e^{-2i\phi}}{1 - e^{-4i\phi}\tanh^2 r}.$$

Following the outline at the beginning of the lecture

$$\begin{aligned}\mathcal{C}_\phi &= \frac{1}{\sqrt{2}}\sqrt{\ln^2\left|\frac{1 + e^{-2i\phi}\tanh r}{1 - e^{-2i\phi}\tanh r}\right| + \arctan^2(\sin 2\phi \sinh 2r)} \\ &= \frac{1}{\sqrt{2}}\sqrt{\ln^2\left|\frac{1 + e^{-2i\phi}e^{-\beta\omega/2}}{1 - e^{-2i\phi}e^{-\beta\omega/2}}\right| + \arctan^2\left(2\sin 2\phi \frac{e^{-\beta\omega/2}}{1 - e^{-\beta\omega}}\right)}\end{aligned}$$

Complexity of the pure state

Complexity of Purification

A thermal state ⟹ A pure state

For any mixed state ρ_{mix} on the Hilbert space H, one can construct a purification of ρ_{mix} which consists of a pure state $|\Psi\rangle$ in an enlarged Hilbert space

$$\mathcal{H}_{\text{pure}} = \mathcal{H} \otimes \mathcal{H}_{\text{anc}};$$

ancillary d.o.f.

Trace of the density matrix of this *pure state* $|\Psi\rangle$ over the ancillary degrees of freedom gives the original mixed state

$$\text{Tr}_{\text{anc}}\left(|\Psi\rangle\langle\Psi|\right) = \hat{\rho}_{\text{mix}};$$ ⟹ $|\Psi\rangle$ is a "purification" of ρ_{mix}.

Note that expectation values of operators acting in H are preserved under purification,

$$\langle\hat{\mathcal{O}}\rangle = \text{Tr}_{\text{anc}}\left(\langle\Psi|\hat{\mathcal{O}}|\Psi\rangle\right) = \text{Tr}\left(\hat{\rho}_{\text{mix}}\hat{\mathcal{O}}\right)$$

Observables are preserved by purification.

Complexity of Purification

The purification is not unique $\Longrightarrow$ Many choices for the ancillary Hilbert space

H_{anc} is arbitrary $\Longrightarrow$ Just needs to meet the purification requirement.

Example: There may be a set of pure states $\{|\Psi\rangle_{\alpha,\beta,\ldots}\}$, parameterized by $\alpha, \beta, \ldots$;

To distinguish among the set of purifications $\Rightarrow$ Minimize a quantity with respect to the parameters.

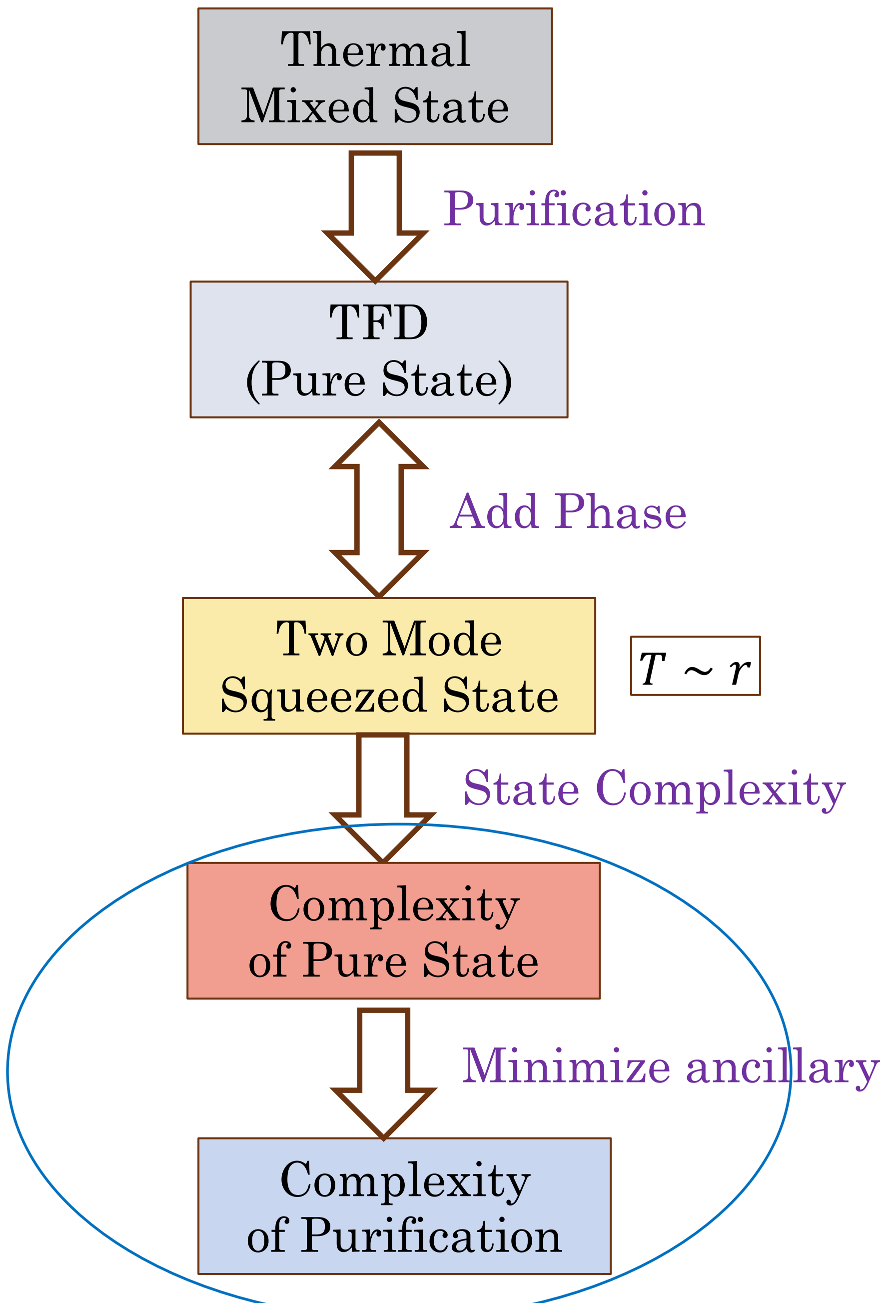
E. E. or Complexity

We are interested in the **complexity** of the mixed thermal state

So, we will minimize the complexity

Complexity of purification $\mathcal{C}_{\text{th}}(\beta) = \min\limits_{\alpha,\beta,\ldots} \mathcal{C}\left(|\Psi\rangle_{\alpha,\beta,\ldots}, |\psi_R\rangle\right)$

SO, WHAT ARE WE DOING?



Why?
Because sensitive to squeezing angle

Squeezed States, Thermal Density Matrix and TFD

Complexity for
The purified state $\mathcal{C}_\phi = \frac{1}{\sqrt{2}}\sqrt{\ln^2\left|\frac{1+e^{-2i\phi}e^{-\beta\omega/2}}{1-e^{-2i\phi}e^{-\beta\omega/2}}\right| + \arctan^2\left(2\sin 2\phi\frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}}\right)}$ $\tanh r = e^{-\beta\omega/2}$

COP for Thermal State

C is minimized at $\frac{\pi}{4}$

$$\mathcal{C}_{\rm th}(\beta) = \mathcal{C}_\phi\big|_{\phi=\pi/4} = \frac{1}{\sqrt{2}}|\arctan(\sinh 2r)| = \frac{1}{\sqrt{2}}\left|\arctan\left(2\frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}}\right)\right|$$

$$\approx \begin{cases} \sqrt{2}\,e^{-\beta\omega/2} & \text{low temperature limit } \beta\omega \gg 1 \quad r\to 0 \\ \frac{\pi}{2\sqrt{2}} & \text{high temperature limit, } \beta\omega \to 0 \quad r \gg 1 \end{cases}$$

System and ancillary not entangled, $B \sim 0$

Wavefunction is off diagonal $A \sim 0, B \sim i$

important when we study decoherence in the next lecture

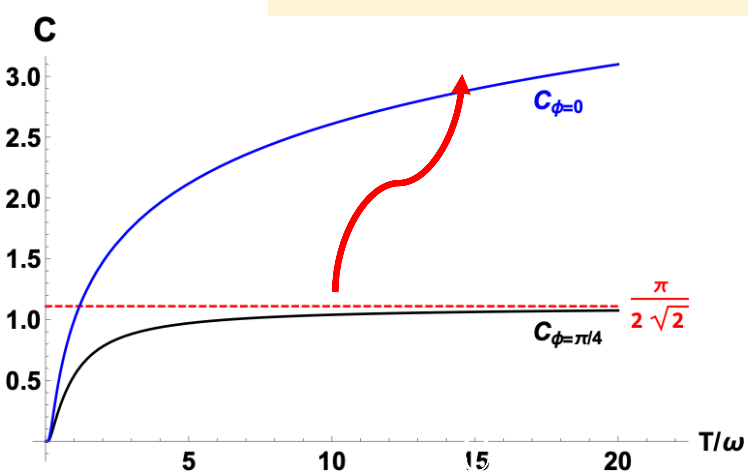
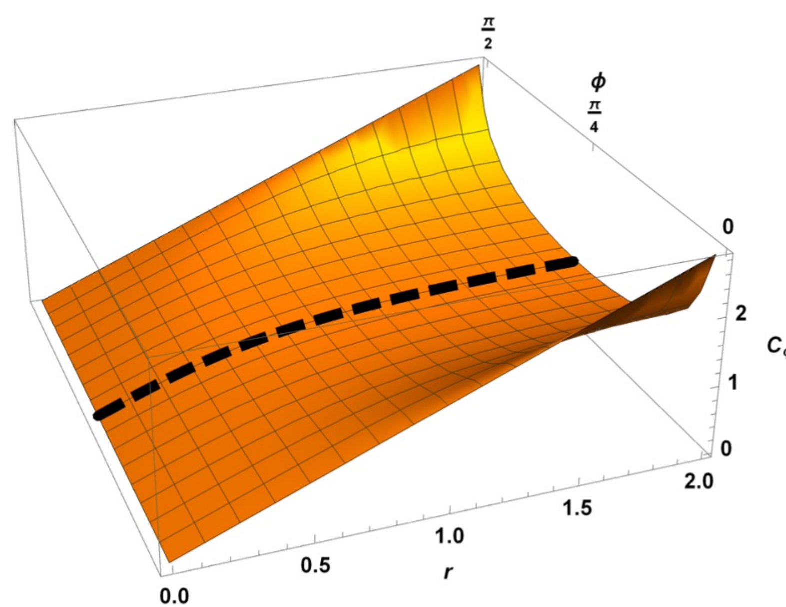


Fig: Complexity of purification COP, as a function of temperature $T = \beta^{-1}$ for $\phi = 0$ and $\phi = \pi/4$.



End of Lecture 2

Thank You