Introduction to Various Measures of Quantum Correlations

QIW - 2024

5-7 February 2024, CAPP, University of Johannesburg

Khushboo Dixit

Centre for Astro-Particle Physics (CAPP) & Department of Physics
University of Johannesburg
South Africa

Foundational issues in quantum mechanics

MAY 15, 1935 PHYSICAL REVIEW VOLUME 47

Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

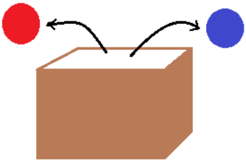
A. Einstein, B. Podolsky and N. Rosen, *Institute for Advanced Study, Princeton, New Jersey*

(Received March 25, 1935)

In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

- Quantum information theory is built upon the realisation that quantum resources like coherence and entanglement can be exploited for novel or enhanced ways of transmitting and manipulating information, such as quantum cryptography, teleportation, and quantum computing.
- These features were initially introduced to elegantly abase the opponents of quantum mechanics.
- Owing to the development of the quantum information science, these quantum mechanical features were reassessed and elevated as resources that may be exploited to achieve tasks that are not possible within the realm of classical physics.

Spatial Quantum Correlations



Classical Correlation

Quantum Correlation

Measurement process discerns the two types of correlations.

Observables don't have preassigned values. The measurement process forces an observable to take a particular value.

Entanglement $\equiv$ nonseparability

$$\underline{\textit{Separable states}} : |1\rangle_A \otimes |0\rangle_B \equiv |10\rangle_{AB} ;$$

$$\frac{1}{2}\left(|00\rangle_{AB} + |01\rangle_{AB} + |10\rangle_{AB} + |11\rangle_{AB} \right) = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle \right)_A \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle \right)_B$$

$$\underline{\textit{Entangled states}} : |\psi\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle \pm |10\rangle \right)$$

Spatial Quantum Correlations

A system is separable if

$$\rho_{\alpha\beta} = \rho_\alpha \otimes \rho_\beta$$

where ρ_α and ρ_β $\rightarrow$ pure states
System is entangled if

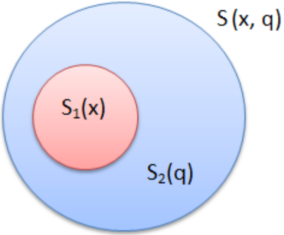
$$\rho_{\alpha\beta} \neq \rho_\alpha \otimes \rho_\beta$$

- **Von-Neumann Entanglement Entropy:**

  $$S(\rho_j) = -Tr(\rho_j \log_2(\rho_j))$$

  and

  $$S(\rho_j) = \begin{cases} 0 & \textit{for pure state} \\ \log_2 d & \textit{for mixed state}, \end{cases}$$



For a system defined by $\rho_{\alpha\beta}$, entanglement can be measured in the form of the von-Neumann entropy of the reduced density ρ_α matrix representing one of the sub-systems.

Spatial Quantum Correlations

LOCALITY: A measurement made on a system cannot influence other systems instantaneously.
REALISM: A system has well defined values of an observable whether someone measures it or not. Measurement process simply reveals these values to us.

Bell's inequality

Probability of a coincidence between separated measurements of particles with correlated (e.g. identical or opposite) orientation properties

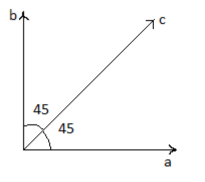
$$P(a, b) = \int d\lambda \rho(\lambda) p_A(a, \lambda) p_B(b, \lambda)$$

where, $p_A(a, \lambda)$ is the probability of detection of particle A with hidden variable λ by detector A, set in direction a, and similarly $p_B(b, \lambda)$ is the probability at detector B, set in direction b, for particle B, sharing the same value of λ. The source is assumed to produce particles in the state λ with probability $\rho(\lambda)$

The inequality $P(a, b) - P(a, c) \leq 1 + P(b, c)$
or
$|\langle M_a M_b \rangle - \langle M_a M_c \rangle| \leq 1 + \langle M_b M_c \rangle$

Non-locality

Local Realism restricts the correlations:



Quantum correlations $\rightarrow$ *product of the outcomes on the two sides*
$|\langle M_a M_b\rangle - \langle M_a M_c\rangle| \leq 1 + \langle M_b M_c\rangle$ Bell's Inequality.
For arbitrary orientations, quantum Mechanics predicts $\langle M_a M_b\rangle = -\mathbf{a}.\mathbf{b}$
$|0 - \frac{1}{\sqrt{2}}| \nleq 1 - \frac{1}{\sqrt{2}}$

Nonlocality Measures for Tripartite Systems

Tripertite nonlocality :

- A three qubit system may be nonlocal if nonclassical correlations exist between two of the three qubits. Such a state will be absolute nonlocal and will violate the Mermin inequality for a detector setting A, B and C. Mermin inequalities are:

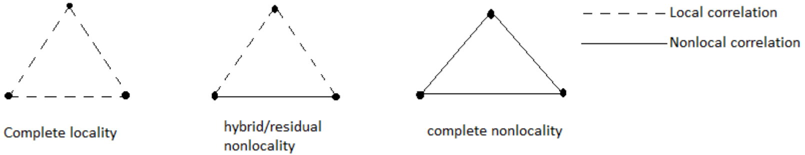
$$M_1 \equiv \langle ABC' \rangle + \langle AB'C \rangle + \langle A'BC \rangle - \langle A'B'C' \rangle \leq 2$$

$$M_2 \equiv \langle ABC \rangle - \langle A'B'C \rangle - \langle A'BC' \rangle - \langle AB'C' \rangle \leq 2$$

with

$$P(a,b,c) = \int d\lambda \rho(\lambda) P_1(a|\lambda) P_2(b|\lambda) P_3(c|\lambda), \tag{1}$$

PRL 65 (1990) 15, 1838



Nonlocality Measures for Tripartite Systems

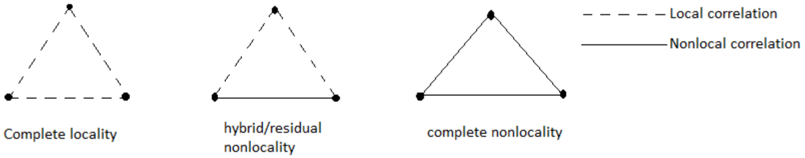
- To probe genuine nonlocal correlations, we make use of the Svetlichny inequality which is based on hybrid nonlocal-local realism as follows

$$P_B(a_1a_2a_3) = \sum_{k=1}^{3} P_k \int d\lambda \rho_{ij}(\lambda) P_{ij}(a_i a_j|\lambda) P_k(a_k|\lambda). \quad (2)$$

A state violating a Mermin inequality may fail to violate a Svetlichny inequality which provides a sufficient condition for genuine tripartite nonlocality. Svetlichny inequality is:

$$\begin{aligned}\sigma &\equiv M_1 + M_2\\ &= \langle ABC' \rangle + \langle AB'C \rangle + \langle A'BC \rangle - \langle A'B'C' \rangle + \\ &\quad \langle ABC \rangle - \langle A'B'C \rangle - \langle A'BC' \rangle - \langle AB'C' \rangle \leq 4\end{aligned}$$

PRD 35 (1987) 10, 3066



Quantum Coherence

- A coherence measure is defined by a distance to the closest incoherent state.

For a d-dimensional state

$$\rho = |\psi\rangle \langle\psi| \,,$$

the l_1-norm of coherence parameter is formulated as

$$\chi = \sum_{i \neq j} |\rho_{ij}| \leq d - 1$$

$d \rightarrow$ dimension of system

- satisfies basic requirements i.e., non-negativity ($\chi(\rho) = 0$ iff $\rho \in \mathcal{I}$), convexity (i.e., $\chi(\sum_k p_k \rho_k) \leq \sum_k p_k \chi(\rho_k)$, where $\rho_k = Q_k \rho Q_k^\dagger / p_k$ (Q_k is the Kraus operator) and $p_k = Tr(Q_k \rho Q_k^\dagger)$) and monotonicity (i.e., nonicreasing under the incoherent operations).

Steering

- The possibility of manipulating the state of one subsystem by making measurements on the other.
- An asymmetric form of correlations, which means that some states can be steered from A to B but not the other way around.

- Consider a bipartite situation composed by Alice and Bob sharing an unknown quantum state ρ^{AB}.
- Measurements made by Alice $\{M_{a|x}\}$ are also completely unknown.
- *One-sided device-independent scenario:* Bob has full control of his measurements and can thus access the conditional states $\rho_{a|x}$ (with corresponding probability $p_{a|x}$).
- Given that Alice decides to apply measurement x, the variable λ instructs Alice's measurement device to output the result a with probability $p(a|x,\lambda)$.
- If assemblage $\sigma_{a|x} = tr_A[(M_{a|x} \otimes \mathbb{I})\rho_{AB}] \neq \int d\lambda \mu(\lambda) p(a|x,\lambda)\rho_\lambda \Rightarrow$ Steerable state ρ_{AB}.

Rep. Prog. Phys. 80 024001 (2017)

Non-Local Advantage of Quantum Coherence (NAQC)

- The l_1-norm of coherence in the basis of eigenvectors of Pauli spin observables σ_i $(i = x, y, z)$ is reformulated as

$$\mathcal{C}_{l_1}^{\sigma_i}(\rho) = \sum_{R \neq S} \langle R|\rho|S\rangle; \qquad |R\rangle \text{ and } |S\rangle \rightarrow \text{ eigenstates of } \sigma_i$$

- The complementarity relation of coherence $\sum_{i=x,y,z} \mathcal{C}_{l_1}^{\sigma_i}(\rho) \leq C_{max} \approxeq \sqrt{6}$
- Suppose that Alice and Bob are two game participants and share qubits A and B with state ρ^{AB}, respectively.
    - Alice randomly performs one of the measurements M_i^a on qubit A with probability $P_{M_i^a} = Tr[(M_i^a \otimes \mathbb{I})\rho_{AB}]$.
    - The measured state for the two-qubit state can be obtained as $\rho_{AB|M_i^a} = (M_i^a \otimes \mathbb{I})\rho_{AB}(M_i^a \otimes \mathbb{I})/P_{M_i^a}$
    - The conditional state for qubit B is $\rho_{B|M_i^a} = Tr_A(\rho_{AB|M_i^a})$
    - Alice then tells Bob to her measurement choice and outcome, and Bob's task is to measure the coherence of qubit B at random in the eigenbasis of the other two of the three Pauli matrices σ_j and σ_k.
    - The violation inequality implies that a single-system description of the coherence of subsystem B does not exist.

PRA 95, 010301 (2017)

Non-Local Advantage of Quantum Coherence (NAQC)

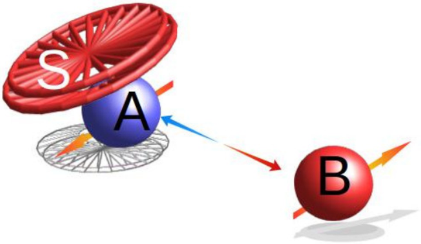


Figure: Coherence of Bob's particle is being steered beyond what could have been achieved by a single system, only by local projective measurements on Alice's particle and classical communications.

PRA 95, 010301 (2017)

Trade-off Relations

- Complementarity relations provide an effective way to characterize quantum correlations in bi- and multi-partite systems.
- Coherence and Mixedness

$$\text{Complimentarity rel} \quad \beta(\rho) = \frac{\chi^2(\rho)}{(n-1)^2} + \eta(\rho) \leq 1$$

with

$$\text{Coherence} \quad \chi(\rho) = \sum_{i \neq j} |\rho_{ij}|,$$

$$\text{and mixedness} \quad \eta(\rho) = \frac{n}{n-1}(1 - Tr(\rho^2)).$$

- Upperbound the maximum quantum coherence for fixed mixedness in a system.
  PRA **91**, 052115 (2015)

Trade-off Relations

- Predictability, Local Coherence and Nonlocal Coherence

$$P_{hs}(\rho_A) + C_{hs}(\rho_A) + C_{hs}^{nl}(\rho_{A|B}) = \frac{d_A - 1}{d_A}$$

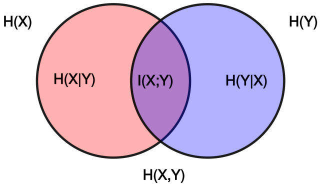
$P_{hs}(\rho_A) = \sum_{i=0}^{d_A-1} (\rho_{ii}^A)^2 - \frac{1}{d_A}$
$C_{hs}(\rho_A) = \sum_{i\neq k}^{d_A-1} |\rho_{ik}^A|^2$
$C_{hs}^{nl}(\rho_{A|B}) = \sum_{i\neq k, j<l} |\rho_{ij,kl}|^2 - 2\sum_{i\neq k, j\neq l} \Re(\rho_{ij,kj}\rho_{il,kl}^*)$

J. Phys. A: Math. Theor. 53 (2020) 465301

Quantum Discord

- The concept of quantumness as measured by quantum discord can be essentially explained as the impossibility of measuring a quantum state without disturbing it. Rep. Prog. Phys. 81 024001 (2018)

- Classical mutual information (total correlations) $\mathcal{I}(X:Y) = H(X) + H(Y) - H(X,Y) = H(X) - H(X|Y)$,
- Quantum mutual information $\mathcal{I}_{AB} = S(\rho_A) + S(\rho_B) - S(\rho_{AB}) \neq S(\rho_A) - S_{A|B}$,
- Quantum conditional entropy of ρ_{AB} $S(\rho_{A|B}) = min\ over\{\pi_K^B\}\epsilon M^B \sum_k p_k S(\rho_{A|k})$
- Classical correlations $\mathcal{J}_{A|B} = S(\rho_A) - S_{A|B}$
- *Quantum Discord* $= \mathcal{I}_{AB} - \mathcal{J}_{A|B}$



$H(X)$ and $H(Y)$ $\rightarrow$ Shannon entropies, $H(A) = -\sum_a p_a \log_2 p_a$ for marginal distributions $p_{x,.}$ and $p_{.,y}$
$H(X,Y) \rightarrow$ for joint distributions $p_{x,y}$,
$H(X|Y) \rightarrow$ conditional entropy.

Temporal quantum correlations

Leggett-Garg inequality (LGI) (PRL **54**, 857 (1985)) follows two concepts:

- *macrorealism (MR):* the system which has available to it two or more macroscopically distinct states, pertaining to an observable $\hat{Q}$, always exists in one of these states irrespective of any measurement performed on it.
- *noninvasive measurability (NIM):* we can perform the measurement without disturbing the future dynamics of the system.

The simplest form of LGI is the one involving three measurements performed at time t_0, t_1 and t_2 ($t_0 \leq t_1 \leq t_2$) (three-time measurement)

$$K_3 = C_{01} + C_{12} - C_{02}$$

where, $C_{ij} = \langle Q(\hat{t}_i) Q(\hat{t}_j) \rangle$ (the two-time correlation function) and bounds on K_3 are obtained as $-3 \leq K_3 \leq 1$. $\hat{Q} \rightarrow$ dichotomic observable (with possible outcomes ± 1), $\hat{Q}^\dagger = \hat{Q}$.

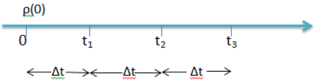
Temporal quantum correlations

Leggett-Garg-type inequality (PRL 107, 090401 (2011); PRD **97**, 053008 (2018))

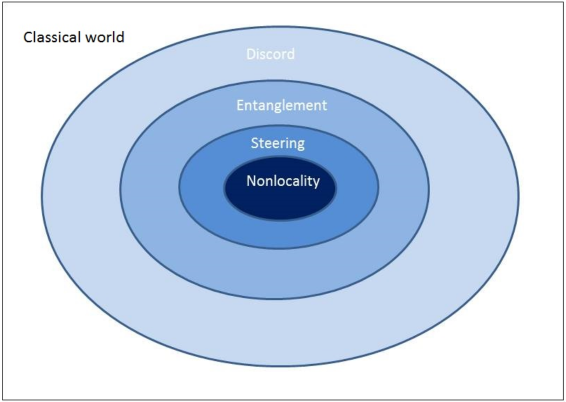
- Constructed by replacing the NIM condition with *stationarity*, implying that correlations between different measurements only depend on time differences instead of specific time instants, i.e., all the intermediate nonmeasurable correlations are replaced by measurable ones

$$\tilde{K}_3 = 2C_{01} - C_{02} \leq 1$$

- More suitable to be verified experimentally



Hierarchy of Quantum Correlations

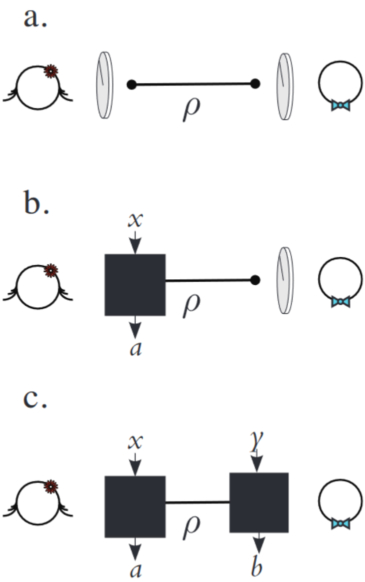


Reference : J. Phys. A: Math. Theor. 49, 473001 (2016)

Thanks for your attention!

THANK YOU

BACKUP SLIDES


a.
ρ
b.
x
ρ
a
c.
x
y
ρ
a
b