Quantum Information
Lecture #3

Recall our postulate #4:

The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. If the systems are labelled by numbers 1 through n and the i-th system has been prepared in the state $|\psi_i\rangle$ then joint state of the system is $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$

Tensor Product:

Let A & B be two quantal systems with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. If we prepare system A & B in states $|\psi_A\rangle$ and $|\varphi_B\rangle$ the joint state of the system is $|\psi_A, \varphi_B\rangle \equiv |\psi_A\rangle \otimes |\varphi_B\rangle$.

Comment: A tensor product can be defined between <u>any</u> two vector spaces. For tensor product between two Hilbert spaces we choose an inner product on the tensor product space that's naturally inherited from the component Hilbert spaces.

Suppose $\{|\alpha_i\rangle\}$ and $\{|\beta_m\rangle\}$ are orthonormal bases for A & B. We can then perform projective measurements $\{|\alpha_i\rangle\langle\alpha_i|\}$ & $\{|\beta_m\rangle\langle\beta_m|\}$
Then the probabilies are $p^{AB}(i,m) = p^A(i)\, p^B(m)$
where $p^A(i) = |\langle\alpha_i|\psi_A\rangle|^2$ and $p^B(m) = |\langle\beta_m|\varphi_B\rangle|^2$.

But we also expect $p^{AB}(i,m) = |\langle\alpha_i,\beta_m|\psi_A,\varphi_B\rangle|^2$.

This motivates: $\langle\alpha_i,\beta_m|\psi_A,\varphi_B\rangle = \langle\alpha_i|\psi_A\rangle\langle\beta_m|\varphi_B\rangle$.

Defn: Tensor Product

If $\mathcal{H}_A$ & $\mathcal{H}_B$ are two Hilbert spaces, we define their tensor product $\mathcal{H}_{AB}$ as the Hilbert space whose elements are given by the bilinear map $\otimes : \mathcal{H}_A \times \mathcal{H}_B \rightarrow \mathcal{H}_{AB}$ with the following properties:

i) If $|\alpha\rangle \in \mathcal{H}_A$ & $|\beta\rangle \in \mathcal{H}_B$ then $|\alpha, \beta\rangle \equiv |\alpha\rangle \otimes |\beta\rangle \in \mathcal{H}_{AB}$.

ii) $|\alpha\rangle \otimes (c_1|\beta_1\rangle + c_2|\beta_2\rangle) = c_1(|\alpha\rangle \otimes |\beta_1\rangle) + c_2(|\alpha\rangle \otimes |\beta_2\rangle)$

$(d_1|\alpha_1\rangle + d_2|\alpha_2\rangle) \otimes |\beta\rangle = d_1(|\alpha_1\rangle \otimes |\beta\rangle) + d_2(|\alpha_2\rangle \otimes |\beta\rangle)$

iii) If $|\alpha_1, \beta_1\rangle$ & $|\alpha_2, \beta_2\rangle \in \mathcal{H}_{AB}$, then their inner product

is given by $\langle \alpha_1, \beta_1 | \alpha_2, \beta_2\rangle = \langle \alpha_1 | \beta_1\rangle \langle \alpha_2 | \beta_2\rangle$

where $\langle \alpha_1, \beta_1| = \langle \alpha_1| \otimes \langle \beta_1|$.

iv) If $|\alpha_1, \beta_1\rangle$ & $|\alpha_2, \beta_2\rangle \in \mathcal{H}_{AB}$ then $c_1|\alpha_1, \beta_1\rangle + c_2|\alpha_2, \beta_2\rangle \in \mathcal{H}_{AB}$

Comments:

1. States of the form $|\psi_A, \varphi_B\rangle = |\psi_A\rangle \otimes |\varphi_B\rangle$ are called product states.

2. States $|\Psi_{AB}\rangle$ that cannot be written as product states are called **entangled states**.

3. $|\alpha\rangle \otimes \{b|\beta\rangle\} = b\{|\alpha\rangle \otimes |\beta\rangle\} = \{b|\alpha\rangle\} \otimes |\beta\rangle$

4. Suppose $|\alpha_1\rangle$ & $|\alpha_2\rangle \in \mathcal{H}_A$ st $\langle\alpha_1|\alpha_2\rangle = 0$. Then $\langle\alpha_1, \beta_1|\alpha_2, \beta_2\rangle = 0$, irrespective of $\langle\beta_1|\beta_2\rangle$.

5. If $\{|\alpha_i\rangle\}$ and $\{|\beta_m\rangle\}$ are orthonormal bases for $\mathcal{H}_A$ & $\mathcal{H}_B$ then $\{|\alpha_i, \beta_m\rangle\}$ furnishes an orthonormal basis for $\mathcal{H}_{AB}$.

6. $\mathrm{Dim}\,\mathcal{H}_{AB} = (\mathrm{Dim}\,\mathcal{H}_A) \times (\mathrm{Dim}\,\mathcal{H}_B)$.

7. Extending linear maps on $\mathcal{H}_A$ & $\mathcal{H}_B$ onto $\mathcal{H}_{AB}$. Suppose $\mathcal{O}_A$ ($\mathcal{O}_B$) is a linear map on $\mathcal{H}_A$ ($\mathcal{H}_B$):

$$\mathcal{O}_A : \mathcal{H}_A \to \mathcal{H}_A \quad (\mathcal{O}_B : \mathcal{H}_B \to \mathcal{H}_B)$$

Then we can extend their actions to $\mathcal{H}_{AB}$ by defining:

$$\tilde{\mathcal{O}}_A (|\alpha\rangle \otimes |\beta\rangle) \equiv (\mathcal{O}_A |\alpha\rangle) \otimes |\beta\rangle$$

$$\tilde{\mathcal{O}}_B (|\alpha\rangle \otimes |\beta\rangle) \equiv |\alpha\rangle \otimes (\mathcal{O}_B |\beta\rangle)$$

The product of operators $\mathcal{O}_A \otimes \mathcal{O}_B$ is defined by

$$\mathcal{O}_A \otimes \mathcal{O}_B (|\alpha\rangle \otimes |\beta\rangle) \equiv (\mathcal{O}_A |\alpha\rangle) \otimes (\mathcal{O}_B |\beta\rangle)$$

E.g. $|\alpha_1, \beta_1\rangle\langle\alpha_2, \beta_2| = |\alpha_1\rangle\langle\alpha_2| \otimes |\beta_1\rangle\langle\beta_2|$

Example: Two qubits

Consider two qubits A & B. The product basis in the computational basis is given by:

$$|0,0\rangle, |0,1\rangle, |1,0\rangle, |1,1\rangle.$$

Explicitly, since $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ & $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$$|00\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad |0,1\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$|1,0\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad |11\rangle = \begin{pmatrix}0\\1\end{pmatrix} \times \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

However there are orther bases in which the basis states are entangled.

The Bell basis:

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\{|00\rangle + |11\rangle\}$$
$$|\beta_{01}\rangle = \frac{1}{\sqrt{2}}\{|01\rangle + |10\rangle\}$$

$$|\beta_{10}\rangle = \frac{1}{\sqrt{2}}\{|00\rangle - |11\rangle\}$$
$$|\beta_{11}\rangle = \frac{1}{\sqrt{2}}\{|01\rangle - |10\rangle\}$$

Also: $\vec{S} = \vec{S}_1 + \vec{S}_2$,

$\vec{S}^2, S_z$

$$\bar{3}\begin{cases} |1,1\rangle = |00\rangle \\ |1,0\rangle = \dfrac{|01\rangle + |10\rangle}{\sqrt{2}} \\ |1,-1\rangle = |11\rangle \end{cases}$$

$$\bar{1} \to |0,0\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

The states of the Bell basis are examples of maximally entangled states. Consider the state:

$$|\beta_{11}^{\alpha}\rangle = \frac{\alpha|01\rangle - (1-\alpha)|10\rangle}{N_\alpha}$$

where $\alpha \in [0,1]$. This state is unentangled when $\alpha = 0$ & $\alpha = 1$ but not in-between.

Now suppose, given $|\beta_{11}^{\alpha}\rangle$ we make a measurement on A in the computational basis. Then the conditional state of qubit B will be

Result of A measurement is 0, prob. state of B is $|1\rangle$ is $\left(\frac{\alpha}{N_\alpha}\right)^2$.

Result of B " " 1, " " " B " $|0\rangle$ is $\left(\frac{1-\alpha}{N_\alpha}\right)^2$.

Thus we see that the conditional state of B qubit is a mixed state but only when the state of the combined AB system is entangled.

Thus the state of the qubit B is:

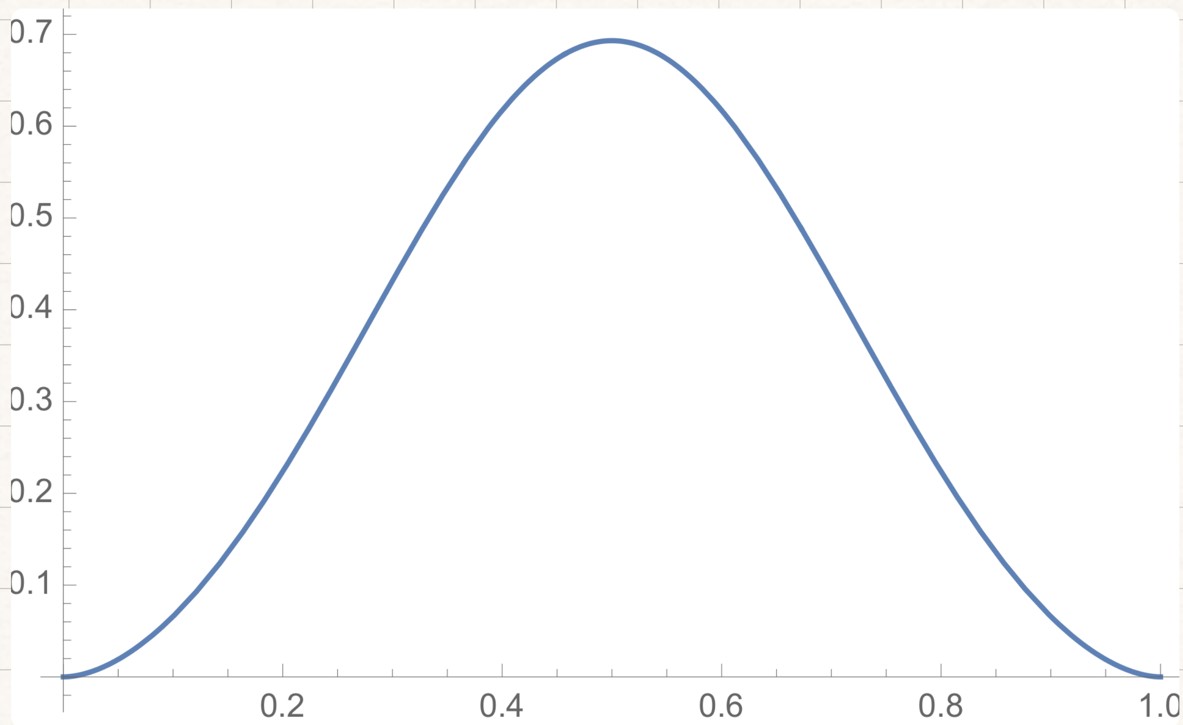
$$\rho_B = \left(\frac{\alpha}{N_\alpha}\right)^2 |0\rangle\langle 0| + \left(\frac{1-\alpha}{N_\alpha}\right)^2 |1\rangle\langle 1|.$$

$$\mathrm{Tr}\,\rho = 1 \Rightarrow \frac{\alpha^2 + 1 + \alpha^2 - 2\alpha}{N_\alpha^2} = 1 \Rightarrow N_\alpha = \sqrt{2\alpha^2 - 2\alpha + 1}$$

If we compute $S_\alpha = -\mathrm{Tr}(\rho \log \rho) = -\left[\frac{\alpha^2}{N_\alpha^2} \cdot \log \frac{\alpha^2}{N_\alpha^2} + \frac{(1-\alpha)^2}{N_\alpha^2} \log \frac{(1-\alpha)^2}{N_\alpha^2} \right]$

$$= -\, 2\left[\frac{\alpha^2}{2\alpha^2 - 2\alpha + 1} \log \frac{\alpha}{N_\alpha} + \frac{1+\alpha^2-2\alpha}{2\alpha^2-2\alpha+1} \log \frac{(1-\alpha)}{N_\alpha} \right]$$

If we plot S_α as a function of α we see that S_α reaches its maximum value when $\alpha = \frac{1}{2}$ with $S_\alpha = 0$ when $\alpha = 0$ & 1.



Thus we see that von Neumann entropy is a good measure of bi-partite entanglement entropy.

The No communication Theorem:

Suppose Alice & Bob share an entangled state:

$$|\beta_{11}\rangle = \frac{1}{\sqrt{2}}\{|01\rangle - |10\rangle\}$$

If Alice makes a measurement then it influences the result of Bob's measurement. But entanglement can not be used to send information by Alice to Bob in a way that violates the principle of special relativity.

Furthermore Alice's choice of measurement does not influence the probability of Bob's measurement outcomes.

Let us make those ideas more concrete. Let $|\Psi^{(AB)}\rangle$ be an entangled state shared by Alice & Bob:

$$|\Psi^{(AB)}\rangle = \sum_{a,b} \Psi_{ab} |a\rangle \otimes |b\rangle = \sum_{a} |a\rangle \otimes \Big(\sum_{b} \Psi_{ab} |b\rangle\Big) = \sum_{a} |a\rangle \otimes |\Psi_a^{(B)}\rangle$$

where $\{|a\rangle\}$ & $\{|b\rangle\}$ are orthonormal bases for $\mathcal{H}_A$ & $\mathcal{H}_B$.

$\{|\Psi_a^{(B)}\rangle\}$ are states that belong to $\mathcal{H}_B$. Note that

$$|\Psi_a^{(B)}\rangle = \langle a|\Psi^{(AB)}\rangle.$$

Now suppose Alice and Bob decide to make projective measurements in the $\{|a\rangle\}$ and $\{|b\rangle\}$ bases.

Then we can compute the joint probability $p(a,b)$ by:

$$p(a,b) = |\langle a,b|\Psi\rangle|^2$$

$$= |\langle a,b| \sum_{a'} |a', \Psi_{a'}^{(B)}\rangle|^2$$

$$= |\sum_{a} \delta_{aa'} \langle b|\Psi_{a'}^{(B)}\rangle|^2$$

$$\boxed{p(a,b) = |\langle b|\Psi_a^{(B)}\rangle|^2}$$

Similarly we can write:

$$\boxed{p(a,b) = |\langle a|\Psi_b^{(A)}\rangle|^2}$$

Now let us compute the probability for Bob to get $|b\rangle$ as a result of his measurement:

$$p(b) = \sum_a p(a,b) = \sum_a |\langle a|\Psi_b^{(A)}\rangle|^2$$

$$= \sum_a \langle \Psi_b^{(A)}|a\rangle\langle a|\Psi_b^{(A)}\rangle$$

$$= \langle \Psi_b^{(A)}|\Psi_b^{(A)}\rangle$$

Thus we see that $p(b)$ is independent of the choice of measurement by Alice.

Since $p(b)$ involves $|\Psi_b^{(A)}\rangle \in \mathcal{H}_A$ if we make a change of basis in $\mathcal{H}_A$ then

$$|\Psi_b^{(A)}\rangle \to |\Psi_b^{(A)'}\rangle = U|\Psi_b^{(A)}\rangle.$$

This may seem to give a different probability distribution $p'(b) = \langle \Psi_b^{(A)'} | \Psi_b^{(A)'} \rangle$ but since

$$|\Psi_b^{(A)'}\rangle = U|\Psi_b^{(A)}\rangle$$

we get $p'(b) = \langle \Psi_b^{(A)} | U^\dagger U | \Psi_b^{(A)} \rangle = \langle \Psi_b^{(A)} | \Psi_b^{(A)} \rangle = p(b)$.

Thus we see that $p(b)$ is independent of the choice of basis for $\mathcal{H}_A$

This is the content of the no-communication Theorem:

Two parties who share a quantum state cannot communicate by :

i) either a choice of local measurement

ii) or by making a local unitary transformation.

Conditional states:

Although Alices choice of measurement or choice of states do not influence Bob's probabilities $p(b)$, The result of Alice's measurement does influence Bob's measurement out comes.

This is most easily seen if we take the Singlet state and Alice measures in The $\{|0\rangle, |1\rangle\}$ basis. Then $p(b=0|a=0)=0$ $p(b=1|a=0)=1$.

According to Baye's Theorem :

$$p(b|a) = \frac{p(a,b)}{p(a)}$$
$$= \frac{|\langle b|\Psi_a^{(B)}\rangle|^2}{p(a)}$$

This probability is identical to that obtained by Bob having the conditional state:

$$|\hat{\Psi}_a^{(B)}\rangle = \frac{|\Psi_a^{(B)}\rangle}{\sqrt{p(a)}}$$

Density Operators for a subsystem:

Now consider a subsystem B of a compositive system AB. The states of B are given by the conditional states:

$$|\hat{\Psi}_a^{(B)}\rangle = \frac{|\Psi_a^{(B)}\rangle}{\sqrt{p(a)}} = \frac{\langle a|\Psi^{(AB)}\rangle}{\sqrt{p(a)}}$$

If we now compute the density operator for system B:

$$\rho^{(B)} = \sum_a p(a) |\hat{\Psi}_a^{(B)}\rangle\langle\hat{\Psi}_a^{(B)}|$$

$$= \sum_a \langle a|\Psi^{(AB)}\rangle\langle\Psi^{(AB)}|a\rangle$$

$$= \sum_a \langle a|\rho^{(AB)}|a\rangle$$

Thus we see that $\rho^{(B)}$, the density operator for the subsystem B is given by tracing over the subsystem A.

Partial Trace:

Tracing over a system involves the mathematical operation of partial tracing which is defined using product states:

If $\quad \rho^{AB} = |\alpha^{(A)}, \phi^{(B)}\rangle\langle\beta^{(A)}, \psi^{(B)}|$

Then $\quad \rho^{A} = \mathrm{Tr}_B\, \rho^{AB} = \langle\phi^{(B)}|\psi^{(B)}\rangle\, |\alpha^{(A)}\rangle\langle\beta^{(A)}|$

$$= \sum_b \langle\phi^{(B)}|b\rangle\langle b|\psi^{(B)}\rangle\, |\alpha^{(A)}\rangle\langle\beta^{(A)}|$$

$$= \sum_b \langle b | \rho^{(A)}, \psi^{(B)} \rangle \langle \alpha^{(A)}, \phi^{(B)} | b \rangle$$

$$= \sum_b \langle b | Q^{AB} | b \rangle.$$

Expectation Values of Operations of A Subsystem:

Suppose O_A is an operator / Observable of the subsystem A. If we compute $\langle O_A \rangle$ then we first extend O_A to the system AB by $O_A \to O_A \otimes \mathbb{1}_B$.

Then if the system in the joint state $|\psi^{(AB)}\rangle$ then

$$\langle O_A \rangle = \langle \psi^{(AB)} | O_A \otimes \mathbb{1}_B | \psi^{(AB)} \rangle$$

$$= \sum_b \langle \psi^{(AB)} | O_A \otimes |b\rangle\langle b| \, | \psi^{(AB)} \rangle$$

$$= \sum_b \underbrace{\langle \psi^{(AB)} | b \rangle}_{\mathcal{H}_A^*} O_A \underbrace{\langle b | \psi^{(AB)} \rangle}_{\in \mathcal{H}_A}$$

$$= \mathrm{Tr}_A \underbrace{\sum_b \langle b | \psi^{(AB)} \rangle \langle \psi^{(AB)} | b \rangle}_{\rho^{(A)}} O_A \;\; = \mathrm{Tr}_A \, \rho^{(A)} O_A$$

The Two Interpretations of Density Operators:

Interpretation 1: Density operator for a system describes our lack of knowledge about how the state was prepared. This is the statistical ensemble picture.

Interpretation 2: If systems A & B share an entangled state but the two systems cannot communicate then $\rho^{(A)} = \mathrm{Tr}_B\, \rho^{(AB)}$

describes the state of the subsystem A.

The two interpretations are related: If Bob makes a measurement on B but cannot communicate the result of his measurement to Alice then we see that Interpretation 2 $\longrightarrow$ Interpretation 1.

Schmidt Decomposition:

Suppose we have a density matrix ρ_P defined on a system P. We can then diagonalize ρ_P in some orthonormal basis $\{|k^P\rangle\}$:

$$\rho_P = \sum_k \lambda_k |k^P\rangle\langle k^P|$$

where $\lambda_k \geq 0$ with $\sum_{k=1}^{\dim \mathcal{H}_P} \lambda_k = 1$. This is just the spectral decomposition of ρ_P. Now suppose that there exists an auxiliary system Q such that the combined system PQ admits an entangled state $|\Psi^{PQ}\rangle \in \mathcal{H}_P \otimes \mathcal{H}_Q$ st that with $\rho_{PQ} = |\Psi^{PQ}\rangle\langle\Psi^{PQ}|$ we have:

$$\rho_P = \mathrm{Tr}_Q \, \rho_{PQ}$$

For a generic basis $\{|\phi^Q\rangle\}$ of Q we can write:

$$|\Psi^{PQ}\rangle = \sum_{\phi,k} c_{\phi k} |k^P\rangle |\phi^Q\rangle$$

$$= \sum_k |k^P\rangle \sum_\phi c_{\phi k} |\phi^Q\rangle$$

$$\boxed{|\Psi^{PQ}\rangle = \sum_k |k^P\rangle |\Psi_k^Q\rangle}$$

where $|\Psi_k^Q\rangle \equiv \sum_\phi c_{\phi k} |\phi^Q\rangle$.

So now $\rho_P = \mathrm{Tr}_Q |\Psi^{PQ}\rangle\langle\psi^{PQ}|$

$$= \sum_{kk'} |k^P\rangle\langle k'^P| \langle \psi^Q_k | \psi^Q_{k'}\rangle$$

Comparing this with $\rho_P = \sum_k \lambda_k |k\rangle\langle k|$ we see that $\langle \psi^Q_k | \psi^Q_{k'}\rangle = \lambda_k \delta_{kk'}$

We can then introduce the orthonormal set: $|\psi^Q_k\rangle = \sqrt{\lambda_k}\, |k^Q\rangle$

Then we write $|\psi^{PQ}\rangle$ as:

$$\boxed{|\Psi^{PQ}\rangle = \sum_k \sqrt{\lambda_k}\, |k^P\rangle |k^Q\rangle}$$

Schmidt Decomposition

<u>Comments:</u>

1. If $\dim \mathcal{H}^P = \dim \mathcal{H}^Q = d$, then the expansion of an entangled state in a generic product basis $\{|\phi^P, \chi^Q\rangle\}$ would have d^2 terms in general. The Schmidt decomposition has at most only d terms with real coefficients $\sqrt{\lambda_k}$. These coefficients are known as the Schmidt coefficients. The Schmidt decomposition is specific to the entangled state we have chosen.

2. For an entangled state at least <u>two</u> Schmidt coefficients must be non-zero.

3. If the dimensions of the two Hilbert spaces are unequal then the number of terms in the decomposition will be determined by the dimension of the smaller dimensional Hilbert space.

4. The Schmidt basis is a special basis (when the two dimensions are the same).

$\{|k^P\rangle\}$ & $\{|k^Q\rangle\}$ are eigenbases for ρ_P & ρ_Q, respectively.

5. $|\psi^{PQ}\rangle$ is known as the purification of ρ_P. For a given mixed state ρ_P one can always find an auxiliary quantum system Q such that there exists a pure state $|\psi^{PQ}\rangle \in \mathcal{H}_P \otimes \mathcal{H}_Q$ s.t. $\rho_P = \mathrm{Tr}_Q |\psi^{PQ}\rangle\langle\psi^{PQ}|$.

<u>Example:</u> For a pair of qubits, find the Schmidt decomposition for the state

$$|\Upsilon\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle + |1,+\rangle)$$

<u>Solution:</u> If we find the eigenbasis for the first qubit (qubit A) then we can find the eigenbasis for the second qubit B.

$$\rho_A = \mathrm{Tr}_B |\Upsilon\rangle\langle\Upsilon| = \frac{1}{2}\left[|0\rangle\langle 0|\langle 0|0\rangle + |0\rangle\langle 1|\langle 0|+\rangle + |1\rangle\langle 0|\langle +|0\rangle + |1\rangle\langle 1|\langle +|+\rangle\right]$$

$$= \frac{1}{2}\left[|0\rangle\langle 0| + \frac{|0\rangle\langle 1|}{\sqrt{2}} + \frac{|1\rangle\langle 0|}{\sqrt{2}} + |1\rangle\langle 1| \right]$$

In the computational basis: $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, ρ_A can be written as

$$\rho_A = \frac{1}{2}\begin{pmatrix} 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 \end{pmatrix}$$

We find the eigenvalues:

$$\det\left[\frac{1}{2}\begin{pmatrix} 1-2\lambda & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1-2\lambda \end{pmatrix} \right] = 0$$

$$\Rightarrow \quad (1-2\lambda)^2 = \frac{1}{2}$$

$$\Rightarrow \quad 2\lambda - 1 = \pm \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \lambda = \frac{1}{2}\left(\pm \frac{1}{\sqrt{2}} + 1\right)$$

$$\Rightarrow \quad \lambda = \frac{1}{2} \pm \frac{1}{2\sqrt{2}}$$

$$\lambda_+ = \frac{1}{2} + \frac{1}{2\sqrt{2}}$$

$$|\lambda_+\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\rho_A |\lambda_+\rangle = \lambda_+ |\lambda_+\rangle$$

$$\frac{1}{2}\left[\cancel{a} + \frac{b}{\sqrt{2}} \right] = \left(\cancel{\frac{1}{2}} + \frac{1}{2\sqrt{2}} \right) a$$

$$\Rightarrow \quad a = b$$

$$|\lambda_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|\lambda_+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$= |+\rangle$$

$$\lambda_- = \frac{1}{2} - \frac{1}{2\sqrt{2}}$$

$$|\lambda_-\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\rho_A |\lambda_-\rangle = \lambda_- |\lambda_-\rangle$$

$$\frac{1}{2}\left[\cancel{a} + \frac{b}{\sqrt{2}} \right] = \left[\cancel{\frac{1}{2}} - \frac{1}{2\sqrt{2}} \right] a$$

$$\Rightarrow \quad a = -b$$

$$|\lambda_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$|\lambda_-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$= |-\rangle$$

Thus $\rho_A = \lambda_+ |+\rangle\langle +| + \lambda_- |-\rangle\langle -|$, with $\lambda_+ + \lambda_- = 1$ & $\lambda_\pm > 0$

Thus we can write $\langle +|\Upsilon\rangle = \frac{1}{\sqrt{2}} (\langle +|0\rangle |0\rangle + \langle +|1\rangle |+\rangle)$

$$\lambda_+ = \frac{1}{2} + \frac{1}{2\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|+\rangle\right)$$

$$= \frac{1}{2} (|0\rangle + |+\rangle) = \sqrt{\lambda_+} \left[\sqrt{\frac{2\sqrt{2}}{\sqrt{2}+1}} \cdot \frac{1}{2} (|0\rangle + |+\rangle)\right]$$

$$|\Upsilon_+^B\rangle = \left\{\frac{\sqrt{2}}{2(1+\sqrt{2})}\right\}^{\frac{1}{2}} (|0\rangle + |+\rangle)$$

$$\langle -|\Upsilon\rangle = \frac{1}{\sqrt{2}} (\langle -|0\rangle |0\rangle + \langle -|1\rangle |+\rangle)$$

$$= \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |+\rangle\right) = \sqrt{\lambda_-} \left\{\sqrt{\frac{2\sqrt{2}}{\sqrt{2}-1}} \ \frac{1}{2} (|0\rangle - |+\rangle)\right.$$

$$|\Upsilon_-^B\rangle = \left\{\frac{\sqrt{2}}{2(\sqrt{2}-1)}\right\}^{\frac{1}{2}} (|0\rangle - |+\rangle)$$

Orthogonality $\langle \Upsilon_-^B|\Upsilon_+^B\rangle = c \ (\langle 0| - \langle +|)(|0\rangle + |+\rangle) = c \ (\langle 0|0\rangle - \langle +|+\rangle$
$+ \langle 0|+\rangle - \langle +|0\rangle)$

$$= c \left(1 - 1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = 0$$

Normalization: $\langle \gamma_+^B | \gamma_+^B \rangle = \dfrac{\sqrt{2}}{2(\sqrt{2}+1)} \left\{ \langle 0|0\rangle + \langle 0|+\rangle + \langle +|0\rangle + \langle +|+\rangle \right\}$

$$= \frac{1}{\sqrt{2}(1+\sqrt{2})} \left\{ 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 1 \right\} = \frac{1}{\sqrt{2}(1+\sqrt{2})} \, \frac{\sqrt{2}+1+1+\sqrt{2}}{\sqrt{2}}$$

$$= \frac{2(1+\sqrt{2})}{(1+\sqrt{2})\cdot 2} = 1$$

$$\langle \gamma_-^B | \gamma_-^B \rangle = \frac{\sqrt{2}}{2(\sqrt{2}-1)} \left\{ \langle 0|0\rangle - \langle 0|+\rangle - \langle +|0\rangle + \langle +|+\rangle \right\}$$

$$= \frac{1}{\sqrt{2}(\sqrt{2}-1)} \left\{ 1 - \frac{2}{\sqrt{2}} + 1 \right\}$$

$$= \frac{1}{\sqrt{2}(\sqrt{2}-1)} \cdot \frac{2\sqrt{2}-2}{\sqrt{2}} = \frac{2(\sqrt{2}-1)}{2(\sqrt{2}-1)} = 1.$$

Thus $|\gamma\rangle = \lambda_+ |+\rangle |\gamma_+^B\rangle + \lambda_- |-\rangle |\gamma_-^B\rangle$