Quantum Information
lecture #4

The EPR Critique of Quantum Mechanics:

In 1935 Albert Einstein, Boris Podolsky, and Nathan Rosen (EPR) offered an argument that quantum mechanics is an incomplete Theory. The EPR argument, if true, would imply that there there must exist hidden variables which are not part of quantum mechanics. Such theories are called 'hidden variable theories.'

Here we present a version of the EPR argument that is due to John Bell who derived a testable version of the argument which led to the Bell inequalities.

Suppose we have two qubits (say, two particles with spin $\frac{1}{2}$) which are in an entangled state given by the Bell state:

$$|\beta_{11}\rangle = \frac{1}{\sqrt{2}}\{|01\rangle - |10\rangle\}$$

For two spin $\frac{1}{2}$ particles in this state it can be shown that the total angular momentum operator $\vec{S} = \vec{S}_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes \vec{S}_B$ has eigenvalue given by $\vec{S}^2 = s(s+1)\hbar$

with $\delta = 0$. This state is called a singlet state.

The nice thing about the singlet state is that it has the same form in any basis. So if we express it in the X basis it becomes:

$$|\beta_{11}\rangle = \frac{1}{\sqrt{2}}\left\{\frac{1}{2}[|+\rangle + |-\rangle][|+\rangle - |-\rangle] - \frac{1}{2}[|+\rangle - |-\rangle][|+\rangle + |-\rangle]\right\} = \frac{1}{\sqrt{2}}\left\{\frac{1}{2}(\cancel{|++\rangle} + |-+\rangle - |+-\rangle - \cancel{|--\rangle}) - \frac{1}{2}(\cancel{|++\rangle} - |-+\rangle + |+-\rangle - \cancel{|--\rangle})\right\} = \frac{1}{\sqrt{2}}(|-+\rangle - |+-\rangle).$$

Now suppose the particle A ends up in Alice's lab while particle B ends up in Bob's lab. Alice and Bob's labs can be far apart. EPR argued that any measurement that Alice made on her qubit must be independent of any measurement that Bob did and vice versa. We may call this assumption the locality assumption.

Now according to quantum mechanics Alice can do a bunch of incompatible measurement on her qubit. Suppose Alice has a choice of two measurements X_A or Z_A. Suppose Bob also has the same choice: X_B or Z_B. But according to QM the 'value' of these variables do not exist before Alice or Bob makes the measurement.

If Alice chooses to measure X_A then the value of Bob's qubit's X_B value is determined. On the other hand a measurement of Z_A will yield the value of Z_B. But the (apparently) reasonable assumption of locality means that Alice's choice

of measurement doesn't influence the measurement that Bob does. Thus the values of X_B & Z_B must exist even though they are not simultaneously measureable according to quantum mechanics. An aspect of a physical system which can be measured without disturbing it is called 'an element of reality.' This assumption is known as the reality assumption.

This version of local realism seems compelling since there is no known 'mechanism' by which the two particles can interact over vast distances. Furthermore Alice and Bob can even do their measurements so that the elapsed between the events of measurement shorter than the time taken for a beam of light to traverse the distance between them. In the language of special relativity the two events are space-like separated.

Criticism of the EPR argument involves:

i) It's counterfactual. So Alice can never do both X_A & Z_A measurements and so making the statement the values of both X_B & Z_B exist is not a factual statement.

ii) Niels Bohr argued that there needn't be a physical mechanism by which the two qubits can communicate with each other. He argued that the two different choices of measurements were complementary to each other in the same way the wave aspect and the particle aspect of a quantum particle are complementary. The latter is the statement of principle of complementarity which says that whether we see the particle or wave nature of a quantum particle depends on the experimental setup.

The Bell Inequalities or the Bell Theorem

In 1964, John Bell proposed a statistical experimental test of the EPR argument. Here we present a version of the argument due to John Clauser, Michael Horne, Abner Shimony, and Richard Holt (CHSH).

Let Alice and Bob have choice of making measurements A_1 or A_2 and B_1 or B_2, respectively. Thus jointly there are four possible measurements: (A_1, B_1), (A_1, B_2), (A_2, B_1), (A_2, B_2). Suppose we choose units such that these measurements can take the values $+1$ or -1. For our case we take the state to be an entangled singlet state and so we are measuring in units of $\pi/2$.

Note that each pair of these observables are mutually exclusive as a set but we can built us a statistics of their joint values: $A_i \cdot B_j$. This quantity takes the value either $+1$ or -1. CHSH then proposed to measure the average value of $Q = A_1(B_1 - B_2) + A_2(B_1 + B_2)$. Q then can take values between $+2$ and -2.

This means that the average value of Q lies between -2 & $+2$.

$$\Rightarrow \boxed{-2 \leq \langle A_1 B_1 \rangle - \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle + \langle A_2 B_2 \rangle \leq +2}$$

This is known as the CHSH inequality and it is an example of a Bell inequality. It was derived assuming that A_i & B_j can take values independent of each other.

What values does quantum mechanics predict?

The values of $\langle A_i B_j \rangle$ in quantum mechanics will depend on <u>both</u> our choice of A_i & B_j as well as the state with respect to which we take the average.

For measurement we consider a spin measurement in the x-z plane in which the angle of the axis of measurement makes an angle θ from the z-axis. This direction is defined by the unit vector $\hat{n} = (\sin\theta, 0, \cos\theta)$ and the measurement is $W_\theta = \frac{2}{\hbar}\hat{n}\cdot\vec{S} = \sin\theta\, X + \cos\theta\, Z$.

We can now calculate $\langle W_{A\theta} W_{B\theta'} \rangle$ for the singlet Bell state:

We first observe:

$$X|0\rangle = |1\rangle \quad \& \quad X|1\rangle = |0\rangle$$

And so

a) $X_A X_B |\beta_{11}\rangle = X_A X_B \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle) = -\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = -|\beta_{11}\rangle$

b) $X_A Z_B |\beta_{11}\rangle = X_A Z_B \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}}(-|11\rangle - |00\rangle) = -\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = -|\beta_{00}\rangle$

c) $Z_A Z_B |\beta_{11}\rangle = Z_A Z_B \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{-1}{\sqrt{2}}(|01\rangle - |10\rangle) = -|\beta_{11}\rangle$

d) $Z_A X_B |\beta_{11}\rangle = Z_A X_B \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\beta_{00}\rangle$

Thus $\langle W_{A_\theta} W_{B\theta'} \rangle = \sin\theta \sin\theta' \langle X_A X_B \rangle + \sin\theta \cos\theta' \langle X_A Z_B \rangle$

$$+ \cos\theta \sin\theta' \langle Z_A X_B \rangle + \cos\theta \cos\theta' \langle Z_A Z_B \rangle .$$

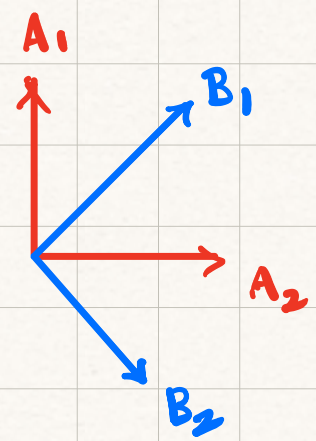
$$= -\sin\theta \sin\theta' + 0 + 0 - \cos\theta \cos\theta' = -\cos(\theta - \theta')$$

Let us pause for a moment and see if this result makes sense. For $|\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ we see that Z_A & Z_B are anti-correlated. This agrees with $\theta = \theta' \Rightarrow \langle W_{A\theta} W_{B\theta} \rangle = -1$.

Now for A_i & B_j we choose

$$A_1 = W_0 , \quad B_1 = W_{\frac{\pi}{4}}$$

$$A_2 = W_{\frac{\pi}{2}} , \quad B_2 = W_{\frac{3\pi}{4}}$$



And so

$$\langle Q \rangle = \langle W_0 W_{\frac{\pi}{4}} \rangle - \langle W_0 \, W_{\frac{3\pi}{4}} \rangle + \langle W_{\frac{\pi}{2}} W_{\frac{\pi}{4}} \rangle + \langle W_{\frac{\pi}{2}} W_{\frac{3\pi}{4}} \rangle$$

$$= \left(-\tfrac{1}{\sqrt{2}}\right) - \left(\tfrac{1}{\sqrt{2}}\right) + \left(-\tfrac{1}{\sqrt{2}}\right) + \left(-\tfrac{1}{\sqrt{2}}\right)$$

$$= -2\sqrt{2} < -2$$

Thus we see that in QM $\langle Q \rangle$ violates the CHSH inequality. By changing θ & θ' we can also obtain $\langle Q \rangle = 2\sqrt{2} > 2$.

Thus we see that QM violates the prediction of locally realistic hidden variable theory.

We have proved Bell's theorem.

Tripartite Entanglement:

So far we have only dealt with bi-partite entanglement. Here we have seen that entanglement entropy is a good measure of entanglement.

However when we go to higher order entanglement there is no one choice of measure of entanglement. For example, consider following three qubit entangled states:

1. The GHZ state: $|GHZ\rangle = \dfrac{|000\rangle - |111\rangle}{\sqrt{2}}$

2. The W-state: $|W\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right)$

Note that $|GHZ\rangle$ has the same form as the Bell states. So one might be tempted to label it as a maximally entangled state. But an observation of any of the three qubits destroys entanglement.

On the other hand with the W-state observation of any one of the qubits do not destroy entanglement completely. Thus from a certain perspective $|W\rangle$ is a maximally entangled state.

The Area Law of Entanglement Entropy:

Entanglement entropy is an important quantity in many body physics. It is a difficult quantity to measure. Questions have been raised about the relationship between entanglement entropy and thermal entropy.

Consider the thermal state $\rho = \dfrac{\sum_n |n\rangle\langle n| \, e^{-\beta E_n}}{Z}$.

The absence of off-diagonal terms in ρ imply lack of coherence. The emergence of the thermal state of a quantum system that starts off in a pure state in contact with its environment (another closed quantum system, albeit very large) is a non-trivial problem.

There seems to be some relationship between entanglement entropy and thermal entropy. However, whereas thermal entropy is assumed to be extensive and hence proportional to the volume of the system, entanglement entropy between two subsystems A & B is proportional to the boundary area of the two subsystems. In this way entanglement entropy is like the thermal entropy of a black hole.