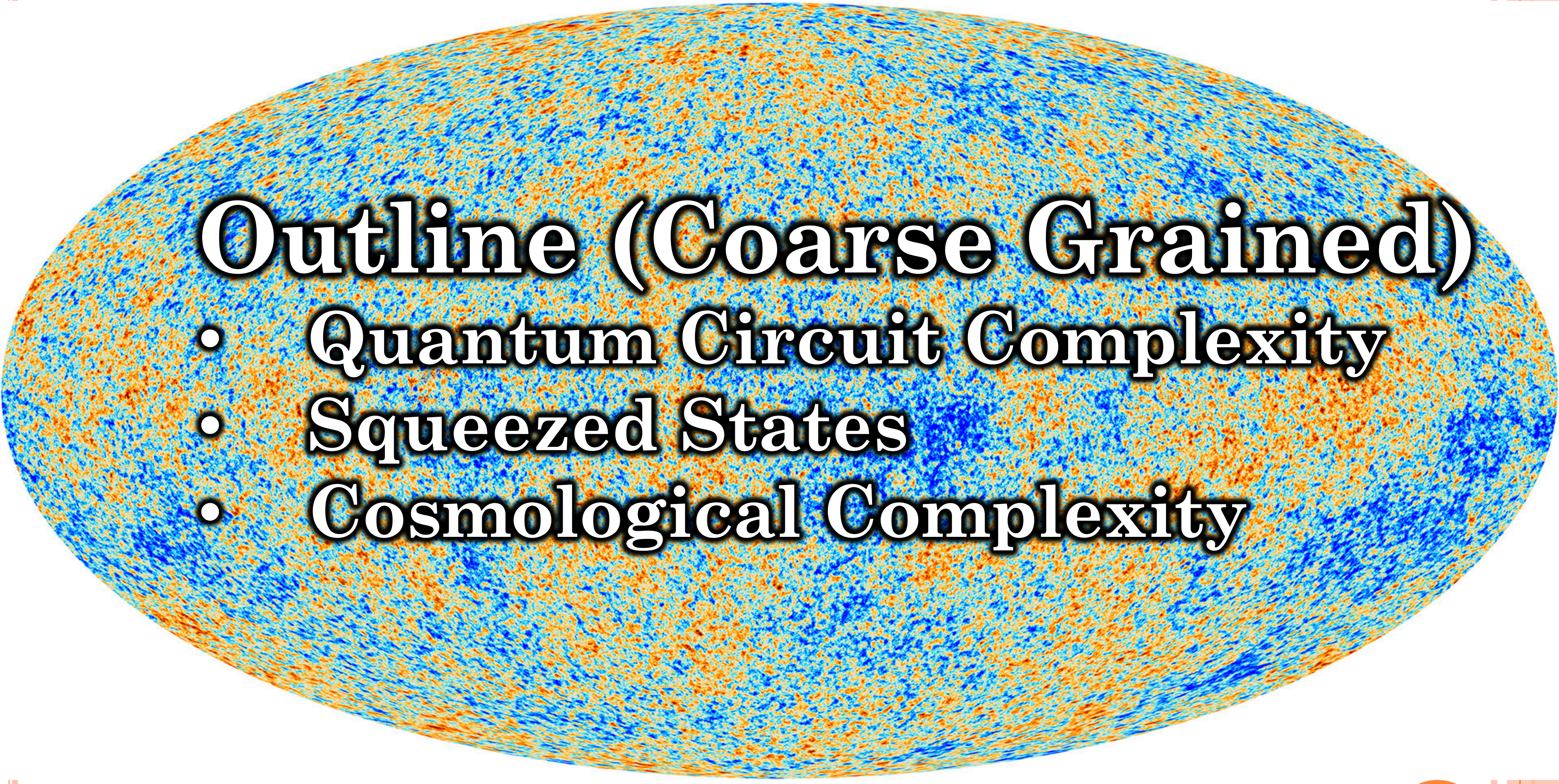
Outline (Coarse Grained)

- Quantum Circuit Complexity
- Squeezed States
- Cosmological Complexity



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Digression

Cosmology: A Quick Review

I will use David Tong and D. Baumann's Lecture notes.

- What is Cosmology?
- The Big Bang Theory

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- Our Observable Universe
- Cosmological Principle

↓

- Geometry of Spacetime

↓

- Geodesic in GR

- Dynamics of spacetime

  -Equation of State

  -Sources

  -Continuity Equation

  -Friedmann Equation

↓

- Cosmological Solutions

↓

- Inflation

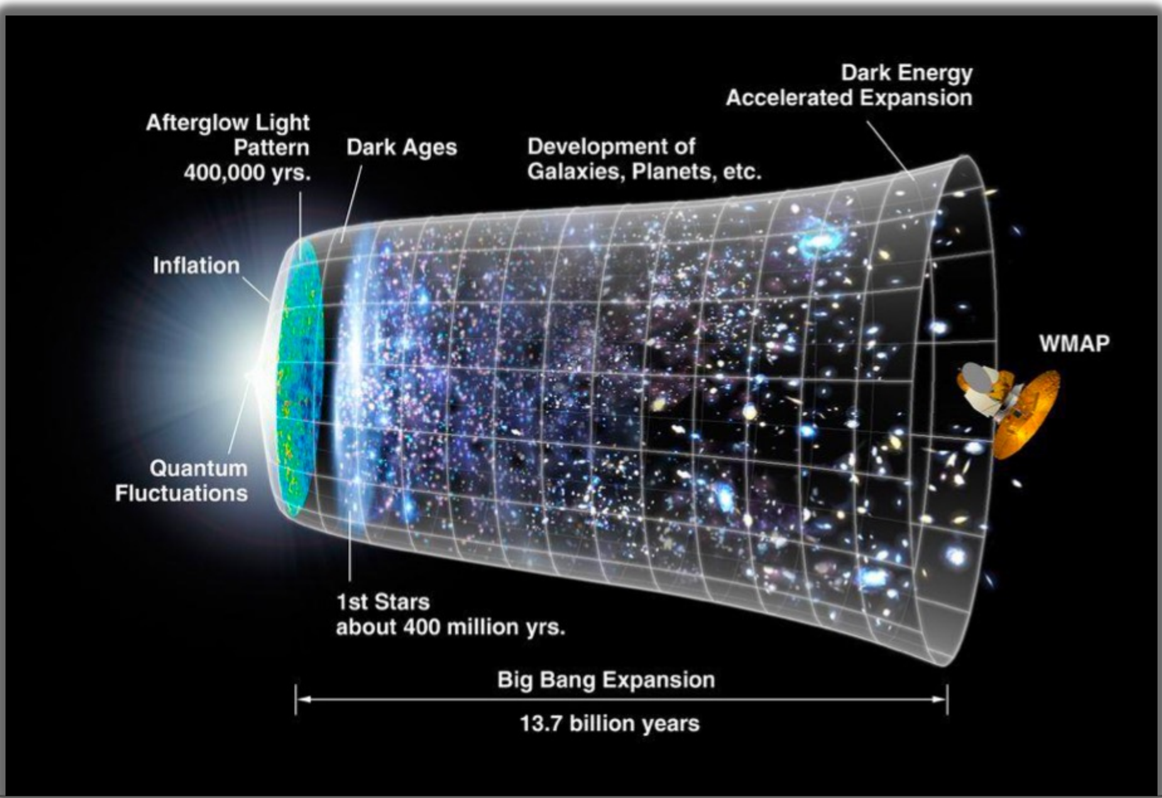
What is Cosmology?

The Big Bang Theory

COSMOLOGY: A QUICK REVIEW

Timeline of the expansion of the universe, where space, is represented at each time by the circular sections. On the left, the dramatic expansion occurs in the inflationary epoch; and at the center, the expansion accelerates (neith er time or size are to scale).

--Wikipedia



The Big bang theory basically tells us about our universe at an earlier time, when the universe was much younger.

The idea is based on an observation that our universe is expanding. That means in the past everything was nearby. When objects are close by, they got hotter. So Big Bang is essentially telling us that there was a time when our universe was very hot and matter, atoms and nuclei melted to a **fireball**.

Cosmology: A quick Review

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The Big Bang theory is a collection of ideas, calculations and predictions that explain what happened in that fireball and how it evolved into the current universe. **The subject that deals with this topic is cosmology.**

Oh, I thought it's a TV show….

Cosmology: A quick Review

We start with an assumption:

On the largest scales, the universe is spatially homogeneous and isotropic.

Homogeneity: Universe looks identical at every point in space

Isotropy: Universe looks the same in every direction.

Why do we make this assumption?

For convenience we consider the description of the Universe on a very large scale where things are simple.

Is it reasonable to do so?

There are **observational evidences:**

Note: Cosmological principle refers only to space. Universe is neither homogenous nor isotropic in time.

Our Observable Universe
Cosmological Principle

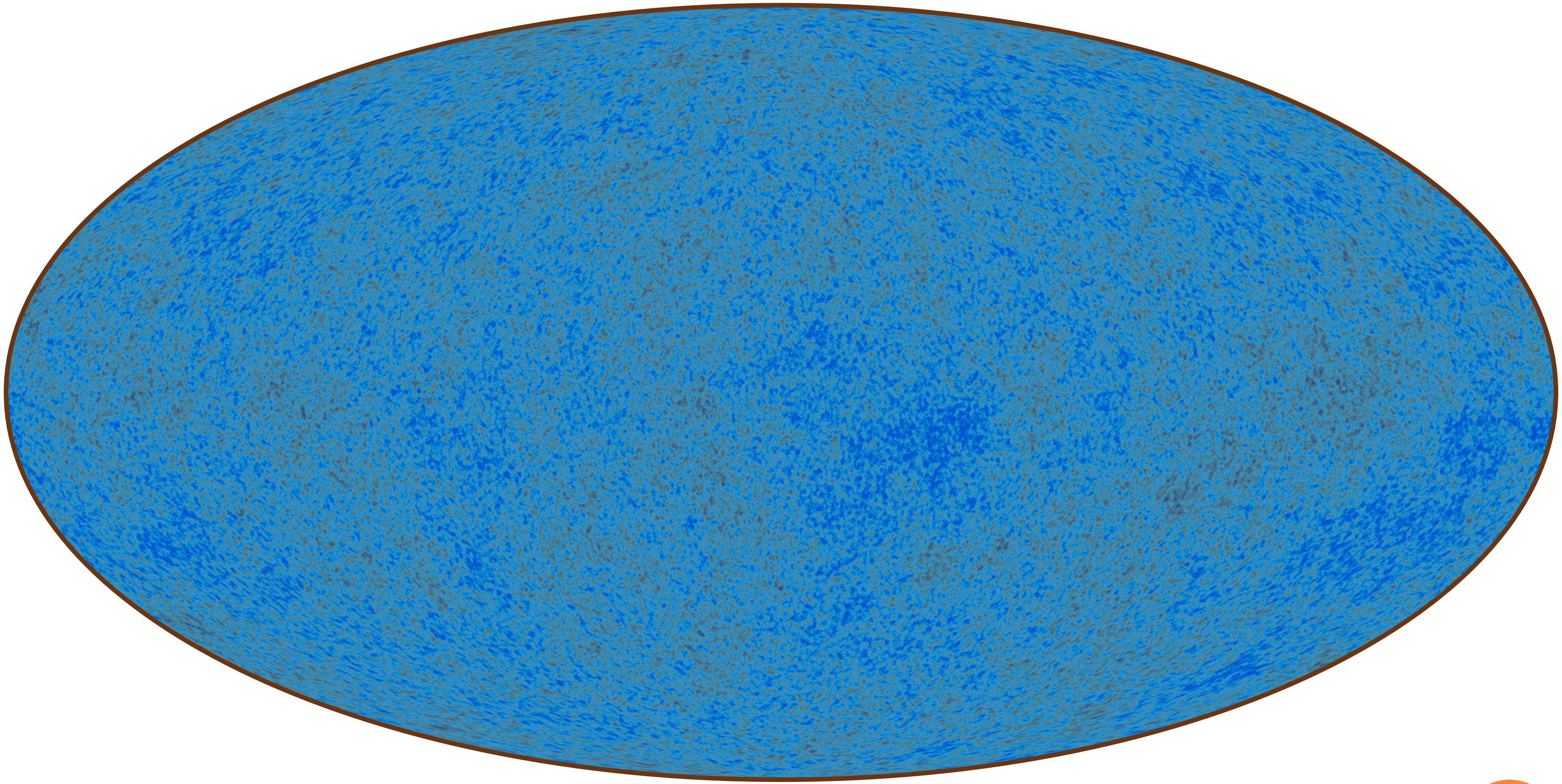
Cosmology: A quick Review

On the largest scales, the universe is spatially homogeneous & isotropic.

Homogeneity: Universe looks identical at every point in space,
Isotropy: Universe looks the same in every direction.

- The CMB is an almost uniform sea of photons which fills all of space and provides a snapshot of the universe from almost 13 billion years ago. The temperature of the CMB is 2.73 K.

Cosmic Microwave Background



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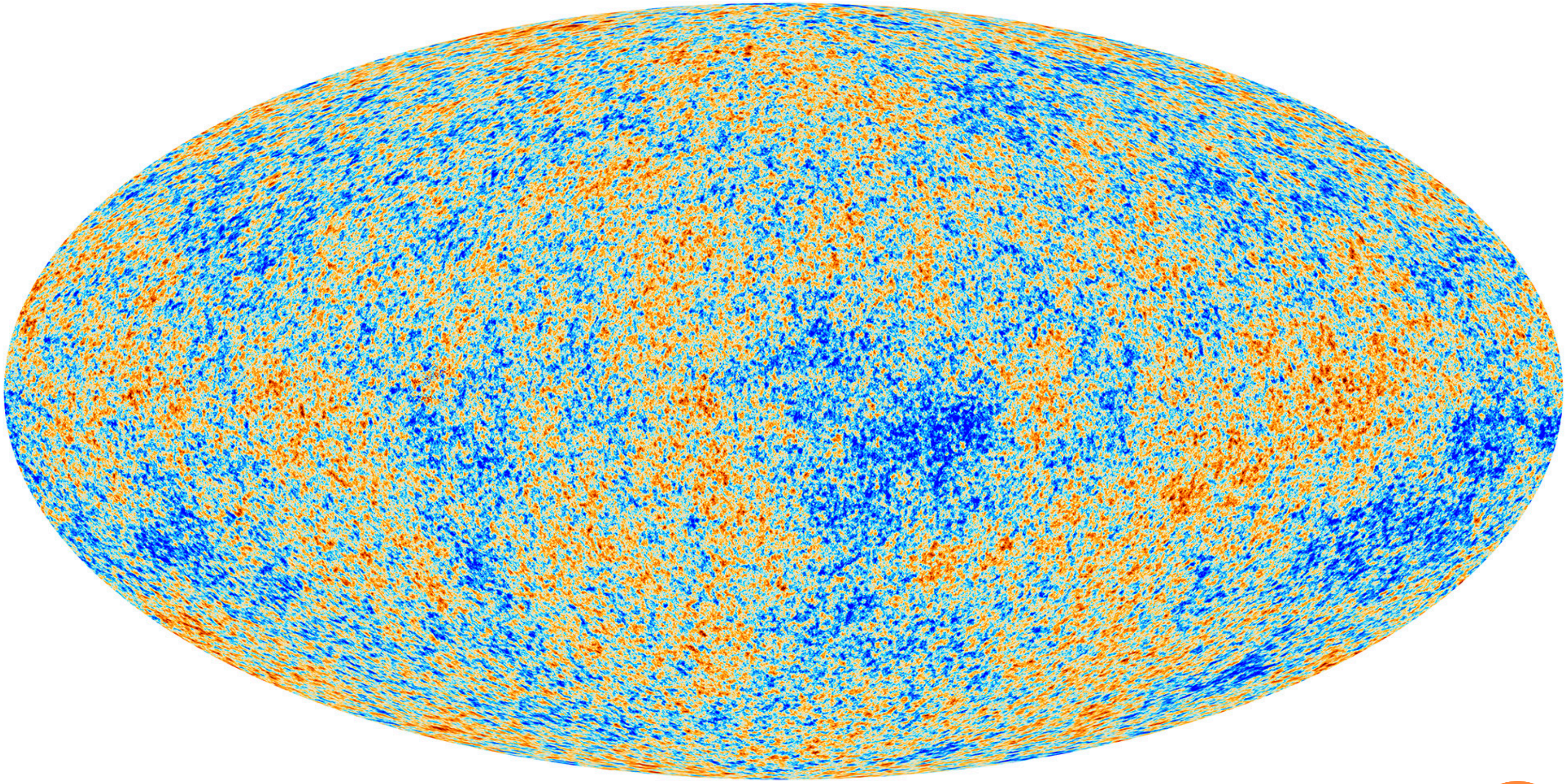
Cosmic Microwave Background



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Our Observable Universe
Cosmological Principle

On the largest scales, the universe is spatially homogeneous & isotropic.

Homogeneity: Universe looks identical at every point in space,
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- The CMB is an almost uniform sea of photons which fills all of space and provides a snapshot of the universe from almost 13 billion years ago. The temperature of the CMB is 2.73 K. However, it's not absolutely uniform. There are small fluctuations:

$$\frac{\delta T}{T_{\mathrm{CMB}}} \sim 10^{-5}$$

  This tiny temperature fluctuations tells us that the early universe was extremely smooth.

- Observations indicate that, although exhibiting clumpiness on smaller scales, the arrangement of galaxies appears to be approximately uniform at distances exceeding $\sim 3 \times 10^8$ light-years.

Geometry of Spacetime

There are three such geometries:

$$ds^2 = -c^2\,dt^2 + a^2(t)\left[\frac{1}{1-kr^2/R^2}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

$$\begin{cases} k = +1 & \text{Spherical} \\ k = 0 & \text{Flat} \\ k = -1 & \text{Hyperbolic} \end{cases}$$

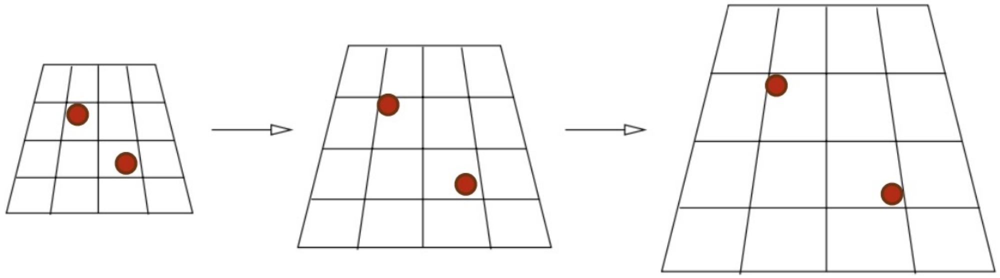
This is known as the **FRW** metric.

Role of the scale factor?

The scale factor a(t) changes the distances over time.

FRW metric is not invariant under Lorentz transformation ⇒ Universe picks out a preferred **rest frame**, described by co-moving coordinates.

The physical distance of the co-moving observers increases with time: $\dot{a} > 0$



The universe is not expanding into anything; rather, the geometry of spacetime is increasing in size without reference to anything external.

Consider a free particle in Cartesian coordinates: $\frac{d^2x^i}{dt^2} = 0$

For Cartesian coordinates the Euclidean metric is: $\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

For some arbitrary coordinates (say, spherical) we replace this by: $g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}.$

What's the analogue of $\frac{d^2x^i}{dt^2} = 0$?

Start with the Lagrangian: $L = \frac{m}{2} g_{ij}(x^k)\dot{x}^i\dot{x}^j$

The Euler-Lagrange Equation is:

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{ab}\frac{dx^a}{dt}\frac{dx^b}{dt}$$

where $\Gamma^i_{ab} = \frac{1}{2} g^{ij}(\partial_a g_{jb} + \partial_b g_{aj} - \partial_j g_{ab})$

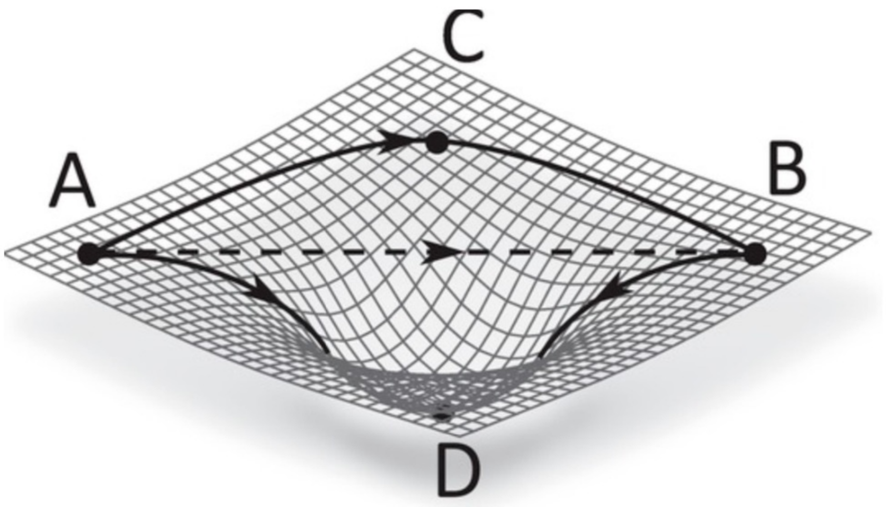
Christoffel Symbol

The EOM of massive particle in GR will take the same form.

Cosmology: A quick Review

The EOM of massive particle in GR will take the same form.

The difference is the RHS cannot be removed even if we go back to Cartesian coordinates.



A free-falling particle in a curved spacetime moves along special trajectories called **geodesics.** For a massive particle, the geodesic is a time like curve $X^\mu(t)$ that extremizes the proper time $\Delta\tau$ between 2 points in the spacetime.

Geodesic Equation:

$$\frac{d^2X^\mu}{d\tau^2} = -\Gamma^\mu_{\alpha\beta}\frac{dX^\alpha}{d\tau}\frac{dX^\beta}{d\tau}$$

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}\, g^{\mu\nu}(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta})$$

Dynamics of spacetime

-Equation of State

-Sources

-Continuity Equation

-Friedmann Equation

Now we want to calculate $a(t)$. For this we will use GR.

namely Einstein's Field Equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Ricci tensor

Ricci scalar

Stress energy tensor

$G_{\mu\nu}$ Einstein tensor

Cosmological constant

We can derive the LHS from the metric

What is $T_{\mu\nu}$?

Once again, we will go back to the **cosmological principle**.

It guides us to model the matter side as a homogeneous and isotropic fluid.

But that sounds strange!

Our Universe does not look like this!

Again, it is about the scale. The lumpy, clumpy looking galaxies is a consequence of the scale.

If we look at it at a much bigger scale the galaxies will look like an atom in a cosmological fluid.

We will treat our Universe as a perfect fluid (homogeneous and isotropic).

Such a perfect fluid can be described by 2 quantities: Energy density ρ and pressure P

For any fluid there is a relation between Energy and Pressure

$$P = w\rho$$

Equation of State

Sources?

- Matter (ordinary and dark) ($w = 0$)
- Radiation $\left(w = \frac{1}{3}\right)$
- Dark Energy ($w = -1$)

Next: Let's explore the Einstein Equation

By using the curvature information of the FRW metric we can write down the Einstein Equation as:

$$H^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\, T_{oo} + \frac{\Lambda\, c^2}{3}$$

$$g_{ij}\,(H^2 + 2\frac{\ddot{a}}{a} + \frac{kc^2}{a^2} - \Lambda c^2) = -\frac{8\pi G}{c^2}\, T_{ij}$$

$$R_{00} = -\frac{3}{c^2}\frac{\ddot{a}}{a}$$

$$R_{0i} = 0$$

$$R_{ij} = \frac{1}{c^2} g_{ij}\left(2H^2 + \frac{\ddot{a}}{a} + 2\frac{kc^2}{a^2}\right)$$

$$R = \frac{6}{a^2}\left(\frac{\ddot{a}}{a} + H^2 + \frac{kc^2}{a^2}\right)$$

The stress energy tensor $T_{\mu\nu}$ for such a fluid can be written as:

$$T_{\mu\nu} = (\rho + P)u_\mu\, u_\nu + \mathrm{P}\, g_{\mu\nu}$$

$u_\mu = \frac{\mathrm{d}x_\mu}{\mathrm{d}\tau}$ is the 4-velocity of the fluid

ρ and P are energy density and pressure in the rest frame.

For a comoving observer

$$T^\mu_\nu = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

Question: How do the energy density and pressure evolve with time?

In flat Minkowski space, energy and momentum are conserved

$$\partial_\mu T^\mu_\nu = 0$$

In GR, the above equation is promoted to the covariant form:

$$\nabla_\mu T^\mu_\nu = 0 \qquad \text{where} \quad \nabla_\mu V^\alpha = \partial_\mu V^\alpha + \Gamma^\alpha_{\mu\beta} V^\mu$$

For $\nu = 0$, we get $\dot{\rho} + 3\,H\,(\rho + P) = 0$ Continuity Equation

Let's get back to the Einstein equations:

$$H^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\,T_{oo} + \frac{\Lambda\,c^2}{3}$$

$$g_{ij}\,(H^2 + 2\frac{\ddot{a}}{a} + \frac{kc^2}{a^2} - \Lambda c^2) = -\frac{8\pi G}{c^2}\,T_{ij}$$

where $T_{oo} = \rho c^2$ and $T_{ij} = g_{ij} P(t)$

We end up with the following

Important Equations:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\,\rho - \frac{kc^2}{a^2} + \frac{\Lambda\, c^2}{3}$$ Friedmann Equation

$$\dot{\rho} + 3\,H\,(\rho + P) = 0$$ Continuity Equation

$$P = w\,\rho$$ Equation of State

To understand the history and fate of our universe we need to solve the Friedmann equation.

To solve the Friedmann equation, we need to identify the fluids present in the universe.

Cosmology: A quick Review

If different fluids have the same equation of state, we can treat them as one. However, if there are fluids with distinct equations of state, we need to include each of them in our analysis.

$$\rho = \sum_w \rho_w \quad \xrightarrow{\text{Each scales as}} \quad \rho_w = \frac{\rho_{w,0}}{a^{3(1+w)}}$$

- Matter ($w = 0$): $\rho_m \sim \frac{1}{a^3}$
- Radiation $\left(w = \frac{1}{3}\right)$: $\rho_r \sim \frac{1}{a^4}$

Significance?

To make life simple

If we restrict attention to a flat $k = 0$ universe with just a single fluid

$$D^2 = \frac{8\pi G \rho_0}{3}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{D^2}{a^{3(1+w)}} \quad \Rightarrow \quad a(t) = \left(\frac{t}{t_0}\right)^{2/(3+3w)}$$

Example:

For a flat and filled with dust-like matter ($w = 0$), we have

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3}$$

Another Example:

Start with the Friedmann Equation:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\,\rho - \frac{k}{a^2} + \frac{\Lambda}{3}$$

Consider no matter content: $\rho = 0$ and $\Lambda > 0$.

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = -\frac{k}{a^2} + \frac{\Lambda}{3}$$

The scale factor:

$$a(t) = \begin{cases} A\cosh\left(\sqrt{\Lambda/3}\,t\right) & k = +1 \\ A\exp\left(\sqrt{\Lambda/3}\,t\right) & k = 0 \\ A\sinh\left(\sqrt{\Lambda/3}\,t\right) & k = -1 \end{cases}$$

This spacetime is known as de Sitter space.

Why Inflation?

Our cosmological model has many problems.

Flatness Problem

Our Universe shows no sign of spatial curvature, $k = 0$

Actually, there is a bound

$$|\Omega_k| < 0.01$$

A Universe with $k = 0$ is an unstable fixed point. Any small curvature in the early Universe will grow over time

⇩

Bound in early Universe was much tighter

Why should the early Universe be so flat?

Horizon Problem

Different parts of the sky were outside each others particle horizon at the time the CMB was formed.

So, what's the problem?

They were not in causal contact

Still have the same exact temperature

Solution?

Inflation

Our early Universe went through a period of accelerated expansion.

Inflation

Our early Universe went through a period of accelerated expansion.

How early?

We don't know precisely how early. The best we can date is before the Electroweak phase transition.

How does an accelerating solution look like?

Basically 2 types:

1) Scale factor is power law $a(t) \sim t^n, n > 1$

2) Scale factor is exponential (de Sitter), $a(t) \sim e^{Ht}$

Why Inflation?

Our cosmological model has many problems.

Flatness Problem

Why should the early Universe be so flat?

Consider $\rho_{inf} \sim \frac{1}{a^{2/n}}$, $a(t) \sim t^n$, $n > 1$

Curvature term $\sim \frac{1}{a^2}$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}$$

ρ_{inf} will dilute slower than the curvature term.

If we have enough inflation, then we can generate whatever flatness we want.

Why Inflation?

Our cosmological model has many problems.

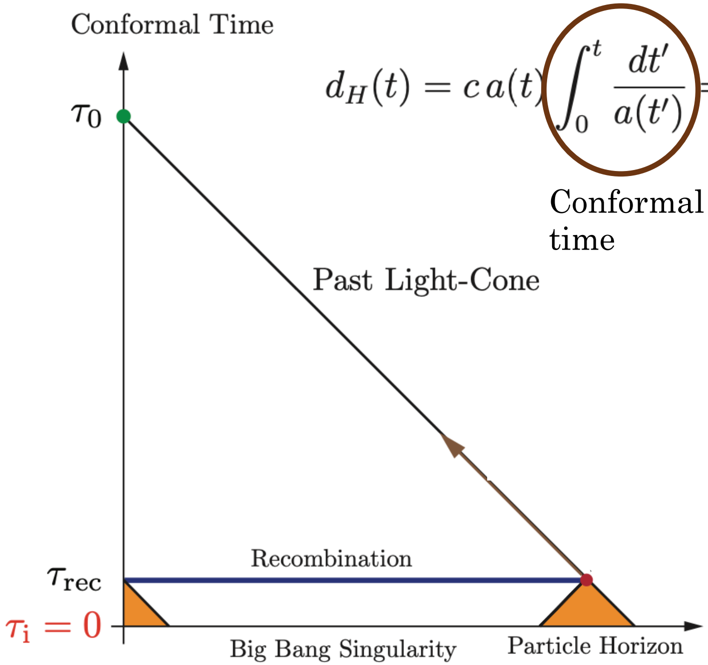
Horizon Problem

Different parts of the sky are outside each others particle horizon, but still have the same exact temperature

For a purely matter-dominated universe $a(t) = (t/t_0)^{2/3}$

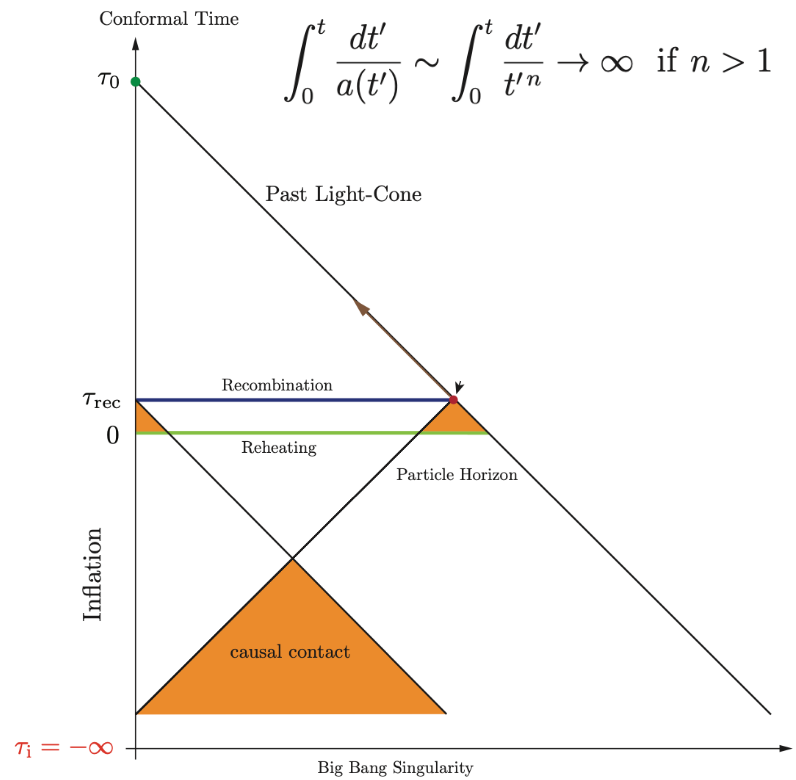
$$d_H(t) = c\,a(t)\int_0^t \frac{dt'}{a(t')} = 3ct$$

Conformal time



For $a(t) \sim t^n,\ n > 1$

$$\int_0^t \frac{dt'}{a(t')} \sim \int_0^t \frac{dt'}{t'^n} \to \infty \quad \text{if } n > 1$$



An early accelerating phase **buys us conformal time** and allows for far flung regions of the early Universe to be in causal contact

Implementing an inflationary phase in the early universe typically involves introducing a new Scalar field known as the inflaton, denoted as $\phi(x,t)$.

The dynamics of this scalar field are best described by the Action:

$$S = \int d^3x\, dt\; a^3(t) \left[\frac{1}{2}\dot{\phi}^2 - \frac{c^2}{2a^2(t)}\nabla\phi\cdot\nabla\phi - V(\phi)\right]$$

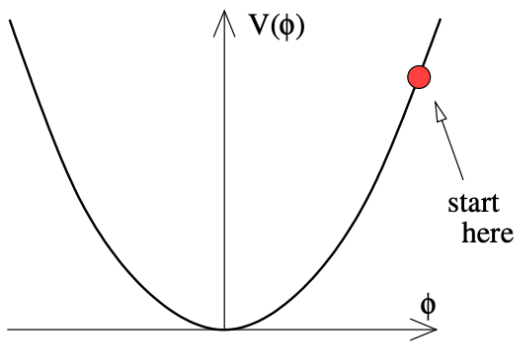
$a^3(t) \to \sqrt{-g}$

EOM (KG): $\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0$

$$H^2 = \frac{8\pi G}{3c^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)$$

Friedmann Equation

energy density



Let's talk about the SETUP

The scalar field sitting high on some potential, with $\dot{\phi}$ small. This will give rise to inflation. As the scalar rolls down the potential, it will pick up kinetic energy and we will exit the inflationary phase.

The presence of the Hubble friction term means that the scalar can ultimately come to rest, rather than eternally oscillating back and forth

But what kind of fluid is Inflaton?

$P = -\rho \;\Rightarrow w = -1$ same as dark energy!

What is the scale factor?

$$H^2 = \frac{8\pi G}{3c^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)$$ We assume that $V(\phi) \gg \frac{1}{2}\dot{\phi}^2$ $\Rightarrow$ $$H^2 \approx \frac{8\pi G}{3c^2} V(\phi)$$

Slow roll condition

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0$$ $\ddot{\phi} \ll H\dot{\phi}.$ $\Rightarrow$ $$3H\dot{\phi} \approx -\frac{\partial V}{\partial \phi}$$

Inflation needs to last long enough: No rapid gain in speed

For $V = \frac{1}{2}m^2\phi^2$ $\quad H = \alpha\phi$ and $\dot{\phi} = -\frac{m^2}{3\alpha}$ with $\alpha^2 = \frac{4\pi G m^2}{3c^2}$

$\Downarrow$ Integrating

$$\phi(t) = \phi_0 - \frac{m^2}{3\alpha}t$$

where we have taken the scalar field to start at some initial value ϕ_0 at $t = 0$.

Then scale factor

$$a(t) = a(0)\,\exp\left[\frac{2\pi G}{c^2}(\phi_0^2 - \phi(t)^2)\right]$$

A quasi de Sitter phase

End of Review