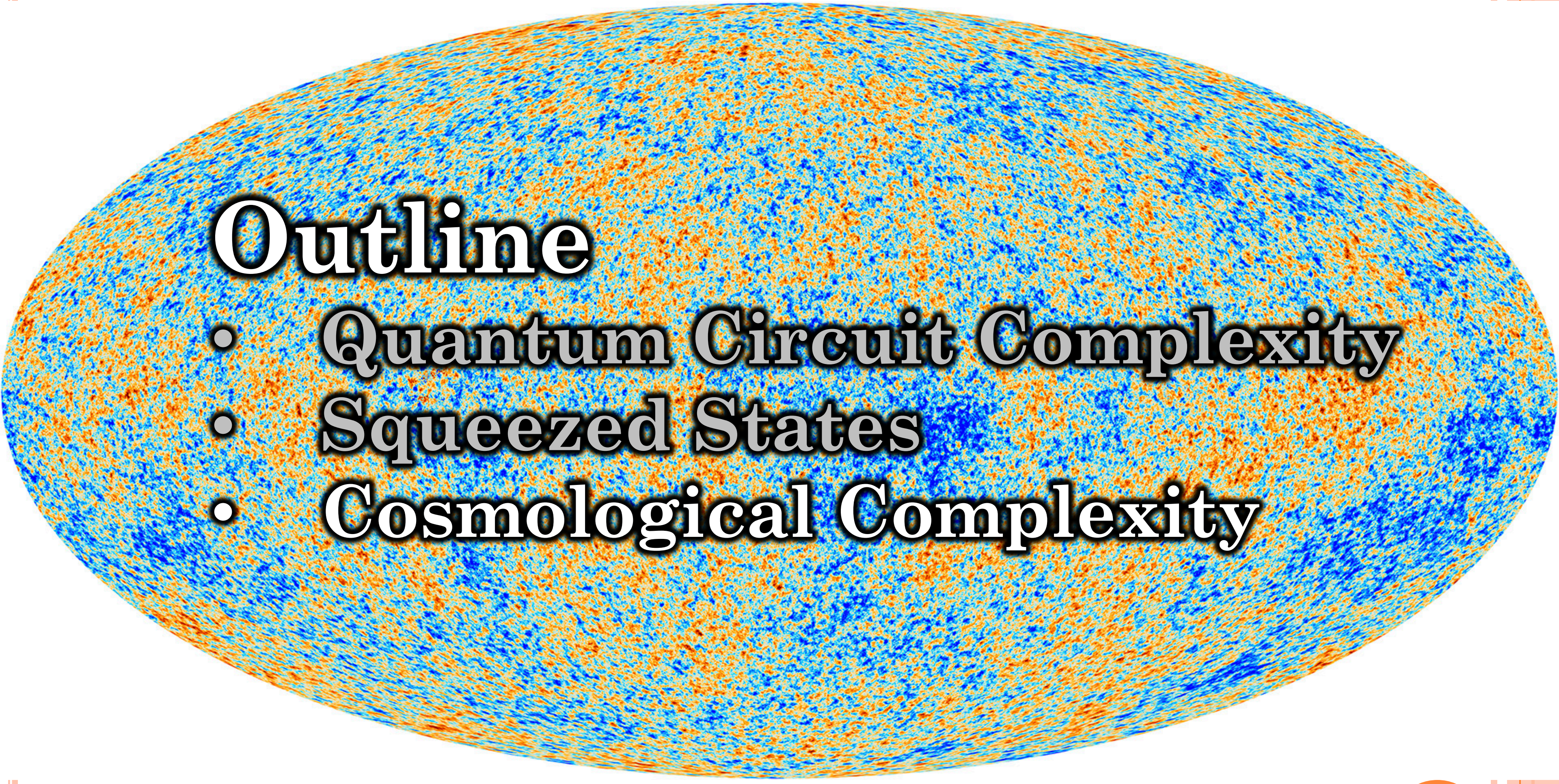
Lecture 4



Outline

- Quantum Circuit Complexity
- Squeezed States
- Cosmological Complexity

ESA and Planck Collaboration

- Growth of complexity with time?
- Bounds on the growth of complexity?
- Complexity for decohered state?

Lecture 4

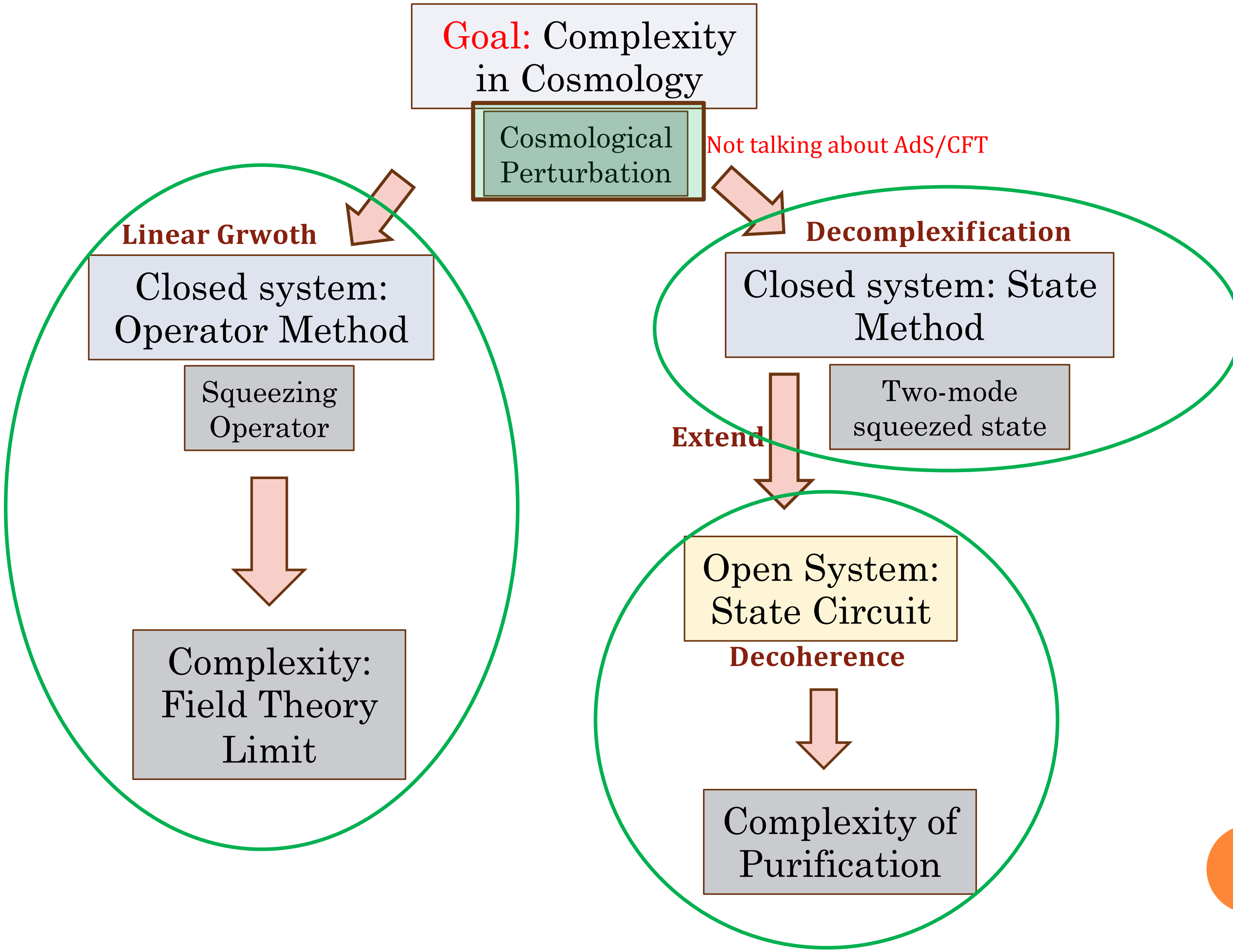
Last time/Lecture 2

- Examples: Displacement operator, Harmonic Oscillator, Free field Theory
- What is Squeezed States?
- Complexity of Purification

Today

- Cosmological Perturbation Model
- Cosmological Squeezed states
- Operator Complexity for cosmological perturbation
- Sate Complexity for cosmological perturbation
- Open System: Complexity of Purification for Cosmological Perturbation

OUTLINE


Goal: Complexity in Cosmology
Cosmological Perturbation
Not talking about AdS/CFT
Linear Grwoth
Closed system: Operator Method
Squeezing Operator
Complexity: Field Theory Limit
Decomplexification
Closed system: State Method
Two-mode squeezed state
Extend
Open System: State Circuit
Decoherence
Complexity of Purification

COSMOLOGICAL PERTURBATION MODEL

To get the Action for linearized cosmological perturbations → We expand the action to quadratic order in the fluctuating d.o.f.

The **linear terms cancel** because the background is taken to satisfy the background equations of motion.

The Einstein-Hilbert action for gravity and the action of a scalar matter field

Mukhanov, Brandenberger

$$S = \int d^4x \sqrt{-g}[-\frac{1}{16\pi G}R + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)]$$

The simplest way to proceed is to work in a fixed gauge, longitudinal gauge

The metric & matter take the form

$$\begin{aligned} ds^2 &= a^2(\eta)\left[(1+2\phi(\eta,\mathbf{x}))d\eta^2 - (1-2\psi(t,\mathbf{x}))d\mathbf{x}^2\right] \\ \varphi(\eta,\mathbf{x}) &= \varphi_0(\eta) + \delta\varphi(\eta,\mathbf{x})\,. \end{aligned}$$

Ansatz

- The off-diagonal spatial Einstein equations force $\psi = \phi$ since $\delta T_{ij} = 0$ for scalar field (no anisotropic stresses to linear order).

- The two remaining fluctuating variables $\delta\varphi$ and ϕ must be linked by the Einstein constraint equations since there cannot be matter fluctuations without induced metric fluctuations.

Calculation:

- insert the ansatz into the action,
- expand the result to second order in the fluctuating fields,
- make use of the background and the constraint equations, and dropping total derivative terms from the action

Then the action quadratic in the perturbations:

$$S^{(2)} = \frac{1}{2}\int d^4x\left[v'^2 - v_{,i}v_{,i} + \frac{z''}{z}v^2\right]$$

Mukhanov variable $\quad v = a\left[\delta\varphi + \frac{\varphi_0'}{\mathcal{H}}\phi\right], \qquad \mathcal{H} = a'/a, \qquad z = \frac{a\varphi_0'}{\mathcal{H}} \quad \Rightarrow \quad z(\eta) \sim a(\eta)\,.$

Note: $v = z\,\mathcal{R}$ (curvature perturbation $\mathcal{R} = \psi + \frac{H}{\dot{\varphi}_0}\,\delta\varphi$)

Cosmological Perturbation Model

The action : $S^{(2)} = \frac{1}{2}\int d^4x\left[v'^2 - v_{,i}v_{,i} + \frac{z''}{z}v^2\right]$

k^{th} Fourier Mode

The EOM in momentum space $\quad v_k'' + k^2 v_k - \frac{z''}{z} v_k = 0$

Here $\quad \frac{z''}{z} \simeq H^2$

Small length scales

$k \gg H$

⇒ v_k constant amplitude oscillations

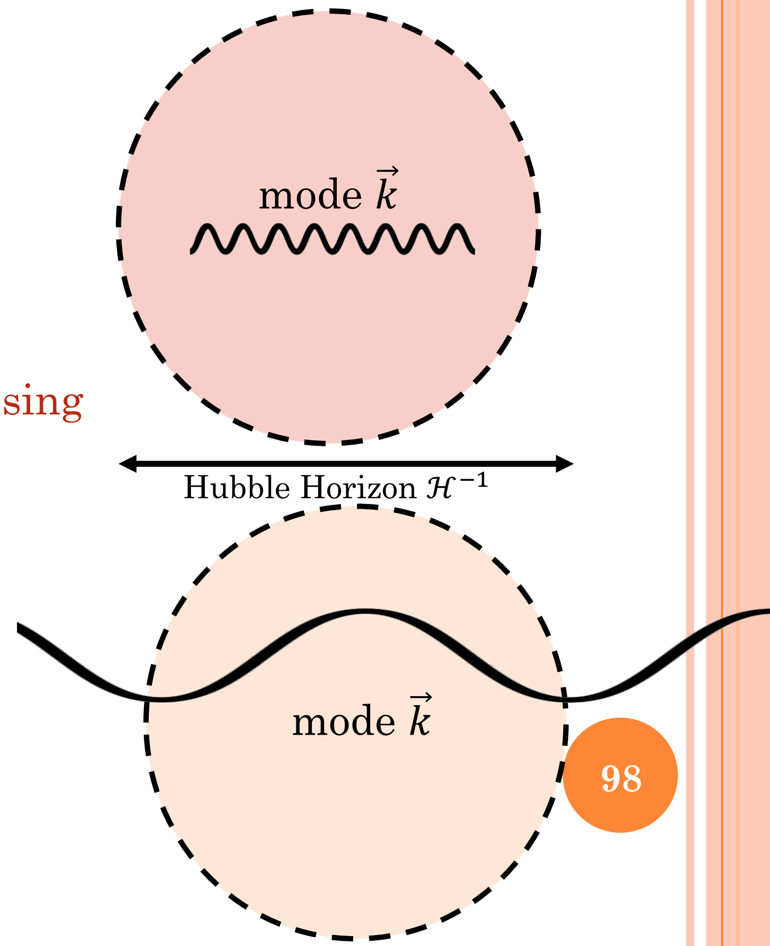
Oscillations freeze out at Hubble radius crossing

$k = H$

Longer length scales

$k \ll H$

⇒ solution goes as: $v_k \sim z$



Cosmological Squeezed States

Background metric – Cosmology

Brandenberger, Mukhanov

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2 = a(\eta)^2\left(-d\eta^2 + d\vec{x}^2\right)$$

equation of state $a(\eta) \sim \left(\frac{\eta}{\eta_0}\right)^{2/(1+3w)}$

Cosmological Perturbations

$$ds^2 = a(\eta)^2\left(-(1+2\psi(x,\eta))d\eta^2 + (1-2\psi(x,\eta))d\vec{x}^2\right)$$

Scalar Field

$$\varphi(x) = \varphi(\eta) + \delta\varphi(\eta, x)$$

Action: $S = \frac{1}{2}\int d\eta\, d^3x \left[v'^2 - (\partial_i v)^2 + \frac{z''}{z}v^2\right]$

Hamiltonian: $\hat{H}_{\rm cosmo} = \int d^3k\, \hat{\mathcal{H}}_{\vec{k}} = \int d^3k \left[k\left(\hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}} + \hat{a}_{-\vec{k}}\hat{a}^\dagger_{-\vec{k}}\right) - i\frac{z'}{z}\left(\hat{a}_{\vec{k}}\hat{a}_{-\vec{k}} - \hat{a}^\dagger_{\vec{k}}\hat{a}^\dagger_{-\vec{k}}\right)\right]$

Free-particle

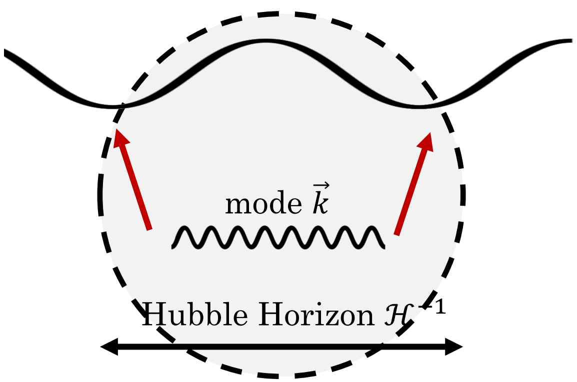
Inverted Oscillator

Dominates when mode outside horizon $k \ll \mathcal{H}$

Time-dependent frequency

Two-mode squeezed state $(\vec{k}, -\vec{k})$

Grishchuk, Sidorov
Albrecht, Ferreira, Joyce, Prokopec



Accelerating background
Stretches modes outside
horizon (horizon-exit)

I will discuss de Sitter, $w = -1$

Cosmological Complexity

The complexity of quantum cosmological perturbations

$$|\psi_T\rangle = \hat{\mathcal{U}}_{\text{cosmo}}\,|\psi_R\rangle$$

Squeezed Cosmological Perturbations $\quad |r_k, \phi_k, \theta_k\rangle = \hat{\mathcal{S}}(r_k, \phi_k)\,\hat{\mathcal{R}}(\theta_k)\,\left|0_{\vec{k}}, 0_{-\vec{k}}\right\rangle$

Operator Circuit Complexity

SH, Jana, Underwood

$$\hat{\mathcal{U}}_{\text{target}} = \overleftarrow{P}\exp\left[\int_0^1 V^I(s)\,\hat{\mathcal{O}}_I\,ds\right]$$

$$\hat{\mathcal{S}} = \exp\left[\frac{r_k}{2}\left(e^{-2i\phi_k}\,\hat{a}_{\vec{k}}\hat{a}_{-\vec{k}} - e^{2i\phi_k}\hat{a}^\dagger_{\vec{k}}\hat{a}^\dagger_{-\vec{k}}\right)\right]$$ **squeezing operator**

$$\hat{\mathcal{R}} = \exp\left[-i\theta_k\left(\hat{a}_{\vec{k}}\,\hat{a}^\dagger_{\vec{k}} + \hat{a}^\dagger_{-\vec{k}}\hat{a}_{-\vec{k}}\right)\right]$$ **rotation operator**

- Characterize gates by structure constants

$$\left[\hat{\mathcal{O}}_I, \hat{\mathcal{O}}_J\right] = i\,f_{IJ}^K\hat{\mathcal{O}}_K$$

$$\left.\begin{aligned}
\hat{\mathcal{O}}_1 &= \frac{\hat{a}_{\vec{k}}\,\hat{a}_{-\vec{k}} + \hat{a}^\dagger_{\vec{k}}\hat{a}^\dagger_{-\vec{k}}}{2}\\
\hat{\mathcal{O}}_2 &= i\,\frac{\hat{a}_{\vec{k}}\,\hat{a}_{-\vec{k}} - \hat{a}^\dagger_{\vec{k}}\hat{a}^\dagger_{-\vec{k}}}{2}\\
\hat{\mathcal{O}}_3 &= \frac{\hat{a}_{\vec{k}}\,\hat{a}^\dagger_{\vec{k}} + \hat{a}^\dagger_{-\vec{k}}\hat{a}_{-\vec{k}}}{2}
\end{aligned}\right\}\ \text{SU(1,1)}$$

Two-mode Operators can be generated from this fundamental operators.

- Minimization:
  ⇒ Euler-Arnold eq on group manifold

$$G_{IJ}\frac{dV^J}{ds} = f_{IJ}^K\,G_{KL}\,V^J\,V^L$$

$V^I(s)$: tangent vectors

⇒ solve for $V^J(s)$, construct $\mathcal{U}_{\text{target}}$

- Operator Circuit Complexity

$$\mathcal{C}^{(\text{o})} = \sqrt{G_{IJ}V^IV^J}$$

Operator Circuit Complexity

Geodesic Equations

$$\frac{dV^1}{ds} = -2V^2V^3; \quad \frac{dV^2}{ds} = 2V^1V^3; \quad \frac{dV^3}{ds} = 0.$$

$\Rightarrow$

$$V^1(s) = v_1\cos(2v_3s) - v_2\sin(2v_3s)$$
$$V^2(s) = v_1\sin(2v_3s) + v_2\cos(2v_3s)$$
$$V^3(s) = v_3,$$

The resulting circuit complexity along this minimal path is then simply

$$\mathcal{C}_{\rm target} = \min_{\{V^I\}}\mathcal{D}\left[V^I\right] = \min_{\{V^I\}}\int_0^1\sqrt{G_{IJ}V^IV^J}\,ds = \sqrt{v_1^2+v_2^2+v_3^2}\,.$$

Now let's construct the $U(s)$

Target operator is the $s = 1$ boundary condition of the s dependent unitary operator

$$\hat{U}(s) = \mathcal{P}\exp\left[-i\int_0^s V^I(s')\hat{e}_I\,ds'\right]$$

We use this representation

$$\hat{e}_1 = \frac{1}{2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\ \hat{e}_2 = \frac{1}{2}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix},\ \hat{e}_3 = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

A general element of SU(1,1)

$$\hat{U}(s) = \begin{pmatrix}q(s) & p(s)^*\\ p(s) & q(s)^*\end{pmatrix}$$

$$|q|^2 - |p|^2 = 1\,.$$

Operator Circuit Complexity

with this group element and the sols of the E A equations in

$$\frac{d\hat{U}(s)}{ds} = -iV^I(s)\,\hat{e}_I\,\hat{U}(s)$$

solved by the parametrization

$$\begin{cases} q(s) = e^{-iv_3 s}(c_1 e^{\lambda s/2} + c_2 e^{-\lambda s/2})\,; \\ p(s) = \frac{v_2 - iv_1}{v^2} e^{iv_3 s}\left(c_1(\lambda - iv_3)e^{\lambda s/2} - c_2(\lambda + v_3)e^{-\lambda s/2}\right) \end{cases}$$

$$v_1^2 + v_2^2 = v^2$$
$$\lambda = \sqrt{v^2 - v_3^2}$$

Then

$$\hat{U}(s) = \begin{pmatrix} q(s) & p(s)^* \\ p(s) & q(s)^* \end{pmatrix}$$

$$s = 0,\ \hat{U}(s=0) = \hat{\mathbb{1}}$$

$$\hat{U}(s) = \begin{pmatrix} e^{-iv_3 s}\left(\cosh\left(\frac{\lambda s}{2}\right) + i\frac{v_3}{\lambda}\sinh\left(\frac{\lambda s}{2}\right)\right) & e^{-iv_3 s}\frac{(v_2+iv_1)}{\lambda}\sinh\left(\frac{\lambda s}{2}\right) \\ e^{iv_3 s}\frac{(v_2-iv_1)}{\lambda}\sinh\left(\frac{\lambda s}{2}\right) & e^{iv_3 s}\left(\cosh\left(\frac{\lambda s}{2}\right) - i\frac{v_3}{\lambda}\sinh\left(\frac{\lambda s}{2}\right)\right) \end{pmatrix}$$

We will determine the constants v_i by applying the boundary conditions:

$$s = 1,\ \hat{U}(s=1) = \hat{\mathcal{U}}_{\text{target}}; \qquad \hat{\mathcal{U}}_{\text{target}} = \begin{pmatrix} e^{-i\theta}\cosh r & e^{i(2\phi+\theta)}\sinh r \\ e^{-i(2\phi+\theta)}\sinh r & e^{i\theta}\cosh r \end{pmatrix}$$

Cosmological Squeezed States

The complexity of quantum cosmological perturbations

$$|\psi_T\rangle = \hat{\mathcal{U}}_{\text{cosmo}} \, |\psi_R\rangle$$

Squeezed Cosmological Perturbations

$$|r_k, \phi_k, \theta_k\rangle = \hat{\mathcal{S}}(r_k, \phi_k)\, \hat{\mathcal{R}}(\theta_k)\, |0_{\vec{k}}, 0_{-\vec{k}}\rangle$$

$$\hat{\mathcal{S}} = \exp\left[\frac{r_k}{2}\left(e^{-2i\phi_k}\, \hat{a}_{\vec{k}}\hat{a}_{-\vec{k}} - e^{2i\phi_k}\hat{a}^\dagger_{\vec{k}}\hat{a}^\dagger_{-\vec{k}}\right)\right]$$ **squeezing operator**

$$\hat{\mathcal{R}} = \exp\left[-i\theta_k\left(\hat{a}_{\vec{k}}\, \hat{a}^\dagger_{\vec{k}} + \hat{a}^\dagger_{-\vec{k}}\hat{a}_{-\vec{k}}\right)\right]$$ **rotation operator**

Complexity?

- Unsqueezed limit $r_k \to 0$ is the Minkowski vacuum $|0\rangle_{\vec{k},-\vec{k}}$ (Bunch Davies)
- EOM for r_k, ϕ_k, θ_k come from Hamiltonian

$$\hat{H}_{\vec{k}} = k\left(\hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}} + \hat{a}_{-\vec{k}}\hat{a}^\dagger_{-\vec{k}}\right) - i\frac{z'}{z}\left(\hat{a}_{\vec{k}}\hat{a}_{-\vec{k}} - \hat{a}^\dagger_{\vec{k}}\hat{a}^\dagger_{-\vec{k}}\right)$$

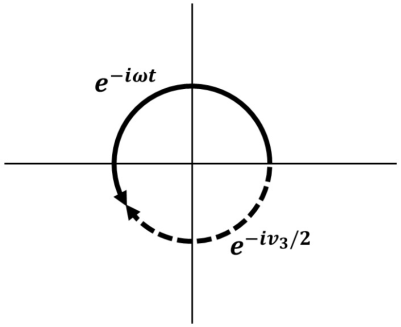
Complexity

$$\mathcal{C}_{\vec{k}}(\eta) = \sqrt{4 r_k(\eta)^2 + v_3(\eta)^2}$$

where v_3 is given in terms of θ_k

$$|v_3| = \begin{cases} 2\theta_{\min} & \text{for } r \ll 1 \\ \theta_{\min} & \text{for } r \gg 1 \end{cases}$$

$$\theta_{\min} = \begin{cases} \theta - 2\pi n & \text{for } 2\pi n < \theta < \pi(2n+1) \\ 2\pi n - \theta & \text{for } \pi(2n-1) < \theta < 2\pi n \end{cases}$$



Time Dependence?

Cosmological Squeezed States

$$|\psi_T\rangle = \hat{\mathcal{U}}_{\text{cosmo}} \, |\psi_R\rangle$$

Bhattacharyya, Das, **SSH,** Underwood

Inside Horizon:

$\frac{k}{\mathcal{H}} \gg 1$

- r_k is fixed
- ϕ_k, θ_k evolve

Two-mode Squeezed States

$$|r_k, \phi_k, \theta_k\rangle = \hat{\mathcal{S}}(r_k, \phi_k)\hat{\mathcal{R}}(\theta_k)|0\rangle_{\vec{k}}$$

- Unsqueezed limit $r_k \to 0$ is the Minkowski vacuum $|0\rangle_{\vec{k},-\vec{k}}$ (Bunch Davies)
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Outside Horizon:

$\frac{k}{\mathcal{H}} \ll 1$

- $r_k \sim \ln a$
- ϕ_k, θ_k fixed

$$\hat{H}_{\vec{k}} = k\left(\hat{a}^{\dagger}_{\vec{k}}\hat{a}_{\vec{k}} + \hat{a}_{-\vec{k}}\hat{a}^{\dagger}_{-\vec{k}}\right) - i\frac{z'}{z}\left(\hat{a}_{\vec{k}}\hat{a}_{-\vec{k}} - \hat{a}^{\dagger}_{\vec{k}}\hat{a}^{\dagger}_{-\vec{k}}\right)$$

Heisenberg Equations

$$\begin{aligned}
\frac{dr_k}{d\eta} &= -\frac{z'}{z}\cos(2\phi_k)\,; \\
\frac{d\phi_k}{d\eta} &= k + \frac{z'}{z}\coth(2r_k)\sin(2\phi_k)\,; \\
\frac{d\theta_k}{d\eta} &= k - \frac{z'}{z}\tanh(r_k)\sin(2\phi_k)\,.
\end{aligned}$$

Cosmological Squeezed States

$$|\psi_T\rangle = \hat{U}_{\text{cosmo}}\,|\psi_R\rangle$$

Bhattacharyya, Das, **SSH,** Underwood

Two-mode Squeezed States

$$|r_k,\phi_k,\theta_k\rangle = \hat{\mathcal{S}}(r_k,\phi_k)\hat{\mathcal{R}}(\theta_k)|0\rangle_{\vec{k}}$$

Inside Horizon:

$\frac{k}{\mathcal{H}} \gg 1$

- r_k is fixed
- ϕ_k, θ_k evolve

- Unsqueezed limit $r_k \to 0$ is the Minkowski vacuum $|0\rangle_{\vec{k},-\vec{k}}$ (Bunch Davies)
- EOM for r_k, ϕ_k, θ_k come from Hamiltonian

Outside Horizon:

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\frac{d\theta_k}{d\eta} &= k - \frac{z'}{z}\tanh(r_k)\sin(2\phi_k)\,.
\end{aligned}$$

Exact Solutions for de Sitter

$$\begin{aligned}
r_k &= \sinh^{-1}\left(\frac{1}{2|k\eta|}\right) = \sinh^{-1}\left(\frac{a}{2kH_{dS}}\right)\\
\phi_k &= -\frac{\pi}{4} - \frac{1}{2}\tan^{-1}\left(\frac{1}{2|k\eta|}\right) = -\frac{\pi}{4} - \frac{1}{2}\tan^{-1}\left(\frac{a}{2kH_{dS}}\right)\\
\theta_k &= |k\eta| - \tan^{-1}\left(\frac{1}{2|k\eta|}\right) = \frac{kH_{dS}}{a} - \tan^{-1}\left(\frac{a}{2kH_{dS}}\right)
\end{aligned}$$

Cosmological Squeezed States

$$|\psi_T\rangle = \hat{\mathcal{U}}_{\text{cosmo}}\,|\psi_R\rangle$$

Bhattacharyya, Das, **SSH,** Underwood

Two-mode Squeezed States

$$|r_k, \phi_k, \theta_k\rangle = \hat{\mathcal{S}}(r_k, \phi_k)\hat{\mathcal{R}}(\theta_k)|0\rangle_{\vec{k}}$$

- Unsqueezed limit $r_k \to 0$ is the Minkowski vacuum $|0\rangle_{\vec{k},-\vec{k}}$ (Bunch Davies)
- EOM for r_k, ϕ_k, θ_k come from Hamiltonian

<u>Inside Horizon</u>:

$\frac{k}{\mathcal{H}} \gg 1$

- r_k is fixed
- ϕ_k, θ_k evolve

<u>Outside Horizon</u>:

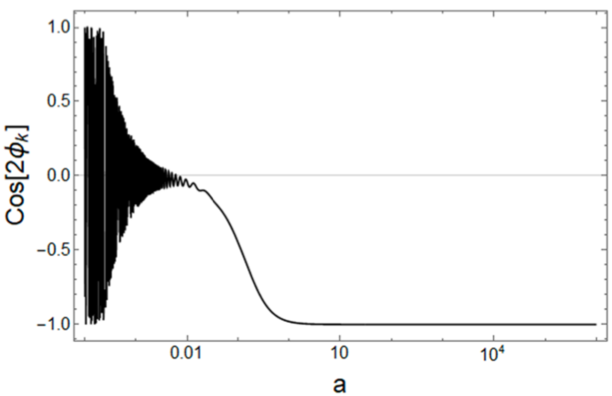
$\frac{k}{\mathcal{H}} \ll 1$

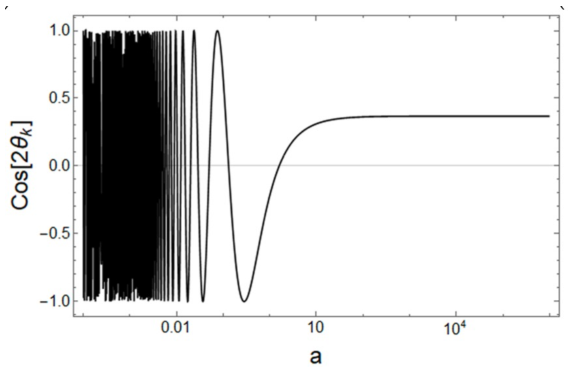
- $r_k \sim \ln a$
- ϕ_k, θ_k fixed

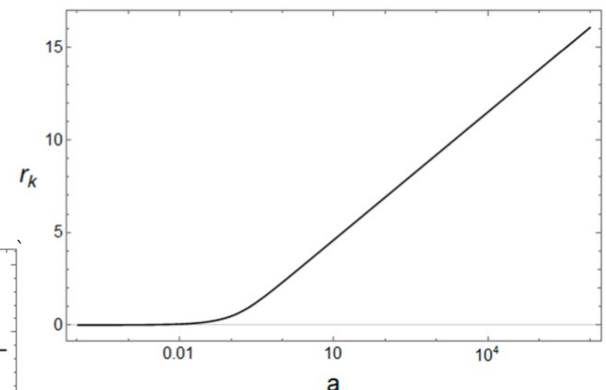
$$\hat{H}_{\vec{k}} = k\left(\hat{a}^{\dagger}_{\vec{k}}\hat{a}_{\vec{k}} + \hat{a}_{-\vec{k}}\hat{a}^{\dagger}_{-\vec{k}}\right) - i\frac{z'}{z}\left(\hat{a}_{\vec{k}}\hat{a}_{-\vec{k}} - \hat{a}^{\dagger}_{\vec{k}}\hat{a}^{\dagger}_{-\vec{k}}\right)$$

<u>When Mode exits the horizon squeezing angle freezes</u>

Squeezing begins to grow when mode exit the horizon







Cosmological Perturbations:

Complexity $\mathcal{C}_{\vec{k}}(\eta) = \sqrt{4r_k(\eta)^2 + v_3(\eta)^2}$

IR (Super-horizon modes) Limit

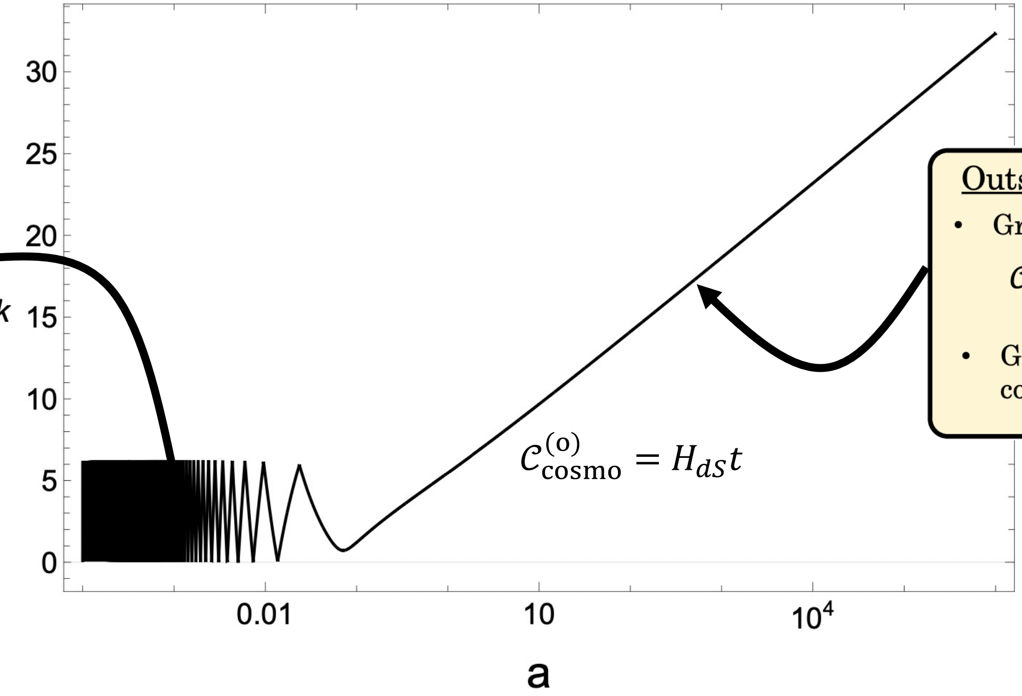
$$\mathcal{C}_{\vec{k}}(\eta) \sim 2r_k \sim 2\ln\left(\frac{1}{2|k\eta|}\right) \sim 2\ln\left(a(\eta)/a_e\right) \sim 2N_e$$

Scale factor at the horizon-exit

No of e-folds after the horizon-exit

Inside Horizon:

- $r_k \ll 1$
- Complexity oscillates $\mathcal{C}^{(o)}_{\text{cosmo}} \leq 2\pi$ due to rotation phase θ



Outside Horizon:

- Grows with squeezing, e-folds $\mathcal{C}^{(o)}_{\text{cosmo}} \approx r_k \approx \ln\left(\frac{a}{a_{\text{exit}}}\right)$
- Growth is linear in cosmic time t, $a(t) = e^{H_{dS}t}$

COSMOLOGICAL PERTURBATIONS: FIELD THEORY LIMIT

$$\mathcal{C}_{\rm op}^{\rm tot} \to L^{3/2}\sqrt{\int^{\Lambda}\left(4r_k(\eta)^2 + v_3(\eta)^2\right) d^3k}\,.$$

IR limit $|k\eta| \ll 1$, ⇩ UV limit $|k\eta| \gg 1$

$$\mathcal{C}_{\rm op}^{\rm tot} \sim L^{3/2}\sqrt{\int_{IR} 4r_k(\eta)^2\, d^3k + \int_{UV} v_3(\eta)^2\, d^3k} \sim L^{3/2}\sqrt{\int_0^{|\eta|^{-1}} k^2 \ln^2\left(\frac{1}{|k\eta|}\right) dk + \int_{|\eta|^{-1}}^{\Lambda} k^2 dk}$$

$$\sim L^{3/2}\sqrt{\frac{\alpha_1}{|\eta|^3} + \alpha_2\Lambda^3},$$

upto some $\mathcal{O}(1)$ factors absorbed in α_i constants

Proportional to growth of the de Sitter volume $\sim H_{ds}^3 a(\eta)^3$

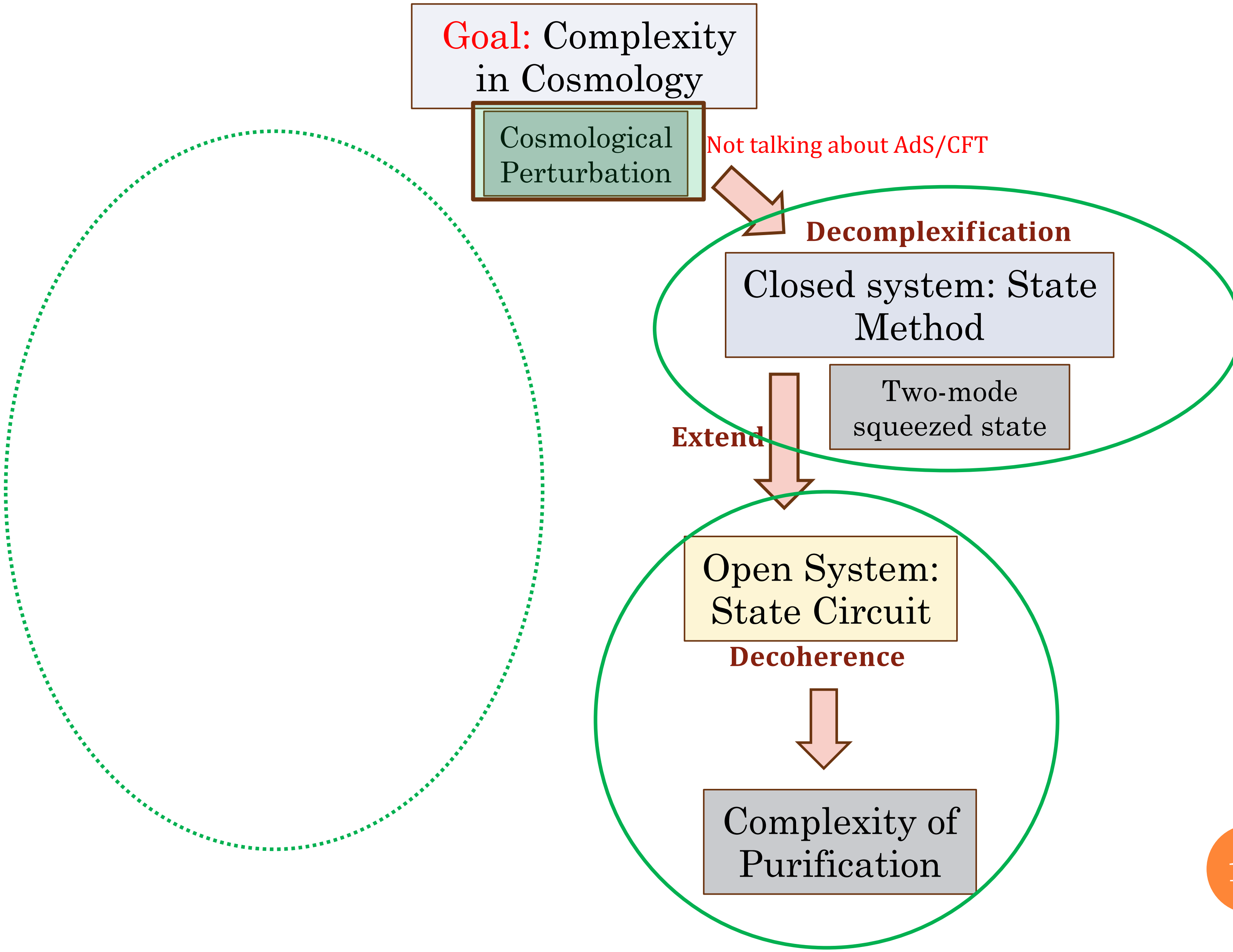
Even for Planck-scale UV cut-off and 60 e-folds, IR terms dominates for Hubble scale down to $H_{ds} > 1$ keV

During inflation, complexity of the universe grows by a factor

$$\frac{\mathcal{C}^{\rm tot}(\eta_f)}{\mathcal{C}^{\rm tot}(\eta_i)} \sim \left(\frac{a_f}{a_i}\right)^{3/2} \sim e^{3N_e/2} \sim e^{90} \sim 10^{39} \text{ for } N_e \sim 60 \text{ e-folds of inflation}$$

<u>Now we will consider the (Gaussian) State Circuit Complexity</u>

Outline



Goal: Complexity in Cosmology
Cosmological Perturbation
Not talking about AdS/CFT
Decomplexification
Closed system: State Method
Two-mode squeezed state
Extend
Open System: State Circuit
Decoherence
Complexity of Purification

Cosmological Complexity

(Gaussian) State Circuit Complexity

the complexity of quantum cosmological perturbations $|\psi_T\rangle = \hat{\mathcal{U}}_{\text{cosmo}}\, |\psi_R\rangle$

Bhattacharyya, Das, **SH**, Underwood

Squeezed Cosmological Perturbations $|r_k, \phi_k, \theta_k\rangle = \hat{\mathcal{S}}(r_k, \phi_k)\, \hat{\mathcal{R}}(\theta_k)\, |0_{\vec{k}}, 0_{-\vec{k}}\rangle$

Reference State
vacuum

$$\psi_R(q_{\vec{k}}, q_{-\vec{k}}) = \langle q_{\vec{k}}, q_{-\vec{k}} | 0 \rangle_{\vec{k}, -\vec{k}} = \left(\frac{k}{\pi}\right)^{1/4} e^{-\frac{k}{2}(q_{\vec{k}}^2 + q_{-\vec{k}}^2)}$$

Target State
cosmological squeezed state

$$\Psi_{sq}(q_{\vec{k}}, q_{-\vec{k}}) = \langle q_{\vec{k}}, q_{-\vec{k}} | \Psi_{sq} \rangle_{\vec{k}} = \frac{e^{A(q_{\vec{k}}^2 + q_{-\vec{k}}^2) - B q_{\vec{k}} q_{-\vec{k}}}}{\cosh r_k \sqrt{\pi} \sqrt{1 - e^{-4i\phi_k} \tanh^2 r_k}}$$

$$A = \frac{k}{2}\left(\frac{e^{-4i\phi_k} \tanh^2 r_k + 1}{e^{-4i\phi_k} \tanh^2 r_k - 1}\right), \qquad B = 2k\left(\frac{e^{-2i\phi_k} \tanh r_k}{e^{-4i\phi_k} \tanh^2 r_k - 1}\right)$$

$$\mathcal{C}_2(k) = \frac{1}{\sqrt{2}} \sqrt{\left(\ln\left|\frac{1 + e^{-2i\phi_k} \tanh r_k}{1 - e^{-2i\phi_k} \tanh r_k}\right|\right)^2 + \left(\tan^{-1}\left(2 \sin 2\phi_k \sinh r_k \cosh r_k\right)\right)^2}$$

We have

$$\frac{dr_k}{d\eta} = -\frac{z'}{z} \cos(2\phi_k)\,;$$

$$\frac{d\phi_k}{d\eta} = k + \frac{z'}{z} \coth(2r_k) \sin(2\phi_k)\,;$$

Cosmological Complexity

(Gaussian) State Circuit Complexity

the complexity of quantum cosmological perturbations

$$|\psi_T\rangle = \hat{\mathcal{U}}_{\text{cosmo}}\,|\psi_R\rangle$$

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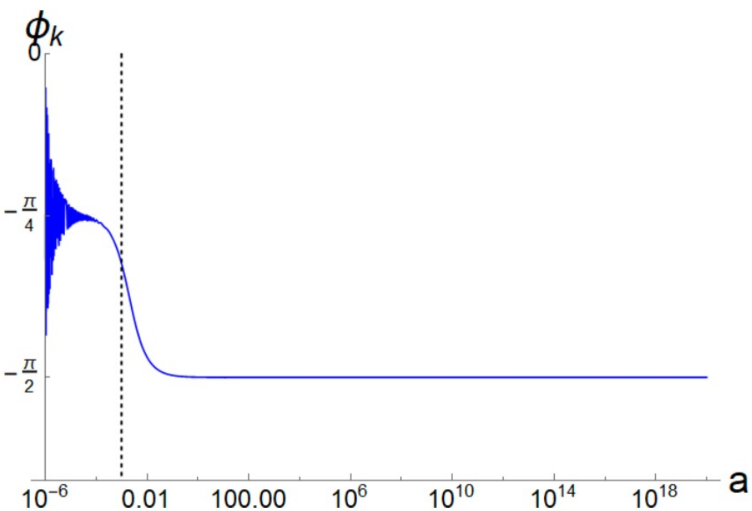
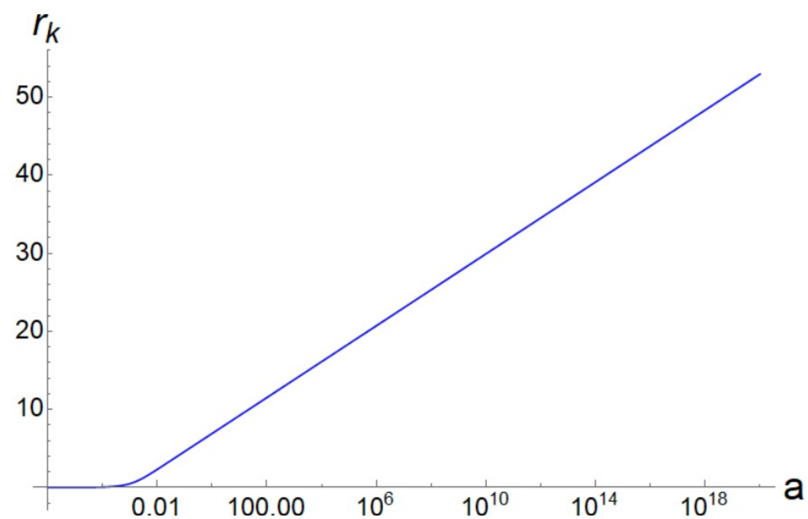
Squeezed Cosmological Perturbations

$$|r_k, \phi_k, \theta_k\rangle = \hat{\mathcal{S}}(r_k, \phi_k)\,\hat{\mathcal{R}}(\theta_k)\,\left|0_{\vec{k}}, 0_{-\vec{k}}\right\rangle$$

$$\mathcal{C}_2(k) = \frac{1}{\sqrt{2}}\sqrt{\left(\ln\left|\frac{1+e^{-2i\phi_k}\tanh r_k}{1-e^{-2i\phi_k}\tanh r_k}\right|\right)^2 + \left(\tan^{-1}\left(2\sin 2\phi_k \sinh r_k \cosh r_k\right)\right)^2}$$





on super-horizon scales we expect

$$\mathcal{C} \approx \frac{1}{\sqrt{2}}\left|\ln\left|\frac{1+e^{-2i\phi_k}\ \tanh r_k}{1-e^{-2i\phi_k}\ \tanh r_k}\right|\right| \approx \frac{1}{\sqrt{2}}\ln\left(\frac{a}{a_{exit}}\right)$$

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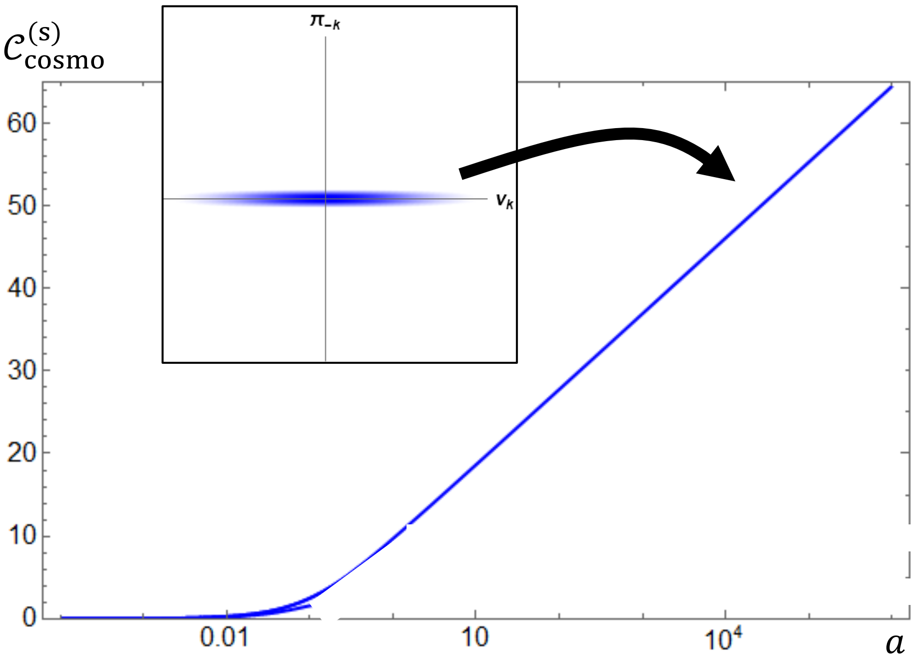

Unbounded growth of complexity depends sensitively on squeezing angle ϕ

Cosmological Complexity

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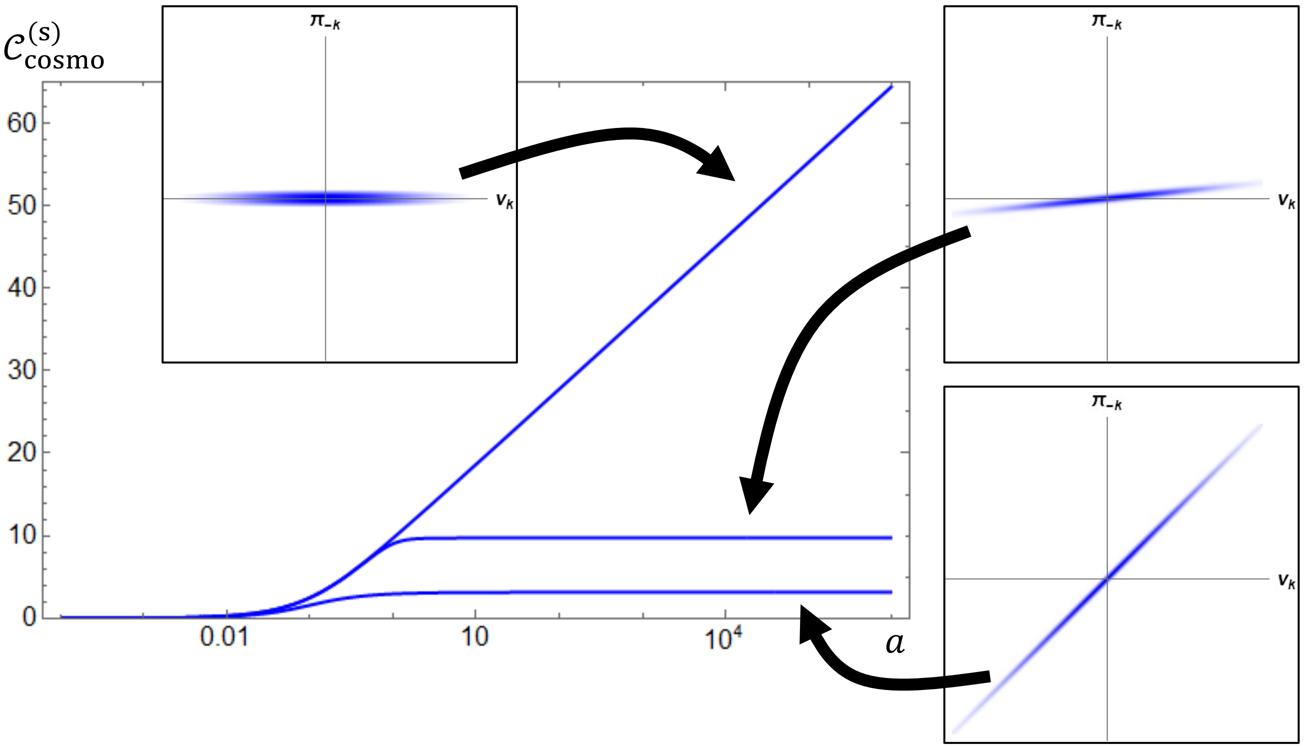

Unbounded growth of complexity depends sensitively on squeezing angle ϕ

- ❑ Complexity of dS is **maximal** w.r.t. ϕ Why?

Cosmological Complexity

the complexity of quantum cosmological perturbations

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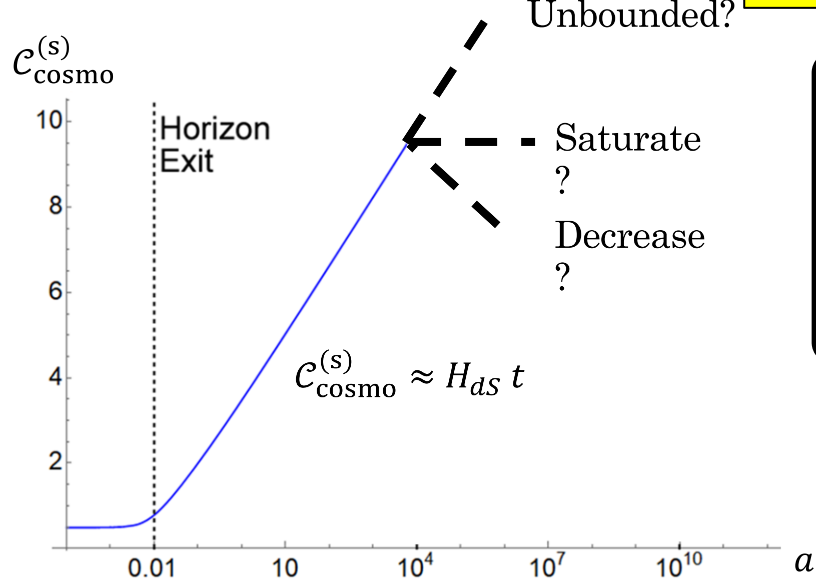
What is the long-term behavior of cosmological complexity?

- Characterize $\hat{\mathcal{U}}_{\text{cosmo}}$ by its action on Gaussian states

$$\Psi_{\text{sq}} = \langle q_{\vec{k}}, q_{-\vec{k}} | r_k, \phi_k, \theta_k\rangle \sim e^{A\left(q_{\vec{k}}^2 + q_{-\vec{k}}^2\right) - B\, q_{\vec{k}}\, q_{-\vec{k}}}$$

Inside Horizon:

- $r_k \ll 1$
- Gaussian State Complexity insensitive to phase $\mathcal{C}^{(s)}_{\text{cosmo}} \ll 1$



Outside Horizon:

- Grows with squeezing, e-folds

$$\mathcal{C}^{(s)}_{\text{cosmo}} \approx r_k \approx \ln\left(\frac{a}{a_{\text{exit}}}\right)$$

- Growth is linear in cosmic time t, $a(t) = e^{H_{dS} t}$

Cosmological Complexity

(Gaussian) State Circuit Complexity

Reference State

vacuum

$$\psi_R(q_{\vec{k}}, q_{-\vec{k}}) = \langle q_{\vec{k}}, q_{-\vec{k}}|0\rangle_{\vec{k},-\vec{k}} = \left(\frac{k}{\pi}\right)^{1/4} e^{-\frac{k}{2}(q_{\vec{k}}^2+q_{-\vec{k}}^2)}$$

Target State

cosmological squeezed state

$$\Psi_{sq}(q_{\vec{k}}, q_{-\vec{k}}) = \langle q_{\vec{k}}, q_{-\vec{k}}|\Psi_{sq}\rangle_{\vec{k}} = \frac{e^{A(q_{\vec{k}}^2+q_{-\vec{k}}^2)-Bq_{\vec{k}}q_{-\vec{k}}}}{\cosh r_k\sqrt{\pi}\sqrt{1-e^{-4i\phi_k}\tanh^2 r_k}}$$

$$A = \frac{k}{2}\left(\frac{e^{-4i\phi_k}\tanh^2 r_k + 1}{e^{-4i\phi_k}\tanh^2 r_k - 1}\right), \quad B = 2k\left(\frac{e^{-2i\phi_k}\tanh r_k}{e^{-4i\phi_k}\tanh^2 r_k - 1}\right)$$

Radiation?

$$r_k = \sinh^{-1}\left(\frac{1}{2k\eta}\right);$$

$$\phi_k = -\frac{\pi}{4} + \frac{1}{2}\tanh^{-1}\left(\frac{1}{2k\eta}\right)$$

At sufficiently early times $\eta \to 0$

a mode will start outside the horizon $k\eta \ll 1$, then re-enter the horizon later.

We expect that the squeezing of the mode will continue to grow while outside of the horizon, then “freeze in” when the mode re-enters the horizon.

Cosmological Complexity

(Gaussian) State Circuit Complexity

Radiation?

$$r_k = \sinh^{-1}\left(\frac{1}{2k\eta}\right) ;$$
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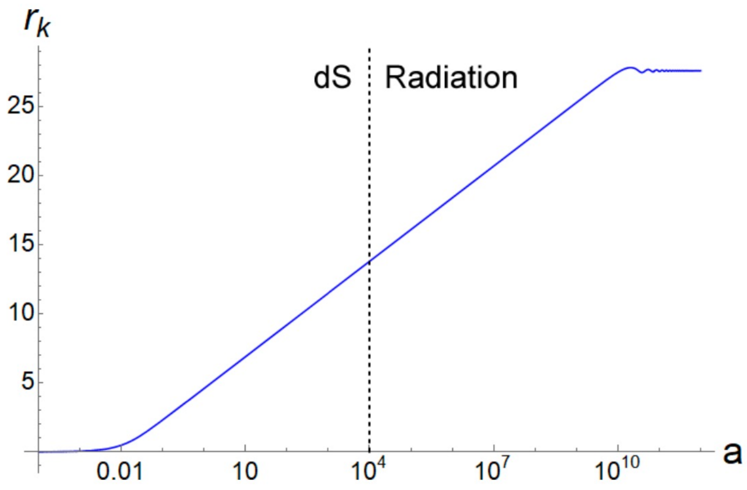
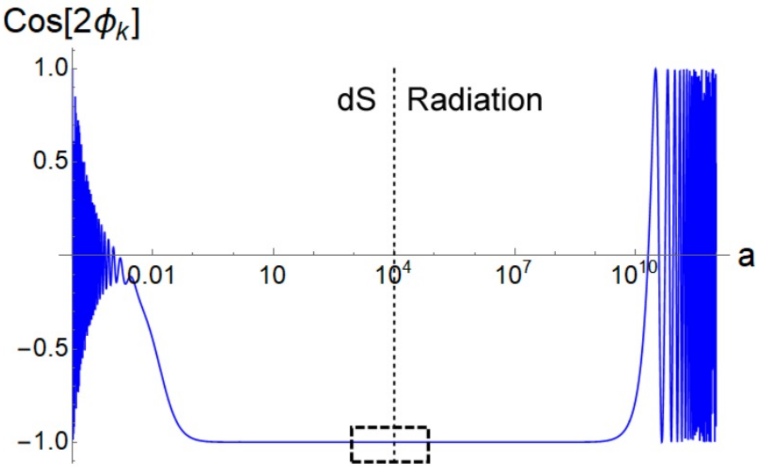


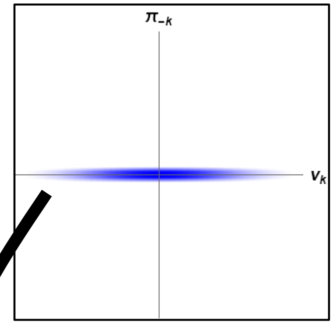


What happens to complexity?

Cosmological Complexity

(Gaussian) State Circuit Complexity

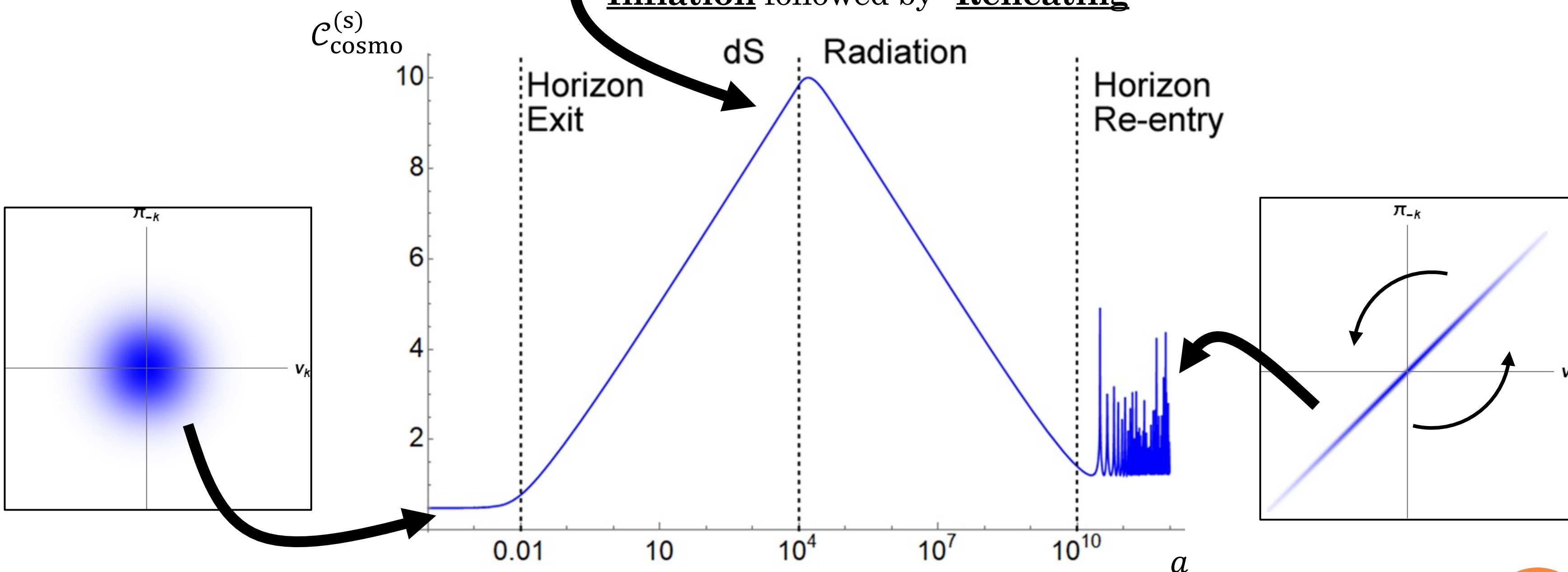
the complexity of quantum cosmological perturbations



"De-Complexification":

- Complexity **decreases** for radiation-dominated, then "freezes-in" upon horizon re-entry!
- Modes are still highly squeezed
- dS → radiation transition cuts off complexity growth

Inflation followed by "**Reheating**"



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Cosmological Complexity

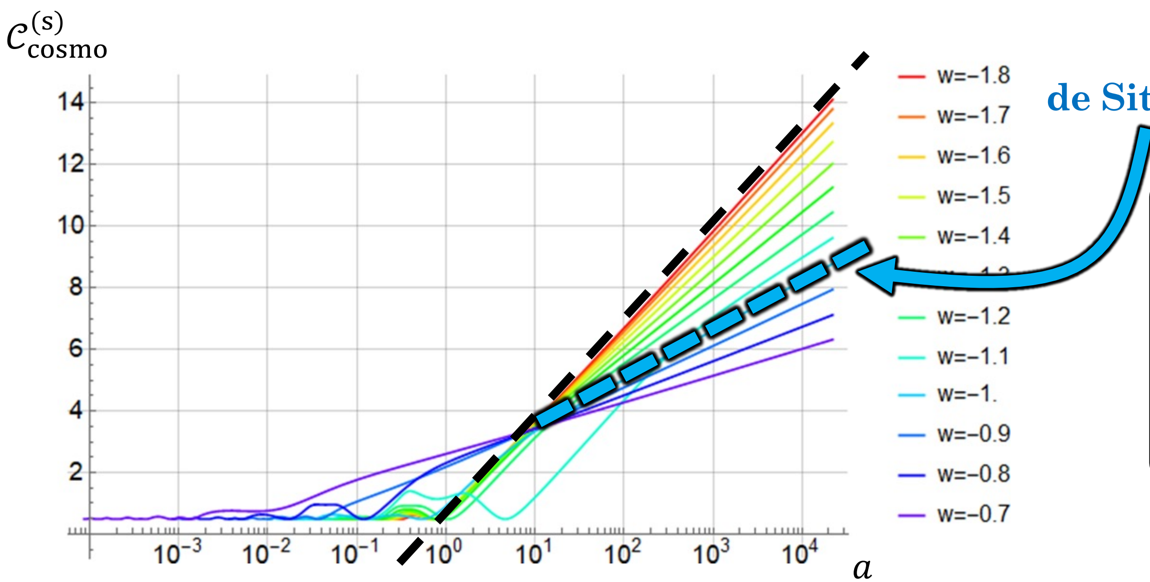
(Gaussian) State Circuit Complexity

the complexity of quantum cosmological perturbations

Accelerating, Expanding Backgrounds

$ds^2 = a(\eta)^2\,(-d\eta^2 + d\vec{x}^2)$ $\quad a(\eta) = \left(\frac{\eta_0}{\eta}\right)^{-2/(1+3w)}$ $\quad$ Equation of state $p = w\rho$

Bound on growth rate $\frac{d\mathcal{C}}{dt} \leq \sqrt{2}H$



de Sitter $w = -1$

Growth rate of complexity **saturates** at $w = -5/3$

- de Sitter is not fastest growth in cosmological complexity among all possible accelerating backgrounds… *(but others violate NEC)*

NEC: $-1 \leq w \leq -\frac{1}{3}$

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Open Quantum System?

From Last time (lecture 2)

Thermal state $\hat{\rho}_{\rm th} = \frac{1}{Z}\sum_{n=0}^{\infty} e^{-\beta E_n}|n\rangle\langle n|$; $\Longrightarrow$ purification $|{\rm TFD}\rangle = \frac{1}{\sqrt{Z}}\sum_{n=0}^{\infty} e^{-\beta E_n/2}|n\rangle \otimes |n\rangle_{\rm anc}$

This is not a unique purification, and it is possible to include an additional phase

$$|\Psi\rangle_\phi = |{\rm TFD}\rangle_\phi = \frac{1}{\sqrt{Z}}\sum_{n=0}^{\infty}(-1)^n e^{-2in\phi} e^{-n\beta\omega/2}|n\rangle \otimes |n\rangle_{\rm anc} ;$$

We recognize this as a two-mode squeezed vacuum state

$$|\Psi_{sq}\rangle = \frac{1}{\cosh r}\sum_{n=0}^{\infty}(-1)^n e^{-2in\phi}\tanh^n r|n\rangle \otimes |n\rangle_{\rm anc} \equiv \hat{S}_{\rm sq}(r,\phi)|0\rangle \otimes |0\rangle_{\rm anc}$$

$$\beta\omega = -\ln\tanh^2 r$$

$$\hat{\rho}_{\rm pure} = |\Psi_{sq}\rangle\langle\Psi_{sq}| = \frac{1}{\cosh^2 r_k}\sum_{n,m=0}^{\infty}(-1)^{n+m} e^{-2i(n-m)\varphi_k}\tanh^{n+m} r_k\, |n_{\vec{k}}, n_{-\vec{k}}\rangle\langle m_{\vec{k}}, m_{-\vec{k}}|$$

We can get a reduced density matrix only with diagonal entries by averaging this density matrix over the squeezing angle

Brandenberger

$$\hat{\rho}_{\rm red} = \frac{1}{\cosh^2 r_k}\sum_{n=0}^{\infty}\tanh^{2n} r_k\, |n_{\vec{k}}, n_{-\vec{k}}\rangle\langle n_{\vec{k}}, n_{-\vec{k}}|$$

Serve as a simple model for decoherence

Cosmological Complexity

(Gaussian) State Circuit Complexity

Decoherence

Reduced density matrix resulting from this simple model of decoherence is thermal

Pure State

$$\hat{\rho}_{\text{pure}} = |r_k, \phi_k, \theta_k\rangle\langle r_k, \phi_k, \theta_k| \longrightarrow$$

Thermal Density Matrix

$$\hat{\rho}_{\text{red}} = \frac{1}{\cosh^2 r_k} \sum_{n=0}^{\infty} \tanh^{2n} r_k \; |n_{\vec{k}}, n_{-\vec{k}}\rangle\langle n_{\vec{k}}, n_{-\vec{k}}|$$

Complexity of Purification

- Assume decoherence occurs at re-entry
- Purification with ancillary dof
  $\mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}_{\text{anc}}$
- Minimize complexity over purification
  $\mathcal{C}_{\text{purif}} = \min_{\{\text{anc}\}} \mathcal{C}_{\text{tot}}$
- Complexity of purification $\mathcal{O}(1)$
  $\mathcal{C}_{\text{purif}} \approx \frac{\pi}{2\sqrt{2}}$

We will calculate the associated thermal complexity of purification of the cosmological perturbations.

To do this, we expand our Hilbert space to include an ancillary copy

$$|\Psi_{\text{cosmo,p}}\rangle_{\vec{k}} = \frac{1}{\cosh r_k} \sum_{n=0}^{\infty} (-1)^n e^{-2in\phi} \tanh^n r_k \, |n_{\vec{k}}\rangle \otimes |n'_{\vec{k}}\rangle$$

(Suppressing the $-\vec{k}$ modes here)

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Cosmological Complexity

(Gaussian) State Circuit Complexity

Decoherence

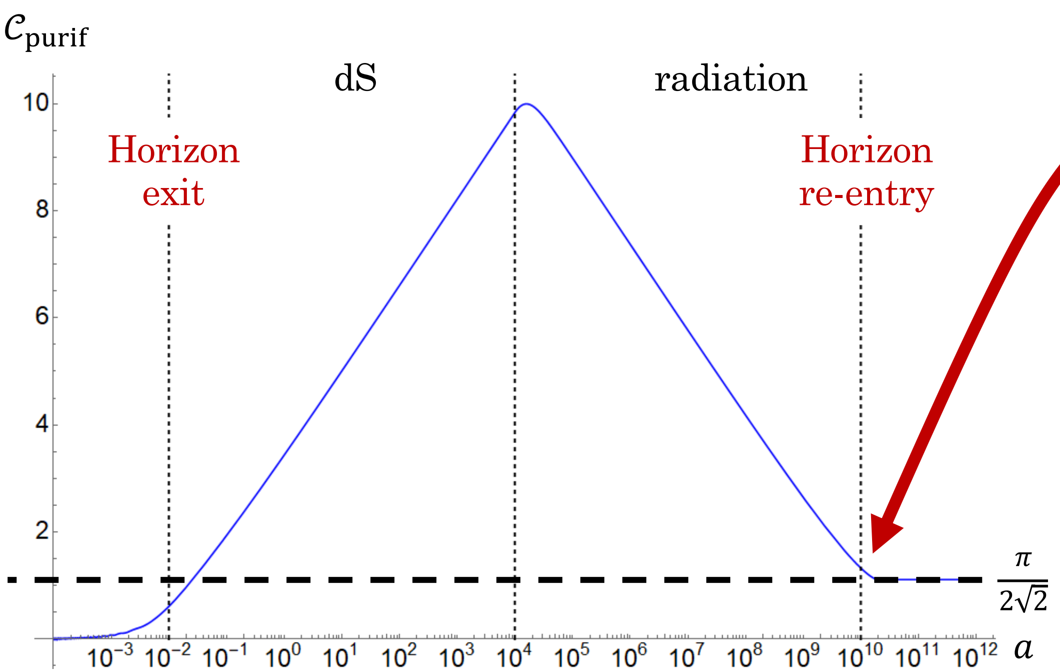
Pure State

$$\hat{\rho}_{\text{pure}} = |r_k, \phi_k, \theta_k\rangle\langle r_k, \phi_k, \theta_k|$$

→

Thermal Density Matrix

$$\hat{\rho}_{\text{red}} = \frac{1}{\cosh^2 r_k} \sum_{n=0}^{\infty} \tanh^{2n} r_k \; |n_{\vec{k}}, n_{-\vec{k}}\rangle\langle n_{\vec{k}}, n_{-\vec{k}}|$$



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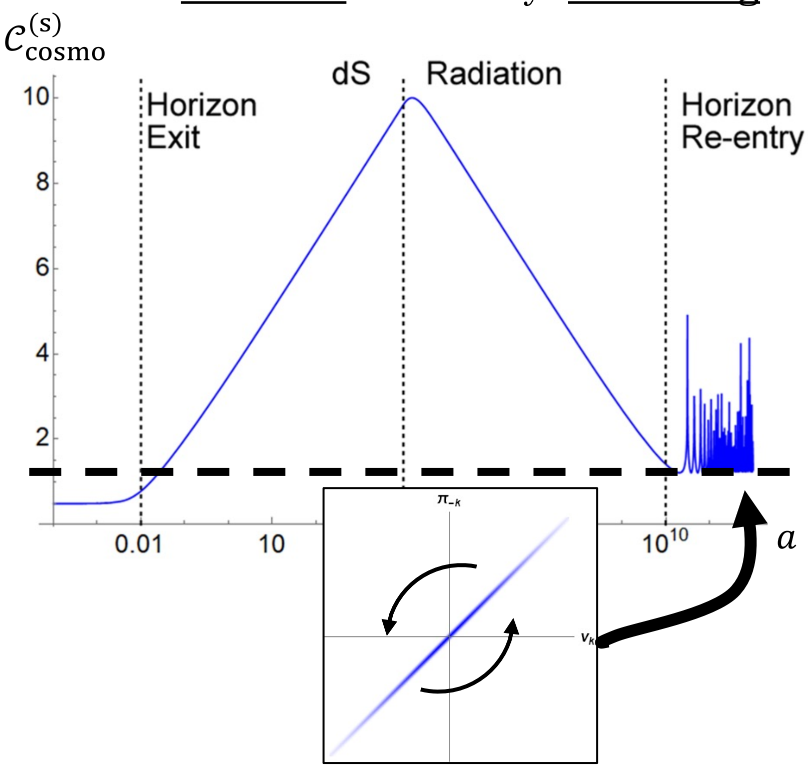
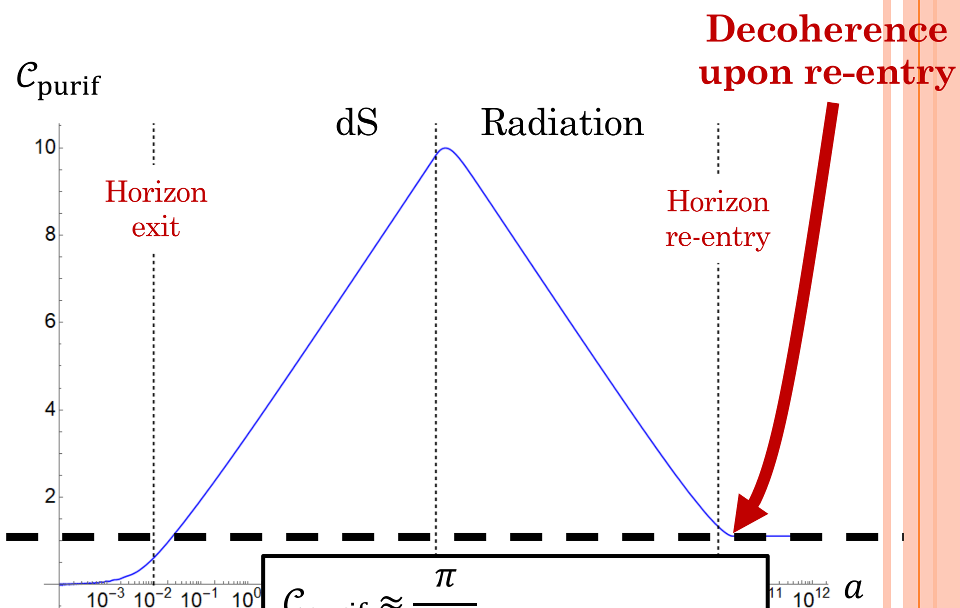
Cosmological Complexity

(Gaussian) State Circuit Complexity

Decomplexification vs Decoherence

Inflation followed by "**Reheating**"



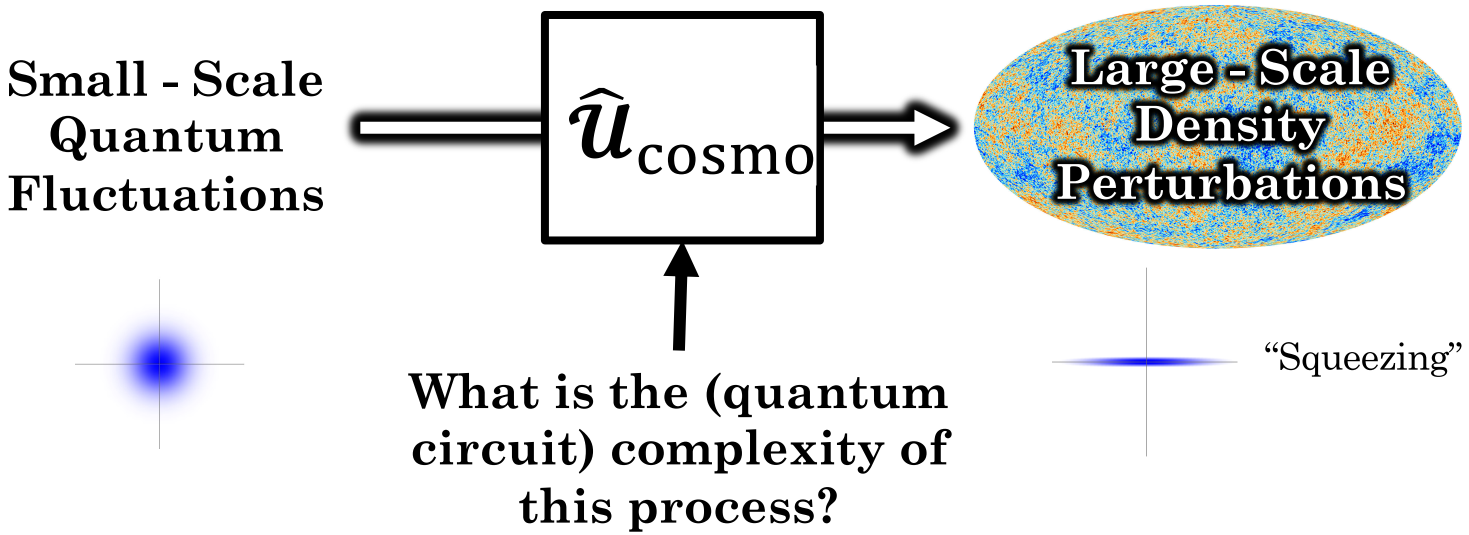


$$\mathcal{C}_{\text{purif}} \approx \frac{\pi}{2\sqrt{2}}$$

Complexity of Purification

Rapidly changing squeezing angle $\approx$ Complexity of decohered mixed state

SUMMARY

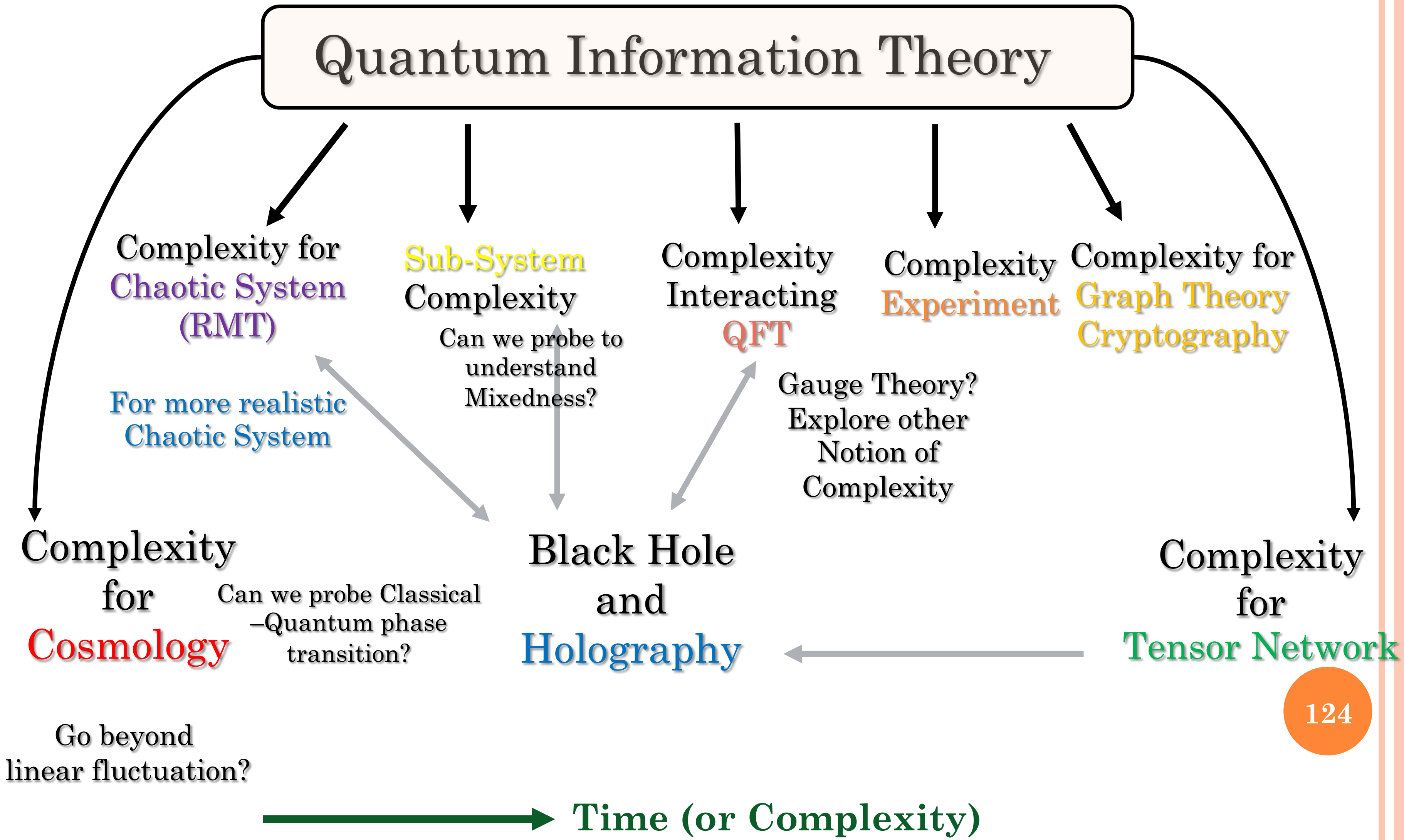


- Cosmological Complexity in dS grows linearly with time $\mathcal{C}_{\text{cosmo}} = H_{dS}\, t$
- Complexity depends sensitively on squeezing angle ϕ
  - Complexity of dS is **<u>maximal</u>** w.r.t. ϕ. Why? **Did our Universe choose to be more complex?**
- Growth rate of complexity $\frac{d\mathcal{C}}{dt}$ is bounded from above for accelerating backgrounds
- **De-complexification** during radiation-domination phase
  - Connection between decomplexification and decoherence?

Complexity can be a sensitive measure for studying different models and implementing decoherence

Future Direction

Understanding Gravity and QFT from QI



ChatGPT

What is Cosmological complexity?

Cosmological complexity refers to the intricate and multifaceted nature of the universe at the largest scales. It encompasses the study of the formation and evolution of cosmic structures like galaxies, galaxy clusters, and the vast cosmic web, as well as the complexities associated with cosmic microwave background radiation, dark matter, and dark energy. This concept highlights the rich interplay of physical processes, gravitational interactions, and cosmic phenomena that have shaped the universe's history and structure over billions of years, making cosmology a complex and fascinating field of study in astrophysics and cosmology.

Collaborators

Helped me to put together the lectures

Bret Underwood

Arpan Bhattacharyya

Ghadir Jafari

Thank You