W-symmetries, anomalies and heterotic backgrounds with SU holonomy

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- σ-models play a central role in string theory as they describe the string propagation in an n-dim spacetime M.
- Given a connection $\hat{\nabla}$ on a manifold M one can investigate its holonomy $hol(\hat{\nabla})$.
- If $\exists\hat{\nabla}$-covariantly constant ℓ-form L on M, $\hat{\nabla}L = 0 \Rightarrow \hat{R}_{\mu\rho}{}^{\sigma}{}_{[\nu_1}L_{|\sigma|\nu_2\ldots\nu_\ell]} = 0$.
- This is the condition for invariance under the group action of $hol(\hat{\nabla})$, following from $L_{\nu_1\ldots\nu_\ell}O^{\nu_1}{}_{\mu_1}\ldots O^{\nu_\ell}{}_{\mu_\ell} = L_{\mu_1\ldots\mu_\ell}$.
- Thus, $\hat{\nabla}$-covariantly constant forms are invariant forms of $hol(\hat{\nabla})$.
- Given a $\hat{\nabla}$-covariantly constant form, the $hol(\hat{\nabla})$ reduces to a subgroup of $U(\frac{n}{2})$, $SU(\frac{n}{2})$, $Sp(\frac{n}{4})$, $Sp(\frac{n}{4})\cdot Sp(1)$, $G_2(n=7)$ and $Spin(7)(n=8)$.
- Such forms generate symmetries in 2-dim supersymmetric σ-models with couplings a metric g, and a Wess-Zumino term b, with $H = db$.

  [Odake (1989); Delius, Rocek, Sevrin, van Nieuwenhuizen (1989); Howe, Papadopoulos (1991 & 1993)]

- As these symmetries arise from the reduction of holonomy of $\hat{\nabla}$ to a subgroup of SO, are called *holonomy symmetries*.
- They satisfy a W-algebra as the structure constants depend on the conserved currents of the theory. [Howe, Papadopoulos (1991 & 1993)]
- Such forms arise naturally in heterotic string backgrounds that preserve some spacetime supersymmetry, because of the gravitino KSE, $\hat{\nabla}_\mu \epsilon = 0$.
[Moore, Nelson; Gaume, Ginsparg; Bagger, Nemeschansky, Yankielowicz (1985)]

$$\hat{\nabla} = \nabla + \frac{1}{2} H \ .$$

- Integrability condition of the gravitino KSE

$$[\hat{\nabla}_\mu, \hat{\nabla}_\nu]\epsilon = \frac{1}{4} \hat{R}_{\mu\nu, AB} \Gamma^{AB} \epsilon = 0 \ .$$

The existence of Killing spinors requires that $hol(\hat{\nabla})$ must be a subgroup of their isotropy group in $Spin(9,1)$.

When the Killing spinors $(\epsilon_1, \ldots, \epsilon_N)$ have a non-trivial isotropy group,

$$hol(\hat{\nabla}) \subseteq Stab(\epsilon_1, \ldots, \epsilon_N) \subset Spin(9,1) \ .$$

Distinction between compact and non-compact holonomy. It has been shown that [Gran, Lohrmann, Papadopoulos (2006); Gran, Papadopoulos, Roest, Sloane (2007)]

- Compact: $hol(\hat{\nabla}) \subseteq SU(2) \ , \quad SU(3) \ , \quad G_2 \ ,$
- Non-compact: $hol(\hat{\nabla}) \subseteq \mathcal{K} \ltimes \mathbb{R}^8$, with

$$\begin{aligned} \mathcal{K} = Spin(7) \ , \quad SU(4) \ , \quad Sp(2) \ , \\ \times^2 Sp(1) \ , \quad Sp(1) \ , \quad U(1) \ , \quad \{1\} \ . \end{aligned}$$

Alternatively, one can characterize the geometry of supersymmetric heterotic backgrounds in terms of spacetime form bilinears of the Killing spinors.

Such bilinears are also $\hat{\nabla}$-covariantly constant, as a consequence of the KSE.

$$\hat{\nabla}_\mu L_{\nu_1 \dots \nu_\ell} = 0 \ .$$

The converse is also true. [Papadopoulos, Tsimpis (2003)]
The isotropy subgroup of Killing spinors does also leave the L invariant.

- Due to the presence of chiral worldsheet fermions, symmetries of heterotic σ-models are anomalous.

  [Moore, Nelson; Gaume, Ginsparg; Bagger, Nemeschansky, Yankielowicz (1985)]

- To preserve the geometric interpretation of these theories, the anomalies of some of these symmetries must cancel.

- We focus on heterotic backgrounds with $hol(\hat{\nabla}) \subseteq SU(2)$ and $SU(3)$.
- The $\hat{\nabla}$-covariantly constant forms constructed from Killing spinors are known.
  [Gran, Lohrman, Papadopoulos (2005)]

Our goal:

- Algebra of symmetries
- Anomalies
- Anomaly cancellation

Symmetries of σ-model

Let the (1,0)-supersymmetric 2-dim σ-model with $(\sigma^{=}, \sigma^{\ddagger}, \theta^{+})$ the coordinates of the superspace $\Xi^{2|1}$. Classical fields: X, ψ

$$S = -i \int d^2\sigma\, d\theta^+ \left((g_{\mu\nu} + b_{\mu\nu})\, D_+ X^\mu\, \partial_= X^\nu + i h_{ab}\, \psi_-^a\, \mathcal{D}_+ \psi_-^b \right) \,,$$

with metric g, 2-form b with $H = db$, $D_+^2 = i\partial_{\ddagger}$, h fibre metric on E and

$$\mathcal{D}_+ \psi_-^a = D_+ \psi_-^a + D_+ X^\mu\, \Omega_\mu{}^a{}_b\, \psi_-^b \,,$$

where Ω connection on E with curvature F. σ-model symmetries:

- Diffeomorphisms of the target space M.
- Gauge transformations u of the gauge sector with connection Ω.

Symmetries of σ-model

- Spacetime frame rotations ℓ, with $g_{\mu\nu} = \eta_{AB} e^A_\mu e^B_\nu$, and ω the frame connection.
- Gauge symmetry of b.
- *Holonomy symmetries* generated by a vector ℓ-form on M

$$\delta_L X^\mu = a_L L^\mu{}_{\nu_1 \dots \nu_\ell} D_+ X^{\nu_1 \cdots \nu_\ell} \ , \quad \Delta_L \psi^a_- \equiv \delta_L \psi^a_- + \delta_L X^\mu \, \Omega_\mu{}^a{}_b \, \psi^b_- = 0 \ ,$$

with $\partial_= a_L = 0$, and provided that

$$\hat{\nabla}_\nu L_{\lambda_1 \dots \lambda_{\ell+1}} = 0 \ , \quad F_{\nu[\lambda_1} L^\nu{}_{\lambda_2 \dots \lambda_{\ell+1}]} = 0 \ .$$

For KS form bilinears, these are consequences of the gravitino and the gaugino KSE.

Conserved (holonomy) currents $J_L = L_{\mu_1 \dots \mu_{\ell+1}} D_+ X^{\mu_1 \cdots \mu_{\ell+1}}$.

For $\ell = 0$, $L = K$ is a parallel 1-form associated to a Killing vector field.

- To express $[\delta_L, \delta_M]$ as a sum of symmetry transformations, we require additional generators.
- E.g. the vector (q+1)-form $S^\mu{}_{\nu\rho_1\ldots\rho_q} = \delta^\mu{}_{[\nu} Q_{\rho_1\ldots\rho_q]}$, with $\hat{\nabla} Q = 0$,

$$\delta_S X_\mu = \alpha_S \hat{\nabla}_+ D_+ X^\nu S_{\nu,\mu Q} D_+ X^Q + \frac{(-1)^q}{q+1} \hat{\nabla}_+ (\alpha_S S_{\mu,\nu Q} D_+ X^{\nu Q}) - \frac{q+3}{q+1} \alpha_S H_{[\mu\nu\rho} Q_{Q]} D_+ X^{\nu\rho Q} ,$$

$$\Delta_S \psi_-^a = -\frac{(-1)^q}{q+1} \alpha_S Q_Q F_{\mu\nu}{}^a{}_b \psi_-^b D_+ X^{Q\mu\nu} ,$$

with conserved current TJ_Q, where T is the right-handed (super) energy-momentum tensor.

Anomaly consistency conditions

- What about anomalies?
- Suppose that the classical theory is invariant under the algebra of symmetries whose variations on the fields satisfy

$$[\delta_A, \delta_B] = \delta_{[A,B]} \ .$$

- If these symmetries are anomalous in the quantum theory, i.e. $\delta_A \Gamma = \Delta(a_A)$, then one gets the Wess-Zumino anomaly consistency conditions

$$\delta_A \Delta(a_B) - \delta_B \Delta(a_A) = \Delta(a_{[A,B]}) \ .$$

- The anomaly associated to the frame rotations is determined by the descent equations. [Zumino (1984)]
- Starting from $P_4(R) = tr(R(\omega) \wedge R(\omega))$ for the curvature R of ω, $dP_4 = 0 \Rightarrow P_4(R) = dQ_3^0(\omega)$ (locally), with $Q_3^0(\omega)$ the Chern-Simons form.
- As $\delta_\ell P_4 \Rightarrow d\delta_\ell Q_3^0(\omega) = 0 \Rightarrow \delta_\ell Q_3^0(\omega) = dQ_2^1(\omega, \ell)$.

Anomaly consistency conditions

- The frame rotation anomaly is given by

$$\Delta(\ell) = \frac{i\hbar}{4\pi}\int d^2 d\theta^+ Q_2^1(\omega,\ell)_{\mu\nu} D_+ X^\mu \partial_= X^\nu \ ,$$

and similarly for the gauge transformation $\Delta(u)$.

- The cancellation of the anomalies follows from assigning an anomalous variation to b at one-loop [Hull-Witten (1985)]

$$\delta_\ell b = \frac{\hbar}{4\pi} Q_2^1(\omega,\ell) \ , \quad \delta_u b = -\frac{\hbar}{4\pi} Q_2^1(\Omega,u) \ .$$

- As $[\delta_\ell,\delta_L] = [\delta_u,\delta_L] = 0$, one can show that

$$\Delta(a_L) = \frac{i\hbar}{4\pi}\int d^2\sigma d\theta^+ Q_3^0(\omega,\Omega)_{\mu\nu\rho}\, \delta_L X^\mu D_+ X^\nu \partial_= X^\rho \ ,$$

where $Q_3^0(\omega,\Omega) = Q_3^0(\omega) - Q_3^0(\Omega)$.

Anomaly cancellation

We adopt two different scenarios to cancel the anomalies.

- First, when L does not receive quantum corrections, one can introduce finite local counterterms to the effective action. As $P_4 = dQ_3^0$, then Q_3^0 is specified up to an exact form $Q_3^0 \to Q_3^0 + dW$.

- Second, when L receives quantum corrections, $L^\hbar$.

$$\delta_L^\hbar \Gamma = \delta_L^\hbar(\Gamma^{(0)} + \hbar\Gamma^{(1)}) = \Delta(a_L) \implies$$
$$- i\int d^2\sigma d\theta^+ (a_L \frac{2(-1)^\ell}{\ell+1} \hat{\nabla}^\hbar_\mu L^\hbar_{L+1} \partial_= X^\mu D_+ X^{L+1}$$
$$- i a_L L_L^{\hbar\mu} F^\hbar_{\mu\nu ab} \psi_-^a \psi_-^b D_+ X^{L\nu} + 2i\Delta_L^\hbar \psi_-^a \mathcal{D}_+^\hbar \psi_{-a}) = 0 + \mathcal{O}(\hbar^2) \ ,$$

where $\hat{\nabla}^\hbar$ the quantum corrected connection with $H^\hbar = H - \frac{\hbar}{4\pi} Q_3^0(\omega, \Omega) + \mathcal{O}(\hbar^2)$.

- Provided that $\hat{\nabla}^\hbar L^\hbar = 0$ and $i_{L^\hbar} F^\hbar = 0$, the anomaly cancels.

- KSEs of heterotic supergravity retain their form up to and including two loops in the σ-model perturbation theory provided one replaces H with $H^\hbar$. [Bergshoeff, de Roo (1989)]

- $\hat{\nabla}^\hbar \epsilon = 0 \Rightarrow \hat{\nabla}^\hbar L^\hbar = 0$. Similarly, the gaugino KSE implies $i_{L^\hbar} F^\hbar = 0$.

Holonomy symmetries algebra

The algebra of holonomy symmetries closes as

$$[\delta_L, \delta_M] = \delta_N + \delta_S + \delta_{JP} \ .$$

- δ_N: symmetry generated by a $\hat{\nabla}$-covariantly constant form N with parameter a_N constructed from a_L and a_M,
- δ_S: it has been given earlier, with parameter a_N constructed from a_L and a_M,
- δ_{JP}: symmetry generated by $\hat{\nabla}$-covariantly constant forms collectively denoted by P, with parameters constructed from a_L, a_M and conserved currents J of the theory.

- The $\hat{\nabla}$-covariantly constant forms are six 1-forms e^a, and three 2-forms I_r.
- The spacetime can be modeled as a principal bundle, $M = P(G, N^4, \pi)$, where N^4 is a 4-dimensional conformally hyperkähler manifold with torsion, principal bundle connection e^a, and metric $g = \eta_{ab} e^a e^b + \delta_{ij} e^i e^j$.
- The symmetries generated by e^a and I_r are

$$\delta_K X^\mu = a_K^a e_a^\mu \ , \quad \delta_I X^\mu = a_I^r (I_r)^\mu{}_\nu D_+ X^\nu$$

For the closure of the algebra we need to include the following symmetry

$$\delta_C X^a = \alpha_C \hat{\nabla}_+ D_+ X^a + \hat{\nabla}_+ (\alpha_C D_+ X^a) \ ,$$

associated to the quadratic Casimir operator of the Lie algebra of isometries, with conserved current $C = \eta_{ab} e_\mu^a e_\nu^b D_+ X^\mu \hat{\nabla}_+ D_+ X^\nu$.

SU(2) holonomy backgrounds

The algebra of the symmetries reads

- $[\delta_K, \delta'_K] = \delta''_K$,

- $[\delta_K, \delta_I] = 0$,

- $[\delta_I, \delta'_I] = \delta_T + \delta_C + \delta_K + \delta''_I$, $\quad a^a_K = a'^s_I a^r_I \delta_{rs} H^a{}_{bc} J^b_K J^c_K$.

  The parameter of δ_K depends quadratically on the currents $J^a_K = e^a_\mu D_+ X^\mu$ associated to isometries. Thus, we have a W-algebra.

- Remark: The δ_K symmetry of the last commutator, could have been written as

$$\delta_{\bar{H}} X^a = a_{\bar{H}} H^a{}_{bc} D_+ X^{bc} \ ,$$

  for $\partial_= a_{\bar{H}} = 0$, generated by the $\hat{\nabla}$-covariantly constant form $\bar{H} = \frac{1}{3!} H_{abc} e^{abc}$.

  Then, we would have a standard Lie algebra instead of a W-algebra.

- The remaining commutators read

$$[\delta_C, \delta_I] = 0 \ , \quad [\delta_K, \delta_C] = \delta'_K \ , \quad [\delta_C, \delta'_C] = \delta''_C + \delta''_K \ ,$$

where the parameters of δ'_K and δ''_K depend again on the current J^a_K.

- Anomaly cancellation and finite local counterterms:

There is a $\tilde{P}_4$ on N^4 such that $P_4 = \pi^* \tilde{P}_4$

As $d\tilde{P}_4 = 0$, there is $\tilde{Q}^0_3$ such that $\tilde{P}_4 = d\tilde{Q}^0_3$, thus

$$Q^0_3 = \pi^* \tilde{Q}^0_3 + dW$$

with W a 2-form on M.

- Adding the finite local counterterm to the effective action

$$\Gamma^{fl}_{(1)} = -\frac{i\hbar}{4\pi}\int d^2\sigma d\theta^+ \, W_{\mu\nu}\, D_+X^\mu\, \partial_= X^\nu \ ,$$

$$\Delta(a_L) + \delta_L \Gamma^{fl}_{(1)} = \frac{i\hbar}{4\pi}\int d^2\sigma d\theta^+ \, (\pi^* \tilde{Q}^0_3)_{\mu\nu\rho}\, \delta_L X^\mu\, D_+X^\nu\, \partial_= X^\rho = 0 \ ,$$

the anomaly cancels for $L = K$ or C, as $i_L \pi^* \tilde{Q}^0_3 = 0$.

Re-expressing the anomalies by replacing Q^0_3 with $\pi^* \tilde{Q}^0_3$ results their cancellation. In a similar way we cancel the anomaly of I.

SU(3) holonomy backgrounds

- The $\hat{\nabla}$-covariantly constant forms are four 1-forms e^a, one 2-form I, and two real 3-forms L_r.

- Additional type of symmetry: $\delta_L X^\mu = a_L^r (L_r)^\mu{}_{\nu_1\nu_2} D_+ X^{\nu_1\nu_2}$

$$[\delta_K, \delta_L] = \delta'_L \ ,$$

with no dependence on currents,

$$[\delta_I, \delta_L] = \delta'_L + \delta_K \ , \quad [\delta_{L_1}, \delta_{L_2}] = \delta_S + \delta_I + \delta_K + \delta_C \ , \quad [\delta_{L_r}, \delta_{L_r}] = \delta_I \ , \quad [\delta_L, \delta_C] = \delta'_L + \delta_K \ .$$

All parameters (excl. α_S) may depend on one or more currents J_K, J_I, J_L or C.

- The anomalies of K, C and I can be canceled as in SU(2) with counterterms.

- All anomalies coming from the $\hat{\nabla}^\hbar$-covariantly constant forms $L^\hbar$ will vanish.

Conclusions

- We showed that the Killing spinor bilinears of heterotic backgrounds with $hol(\nabla) \subseteq SU(2)$ and $SU(3)$, satisfy a W-algebra.
- We calculated the anomaly of holonomy symmetries using the Wess-Zumino consistency conditions.
- We argued that these anomalies can be cancelled either by adding finite local counterterms or with an appropriate quantum correction of the bilinears.
- Heterotic backgrounds with G_2 holonomy. Can we approach them in a similar way?

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Thank you!