Shadow formalism for supersymmetric conformal blocks

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arXiv: 2408.07684 V. Belavin, J. Ramos and B.R.

Conformal symmetry

- Global conformal group
  shifts, rotations, scaling special conformal transformation

$$P_\mu, S_{\mu\nu}, D, K^\mu$$

- Wick rotation

$$\tau = it$$

- 2D: complex coordinates

$$z = \tau + ix\,, \quad \bar{z} = \tau - ix$$

$$(z, \bar{z}) \mapsto (f(z), \bar{f}(\bar{z}))$$

- Local conformal transformations

$$\mathcal{L}_m = -z^{m+1}\partial_z\,, \quad \bar{\mathcal{L}}_m = -\bar{z}^{m+1}\partial_{\bar{z}}$$

- Quantization: Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0}$$

- Energy-momentum tensor

$$T(z) = \sum_{n\in\mathbb{Z}} \frac{L_m}{z^{m+2}}$$

c - *central charge* of the theory

Conformal primaries

- Primary field: tensor-like transformation law

$$\phi(w,\bar{w}) = \left(\frac{dw}{dz}\right)^{-h}\left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}}\phi(z,\bar{z})$$

- Primary states

$$L_k|h,\bar{h}\rangle = \bar{L_k}|h,\bar{h}\rangle = 0\,, \quad k>0$$

$$L_0|h,\bar{h}\rangle = h|h,\bar{h}\rangle\,, \quad \bar{L}_0|h,\bar{h}\rangle = \bar{h}|h,\bar{h}\rangle$$

- State-operator correspondence

$$\phi(0,0)|0\rangle = |h,\bar{h}\rangle|h\rangle, \quad \bar{h}\rangle \leftrightarrow \phi(z,\bar{z})$$

- Vacuum state

$$L_0|0\rangle = L_{-1}|0\rangle = 0\,, L_k|0\rangle = 0 \quad k>0$$

- Identity operator $\mathbb{1}$ $\leftrightarrow$ vacuum state

Correlation functions

- Two-point function

$$\langle \phi_1(z_1)\phi_2(z_2)\rangle = \frac{\delta_{h_1,h_2}}{(z-w)^{-2h_1}}$$

- Three-point function

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = C_{123} v_{123}(z_1,z_2,z_3)$$

$$v_{123}(z_1,z_2,z_3) = z_{12}^{-\gamma_{12}} z_{13}^{-\gamma_{13}} z_{23}^{-\gamma_{23}}$$

$$\gamma_{ij} = h_i + h_j - \sum_{k\neq i,j} h_k$$

- Invariant cross-ratio

$$x = \frac{z_{12}z_{34}}{z_{13}z_{24}}$$

- Many-point functions

$$\langle \phi_1 \dots \phi_n\rangle = \prod_{i<j} z_{ij}^{-\gamma_{ij}} f(x_1, \dots, x_{n-3})$$

Conformal blocks

Operator Product Expansion

$$\phi_i(z)\phi_j(w) \sim \sum_p \sum_K \frac{C_{ijp}\beta_{ij}^{p,[-K]}\phi_p^{(-K)}(w)}{(z-w)^{-h_i-h_j+h_p+|K|}}$$

Four-point function decomposition

$$\langle\phi_1(\infty)\phi_2(1)\phi_3(x)\phi_4(0)\rangle$$
$$= \sum_p C_{34p}C_{12p}x^{-h_3-h_4+h_p}\sum_K C_{12p}^{-1}\beta_{34}^{p[-K]}x^{|K|}\langle\phi_1(\infty)\phi_2(1)\phi_p^{(-K)}(0)\rangle$$

Conformal blocks $\mathcal{F}$

$$\langle\phi_1(\infty)\phi_2(1)\phi_3(x)\phi_4(0)\rangle = \sum_p C_{34p}C_{12p}\mathcal{F}_{43}^{12}(p|x)\bar{\mathcal{F}}_{43}^{12}(p,\bar{x})$$

Conformal blocks are model-independent, i.e. depend on c, h_i but not on C_{ijk}.

Semiclassical limit

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

$$\phi_i(z)\phi_j(w) \sim \sum_p \sum_K \frac{C_{ijp}\beta_{ij}^{p,[-K]}\phi_p^{(-K)}(w)}{(z-w)^{-h_i-h_j+h_p+|K|}}$$

In semiclassical limit the central charge c tends to infinity
Coefficients $\beta_{ij}^{k[-K]}$ are proportional to inverse of Gram matrix of vectors at the level $|K| \Rightarrow \beta_{ij}^{k[-K]} \sim O(c^{-1})$ unless $K = (1, 1, 1, 1, \dots, 1)$
In the limit $c \to \infty$ only global conformal algebra generated by $L_0, L_{\pm 1}$ contributes to conformal block

Semiclassical conformal blocks in CFT_2 corresponds to geodesic Witten diagrams in AdS_3 [Hijano et al (2016), Chen et al (2017), Dyer et al (2017)]

Shadow formalism

Dual shadow operator [Ferrara et al (1972), Simons-Duffin (2014), Rosenhaus (2018)]

$$\phi_h^*(z,\bar z) = \int d^2w \frac{\phi_h(w,\bar w)}{|z-w|^{2-2h}}$$

is a (quasi)primary field of dimension

$$\dim \phi_h^* = h^* = 1-h$$

Two-point function

$$\langle \phi_h^*(z,\bar z)\phi(w,\bar w)\rangle = \delta^2(z-w)$$

Projector property

$$\Pi_h = \int d^2z \phi_h(z,\bar z)|0\rangle\langle 0|\phi_h^*(z,\bar z)$$

$$\Pi_{h_1}\Pi_{h_2} = \delta_{h_1 h_2}\Pi_{h_1}$$

Superspace

Superspace $\mathbb{C}^{1|1}$ is parametrized by coordinates $Z, \bar{Z}$

$$Z = (z, \theta)\,, \quad \bar{Z} = (\bar{z}, \bar{\theta})\,, \quad \theta_1\theta_2 = -\theta_2\theta_1$$

$$Z_{12} = z_{12} - \theta_{12}\,, \quad z_{12} = z_1 - z_2\,, \quad \theta_{12} = \theta_1\theta_2$$

Superderivative

$$D = \partial_\theta + \theta\partial_z\,, \quad \bar{D} = \partial_{\bar{\theta}} + \bar{\theta}\partial_{\bar{z}}$$

Superanalytic functions: $\bar{D}f(Z, \bar{Z}) = 0$ Superconformal transformations $Z \mapsto \tilde{Z} = (\tilde{z}, \tilde{\theta})$ preserve superderivative

$$D = D\tilde{\theta}\tilde{D}$$

Superconformal transformations

- Global superconformal group

$$\tilde{z} = \frac{az + b + \alpha\theta}{cz + d\beta\theta}, \quad \tilde{\theta} = \frac{\bar{\alpha}z + \bar{\beta} + \bar{A}\theta}{cz + d + \beta\theta}$$

$$\bar{\alpha} = \frac{a\beta - c\alpha}{\sqrt{ad - bc}}, \bar{\beta} = \frac{b\beta - d\alpha}{\sqrt{ad - bc}},$$

$$\bar{A} = \sqrt{ad - bc - 3\alpha\beta}$$

- Supervirasoro algebra (NS sector)

$$[L_m, L_n] = (m-n)L_{n+m} + \frac{\hat{c}}{8}\delta_{m+n,0}(m^3 - m)$$

$$[L_m, G_s] = \left(\frac{m}{2} - r\right) G_{m+s}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{\hat{c}}{2}\left(r^2 - \frac{1}{4}\right)$$

m, n integer, r, s half-integer

N=1 SCFT

- Superfield

$$\Phi(Z,\bar{Z}) = \phi(z,\bar{z}) + \theta\psi(z,\bar{z}) + \bar{\theta}\bar{\psi}(z,\bar{z}) + \theta\bar{\theta}\varphi(z,\bar{z})$$

$$\dim\phi = h\,, \dim\psi = h + \frac{1}{2}$$

- Two-point function

$$\langle \Phi_1(Z_1)\Phi_2(Z_2)\rangle = \frac{\delta_{h_1 h_2}}{Z_{12}^{2h_1}}$$

- Invariant cross-ratios

$$X = \frac{Z_{12}Z_{34}}{Z_{13}Z_{24}}$$

$$\eta = \frac{z_{12}\theta_3 - z_{13}\theta_2 + z_{23}\theta_1 - \frac{3}{2}\theta_1\theta_2\theta_3}{(z_{12}z_{13}z_{23})^{\frac{3}{2}}}$$

- Three-point function

$$\langle \Phi_1(Z_1)\Phi_2(Z_2)\Phi_3(Z_3)\rangle = |Z_{12}|^{-2\gamma_{123}}|Z_{13}|^{-2\gamma_{132}}|Z_{23}|^{-2\gamma_{231}}(C + \tilde{C}_{123}\eta\bar{\eta})$$

Supershadow operator

- Shadow operator

$$\Phi_h^*(Z,\bar{Z}) = \frac{1}{N_h}\int d^2w \int d^2\xi \frac{\Phi_h(W,\bar{W})}{|Z-W|^{1-2h}}$$

$$W = (w,\xi)\,, Z - W = z - w - \theta\xi$$

- Two-point function

$$\langle \Phi_h(Z,\bar{Z})\Phi_h^*(W,\bar{W}) = \delta^2(\theta-\xi)\delta^2(z-w)$$

$$\delta(\theta-\xi) = \theta - \xi$$

- Projector

$$\Pi_h = \int d^2z \int d^2\theta \Phi_h(Z,\bar{Z})|0\rangle\langle 0|\Phi_h^*(Z,\bar{Z})$$

- Shadow three-point constants

$$C^*_{hh_1h_2} = 2^{4h-2}(2h-1)^2\tilde{C}_{hh_1h_2}I_0(h-h_{12},h+h_{12})N_h$$

$$\tilde{C}^*_{hh_1h_2} = 2^{4h}C_{hh_1h_2}I_0\left(h+h_{12}+\frac{1}{2}, h-h_{12}+\frac{1}{2}\right)N_h$$

Superconformal blocks

Four-point function $$\langle\prod_{i=1}^{4}\Phi_i(Z_i,\bar{Z}_i)\rangle = |\mathcal{L}_{h_1,h_2,h_3,h_4}(Z_{12},Z_{34},Z_{24},Z_{13})|^2 \sum_h G_h(h_1,h_2,h_3,h_4|X,\bar{X},\eta,\bar{\eta},\eta',\bar{\eta}')$$

$$\eta=\eta_{124}\,,\quad \eta'=(1-X)^{-\frac{1}{2}}\eta_{123}$$

Leg factor $\mathcal{L}_{h_1,h_2,h_3,h_4}$ ensures global superconformal symmetry

$$\mathcal{L}_{h_1,h_2,h_3,h_4}=Z_{12}^{-h_1-h_2}Z_{34}^{-h_3-h_4}Z_{24}^{h_1-h_2}Z_{13}^{-h_3+h_4}Z_{14}^{-h_1+h_2+h_3-h_4}$$

$$\begin{aligned}G_h =&g_h^{(0,0)}+g_h^{(1,0)}\eta\bar{\eta}+g_h^{(0,1)}\eta'\bar{\eta}'+g_h^{(1,1)}\eta\eta'\bar{\eta}\bar{\eta}'\\&+f_h^{(1,-1)}\eta\bar{\eta}'+f_h^{(1,1)}\eta\eta'+f_h^{(-1,-1)}\bar{\eta}\bar{\eta}'+f_h^{(-1,1)}\bar{\eta}\eta'\end{aligned}$$

Using shadow formalism

A 4-pt function can be decomposed into sum of CPWs

$$\langle \prod_{i=1}^{4} \Phi_i(Z_i, \bar{Z}_i) \rangle = \sum_h \Psi_h^{h_1,h_2,h_3,h_4}(Z_1, Z_2, Z_3, Z_4)$$

Inserting identity

$$\mathbb{1} = \sum_h \Pi_h \,, \quad \Pi_h = \int d^2 z \int d^2\theta \Phi_h(Z, \bar{Z})|0\rangle\langle 0|\Phi_h^*(Z, \bar{Z})$$

between Φ_2 and Φ_3 we compute superconformal partial wave

$$\Psi_h^{h_1,h_2,h_3,h_4} = \int d^2 z_0 \int d^2\theta_0 \mathcal{V}_{h_1 h_2 h}(Z_1, Z_2, Z_0)\mathcal{V}_{h^* h_3 h_4}(Z_0, Z_3, Z_4)$$

A CPW contains contributions from superconformal block G_h and shadow superconformal block G_{h^*}

Extracting superconformal blocks

Individual components can be obtained via superanalyticity

$$f(X) = f(x) + \sum_{i<j} \frac{\partial^2 f}{\partial\theta_i\theta_j}\Big|_{\theta=0}\theta_i\theta_j + \frac{\partial^4 f}{\partial\theta_1\partial\theta_2\partial\theta_3\partial\theta_4}$$

$g_h^{(0,0)}(X)$ is a superanalytic continuation of $g^{(0,0)}(x)$, which is obtained at the point $\theta_i = 0$

$$g_h^{(0,0)} + g_{h^*}^{(0,0)} = \tilde{C}_{hh_1h_2}C^*_{hh_3h_4}|X|\mathcal{F}_{4pt}\left(h+\frac{1}{2}, h_{12}, h_{34}|X,\bar{X}\right)$$
$$C_{hh_1h_2}\tilde{C}^*_{hh_3h_4}|X|\mathcal{F}_{4pt}\left(h, h_{12}, h_{34}|X,\bar{X}\right)$$

$$\mathcal{F}_{4pt} = Y(1-h,h_{12})|X^{\frac{1}{2}-h}{}_2F_1(1-h+h_{34},1-h-h_{12},2-2h|X)|^2$$
$$+Y(h,h_{34})|X^{h-\frac{1}{2}}{}_2F_1(h+h_{34},h-h_{12},2h|X)|^2$$

$$Y(h,h') = \pi\frac{\Gamma(1-2h)\Gamma(h+h')\Gamma(h-h')}{\Gamma(2h)\Gamma(1-h-h')\Gamma(1-h+h')}$$

Results for superconformal blocks

$$g_h^{(0,0)}(X,\bar{X}) = \tilde{C}_{120}\tilde{C}_{034}\left|G_{0,0}^{(o)}(h,h_{12},h_{34}|X)\right|^2 + \tilde{C}_{120}\tilde{C}_{034}\left|G_{0,0}^{(e)}(h,h_{12},h_{34}|X)\right|^2$$

$$G_{0,0}^{(e)} = X^h {}_2F_1(h_{34}+h,h-h_{12},2h|X)\,, G_{0,0}^{(o)} = \frac{1}{2h}G_{0,0}^{(e)}\left(h+\frac{1}{2},h_{12},h_{34}|X\right)$$

$$g_h^{(1,0)}(X,\bar{X}) = \tilde{C}_{120}C_{034}\left|G_{1,0}^{(o)}(h,h_{12},h_{34}|X)\right|^2 + C_{120}\tilde{C}_{034}\left|G_{1,0}^{(e)}(h,h_{12},h_{34}|X)\right|^2$$

$$G_{1,0}^{(e)} = X^{h-\frac{1}{2}} {}_2F_1\left(h_{34}+h-\frac{1}{2},h-h_{12},2h|X\right)$$

$$G_{1,0}^{(o)} = \frac{1}{2h}G_{1,0}^{(e)}\left(h+\frac{1}{2},h_{12},h_{34}|X\right)$$

Correlation functions on a torus

Coordinates

$$z = e^{2\pi i w}\,, \quad q = e^{2\pi i \tau}\,, \quad z \sim qz$$

One-point function

$$\langle \Phi_1(Z_1) \rangle_\tau = \sum_{\Delta_1} \mathrm{str}_{\mathcal{H}_{\Delta_1}} \left[q^{L_0} \bar{q}^{\bar{L}_0} \Phi_1(Z_1, \bar{Z}_1) \right]$$

Two-point function

$$\langle \Phi_1(Z_1) \Phi_2(Z_2) \rangle_\tau = \sum_{\Delta_1} \mathrm{str}_{\mathcal{H}_{\Delta_1}} \left[q^{L_0} \bar{q}^{\bar{L}_0} \Phi_1(Z_1, \bar{Z}_1) \Phi_2(Z_2, \bar{Z}_2) \right]$$

Shadow formalism on a torus

$$W^{h_1}_{\Delta_1}(q,\bar{q},Z_1,\bar{Z}_1) = q^{\Delta_1}\bar{q}^{\Delta_1}\int d^2w_1 d^2\xi_1 \mathcal{V}_{\Delta_1^*,h_1,\Delta_1}(W_1,Z_1,q\cdot W_1)$$

$$\begin{aligned} &W^{h_1 h_2}_{\Delta_1\Delta_2}(q,\bar{q},Z_1,\bar{Z}_1,Z_2,\bar{Z}_2) \\ &= q^{\Delta_1}\bar{q}^{\Delta_1}\int d^2w_1 d^2\xi_1 \mathcal{V}_{\Delta_1^*,h_1,\Delta_2}(W_1,Z_1,W_2) \\ &\times \mathcal{V}_{\Delta_2^*,h_2,\Delta_1}(W_2,Z_2,q\cdot W_1) \end{aligned}$$

$$q\cdot W_i = (qw,\sqrt{q}\xi)$$

$$\langle\Phi_1(Z_1,\bar{Z}_1)\rangle_\tau = \sum_{\Delta_1} C_{\Delta_1 h\Delta_1}\left|B_0(h_1,\Delta_1|q)\right|^2 + \tilde{C}_{\Delta_1 h\Delta_1}\left|B_1(h_1,\Delta_1|q)\right|^2$$

$$\begin{aligned} \langle\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\rangle_\tau = &\sum_{\Delta_1,\Delta_2} C_{\Delta_1 h\Delta_2}C_{\Delta_2 h\Delta_1}|B^{(1)}_{00}(q_1,z_1,z_2)|^2 \\ &+ \sum_{\Delta_1,\Delta_2} C_{\Delta_1 h\Delta_2}C_{\Delta_2 h\Delta_1}|B^{(1)}_{00}(q_1,z_1,z_2)|^2 \end{aligned}$$

One-point block on a torus

$$B_0(h_1, \Delta_1|q) = \frac{1}{c_1(h_1, \Delta_1, z_1)} q^{\Delta_1} \int\limits_0^{z_1} b_0(z_1, w_1) dw_1$$

$$B_1(h_1, \Delta_1|q) = \frac{1}{c_1\left(h_1 + \frac{1}{2}, \Delta_1, z_1\right)} q^{\Delta_1} \int\limits_0^{z_1} b_1(z_1, w_1) dw_1$$

$$c_1(h, \Delta, z) = \frac{\Gamma(2\Delta - h)\Gamma(h)(-1)^{2\Delta}(-z)^h}{\Gamma(2\Delta)}$$

$$b_0(z_1, w) = v_{1-\Delta_1, h_1, \Delta_1}(w_1, z_1, qw_1) + \sqrt{q} v_{\frac{1}{2}-\Delta_1, h_1, \frac{1}{2}+\Delta_1}(w_1, z_1, qw_1)$$

$$b_1(z_1, w) = \left(\frac{1}{2} - 2\Delta_1 + h_1\right) v_{1-\Delta_1, h_1+\frac{1}{2}, \Delta_1}(w_1, z_1, qw_1)$$
$$+ \left(2\Delta_1 - \frac{1}{2} + h_1\right) \sqrt{q} v_{\frac{1}{2}-\Delta_1, h_1+\frac{1}{2}, \Delta_1+\frac{1}{2}}(w_1, z_1, qw_1)$$

Two-point block on a torus

$$B_{00}^{(1)}(q,z_1,z_2) = \mathcal{F}^{h_1,h_2}_{\Delta_1\Delta_2}(q,z_1,z_2)$$
$$-\frac{(\Delta_1+\Delta_2-h_1)(\Delta_1+\Delta_2-h_2)}{4\Delta_1\Delta_2}\mathcal{F}^{h_1,h_2}_{\Delta_1+1/2,\Delta_2+1/2}(q,z_1,z_2)$$

$$B_{00}^{(2)} = \mathcal{F}^{h_1,h_2}_{\Delta_1,\Delta_2+1/2}(q,z_1,z_2) - \frac{\Delta_2}{\Delta_1}\mathcal{F}^{h_1,h_2}_{\Delta_1+1/2,\Delta_2}(q,z_1,z_2)$$

$$\mathcal{F}^{h_1,h_2}\Delta_1,\Delta_2(q,z_1,z_2) = \left(\frac{z_1^{h_2}z_2^{h_1}(1-q)^{h_1+h_2}}{z_{12}^{h_1+h_2}(z_2-qz_1)^{h_1+h_2}}\right)$$
$$\times\,\rho_1^{\Delta_1}\rho_2^{\Delta_2}F_4\left[\begin{array}{c}\Delta_1+\Delta_2-h_1,\Delta_1+\Delta_2-h_2\\ 2\Delta_1,2\Delta_2\end{array}\middle|\rho_1,\rho_2\right]$$

$$\rho_1 = \frac{qz_{12}^2}{z_1z_2(1-q)^2}\,,\quad \rho_2 = \frac{(z_2-qz_1)^2}{z_1z_2(1-q)^2}$$

Conclusions and outlook

- (Super)shadow formalism correctly reproduces known results for 4pt block
- New results for 2pt block on a torus
- Generalization to Ramond sector?
- Dual description for many-point superblocks?
- Is it possible to extend formalism to full (super)Virasoro algebra?

Thank you

Thank you!