Nearly Critical Superfluids and Holography

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and work in progress

Outline

- Introduction/Generalities
- Field Theory Constructions ($N \leq \infty$)
- Holographic Model ($N \to \infty$)
- Conclusions

Motivation

- At strong coupling often no quasiparticles
- Conserved charges and Goldstone modes dominate at long wavelengths
- Correlation length ξ diverges close to a transition $\Rightarrow$ Universality?
- Amplitude mode has gap $\sim \xi^{-2}$ $\Rightarrow$ Effective theory?
- Use holography to carry out microscopic computations
- Lessons for EFT?

Field Theory Setup - Microscopics

- Relativistic field theory (with global $U(1)$) at finite temperature T and chemical potential μ
- Charged operator $\mathcal{O}_\psi$ transforms as $\mathcal{O}_\psi \to e^{-iq\alpha}\,\mathcal{O}_\psi$
- Phase transition with $\langle \mathcal{O}_\psi \rangle \neq 0$ at $T < T_c$
- Couple to external gauge field A_μ and scalar source λ

$$\delta S = \int d^n x \left(J^\mu\,\delta A_\mu + \mathcal{O}_\psi^*\,\delta\lambda + \mathcal{O}_\psi\,\delta\lambda^* \right)$$

Field Theory Setup - Microscopics

- Generating function $W[g_{\mu\nu}, A_\mu, \lambda, \lambda^*]$ depends on external gauge field A_μ, background metric $g_{\mu\nu}$ and complex source λ
- Functional differentiation gives the VEVs

$$\langle T^{\mu\nu}\rangle = \frac{i}{\sqrt{-g}}\frac{\delta W}{\delta g_{\mu\nu}}, \qquad \langle J^\mu\rangle = i\frac{\delta W}{\delta A_\mu}, \qquad \langle \mathcal{O}_\psi\rangle = i\frac{\delta W}{\delta \lambda^*}$$

- Invariance under gauge transformations $\delta A_\mu = -\,\partial_\mu \delta\Lambda, \quad \delta\lambda = i\,q\,\lambda\,\delta\Lambda$

$$\nabla_\alpha \langle J^\alpha\rangle = iq\left(\langle \mathcal{O}_\psi\rangle \lambda^* - \langle \mathcal{O}_\psi^*\rangle \lambda\right)$$

- Invariance under coordinates transformations

$$\nabla_\mu \langle T^{\mu\nu}\rangle = F^{\nu\mu}\,\langle J_\mu\rangle + \nabla^\nu \lambda\,\langle \mathcal{O}_\psi^*\rangle + \nabla^\nu \lambda^*\,\langle \mathcal{O}_\psi\rangle$$

Normal Phase Hydro

- Express $T_{\mu\nu}$ and J_μ as functions of the fluctuations T, v^μ, μ

$$T^{\mu\nu} = (\epsilon + P)\, v^\mu\, v^\nu + P\,\eta^{\mu\nu} + T^{\mu\nu}_{\text{diss}}$$

$$J^\mu = \rho\, v^\mu + J^\mu_{\text{diss}}$$

- Ideal part is fixed by diffeo and gauge invariance

- Dissipative corrections $T^{\mu\nu}_{\text{diss}}$ and J^μ_{diss} based on derivative expansion of T, v^μ, μ

- Thermodynamics fixes P to be the free energy

$$dP = S\, dT + \rho\, d\mu, \qquad \epsilon + P = \mu\,\rho + T\,S$$

- Conservation laws form closed system of $d+2$ equations in $d+1$ dimensions

Superfluid Phase Hydro

- Include phase ϑ of condensate in the description through $\xi = d\vartheta + A$

$$T^{\mu\nu} = (\epsilon + P)\, v^\mu\, v^\nu + P\, \eta^{\mu\nu} + \chi_{JJ}\, \xi^\mu\, \xi^\nu + T^{\mu\nu}_{\text{diss}}$$
$$J^\mu = \rho\, v^\mu - \chi_{JJ}\, \xi^\mu + J^\mu_{\text{diss}}$$
$$\mu = v^\mu\, \xi_\mu - \mu_{\text{diss}}$$

- Thermodynamics fixes

$$dP = S\, dT + \rho\, d\mu + \frac{1}{2}\chi_{JJ}\, d\xi^2, \qquad \epsilon + P = \mu\, \rho + T\, S$$

- Closed system of $d+3$ equations in $d+1$ dimensions
- Close to transition $\chi_{JJ} \to 0$ but derivative expansion breaks down

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Donos, Kailidis

Order Parameter - Current Sector

- Ignore dynamics of stress tensor and conjugate variables T and $v^\mu = (1,0,\ldots)$
- Focus on coupled sector of charge and order parameter
- Normal phase

Charge diffusion

$$\delta J^t = \chi_n \, \delta\mu + \cdots \,, \qquad \delta J^i = -\,\lambda_0^m \, (\partial^i \delta\mu - E^i) + \cdots \,,$$

- Superfluid phase

$$\delta J^t = \chi_b \, \delta\mu - \chi_b^2 \, \zeta_3 \, \partial_t^2 \delta\mu + \cdots \,, \qquad \delta J^i = -\,\chi_{JJ} \, (\partial^i \vartheta + A^i) - \lambda_0^m \, (\partial^i \delta\mu - E^i) + \cdots \,,$$

$$\mu = A_t + \partial_t \vartheta$$

Supercurrent

Hydro away from T_c

- Normal phase

$$\omega = -i\frac{\lambda_0^m}{\chi_b}k^2$$

Charge diffusion

- Superfluid phase

$$\omega_\pm = \pm\sqrt{\frac{\chi_{JJ}}{\chi_b}}k - \frac{i}{2\chi_b}\left(\lambda_0^m + \chi_b\,\chi_{JJ}\,\zeta_3\right)\,k^2$$

Second sound

- Current susceptibility χ_{JJ} goes to zero close to the transition
- Third bulk viscosity ζ_3 blows up
- $\chi_{JJ}\,\zeta_3$ stays finite

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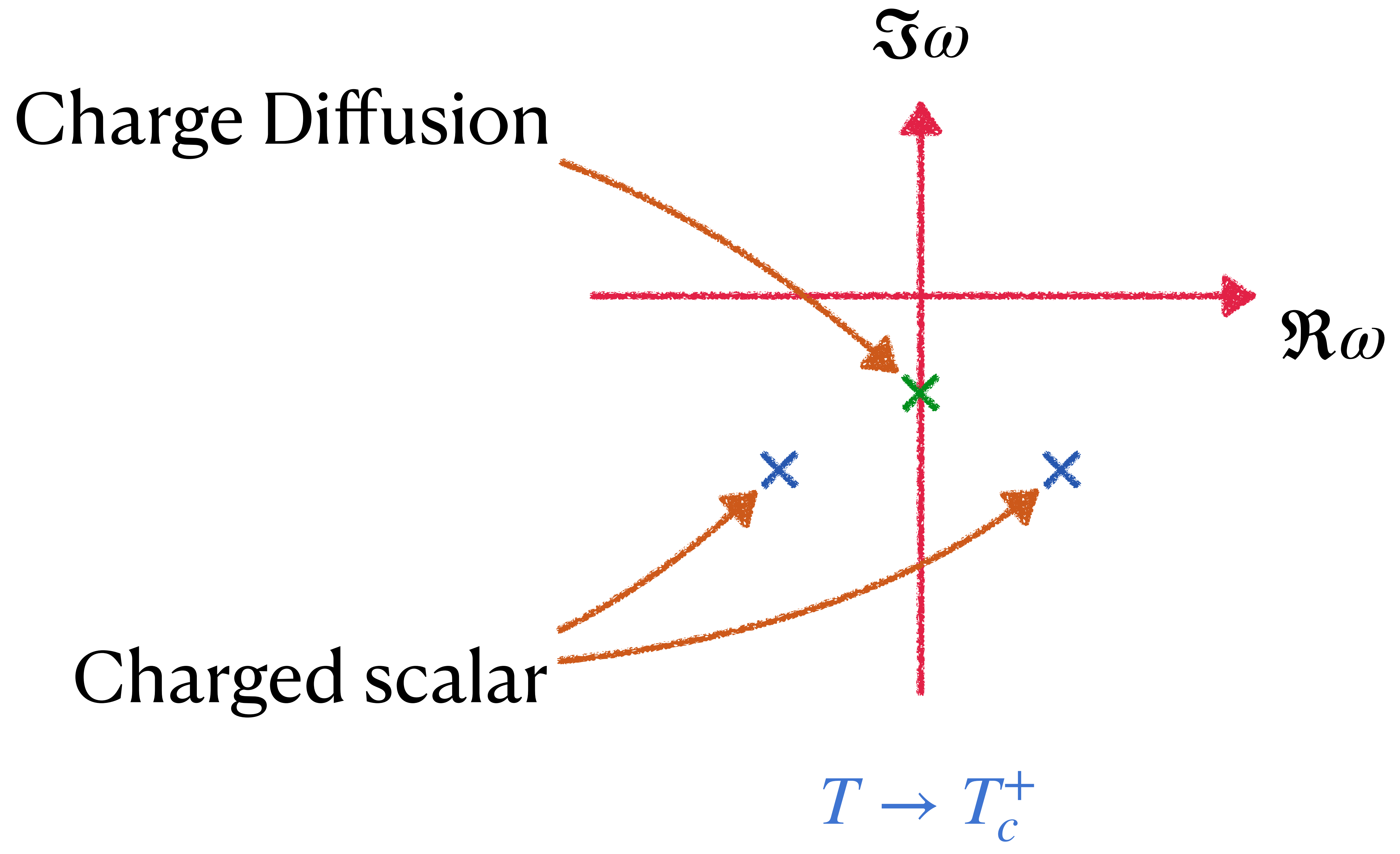
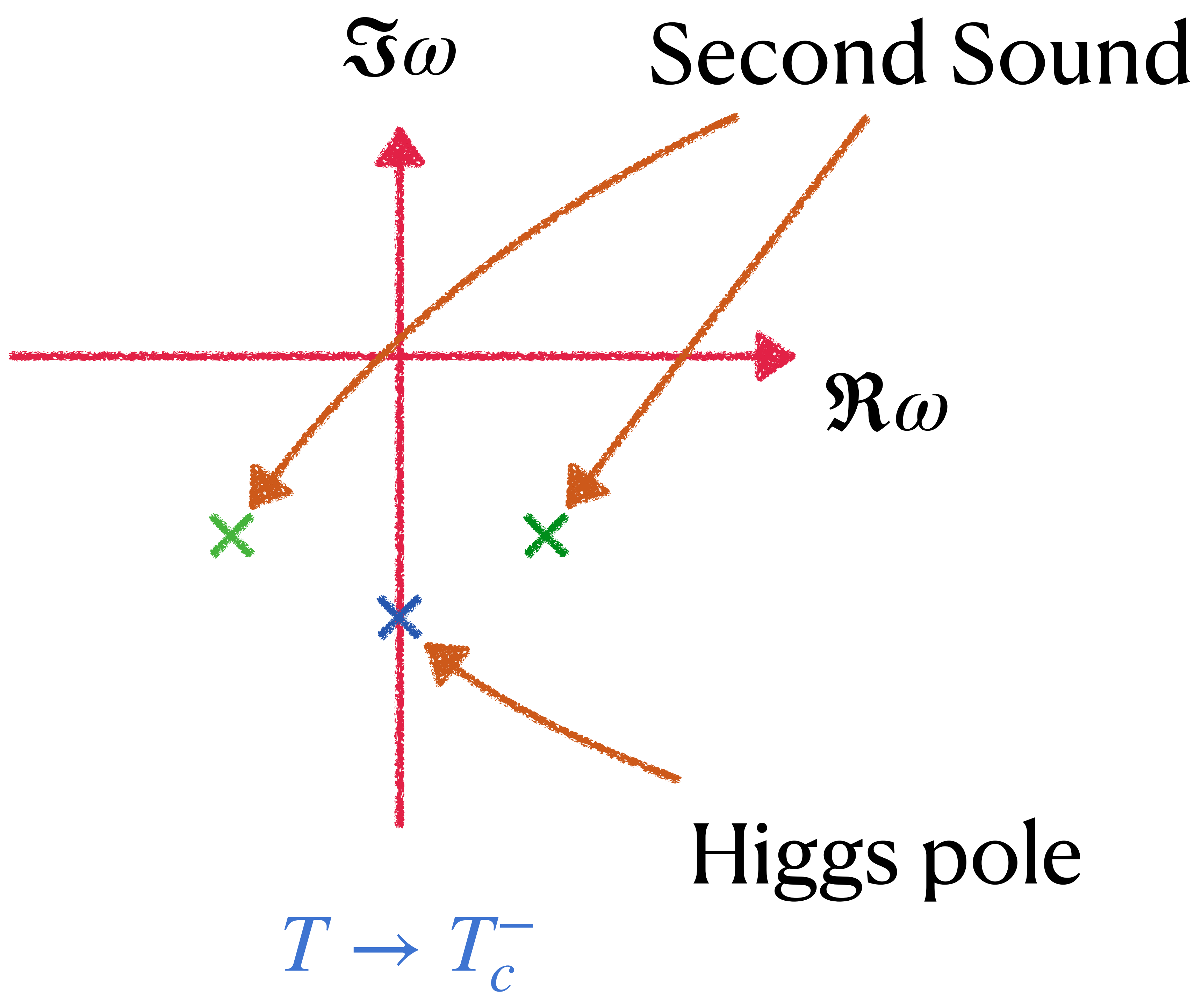
Hydrodynamics as $T \to T_c$





- Away from T_c linear response is dominated by hydro poles
  - Higgs mode is integrated out
- As $T \to T_c^-$ Higgs pole becomes gapless → Transport coefficients blow up
  - Include Higgs mode in hydro description. What is $\partial_t \psi = \cdots$

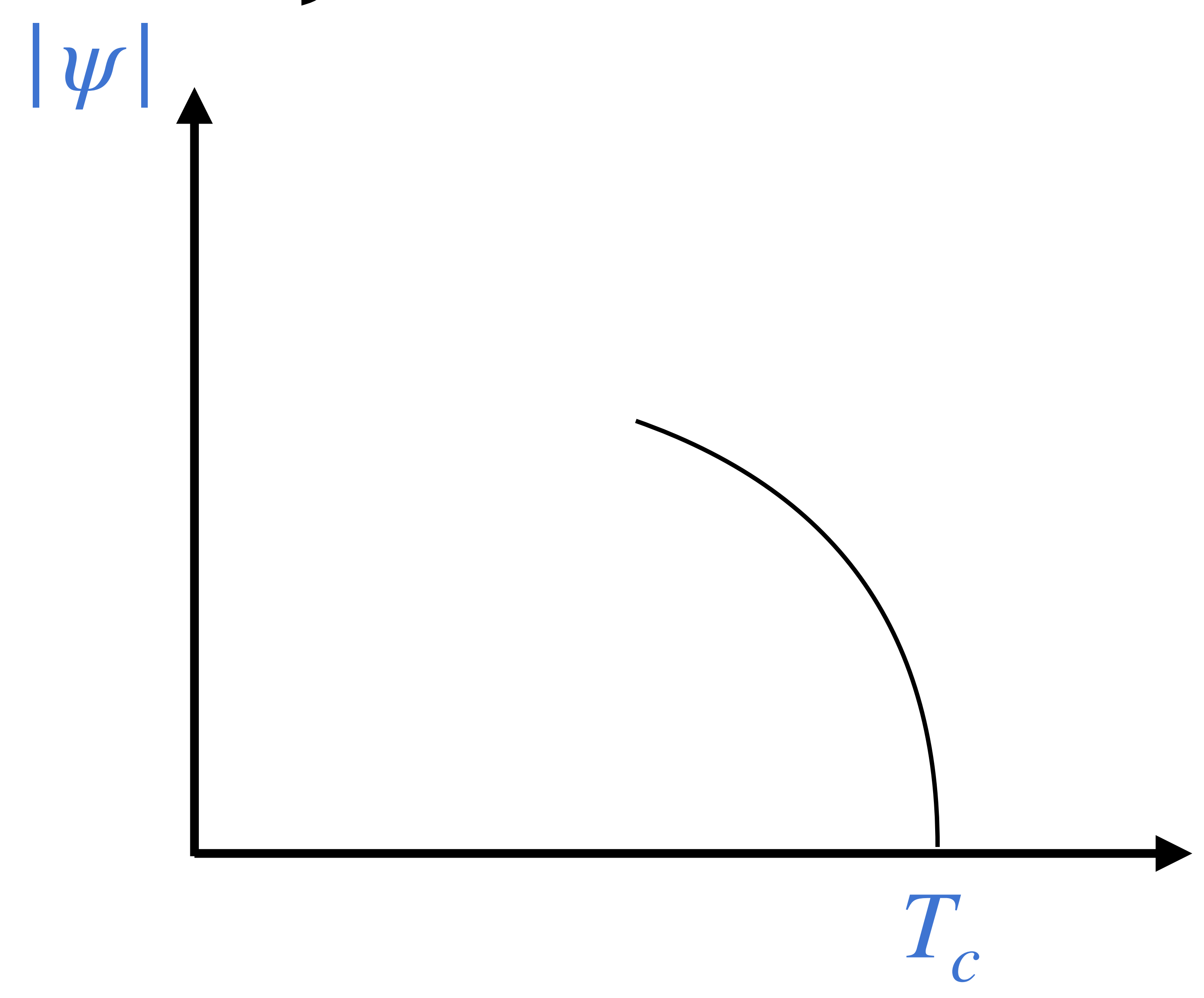
Effective Field Theory

- Identify order parameter ψ and integrate out gapped degrees of freedom
- Introduce UV cutoff Λ
- Long wavelength Boltzmann distribution $P[\psi, \psi^\star] = Z^{-1}\, e^{-\beta\, W_0[\psi,\psi^\star]}$
- Partition function $Z = \int D\psi\, D\psi^\star\, e^{-\beta\, W_0[\psi,\psi^\star]}$
- Free energy $w = -\, k_B T \ln Z$

Ginzburg - Landau Mean Field Theory

$$W_0 = \int d^D x \left(\frac{\hbar^2}{2m(T)} |\vec{\nabla}\psi|^2 + a'(T)|\psi|^2 + \frac{b'(T)}{2}|\psi|^4 \right) + W_n(T, \ldots)$$

$$m(T) = m + \cdots \qquad b'(T) = b' + \cdots \qquad a'(T) = T_c\alpha(1 - T/T_c) + \cdots$$



- Most probable configuration extremises G-L energy potential

$$|\psi_0|^2 = \begin{cases} 0 & , T > T_c \\ -\frac{T_c\alpha}{b'}(1 - T/T_c) & , T < T_c \end{cases}$$

- Second order transition $\Delta F/\mathrm{vol} = -\dfrac{\alpha^2}{2b'}(T - T_c)^2$

Gaussian Approximation

- Expand the G-L potential to second order in fluctuations

$$e^{-\beta w} = e^{-\beta W_0[\psi_0,\psi_0{}^\star]} \int D\delta\psi \, D\delta\psi^\star \, e^{-\beta\,\delta W_0[\delta\psi,\delta\psi^\star]}$$

$$w = W_0[\psi_0, \psi_0^\star] - k_B T \ln \delta f$$

Discontinuity

Divergence for $d < 4$

(or finite N)

- Correlation length $\xi^{-2} \sim m\, b' (T - T_c)$

- Finite wavelengths give diverging corrections as $T \to T_c$

- Interactions become important ⇒ Renormalise by coarse graining $P[\psi, \psi^\star]$

Critical Dynamics - Model A

- Scalar order parameter ϕ_I

- Postulate stochastic kinetic equation

External Force

$$\partial_t \phi_I = -\Gamma_0 \left(\frac{\delta W_0}{\delta \phi_I} - h_I \right) + \theta_I$$

Kawasaki
Halperin, Hohenberg

Damping

Thermal noise

- Introduce finite kinetic coefficient Γ_0

- Thermal fluctuations of modes beyond cutoff Λ captured by Gaussian noise θ_I

$$\langle \theta_I(t, \mathbf{x}) \rangle = 0 \qquad \langle \theta_I(t, \mathbf{x})\, \theta_J(t', \mathbf{x}') \rangle = 2\, \delta_{IJ}\, D\, \delta(t - t')\, \delta(\mathbf{x} - \mathbf{x}')$$

Critical Dynamics - Model A

- Corresponding probability distribution $P[\phi_I]$ follows Fokker-Planck equation

$$\partial_t P = \sum_I \int d^d x \, \frac{\delta}{\delta \phi_I} \left[D \, \frac{\delta P}{\delta \phi_I} + \Gamma_0 \, P \, \frac{\delta W_0}{\delta \phi_I} \right]$$

- System relaxes to Boltzman distribution $P[\phi_I] \sim e^{-\beta F[\phi_I]}$ as long as $D = \beta^{-1} \Gamma_0$
- Sets field space probability current to zero
- Same condition guarantees fluctuation dissipation theorem

Critical Dynamics - Model F

$$\partial_t \psi = -2\,\Gamma_0 \frac{\delta W_0}{\delta \psi} - i g_0\,\psi\,\frac{\delta W_0}{\delta \rho} + \theta$$

Mode Coupling

$$\partial_t \rho = \lambda_0^m\,\nabla^2 \frac{\delta W_0}{\delta \rho} + 2\,g_0\,\mathrm{Im}\left(\psi^\star \frac{\delta W_0}{\delta \psi^\star}\right) + \zeta$$

Halperin, Hohenberg, Ma

- Include conserved charge density ρ $\Rightarrow$ Affects long wavelength physics
- Energy functional at fixed charge density $\Rightarrow \mu = \dfrac{\delta W_0}{\delta \rho}$
- Non-dissipative mode coupling fixed by Josephson relation $\mu = -\,g_0^{-1}\,\partial_t \arg\psi$
- Γ_0 can be complex. What is it?

Critical Dynamics - Model F

$$\partial_t \rho = \lambda_0^m \nabla^2 \frac{\delta W_0}{\delta \rho} + 2\, g_0 \,\mathrm{Im}\left(\psi^\star \frac{\delta W_0}{\delta \psi^\star}\right) + \zeta$$

Define density $W_0 = \int d^d x\, F$

$$\partial_t \rho = -\nabla_i \left(\overbrace{-\lambda_0^m \nabla^i \mu + 2\, g_0 \,\mathrm{Im}\left(\psi^\star \frac{\partial F}{\partial \nabla_i \psi^\star}\right)}^{J^i} \right) + 2\, g_0 \,\mathrm{Im}\left(\psi^\star \frac{\partial F}{\partial \psi^\star}\right) + \zeta$$

Supercurrent

Scalar source

Diffusion

- Model F captures:
  - Current conservation
  - Josephson relation
  - Amplitude mode dynamics

Strategy

Classical /
Mean Field Theory

$$\partial_t \psi = -2\Gamma_0 \frac{\delta W_0}{\delta \psi} - i g_0 \psi \frac{\delta W_0}{\delta \rho} + \theta$$

$$\partial_t \rho = \lambda_0^m \nabla^2 \frac{\delta W_0}{\delta \rho} + 2 g_0 \,\mathrm{Im}\left(\psi^\star \frac{\delta W_0}{\delta \psi^\star}\right) + \zeta$$

Thermal Noise

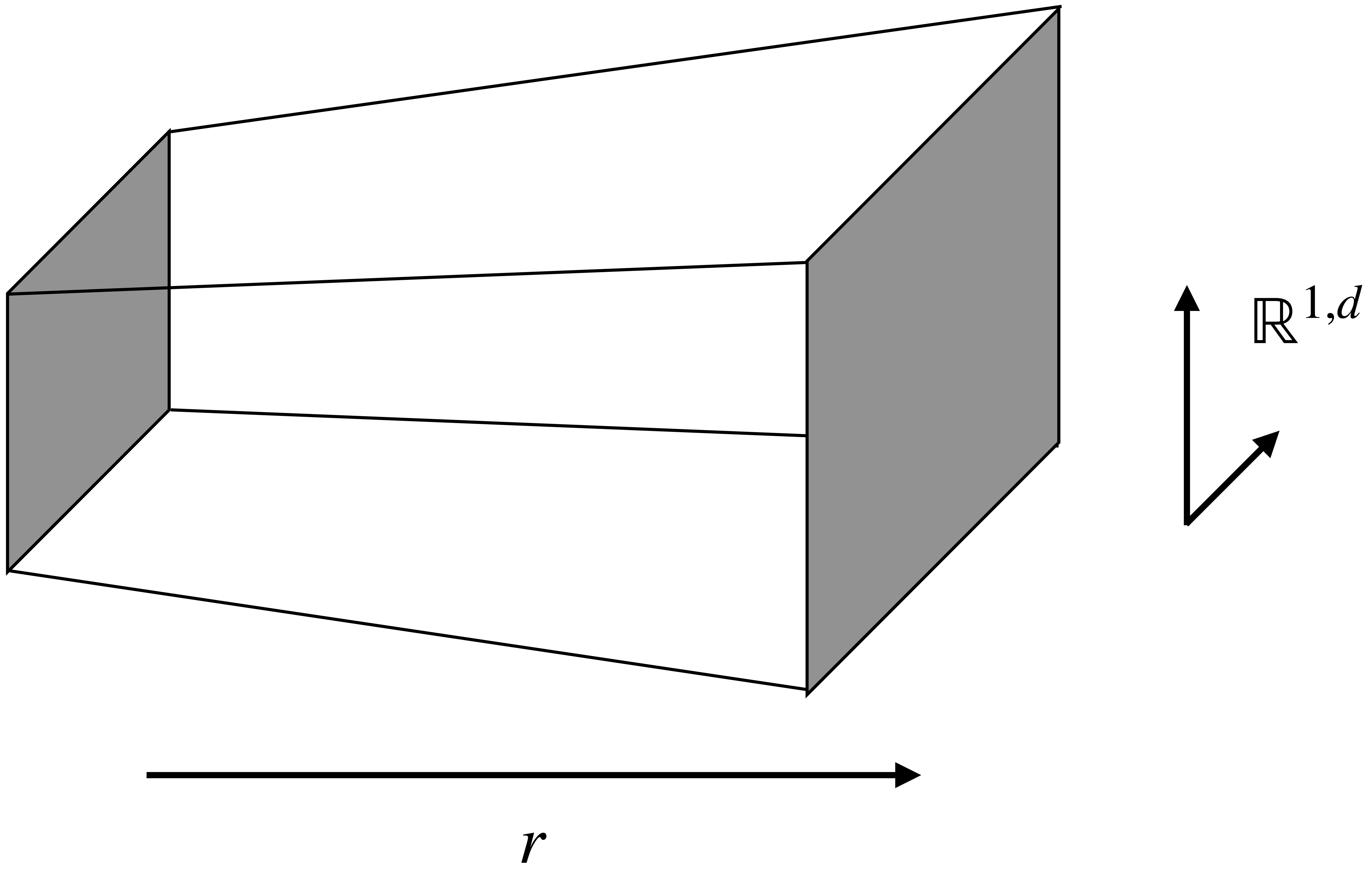
- Classical hydro / holography to develop mean field theory
  - Extract transport coefficients Γ_0 and λ_0^m from black hole horizon data
- Add noise fields to capture thermal/statistical fluctuations
- Develop Keldysh - Schwinger effective action to study thermal noise interactions

Chen-Lin, Delacretaz, Hartnoll

Jain, Kovtun

Donos, Kailidis

AdS/CFT



The vacuum of $CFT_{1,d}$ is modelled by AdS_{d+2}

$$ds^2 = r^2(-dt^2 + d\mathbf{x}_d^2) + \frac{dr^2}{r^2}$$

Setup

Minimal bulk action includes a complex scalar ψ and neutral scalar ϕ

$$\mathscr{L} = -V(\phi, |\psi|^2) - \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2}(D_\mu \psi)(D^\mu \psi)^* - \frac{1}{4}\tau(\phi, |\psi|^2)\, F^{\mu\nu} F_{\mu\nu}$$

$$D_\mu \psi = \nabla_\mu \psi + i\, q\, A_\mu\, \psi$$

- Invariant under $\psi \to e^{-iq\Lambda}\psi,\ A_\mu \to A_\mu + \partial_\mu \Lambda$
- Decouple stress tensor ➜ Fixed background black hole
- Scenario were we deform by neutral scalar ϕ while ψ breaks $U(1)$ spontaneously
- Suppress story about ϕ from now on

Holographic Setup

Boundary conditions of bulk fields correspond to sources in CFT:

- Metric ➜ Source for the stress tensor

$$ds^2 = r^2\,(-dt^2 + d\mathbf{x}_d^2 + \delta g_{\mu\nu}(\mathbf{x})dx^\mu dx^\nu) + \frac{dr^2}{r^2} + \cdots$$

- Gauge Field ➜ Source for $U(1)$ current

$$A = a_\mu(\mathbf{x})\, dx^\mu + \cdots$$

- Massive Scalar ➜ Source for boundary scalar with dimension Δ_ϕ

$$\phi(r, \mathbf{x}) = \frac{\phi_s(\mathbf{x})}{r^{d+1-\Delta_\phi}} + \cdots$$

Holographic Setup

- Boundary theory gets deformed to

$$S[\phi_s, a_\mu, \delta g_{\mu\nu}] = S_{CFT} + \int d^{d+1}x \left(\phi_s(x)\, \mathcal{O}(x) + a_\mu(x)\, J^\mu(x) + \frac{1}{2}\delta g_{\mu\nu}(x) T^{\mu\nu}(x) \right)$$

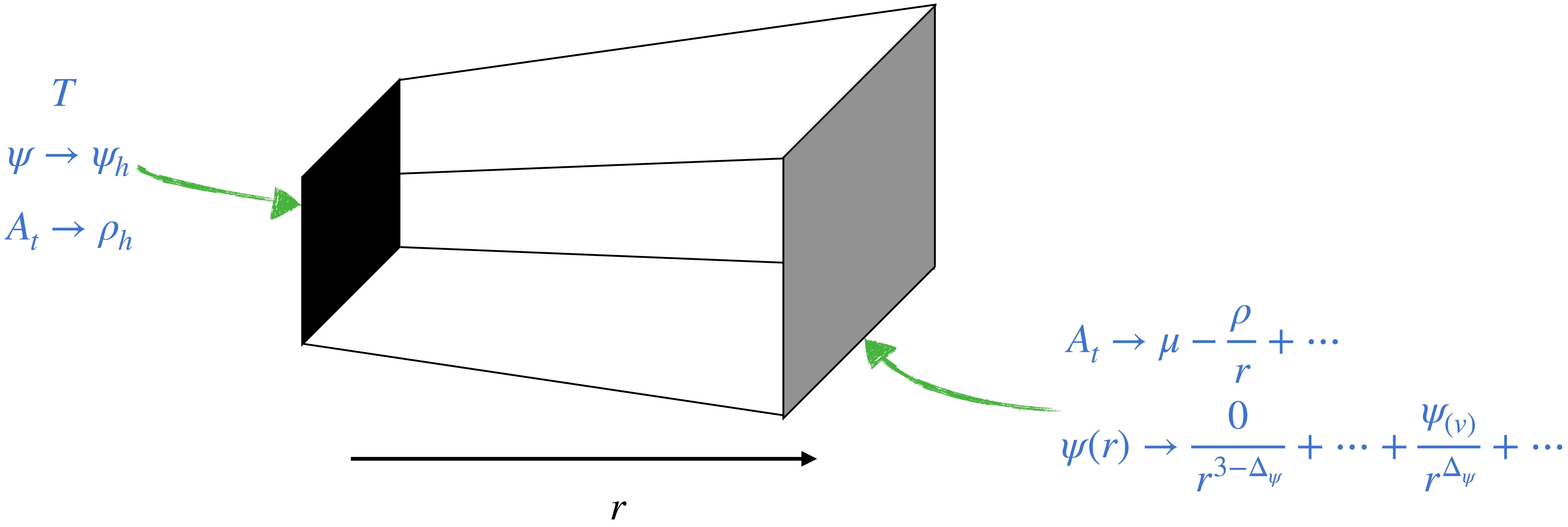
- Holographic conjecture relates partition functions

$$Z_{CFT}[\phi_s, a_\mu, \delta g_{\mu\nu}] = Z_{bulk}[\phi_s, a_\mu, \delta g_{\mu\nu}] \approx e^{iS_{bulk}[\phi_s, a_\mu, \delta g_{\mu\nu}]}$$

- Powerful tool to extract VEVs of operators

$$\langle \mathcal{O}(x) \rangle = \frac{1}{i} \frac{\delta}{\delta \phi_s(x)} \ln Z_{CFT}[\phi_s, a_\mu, \delta g_{\mu\nu}] \approx \frac{\delta}{\delta \phi_s(x)} S_{bulk}[\phi_s, a_\mu, \delta g_{\mu\nu}]$$

Thermal States in AdS/CFT



- Introduce planar event horizon at Hawking temperature T
- Fix μ on the boundary
- Study source free quasinormal modes

Symplectic Current

- Cast the bulk action in terms of first derivatives

$$S_{bulk} = \int_M d^{d+1}x\, \mathscr{L}(\partial\phi, \phi) + counterterms$$

- Vary with respect to bulk field to find

$$\langle \mathcal{O}_\phi \rangle\, \delta\phi_s = \int_{\partial M} d^d x\, \delta\phi\, \frac{\delta \mathscr{L}}{\delta \partial_r \phi} + \cdots$$

- Non-trivial information from knowing on shell value of $\dfrac{\delta \mathscr{L}}{\delta \partial_r \phi}$ close to the boundary
- Think of it as momentum density

Papadimitriou

Symplectic Current

- Need to know variation $\delta\left(\frac{\delta\mathscr{L}}{\delta\partial_r\phi}\right)$ against specific bulk perturbations
- For any two perturbations $\delta_1\phi_A$ and $\delta_2\phi_A$ define the symplectic current density

$$P^\mu = \delta_1\phi_A\,\delta_2\left(\frac{\delta\mathscr{L}}{\delta\partial_\mu\phi_A}\right) - \delta_2\phi_A\,\delta_1\left(\frac{\delta\mathscr{L}}{\delta\partial_\mu\phi_A}\right)$$

- Divergence free when evaluated on-shell

$$\partial_\mu P^\mu = 0$$

- More general way of thinking about the membrane paradigm

Policastro, Son, Starinets

Iqbal, Liu

Donos, Gauntlett

Hydro Modes

- Useful in a hydro/derivative expansion
- Consider set of static solutions e.g. thermodynamic/zero mode perturbations $\delta_1 \phi_A^{(s)}$
- Construct hydro perturbation in derivative expansion

$$\delta_2 \phi_A = e^{-i\epsilon\,\omega\,t + i\epsilon\,k\,x}(\delta_1 \phi_A^{(s)} + \epsilon\,\delta\phi_A^{(1)} + O(\epsilon^2))$$

First dissipative correction

- Construct P^μ out of $\delta_1 \phi_A^{(s)}$ and $\delta_2 \phi_A$
- Expand conservation of P^μ in ϵ, integrate along radial direction to study dynamics of $\delta\phi_A^{(1)}$

Symplectic Current

- Component P^r interesting in holography
- At leading order in boundary derivatives

$$P^r_{\delta_1,\delta_2} = \frac{1}{r^3}\left(\delta_1\varphi^I_{(s)}\,\delta_2\left(\sqrt{-\gamma}\,\langle\mathcal{O}_I\rangle\right) - \delta_2\varphi^I_{(s)}\,\delta_1\left(\sqrt{-\gamma}\,\langle\mathcal{O}_I\rangle\right)\right)$$
$$+\frac{1}{r^3}\frac{1}{2}\left(\delta_1 A_a\,\delta_2\left(\sqrt{-\gamma}\,\langle J^a\rangle\right) - \delta_2 A_a\,\delta_1\left(\sqrt{-\gamma}\,\langle J^a\rangle\right)\right) + \cdots$$
$$+\frac{1}{r^3}\frac{1}{2}\left(\delta_1\gamma_{ab}\,\delta_2\left(\sqrt{-\gamma}\,\langle T^{ab}\rangle\right) - \delta_2\gamma_{ab}\,\delta_1\left(\sqrt{-\gamma}\,\langle T^{ab}\rangle\right)\right) + \cdots$$

- Use known static solutions to read off VEVs of hydro perturbation
- Use to derive effective theories of hydrodynamics

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Donos, Kailidis

Holographic Model F

- Use symplectic current techniques to deduce long wavelength dynamics

Donos, Kailidis

$$\partial_t \psi = -2\Gamma_0 \frac{\delta W_0}{\delta \psi^\star} - i g_0 \psi \frac{\delta W_0}{\delta \rho}\,, \qquad \partial_t \rho = \lambda_0^m \nabla^2 \frac{\delta W_0}{\delta \rho} + 2\, g_0\, \mathrm{Im}(\psi^\star \frac{\delta W_0}{\delta \psi^\star})\,,$$

$$W_0[\psi, \rho] = \int d^2x \left(\frac{w_0}{2} |\nabla \psi|^2 + \frac{\tilde{r}_0}{2} |\psi|^2 + \tilde{u}_0 |\psi|^4 + \frac{1}{2\, C_0} \rho^2 + \gamma_0\, \rho\, |\psi|^2 \right)$$

- Holography reproduces the classical part of Model F
- All constants and transport coefficients fixed by thermodynamics and horizon data

Holographic Model F

$$C_0 = \chi_n \,, \qquad \gamma_0 = -\frac{\Delta\rho}{\chi_n \langle \mathcal{O}_\psi \rangle^2} \,, \qquad w_0 = \frac{\chi_{JJ}}{q_e^2 \langle \mathcal{O}_\psi \rangle^2} \,, \qquad \tilde{u}_0 = -\frac{\Delta E}{\langle \mathcal{O}_\psi \rangle^4} \,, \qquad \tilde{r}_0 = 4\frac{\Delta w_{FE}}{\langle \mathcal{O}_\psi \rangle^2} \,,$$

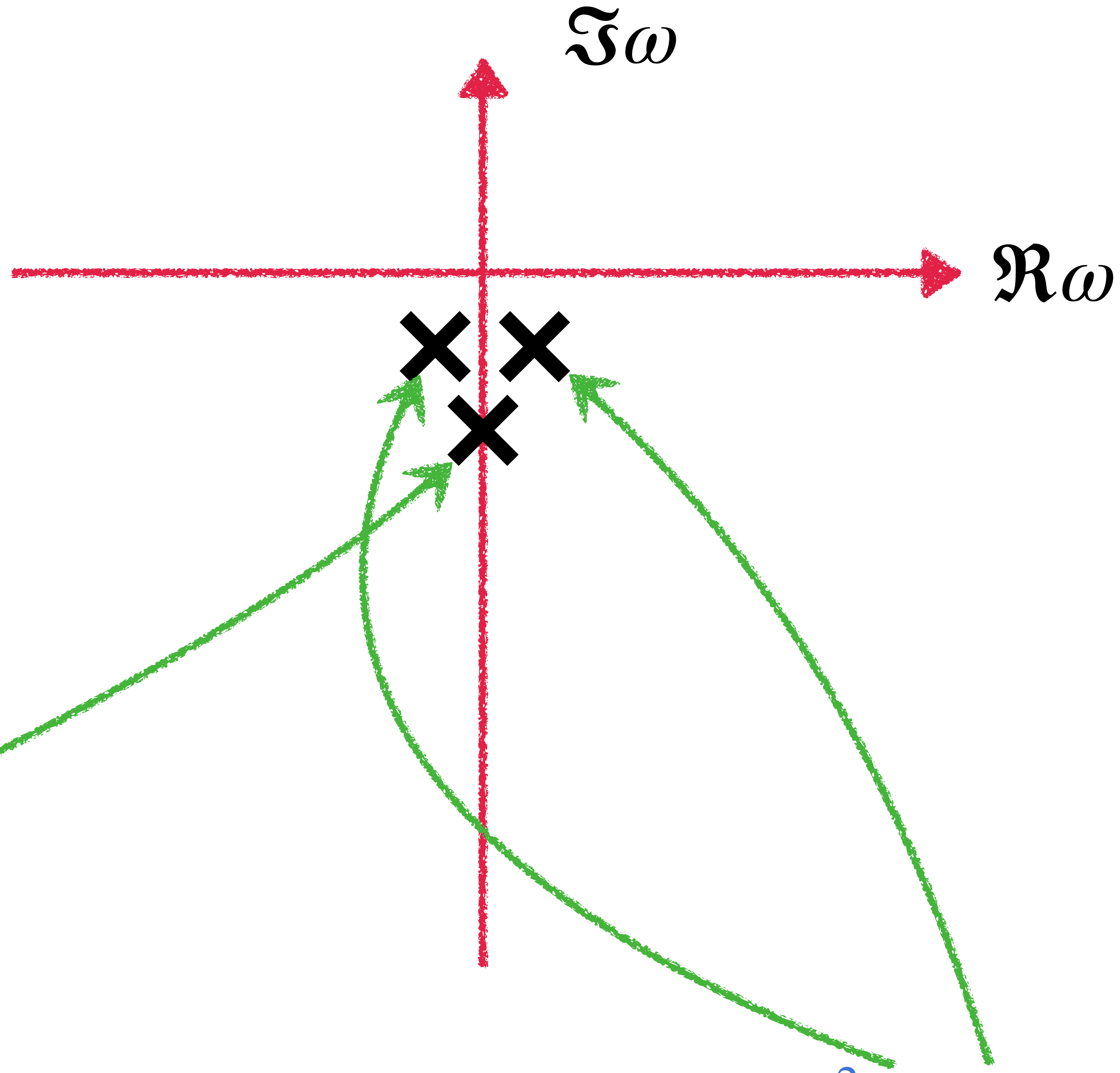
$$\Gamma_0 = \frac{\langle \mathcal{O}_\psi \rangle^2}{\zeta^\star} \,, \qquad \lambda_0^m = \tau_h \,,$$

$$\zeta = \frac{s_c}{4\pi} |\psi_h|^2 + i\,\frac{2}{q_e}\,\left(\varrho_* - \varrho_{h*}\right)$$

- Defined $\Delta R = R(\text{broken phase}) - R(\text{normal phase})$ at fixed T and μ
- The kinetic coefficient Γ_0 is finite at the transition
- Integrate out amplitude to find $\zeta_3 \propto \langle \mathcal{O}_\psi \rangle^{-2}$

Modes away from critical point

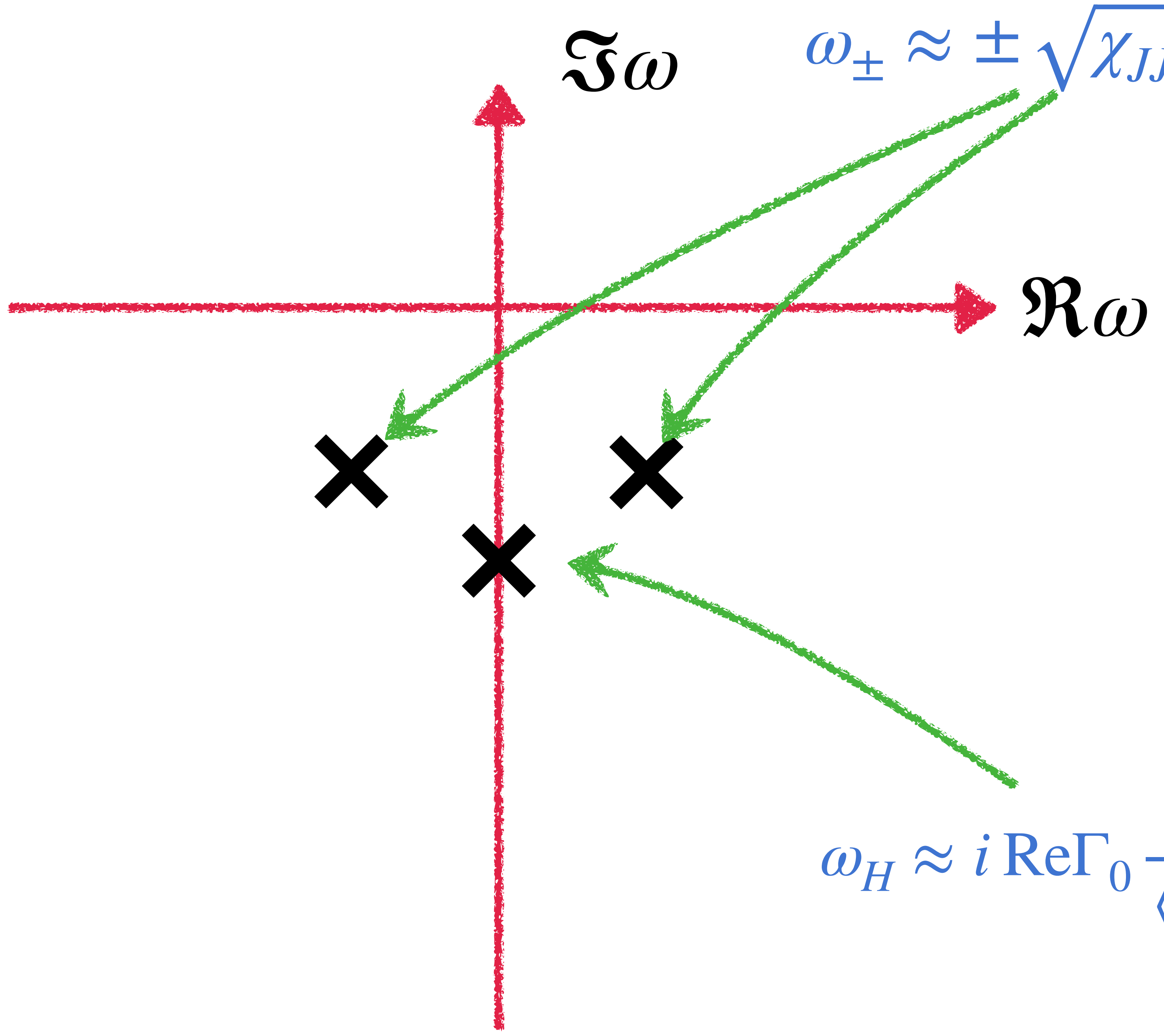
$$T > T_c$$



$$\omega \approx -i\,\frac{\lambda_0^m}{\chi_n}\,k^2$$

$$\omega \approx -i\,w_0\,\Gamma_0\,k^2 + (\pm a - i\,b^2)(T - T_c) + \cdots$$

$$T < T_c, \quad k^2 \ll |T - T_c|$$



$$\omega_\pm \approx \pm\sqrt{\chi_{JJ}/\chi_n}\,k + \cdots$$

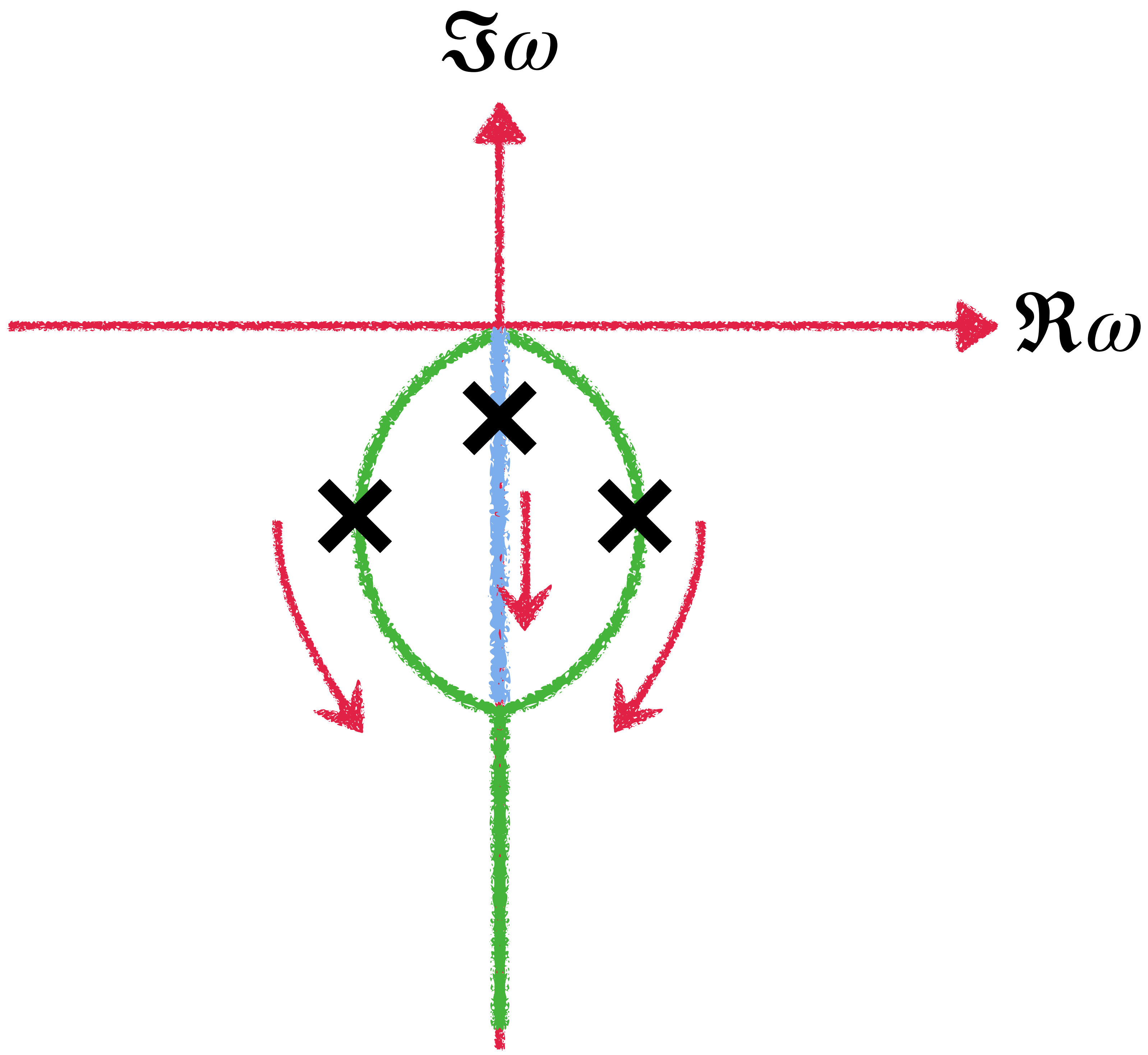
$$\omega_H \approx i\,\mathrm{Re}\Gamma_0\,\frac{8\,\Delta E}{\langle \mathcal{O}_\psi \rangle^2} + \cdots$$

Crossover of Neutral Superfluid

- Gapped mode ω_H and two sound modes $\omega_\pm$ for $k \ll k_c$

- Collision of sound modes at $k_c^2 = \dfrac{4\chi_{JJ}\chi_n}{\left(\lambda_0^m - w_0 \mathrm{Re}\Gamma_0 \chi_n\right)^2}$

- At $k \gg k_c$



$$\omega_\pm \begin{cases} \longrightarrow -i\dfrac{\lambda_0^m}{\chi_n}k^2 + \cdots & \text{Charge Diffusion} \\ \longrightarrow -i\,w_0\Gamma_0 k^2 + \cdots \end{cases}$$

$$\omega_H \longrightarrow -i\,w_0\Gamma_0 k^2 + \cdots$$

Agree with double pole of order parameter fluctuations as $T \to T_c^+$

Numerical Checks

- Check for sound pole dispersion relations

Conclusions & Outlook

- Holographic (probe) nearly critical superfluids captured by (mean field) Model F
- Third bulk viscosity diverges close to the transition. Thermal fluctuations?
- Include normal fluid in the description (KS, holography…)
- New fixed points?