Entanglement Entropy of Scalar Fields in de Sitter space

Dimitrios Katsinis

National & Kapodistrian University of Athens

Xmas Theoretical Physics Workshop @Athens 2024

Relevant Previous Works

1. *Entanglement of harmonic systems in squeezed states*, with G. Pastras & N. Tetradis, JHEP 10 (2023) 039
2. *Entanglement entropy of a scalar field in a squeezed state*, with G. Pastras & N. Tetradis, JHEP 10 (2024) 173
3. *Entanglement in cosmology*, with K. Boutivas, G. Pastras & N. Tetradis, JCAP 04 (2024) 017
4. *Entanglement in (1+1)-dimensional free scalar field theory: Tiptoeing between continuum and discrete formulations*, with G. Pastras, Phys.Rev.D 110 (2024) 8
5. *Entanglement Entropy as a Probe Beyond the Horizon*, with K. Boutivas, G. Pastras & N. Tetradis, under review
6. *A Numerical Calculation of Entanglement Entropy in de Sitter Space*, with K. Boutivas, G. Pastras & N. Tetradis, under review

Introduction

Entanglement Entropy

- Subsystem: A
- Complementary Subsystem / Environment: A^c
- Overall system $A \cup A^c$, described by the Density Matrix ρ
- Reduced Density matrix $\rho_A = \mathrm{Tr}_{A^c}[\rho]$
- Entanglement Entropy: $S_A = -\mathrm{Tr}_A[\rho_A \ln \rho_A]$
- When the overall system lies in a **pure state**, Entanglement Entropy is a measure of quantum Entanglement, i.e. it quantifies quantum correlations.
- For any system in pure state: $S_A = S_{A^c}$

Basic features

- For systems with discrete d.o.f. (say N d.o.f.) that have finite dimensional Hilbert space $\mathcal{H}$, e.g. spins, just apply the definitions. S_A is finite,

$$S_A \leq \min(n, N-n)\, ln\left(\dim\mathcal{H}\right), \qquad n : \#\ \text{of d.o.f. in } A.$$

- For scalar QFT in $d+1$ dimensions the Entanglement Entropy of the **ground state** is divergent and the leading divergence is proportional to the area of the system [Sorkin, Bombelli, Srednicki,. . .]:

$$S = a_{d-1}\frac{R^{d-1}}{\epsilon^{d-1}} + a_{d-3}\frac{R^{d-3}}{\epsilon^{d-3}} + \cdots + a' \ln\frac{R}{\epsilon} + a_0 .$$

The red term exists only for odd d, the coefficient a' is universal. When d is even the blue term is universal.

How to calculate Entanglement Entropy in QFT

Methodologies:

1. Replica trick: Calculate $\mathrm{Tr}\,\rho_A^n$ as an analytic function of n, then

$$S_A = \lim_{n\to 1^+} \frac{\mathrm{Tr}\,\rho_A^n}{1-n}$$

The trace of ρ_A^n can be calculated by glueing multiple copies of the system, hence the name "replica". Works mostly for CFTs. For 2d CFTs in the ground state [Callan, Wilczek, Cardy, Calabrese]:

$$S = \frac{c}{3 \text{ or } 6} \ln\left(\frac{L}{\pi\epsilon} \sin\frac{\pi\ell}{L}\right) + c',$$

where c is the central charge. 6 for 1 boundary, 3 for 2 boundaries.

2. Holography [Ruy, Takayanagi,...]
3. Direct approach / just apply the definitions

Pros and Cons

We will discretize the theory in order to get a quantum mechanical system.

- This approach is in principle applicable in any case.
- For free theories and Gaussian states the problem is solvable.
- We calculate the spectrum of the reduced density matrix, not just the Entanglement Entropy.
- We work with real time.
- More suitable for numerical calculations. A thorough analysis is required in order to make predictions for the continuous theory or derive analytic results.

Outline

1. Entanglement Entropy in Flat Space
2. Entanglement Entropy in de Sitter Space
3. Numerical Calculation - Analysis
4. Results
5. Conclusions and ongoing work (AdS space)

Entanglement in Flat space

Scalar Field in Flat Space

- Subsystem A: the d.o.f inside a spatial region
- Subsystem A^c: the d.o.f outside of this spacial region
- The subsystems are separated by an entangling surface.
- We consider a massless scalar field in flat space:

$$\mathcal{S} = \frac{1}{2} \int dt\, d^3\mathbf{x} \left(\dot{\phi}^2 - (\nabla\phi)^2 \right)$$

- Spherical entangling surface ⇒ spherical symmetry ⇒ expand the field using (real) spherical harmonics

$$\phi_{\ell m}(r) = r \int d\Omega\, Y_{\ell m}(\theta, \varphi)\, \phi(\mathbf{x}),$$

$$\pi_{\ell m}(r) = r \int d\Omega\, Y_{\ell m}(\theta, \varphi)\, \pi(\mathbf{x}).$$

Discretization

- We discretize (only) the radial coordinate, i.e. $r_j = j\epsilon$, where $j = 1, 2, \ldots, N$.
- We have to introduce an IR cutoff $L = (N+1)\epsilon$.
- The d.o.f. are

$$\phi_{\ell m}(j\epsilon) \to \phi_{\ell m,j}, \quad \pi_{\ell m}(j\epsilon) \to \frac{\pi_{\ell m,j}}{\epsilon}.$$

- Subsystem A: $i = 1, \ldots, n$. Subsystem A^c: $i = n+1, \ldots, N$.
- The entangling surface lies between $n\epsilon$ and $(n+1)\epsilon$, thus

$$R = \left(n + \frac{1}{2}\right)\epsilon \equiv n_R \epsilon.$$

Hamiltonian

- The Hamiltonian reads:

$$H = \frac{1}{2\epsilon} \sum_{\ell,m} \sum_{j=1}^{N} \left[\pi_{\ell m,j}^2 + \frac{\ell(\ell+1)}{j^2} \phi_{\ell m,j}^2 + \left(j + \frac{1}{2} \right)^2 \left(\frac{\phi_{\ell m,j+1}}{j+1} - \frac{\phi_{\ell m,j}}{j} \right)^2 \right].$$

- The various angular-momentum sectors do not interact with each other.
- The index m does not appear at all in the dynamics.
- The Entanglement Entropy is given by

$$S(n,N) = \sum_{\ell=0}^{\infty} (2\ell+1) S_\ell(n,N),$$

where S_ℓ is calculated for a single ℓ-sector.

The Discrete System

- The Hamiltonian describes a system of coupled harmonic oscillators

$$H = \frac{1}{2}\mathbf{p}^T\mathbf{p} + \frac{1}{2}\mathbf{x}^T\mathbf{K}\,\mathbf{x}.$$

- The coordinates of these oscillators are the values of the field

$$x_i = \phi_{\ell m,i} = \phi_{\ell m}\left(j\epsilon\right).$$

- For the $(3+1)$-dimensional theory

$$K_{ij} = \left[2 + \frac{\ell\left(\ell+1\right)+\frac{1}{2}}{i^2} - \frac{1}{4}\delta_{i,1}\right]\delta_{i,j} - \frac{\left(i+\frac{1}{2}\right)^2}{i\left(i+1\right)}\delta_{i+1,j} - \frac{\left(j+\frac{1}{2}\right)^2}{j\left(j+1\right)}\delta_{i,j+1}.$$

- For the $(1+1)$-dimensional theory $K_{ij}^{(\mathrm{1d})} = 2\delta_{i,j} - \delta_{i+1,j} - \delta_{i,j+1}.$

The State

- The ground state is

$$\Psi\left(\mathbf{x}\right) \sim \exp\left(-\frac{1}{2}\mathbf{x}^T \Omega\, \mathbf{x}\right), \quad \Omega = K^{1/2}.$$

- We introduce the block-form notation

$$\Omega = \begin{pmatrix} \Omega_A & \Omega_B \\ \Omega_B^T & \Omega_C \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \mathbf{x}_A \\ \mathbf{x}_C \end{pmatrix}.$$

- The reduced density matrix reads

$$\rho_A\left(\mathbf{x}_A; \mathbf{x}'_A\right) \sim \exp\left[-\frac{1}{2}\left(\mathbf{x}_A^T \gamma \mathbf{x}_A + \mathbf{x}_A'^T \gamma \mathbf{x}'_A\right) + \mathbf{x}_A'^T \beta \mathbf{x}_A\right],$$

where $\gamma = \Omega_A - \frac{1}{2}\Omega_B^T \Omega_C^{-1} \Omega_B$ and $\beta = \frac{1}{2}\Omega_B^T \Omega_C^{-1} \Omega_B$.

Entanglement Entropy

- The spectrum can be derived analytically. It reads

$$\lambda_{\{m_1,m_2,\ldots,m_n\}} = \prod_{i=1}^{n} (1-\xi_i)\,\xi_i^{m_i}, \text{ where } \xi_i = \frac{\beta_i}{1+\sqrt{1-\beta_i^2}}.$$

- β_i are the eigenvalues of the matrix $\tilde{\beta} = \gamma^{-1}\beta$.
- The contribution of a single ℓ-sector to the entanglement entropy reads

$$S_\ell = -\sum_{i=1}^{n} \left(\ln(1-\xi_i) + \frac{\xi_i}{1-\xi_i} \ln \xi_i \right).$$

- One can show that $\gamma^{-1}\beta = \frac{M-I}{M+I}$, where $M = \left(\Omega^{-1}\right)_A \Omega_A$.
- Notice that

$$\Omega^{-1}_{ij} \propto \langle x_i x_j \rangle \propto \langle \phi_{\ell m,i} \phi_{\ell m,j} \rangle, \quad \Omega_{ij} \propto \left\langle \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \right\rangle \propto \langle \pi_{\ell m,i} \pi_{\ell m,j} \rangle.$$

Entanglement in dS

Scalar Field in dS

- Consider a FRW background:

$$ds^2 = a^2(\tau)\left(d\tau^2 - dr^2 - r^2 d\Omega^2\right),$$

  where τ is the conformal time.
- We work in comoving coordinates.
- Consider a massless scalar field in this background; after the field redefinition:

$$\phi(\tau, \mathbf{x}) = f(\tau, \mathbf{x}) / a(\tau).$$

- The action reads:

$$\mathcal{S} = \frac{1}{2}\int d\tau\, d^3\mathbf{x}\left(\dot{f}^2 - (\nabla f)^2 + \frac{\ddot{a}}{a} f^2\right).$$

- Time-dependent mass term: $-(\ddot{a}/a)f^2$.

The Discrete System

- We discretize the theory as we did for flat space.
- It turns out that the Hamiltonian reads

$$H = \frac{1}{2\epsilon} \sum_{\ell,m} \sum_{j=1}^{N} \left[\pi_{\ell m,j}^2 + \left(\frac{\ell(\ell+1)}{j^2} - \epsilon^2 \frac{\ddot{a}}{a} \right) f_{\ell m,j}^2 + \left(j + \frac{1}{2} \right)^2 \left(\frac{f_{\ell m,j+1}}{j+1} - \frac{f_{\ell m,j}}{j} \right)^2 \right].$$

- For de Sitter space

$$a(\tau) = -\frac{1}{H\tau}, \quad \frac{\ddot{a}}{a} = \frac{2}{\tau^2}.$$

The State

- The dynamics of each normal mode of the theory is determined by the Hamiltonian

$$H^{\mathrm{dS}}_{\text{normal mode}} = \frac{1}{2}p^2 + \frac{1}{2}\left(\omega_0^2 - \frac{2\epsilon^2}{\tau^2}\right)x^2 .$$

- Bunch - Davies vacuum: The wave function of each mode solves the time dependent Schrödinger equation with the ground state of the flat space as initial condition, i.e. for $\tau \to -\infty$.
- It turns out that the wavefunction of the system reads

$$\Psi\left(\mathbf{x}\right) \propto \exp\left(-\frac{1}{2}x^T W x\right),$$

where the matrix W is **complex** and symmetric:

$$W = \Omega\left[I - \left(I - \frac{i}{\tau}\Omega^{-1}\right)\left(I + \Omega^2\tau^2\right)^{-1}\right], \qquad \Omega = K^{1/2}.$$

- Ω is the same matrix appearing in the flat-space calculation.

The spectrum - Entanglement Entropy

- The spectrum can be calculated by generalizing the flat-space methodology since W is complex [Katsinis, Pastras, Tetradis JHEP 10 (2023) 039].
- The spectrum can also be derived using correlation functions.
- Let $\mathcal{M} = 2iJ\,\mathrm{Re}M$, where the covariance matrix M and J are given by

$$M = \begin{pmatrix} \langle x_i x_j \rangle & \langle x_i \pi_j \rangle \\ \langle x_i \pi_j \rangle_T & \langle \pi_i \pi_j \rangle \end{pmatrix}, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

- The eigenvalues of $\mathcal{M}$ come in pairs $\pm\lambda_i$ and they all satisfy $|\lambda_i| \geq 1$.
- The entanglement entropy is given by

$$S = \sum_{i=0}^{n} \left(\frac{\lambda_i + 1}{2} \ln \frac{\lambda_i + 1}{2} - \frac{\lambda_i - 1}{2} \ln \frac{\lambda_i - 1}{2} \right),$$

  where λ_i are the positive eigenvalues of the matrix $\mathcal{M}$.
- We have to solve a $2n \times 2n$ eigenvalue problem. In flat space $\mathrm{Im}\,W = 0 \Rightarrow \mathrm{Re}\,\langle x_i \pi_j \rangle = 0$, thus the problem is $n \times n$.

Expansion for Early Times

- For the Bunch - Davies vacuum:

$$\mathcal{M} = i \begin{pmatrix} \frac{\left(\Omega^{-3}\right)_A}{\tau^3} & (\Omega)_A - \frac{\left(\Omega^{-1}\right)_A}{\tau^2} + \frac{\left(\Omega^{-3}\right)_A}{\tau^4} \\ -\left(\Omega^{-1}\right)_A - \frac{\left(\Omega^{-3}\right)_A}{\tau^2} & -\frac{\left(\Omega^{-3}\right)_A}{\tau^3} \end{pmatrix}.$$

- We consider the eigenvalue problem of $\mathcal{M}^2$, where it is manifest that the perturbative parameter is $1/\tau^2$.
- In the flat-space limit:

$$\begin{pmatrix} \mathcal{M}^{(0)T} & 0 \\ 0 & \mathcal{M}^{(0)} \end{pmatrix} \begin{pmatrix} v_i^{(0)} \\ w_i^{(0)} \end{pmatrix} = \Lambda_i^{(0)} \begin{pmatrix} v_i^{(0)} \\ w_i^{(0)} \end{pmatrix}, \quad \mathcal{M}^{(0)} = \left(\Omega^{-1}\right)_A (\Omega)_A.$$

- $w_i^{(0)}$ and $v_i^{(0)}$ are the right and left eigenvectors of the matrix $\mathcal{M}^{(0)}$, respectively.

Expansion for Early Times 2

- The eigenvalues in de Sitter case are

$$\lambda_i = \lambda_i^{(0)} + \frac{1}{\tau^2}\frac{\Lambda_i^{(2)}}{2\lambda_i^{(0)}} + \dots , \qquad \lambda_i^{(0)} = \sqrt{\Lambda_i^{(0)}}$$

  where $\lambda_i^{(0)}$ are the eigenvalues of the flat-space problem.

- $\Lambda_i^{(2)}$ are the leading de Sitter corrections:

$$\Lambda_i^{(2)} = \frac{v_i^{(0)T}\left[\left(\Omega^{-3}\right)_A\left(\Omega\right)_A - \left(\Omega^{-1}\right)_A^2\right] w_i^{(0)}}{v_i^{(0)T} w_i^{(0)}} .$$

- The entanglement entropy of each angular-momentum sector assumes the form

$$S_{\ell,\mathrm{dS}} = S_{\ell,\mathrm{flat}} + \frac{1}{\tau^2}\sum_{i=0}^{n}\frac{\Lambda_{\ell,i}^{(2)}}{2\lambda_{\ell,i}^{(0)}}\operatorname{arccoth}\lambda_{\ell,i}^{(0)} + \dots .$$

The $\ell = 0$ sector

- For the $\ell = 0$ sector one can calculate Ω^m analytically:

$$\Omega^m(x, x') = \frac{2}{L} \sum_{k=1}^{\infty} \frac{k^m \pi^m}{L^m} \sin \frac{k\pi x}{L} \sin \frac{k\pi x'}{L}.$$

- Let us focus on $\Omega^{-3}(x, x')$:

$$\Omega^{-3}(x, x') = \frac{L^2}{2\pi^3} \left[\mathrm{Li}_3 \left(e^{-i\frac{\pi}{L}(x-x')} \right) - \mathrm{Li}_3 \left(e^{-i\frac{\pi}{L}(x+x')} \right) + \mathrm{c.c.} \right].$$

- For $L \gg R$ we obtain

$$\Omega^{-3}(x, x') = \frac{1}{2\pi} \left[6xx' + (x - x')^2 \ln \frac{\pi |x - x'|}{L} - (x + x')^2 \ln \frac{\pi |x + x'|}{L} \right].$$

- Neither $\Omega(x, x')$ nor $\Omega^{-1}(x, x')$ depends on L in this limit.
- The leading dS corrections to flat-space Entanglement Entropy depend on the size of the **overall system** as $\ln L$.

Numerical Calculation - Analysis

General Remarks

- The numerical calculation must address the following issues:
  1. The system has finite size, while we want to probe the infinite size limit.
  2. We can't sum the contributions of infinitely many angular-momentum sectors, we have to extrapolate.
  3. We have a finite lattice spacing, while we want to probe the continuum limit.
- We scan dozens of values of the size of the subsystem n, the size of the overall system N and time instants τ, and sum the contributions of hundreds of thousands of angular-momentum sectors.
- It is unfeasible to perform the calculation in a reasonable amount of time using commercial software, such as Wolfram Mathematica or MathWorks Matlab.
- We wrote code in C++ using the package Eigen for linear algebra.
- C++ is extremely faster that Mathematica or Matlab.
- We use floats of 128-bit precision, which corresponds to 33–36 significant digits.

Finite size effect

- For fixed n we study $S_\ell(n, N)$ as a function of N.
- For flat space it turns out that the leading finite size effect $\propto N^{-2(\ell+1)}$ [Lohmayer,Neuberger,Schwimmer,Theisen].
- For dS space

$$S_{\ell,\mathrm{dS}}(n,\tau,N) = S_{\mathrm{IR}}(n,\tau)\,(\ln N)\,\delta_{\ell,0} + S_{\ell,\mathrm{dS},\infty}(n,\tau) + \sum_{\substack{k=0\\ 2\ell+k\neq 0}}^{k_{\max}} \frac{S^{(k)}_{\ell,\mathrm{dS}}(n,\tau)}{N^{(2\ell+k)}}.$$

- The $N^{-2(\ell+1)}$ becomes $N^{-2\ell}$ smoothly as time evolves.
- For $N_0 = 60$ we estimate that finite size effects are significant only for $\ell \leq 220$, i.e.

$$S_{\ell,\mathrm{dS},\infty}(n,\tau) = S_{\ell,\mathrm{dS}}(n,\tau,N_0), \qquad \ell \geq 221.$$

- The Entanglement Entropy of the $\ell = 0$ sector grows as $\ln N$ when $N \to \infty$, thus **the dependence on the size of the overall system is verified both analytically and numerically**.

$\tau = -\infty$ / Flat Space

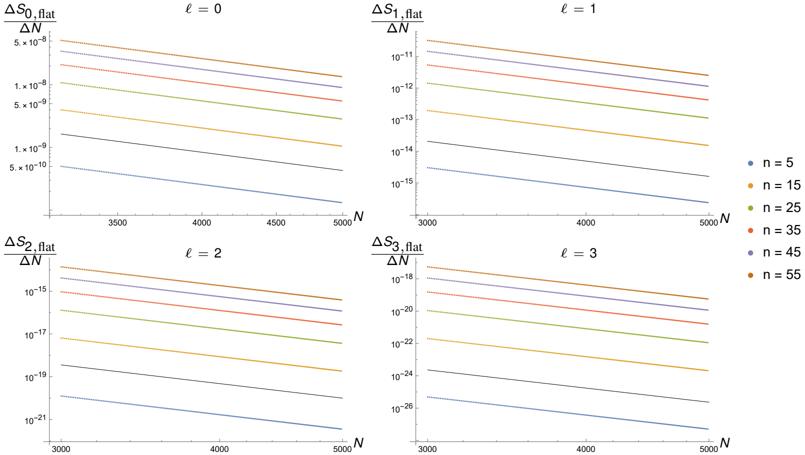

Figure: The finite difference $(S_{\ell,\text{flat}}(n, N+10) - S_{\ell,\text{flat}}(n, N))/10$, which approximates the N-derivative of $S_{\ell,\text{flat}}(n, N)$, for $\ell = 0$ to $\ell = 3$ and various values of n. For comparison, the solid black lines in each plot have slope $-(2\ell + 3)$. It is evident that all lines are parallel, implying that the dominant finite size corrections scale as $N^{-(2\ell+2)}$. A linear fit for each value of n verifies that the slopes are approximately -2.996, -4.992, -6.989, -8.986 for any n and $\ell = 0, 1, 2, 3$, respectively. The small deviations are due to subleading corrections.

$\tau = -3000$

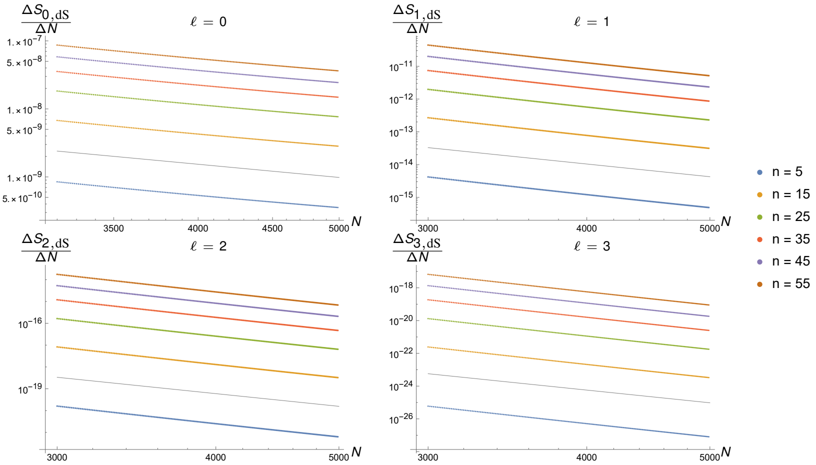

Figure: The finite difference $(S_{\ell,\mathrm{dS}}(n,\tau,N+10) - S_{\ell,\mathrm{dS}}(n,\tau,N))/10$ for various values of n, $\tau = -3000$, and $\ell = 0$ to $\ell = 3$. The slopes are approximately -1.952, -4.214, -6.356 and -8.451 for $\ell = 0$ to $\ell = 3$ respectively. The solid gray lines have slope equal to $-2(\ell + 1)$. All lines of different n exhibit similar behaviour.

$\tau = -1800$

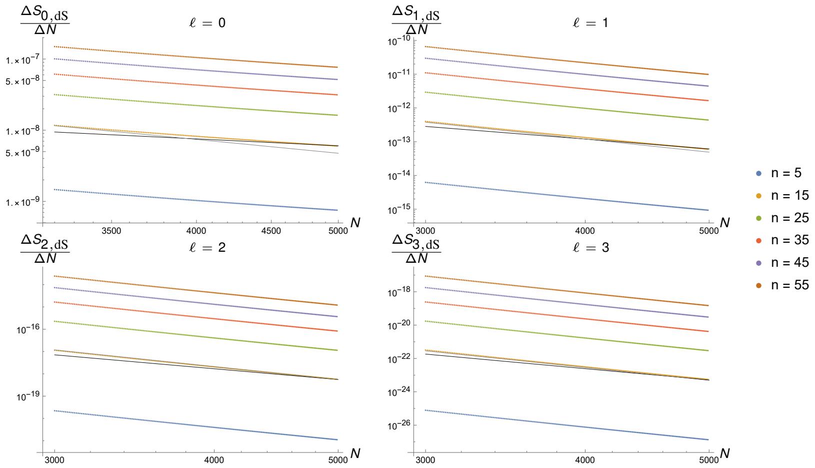

Figure: The finite difference $(S_{\ell,\mathrm{dS}}(n, \tau, N+10) - S_{\ell,\mathrm{dS}}(n, \tau, N))/10$ for various values of n, $\tau = -1800$, and $\ell = 0$ to $\ell = 3$. The slopes are approximately -1.497, -3.721, -5.870 and -7.999 for $\ell = 0$ to $\ell = 3$, respectively. The solid gray lines red have slope equal to $-2(\ell + 1)$, while the black ones $-(2\ell + 1)$. All lines of different n exhibit similar behaviour.

$\tau = -300$

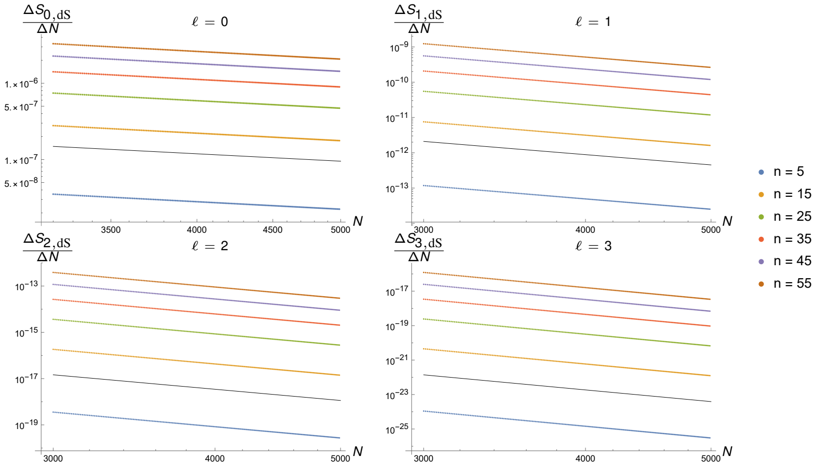

Figure: The finite difference $(S_{\ell,\mathrm{dS}}(n, \tau, N+10) - S_{\ell,\mathrm{dS}}(n, \tau, N))/10$ for various values of n, $\tau = -300$ and $\ell = 0$ to $\ell = 3$. The slopes are approximately -1.037, -3.027, -5.036 and -7.044 for $\ell = 0$ to $\ell = 3$, respectively. The solid black lines have slope equal to $-(2\ell + 1)$. All lines of different n exhibit similar behaviour.

Infinite $\ell_{\max}$ Extrapolation

- We define the truncated sum

$$S_{\mathrm{dS},\infty}(n,\tau;\ell_{\max}) = \sum_{\ell=0}^{\ell_{\max}} (2\ell+1)\, S_{\ell,\mathrm{dS},\infty}(n,\tau).$$

- We are interested in

$$S_{\mathrm{dS},\infty}(n,\tau) = \lim_{\ell_{\max}\to\infty} S_{\mathrm{dS},\infty}(n,\tau;\ell_{\max}).$$

- We study $S_{\mathrm{dS},\infty}(n,\tau;\ell_{\max})$ for $\ell_{\max} = 500k$ to $\ell_{\max} = 1M$ in steps on $10k$.
- It turns out that

$$S_{\mathrm{dS},\infty}(n,\tau;\ell_{\max}) = S_{\mathrm{dS},\infty}(n,\tau) + \sum_{i=1}^{i_{\max}} \frac{1}{\ell_{\max}^{2i}} \left(a_i(n,\tau) + b_i(n,\tau)\ln \ell_{\max}\right).$$

- $S_{\mathrm{dS},\infty}(n,\tau)$ (and $S_{\mathrm{IR}}(n,\tau)$) are the data we are interested in.

Results

Expansion for early times

- After subtracting the finite-size effect and summing the contributions of all angular-momentum sectors

$$S_{\ell,\mathrm{dS}}(n,\tau) = S_{\mathrm{IR}}(n,\tau)\ln N + S_{\mathrm{dS},\infty}(n,\tau) + \ldots .$$

- Our data span instants in time from $\tau = -3000$ to $\tau = -300$ in steps of 50.
- In this regime

$$S_{\mathrm{IR}}(n,\tau) = \sum_{i=0}^{i_{\max}} \frac{S_{\mathrm{IR}}^{(2i)}(n)}{\tau^{2i}} \qquad S_{\mathrm{dS},\infty}(n,\tau) = \sum_{i=0}^{i_{\max}} \frac{S_{\mathrm{dS},\infty}^{(2i)}(n)}{\tau^{2i}} .$$

- As expected, $S_{\mathrm{IR}}^{(0)}(n) = 0$, i.e. no IR effect in flat space. We focus on $S_{\mathrm{IR}}^{(2)}(n)$.
- We also focus on $S_{\mathrm{dS},\infty}^{(0)}(n)$ and $S_{\mathrm{dS},\infty}^{(2)}(n)$.

The $\ell = 0$ sector

- For flat $(1+1)$-dimensional space we expect [Callan, Wilczek, Cardy, Calabrese]

$$S = \frac{c}{6} \ln \frac{r}{\epsilon} + c', \qquad c = 1.$$

- Our flat-space numerical calculation gives $c = 0.999999999975$.
- The flat-space extrapolation of the dS calculation for $\ell = 0$ gives $c = 0.999999999699$.
- The leading coefficient of $\ln N$ is

$$S_{\rm IR}^{(2)}(n) = c_{\rm IR}^{(2)} n_r^2 + \dots, \quad n_r = n + \frac{1}{2}$$

with $c_{\rm IR}^{(2)} = 0.3333366$.

- In general $S_{\rm IR}^{(2i)}(n) \sim n_r^{2i}$, thus

$$S_{\rm IR}(r,\tau) = \sum_{i=1}^{\infty} c_{\rm IR}^{(2i)} \left(\frac{r}{\tau}\right)^{2i}.$$

The $\ell = 0$ sector

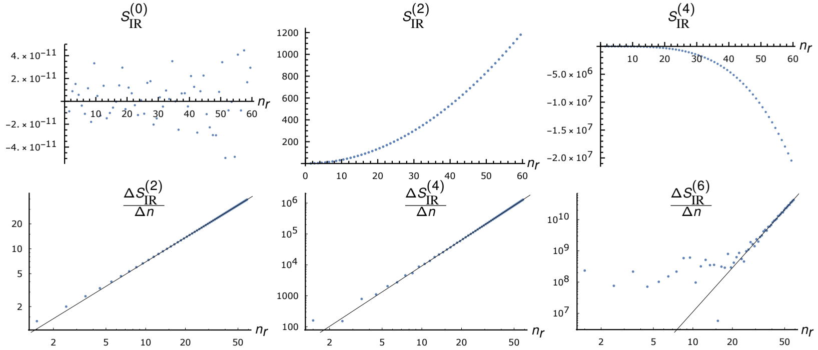

Figure: The first row depicts $S_{\rm IR}^{(0)}(n)$ (left panel), $S_{\rm IR}^{(2)}(n)$ (middle panel) and $S_{\rm IR}^{(4)}(n)$ (right panel). $S_{\rm IR}^{(0)}(n)$ appears as random noise of very small amplitude, in accordance with the fact that no such term is expected in the flat space limit. The second row depicts the finite differences $|S_{\rm IR}^{(2)}(n+1) - S_{\rm IR}^{(2)}(n)|$ (left panel), $|S_{\rm IR}^{(4)}(n+1) - S_{\rm IR}^{(4)}(n)|$ (middle panel) and $|S_{\rm IR}^{(6)}(n+1) - S_{\rm IR}^{(6)}(n)|$ (right panel). The solid black lines are linear fits having slope 0.98, 2.79 and 4.68, respectively, implying that the leading behaviour of the corrections is of the form $(n_r/\tau)^{2i}$. Recall that $n_r = n + 1/2$ and $r = n_r \epsilon$ in the continuous theory.

The $\ell = 0$ sector

- The N-independent part is

$$S^{(2)}_{\mathrm{dS},\infty}(r) = a_2^{(2)\prime} r^2 \ln \frac{r}{\epsilon} + a_2^{(2)} r^2 ,$$

  where $a_2^{(2)\prime} = -0.3334008, \quad a_2^{(2)} = -0.1679215.$
- Combining both the $\ln N$ part and N-independent part

$$S^{(2)}_{\mathrm{dS}}(r) = S^{(2)}_{\mathrm{IR}}(r) \ln \frac{L}{\epsilon} + S^{(2)}_{\mathrm{dS},\infty}(r) = \frac{1}{3} r^2 \ln \frac{L}{r} + a_2^{(2)} r^2 .$$

- Applying the results of [Katsinis, Pastras Phys.Rev.D 110 (2024) 8] to the continuum limit of the equations of the perturbation theory, it can be shown analytically that [Boutivas, Katsinis, Pastras, Tetradis 2407.07824]

$$S^{(2)}_{\mathrm{dS}}(r) = r^2 \left(\frac{1}{3} \ln \frac{L}{2\pi r} + \frac{4}{9} \right) .$$

- Notice that $a_2^{(2)} = -\frac{1}{3} \ln (2\pi) + \frac{4}{9} \simeq -0.16818.$

$(3+1)$-dim theory

- For the $(3+1)$ dimensional flat space we expect [Lohmayer,Neuberger,Schwimmer,Theisen]

$$S^{(3d)}_{\text{flat}} = d_2 \frac{R^2}{\epsilon^2} + d_1 \ln \frac{R}{\epsilon} + d_0 + \ldots, \qquad d_1 = -1/90.$$

- Our flat-space numerical calculation gives $-1/d_1 = 89.999858$.
- The flat-space extrapolation of the dS calculation, i.e. $S^{(0)}_{\text{dS},\infty}(n)$, gives $-1/d_1 = 90.00089$.
- Analyzing $S^{(2)}_{\text{dS},\infty}(n)$ we obtain $S^{(2)}_{\text{dS},\infty}(n) = a_2^{(2)\prime} n_R^2 \ln n_R + a_2^{(2)} n_R^2, \quad n_R = n + \frac{1}{2}$, where $a_2^{(2)\prime} = 0.000025, \quad a_2^{(2)} = -0.142650$.
- The early-time expansion of Entanglement Entropy is

$$S^{(3d)}_{\text{dS}} = \left[d_2 \frac{R^2}{\epsilon^2} - \frac{1}{90} \ln \frac{R}{\epsilon} + d_0 \right] + \frac{R^2}{\tau^2} \left[\frac{1}{3} \ln \frac{L}{\epsilon} + a_2^{(2)\prime} \ln \frac{R}{\epsilon} + a_2^{(2)} \right] + \ldots$$

with $d_2 = 0.295431454$ in agreement with Srednicki.

Conclusions and ongoing work

Conclusion

- We have calculated the leading de Sitter corrections to the flat-space Entanglement Entropy.
- We found

$$S_{\mathrm{dS}}^{(\mathrm{3d})} = \left[d_2 \frac{R_p^2}{\epsilon_p^2} - \frac{1}{90} \ln \frac{R_p}{\epsilon_p} + d_0 \right] + H^2 R_p^2 \left[\frac{1}{3} \ln \frac{L_p}{\epsilon_p} + a_2^{(2)\prime} \ln \frac{R_p}{\epsilon_p} + a_2^{(2)} \right] + \ldots$$

  with $d_2 = 0.295431454$ in agreement with [Srednicki], $d_0 = -0.035290$ [Lohmayer,Neuberger,Schwimmer,Theisen], $a_2^{(2)\prime} \simeq 0$ and $a_2^{(2)} = -0.142650$. H is the Hubble constant and the physical length scales are defined as $\epsilon_p = a\epsilon$ and so on.
- Our calculation indicates that the Entanglement Entopy in de Sitter space has a structure of the form

$$S_{\mathrm{dS}} = \frac{R_p^2}{\epsilon_p^2} f_2 \left(H R_p \right) + f_0 \left(H R_p \right) + g \left(H R_p \right) \ln \frac{R_p}{\epsilon_p} + h \left(H R_p \right) \ln \frac{L_p}{\epsilon_p},$$

  which is more general than the one proposed by Maldacena and Pimentel

$$S_{\mathrm{dS}} = c_1 \frac{A_p}{\epsilon_p^2} + \ln \left(H \epsilon_p \right) \left(c_2 + c_4 A_p H^2 \right) + c_5 A_p H^2 - \frac{c_6}{2} \ln \left(A_p H^2 \right) + \mathrm{const.}$$

Conclusion

- The most significant qualitative outcome of this work is that **the entanglement entropy of scalar fields in de Sitter space depends on the size of the overall system**.
- Each field mode in the Bunch-Davies vacuum is described by a complex Gaussian wave function, which can be interpreted as a squeezed state at any instant in time.
- The wave function probes a larger part of the Hilbert space due to squeezing, which results in enhanced entanglement entropy. $S_{\mathrm{sq}} = z\,\mathrm{min}(n, N-n)$ Katsinis, Pastras, Tetradis
- The IR modes of the theory, which have eigenfrequencies related to the size of the overall system, are the ones that get more squeezed by the expansion of the background.
- This phenomenon is analogous to the EPR paradox. Regions of the space, which were causally connected when inflation began, are entangled even when they are causally disconnected.

Ongoing Work

- Currently we work on Entanglement Entropy in AdS space.
- In global AdS_4 we find a universal logarithmic term of the form

$$-\left[\frac{1}{90}+\left(\frac{\mu^2}{6}+\frac{1}{3}\right)\frac{R^2}{R_{\mathrm{AdS}}^2}\right]\ln\frac{1}{\epsilon}.$$

- For $\mu^2=0$ is the analytic continuation of the analogous term of de Sitter space:

$$R_{\mathrm{AdS}}^2 \leftrightarrow -\frac{1}{H^2}.$$

Extra

Setup

- In the flat-space limit we have the eigenvalue problem

$$\mathcal{M}^{(0)T} v_i^{(0)} = \Lambda_i^{(0)} v_i^{(0)}, \qquad \mathcal{M}^{(0)} w_i^{(0)} = \Lambda_i^{(0)} w_i^{(0)},$$

where $\Lambda_i^{(0)} = \left(\lambda_i^{(0)}\right)^2$. The vectors $v_i^{(0)}$ and $w_i^{(0)}$ are the left and right eigenvectors of $\mathcal{M}^{(0)}$.

- The leading corrections to the flat-space eigenvalues $\lambda_i^{(0)}$ read

$$\lambda_i^{(2)} = \frac{1}{2\lambda_i^{(0)}} \frac{w_i^{(0)T} \mathcal{M}^{(-2)T} v_i^{(0)}}{w_i^{(0)T} v_i^{(0)}}.$$

- The matrices $\mathcal{M}^{(0)}$ and $\mathcal{M}^{(-2)T}$ are

$$\mathcal{M}^{(0)} = I_n - \left(\Omega^{-1}\right)_B (\Omega)_B^T,$$
$$\mathcal{M}^{(-2)T} = \left(\Omega^{-1}\right)_B \left(\Omega^{-1}\right)_B^T - (\Omega)_B \left(\Omega^{-3}\right)_B^T.$$

- There are no UV singularities in the continuum limit.

The Kernels

- The kernels that are relevant for our calculation are

$$\Omega(x,x') = \frac{\pi}{4L^2}\left(\frac{1}{\sin^2\frac{\pi(x+x')}{2L}} - \frac{1}{\sin^2\frac{\pi(x-x')}{2L}}\right),$$

$$\Omega^{-1}(x,x') = -\frac{1}{\pi}\ln\left|\frac{\sin\frac{\pi(x-x')}{2L}}{\sin\frac{\pi(x+x')}{2L}}\right|,$$

$$\Omega^{-3}(x,x') = \frac{L^2}{2\pi^3}\left[\mathrm{Li}_3\left(e^{-i\frac{\pi}{L}(x-x')}\right) - \mathrm{Li}_3\left(e^{-i\frac{\pi}{L}(x+x')}\right)\right] + \mathrm{c.c.}\,.$$

- For $L \gg R$ (R is the radius of the entangling surface)

$$\Omega(x,x') = \frac{1}{\pi}\left(\frac{1}{(x+x')^2} - \frac{1}{(x-x')^2}\right),$$

$$\Omega^{-1}(x,x') = -\frac{1}{\pi}\ln\left|\frac{x-x'}{x+x'}\right|,$$

$$\Omega^{-3}(x,x') = \frac{1}{\pi}\left[xx'\left(3+2\ln\frac{L}{\pi R}\right) + \frac{(x-x')^2}{2}\ln\frac{|x-x'|}{R} - \frac{(x+x')^2}{2}\ln\frac{x+x'}{R}\right].$$

The flat-space Problem

- The solution of the flat-space eigenvalue problem is known [Callan, Wilczek][Katsinis, Pastras].
- The right eigenfunctions $f(x;\omega)$, the left eigenfunctions $g(x;\omega)$, and the eigenvalues $\Lambda^{(0)}(\omega)$ read:

$$f(x;\omega) = \sin\left(\omega u(x)\right),$$
$$g(x;\omega) = \frac{1}{R}\cosh^2\frac{u(x)}{2}\sin\left(\omega u(x)\right),$$
$$\Lambda^{(0)}(\omega) = \coth^2\left(\pi\omega\right),$$

where x is related to the variable u as

$$x(u) = R\tanh\frac{u}{2}, \quad u(x) = \ln\frac{R+x}{R-x}.$$

- We need the inner product $v_i^{(0)T} w_i^{(0)}$, which is divergent in the continuum limit. We impose a cutoff at $x = R - \epsilon$, where $\epsilon \ll R$. We regularize this divergence and at the same time we discretize the spectrum as

$$\omega_k = \frac{k\pi}{u_{\max}}, \qquad u_{\max} = \ln\frac{2R}{\epsilon}, \qquad k \in \mathbb{N}^*.$$

- In the continuum limit

$$w_i^{(0)T} v_i^{(0)} \to \frac{1}{2}\int_0^{u_{\max}} du \sin^2\left(\omega_i u\right) = \frac{1}{4}u_{\max}.$$

Correction to the Eigenvalues 1

- We have to calculate

$$\left\langle M^{(-2)T}\right\rangle = \int_0^R dx \int_0^R dx' f(x;\omega) M^{(-2)T}\left(x,x'\right) g(x';\omega),$$

where $M^{(-2)T}\left(x,x'\right) = \int_R^L dy \left[\Omega^{-1}(x,y)\Omega^{-1}(y,x') - \Omega(x,y)\Omega^{-3}(y,x')\right]$.

- Taking into account that $\Omega^m(x,y) = -\frac{d^2}{dy^2}\Omega^{m-2}(x,y)$ we obtain

$$M^{(-2)T}\left(x,x'\right) = \Omega^{-1}(x,R)\left[\frac{\partial}{\partial R}\Omega^{-3}(R,x')\right] - \left[\frac{\partial}{\partial R}\Omega^{-1}(x,R)\right]\Omega^{-3}(R,x').$$

- Notice that $M^{(-2)T}\left(x,x'\right)$ is finite when $x' \to x$.

Correction to the Eigenvalues 2

- The correction to the eigenvalues is

$$\lambda^{(2)}\left(\omega_i\right) = R^2 \frac{\left[I_1\left(\omega_i\right)\left(3 + 2\ln\frac{L}{\pi R}\right) + I_2\left(\omega_i\right)\right]}{u_{\max}\coth\left(\pi\omega_i\right)} \times \left[I_1\left(\omega_i\right) + \frac{1}{2} I_3\left(\omega_i\right)\right],$$

where

$$I_1\left(\omega\right) = \frac{1}{\pi}\int_0^{u_{\max}} du \sin\left(\omega u\right)\tanh\frac{u}{2},$$

$$I_2\left(\omega\right) = \frac{1}{\pi}\int_0^{u_{\max}} du \sin\left(\omega u\right)\left(\frac{e^{-u/2}}{\cosh\frac{u}{2}}\ln\frac{e^{-u/2}}{\cosh\frac{u}{2}} - \frac{e^{u/2}}{\cosh\frac{u}{2}}\ln\frac{e^{u/2}}{\cosh\frac{u}{2}}\right),$$

$$I_3\left(\omega\right) = \frac{1}{\pi}\int_0^{u_{\max}} du \frac{u\sin\left(\omega u\right)}{\cosh^2\frac{u}{2}}.$$

- For $u_{\max} \to \infty$ we obtain

$$I_1\left(\omega\right) = \frac{1}{\sinh\left(\pi\omega\right)}, \qquad I_1\left(\omega\right) + \frac{1}{2} I_3\left(\omega\right) = \pi\omega\frac{\coth\left(\pi\omega\right)}{\sinh\left(\pi\omega\right)}.$$

Correction to Entanglement Entropy

- At order $1/\tau^2$ the correction the correction to flat-space Entanglement Entropy reads

$$S^{(2)} = \sum_i \lambda_i^{(2)} \operatorname{arccoth} \lambda_i^{(0)} = \sum_i \pi\omega_i \lambda_i^{(2)}.$$

- Substituting $\lambda_i^{(2)}$ we obtain

$$S^{(2)} = R^2 \frac{\pi}{u_{\max}} \sum_i \left[\left(3 + 2\ln\frac{L}{\pi R} \right) \frac{\pi\omega_i^2}{\sinh^2(\pi\omega_i)} + \frac{\pi\omega_i I_2(\omega_i)}{\sinh(\pi\omega_i)} \right].$$

- Since $R \gg \epsilon$ the eigenvalues ω_i become dense and the sum can be approximated by an integral over ω [Callan,Wilczek]

$$S^{(2)} = R^2 \int_0^\infty d\omega \left[\left(3 + 2\ln\frac{L}{\pi R} \right) \frac{\pi\omega^2}{\sinh^2(\pi\omega)} + \frac{\pi\omega I_2(\omega)}{\sinh(\pi\omega)} \right] = R^2 \left[\frac{1}{3}\ln\frac{L}{2\pi R} + \frac{4}{9} \right].$$