An exceptional Cluster Algebra for Higgs+Jet production

Rigers Aliaj

9th Xmas for Theoretical Physics, Athens
18 December 2024

Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

DESY.

e-print: 2408.14544 [hep-th] with Georgios Papathanasiou, to appear in JHEP

Particle Colliders and Scattering Amplitudes

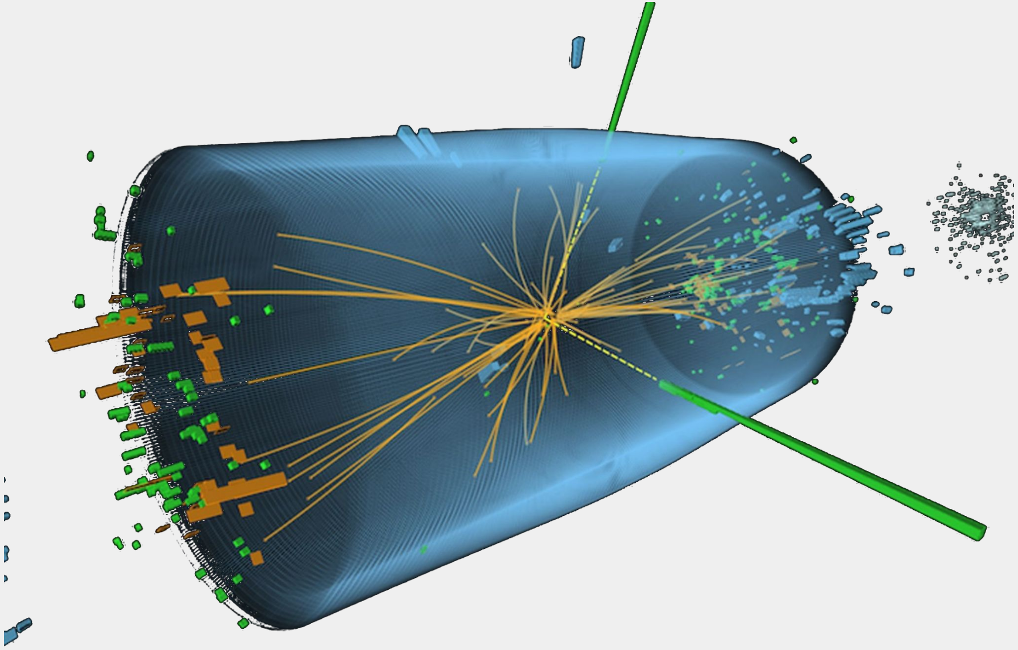

Image: CMS/CERN 2012

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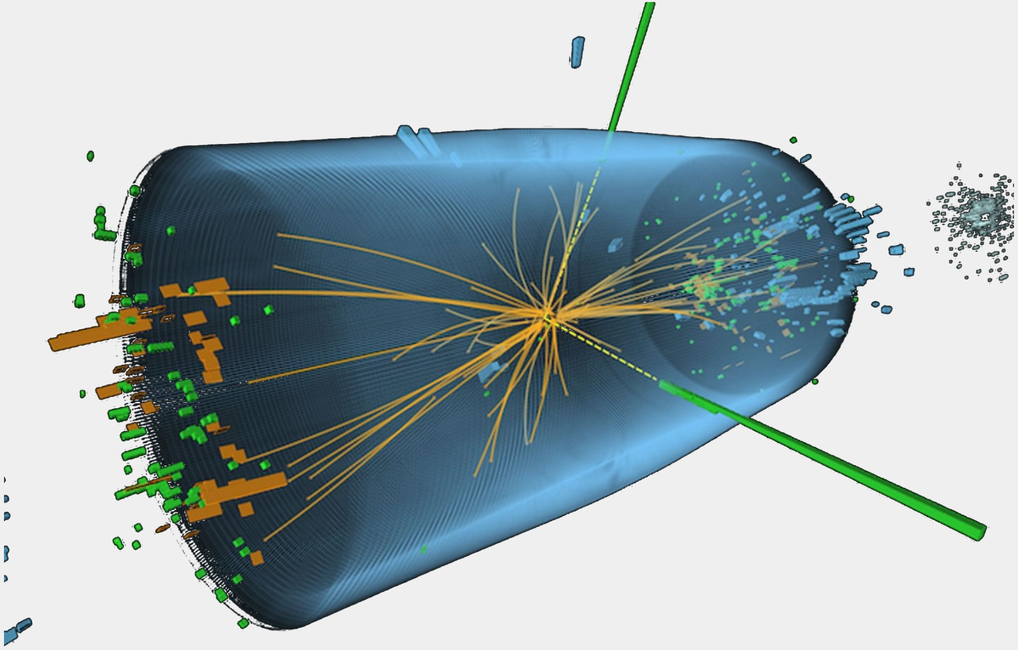

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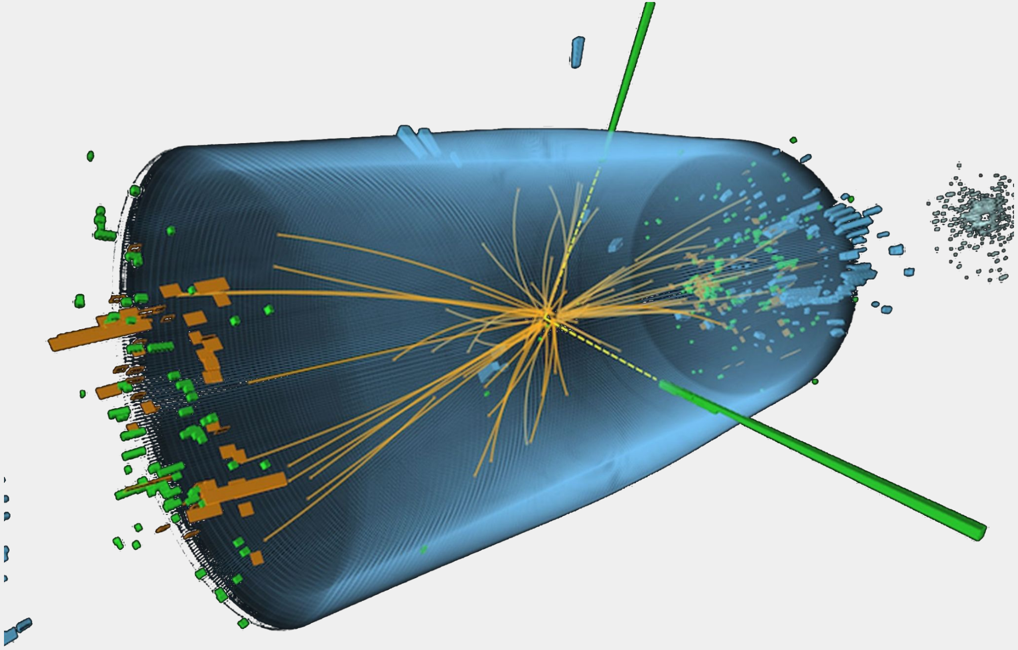

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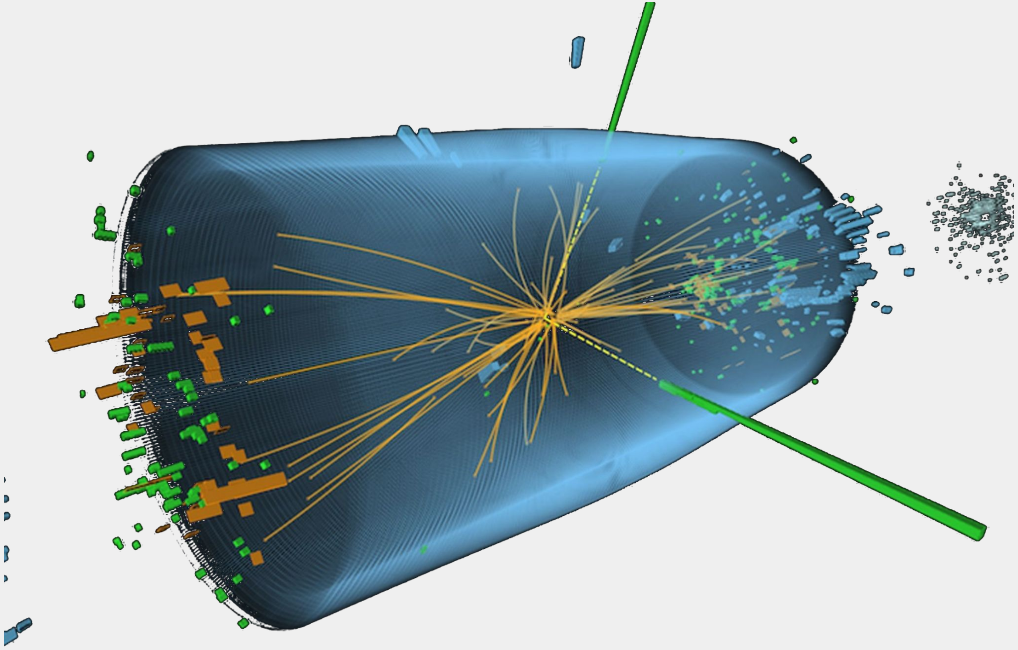

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- At the core of their theoretical description in *Quantum Field Theory* (QFT) lie *Scattering Amplitudes*.
- High-Luminosity Large Hardon Collider (HL-LHC) launching by 2030 requires higher precision in our theoretical predictions.

Amplitudes and Feynman Integrals

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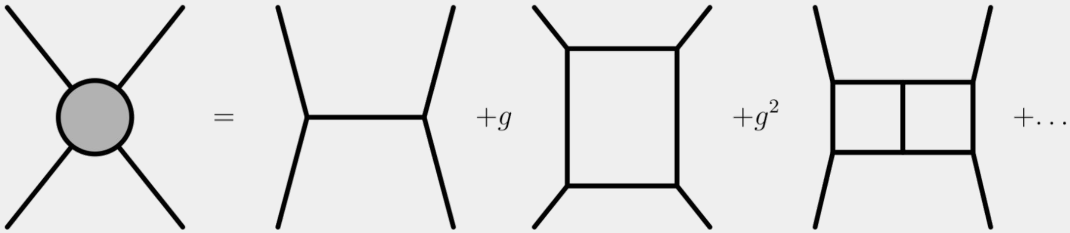

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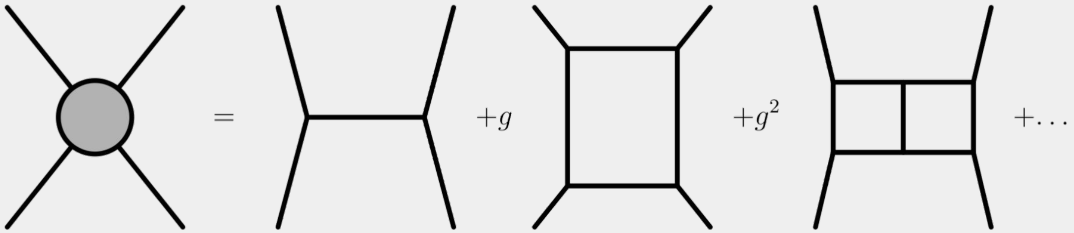

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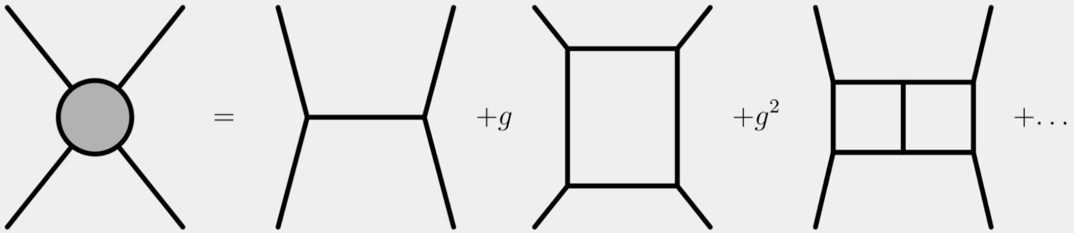

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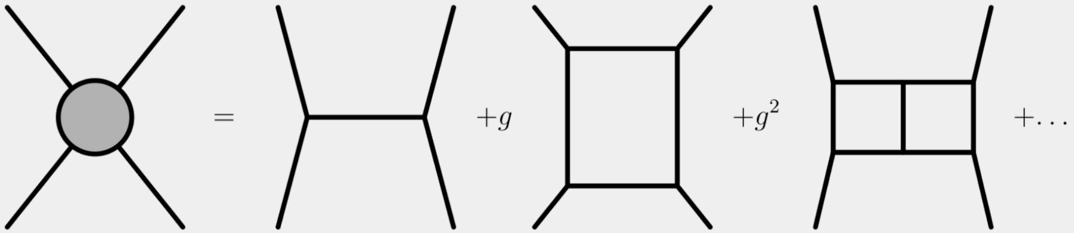

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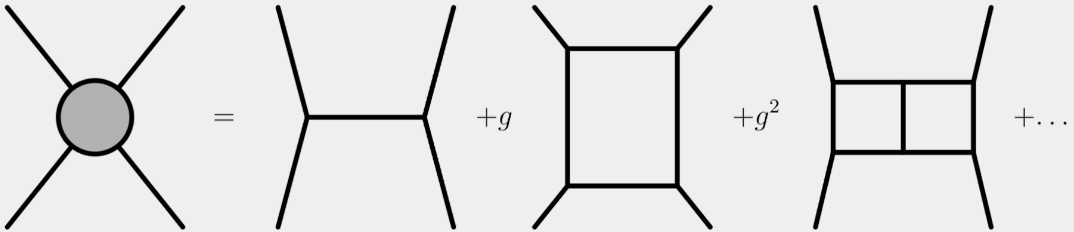

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- State of art: $\mathcal{O}(g^2) \simeq 1\%$

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- 6-point 8-Loop Amplitude! [Drummond, Foster, Gurdogan, Papathanasiou][Dixon Liu]
- 7-point 4-Loop Amplitude! [Caron-Huot, Dixon, Dulat, McLeod, Hippel, Papathanasiou][Dixon Liu]

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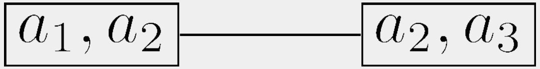

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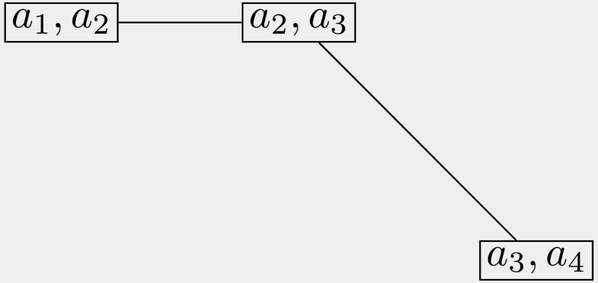

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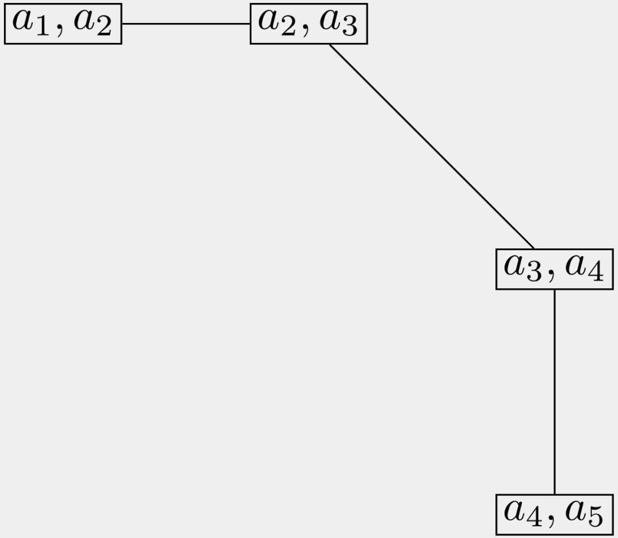

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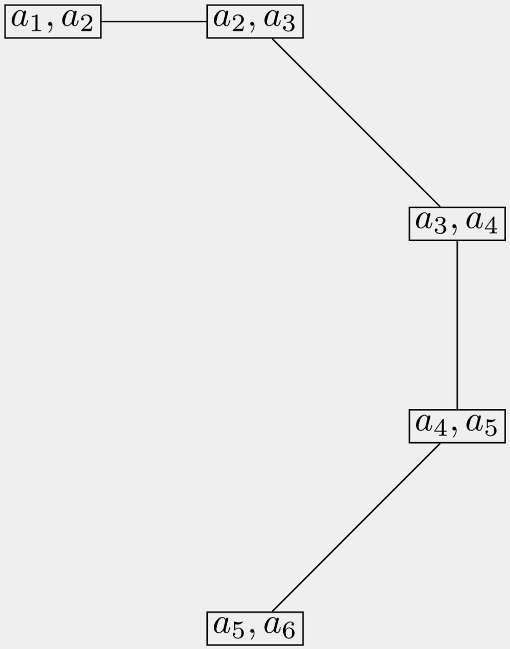

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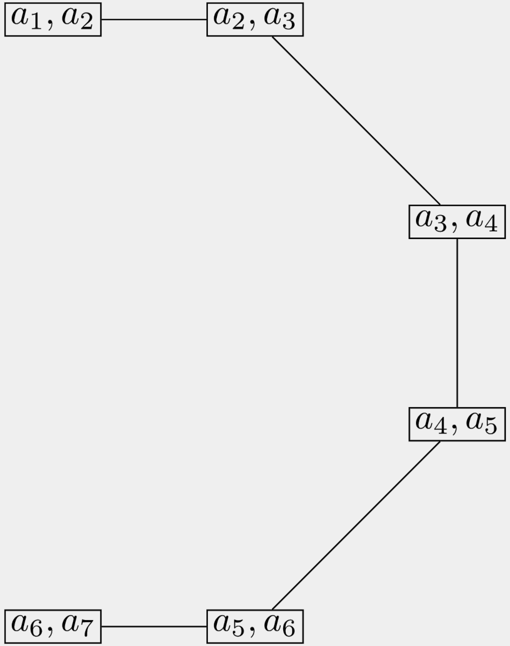

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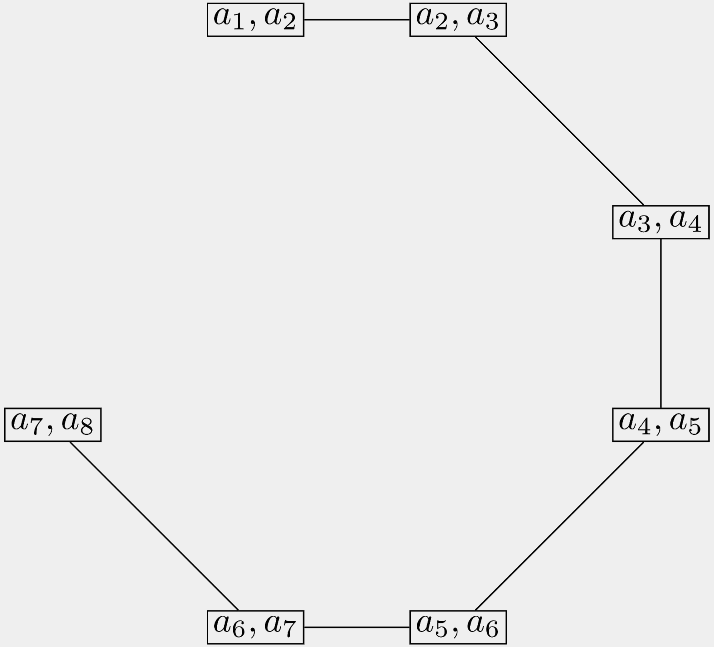

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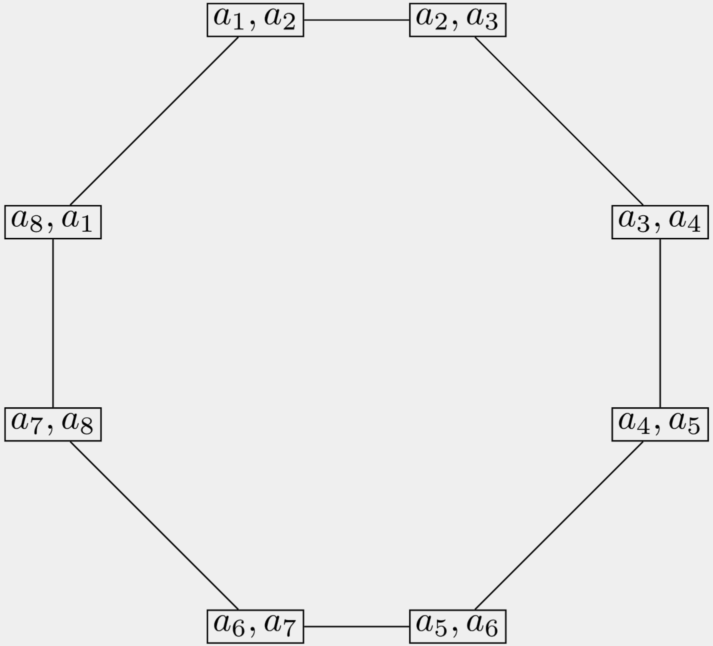

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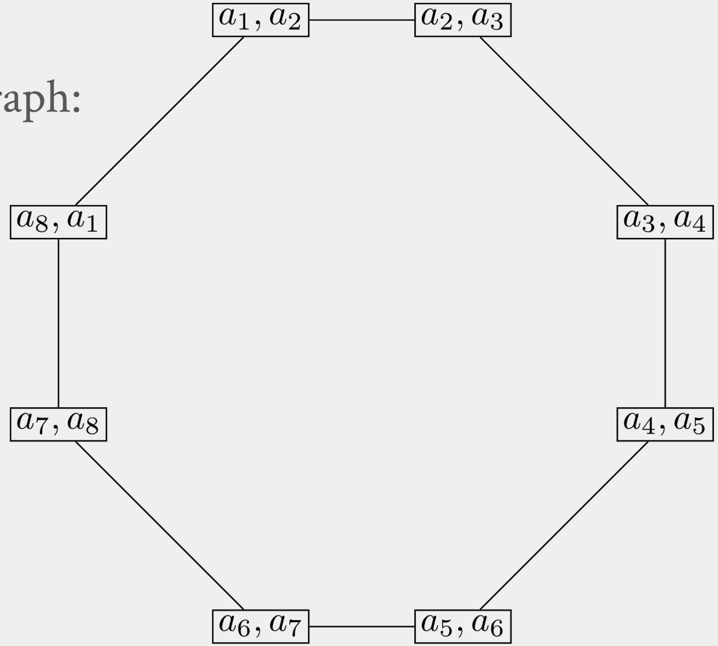
G_2 Exchange graph:



$$\Phi_{G_2} = \{a_1, a_2, \frac{1+a_2^3}{a_1}, \frac{1+a_1+a_2^3}{a_1 a_2}, \frac{1+a_1^3+3a_1^2+3a_1a_2^3+3a_1+a_2^6+2a_2^3}{a_1^2 a_2^3}, \frac{1+a_1^2+2a_1+a_2^3}{a_1 a_2^2}, \frac{1+a_1^3+3a_1^2+3a_1+a_2^3}{a_1 a_2^3}, \frac{1+a_1}{a_2}\}.$$

$$= \{a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,\ a_7,\ a_8\}$$

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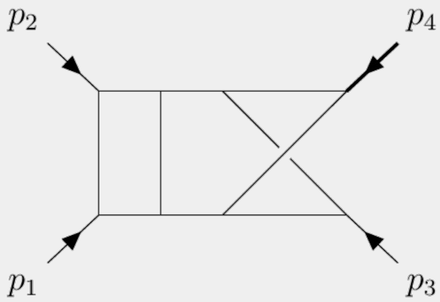

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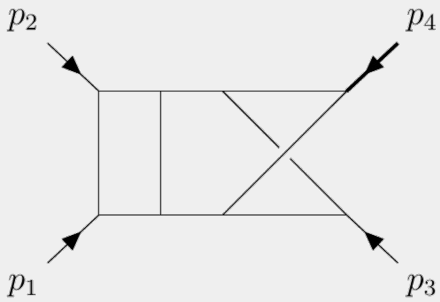

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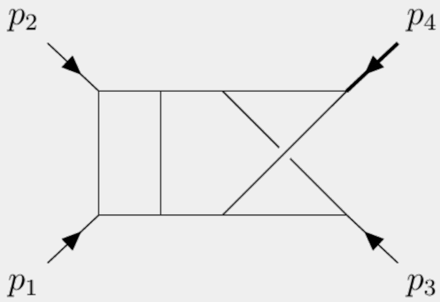

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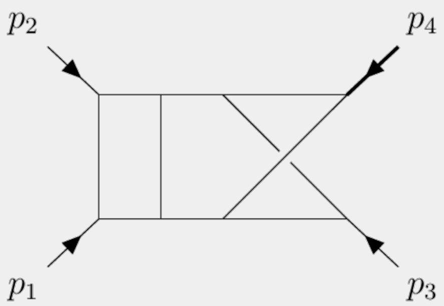

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Is this the end of it?

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- However, the observed Adjacency restrictions (20) are only a subset of the ones this constraint predicts (40)!

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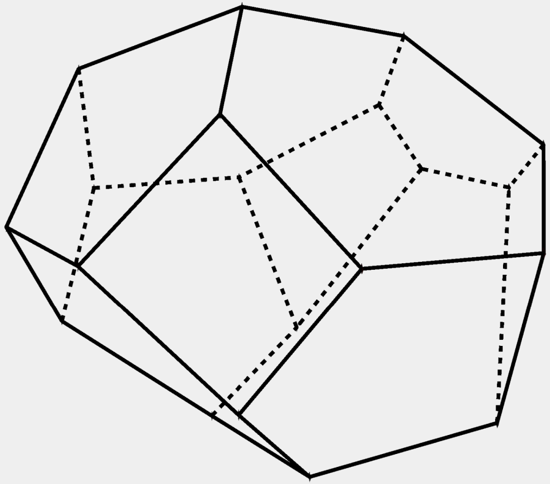

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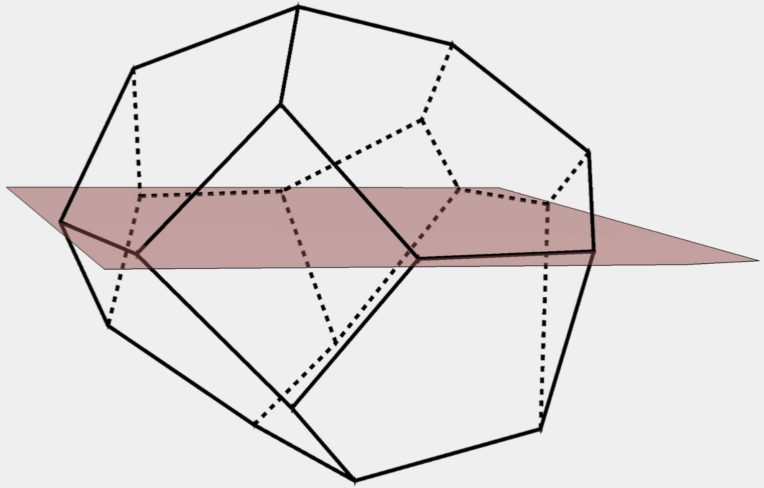

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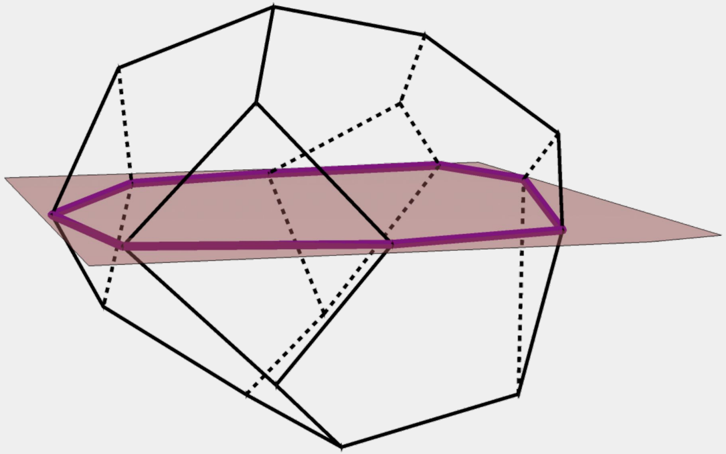

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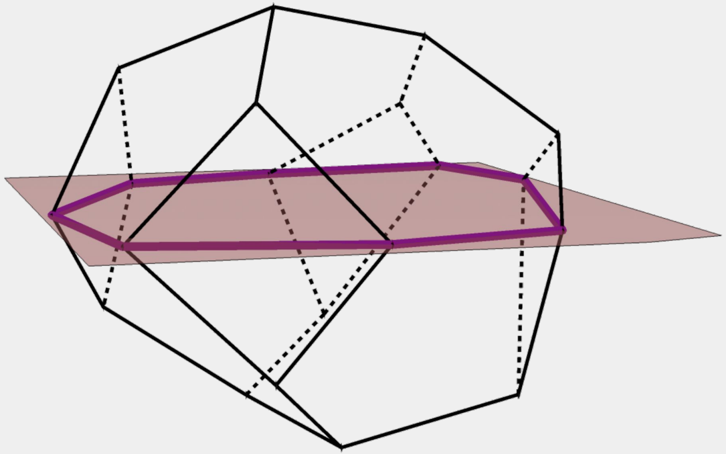

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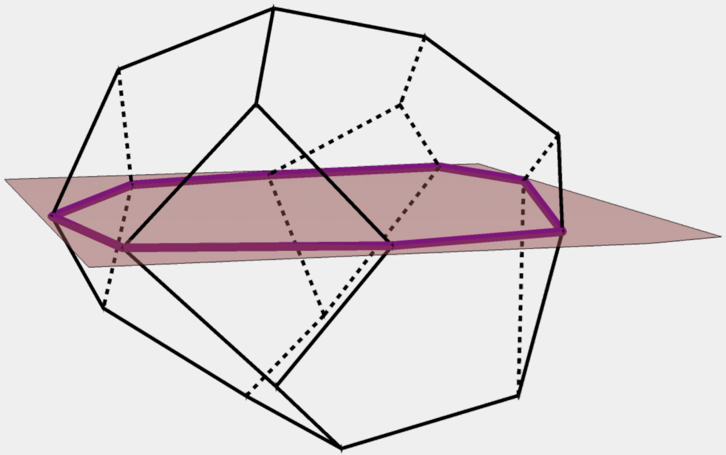

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Future work

- Identify more alphabets described by Cluster Algebras.
- Could we predict their appearance from first principles?
- Can we apply it to bootstrap amplitudes of QCD?

Questions?