The Quantum Spectral Method:

From Atomic Orbitals to Classical Self-Force

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w/ M. Khalaf (HUJI)

Last Talk of The Day

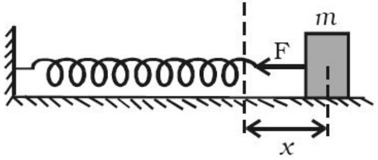
Here's a riddle for you:

What does this song have to do with my talk?

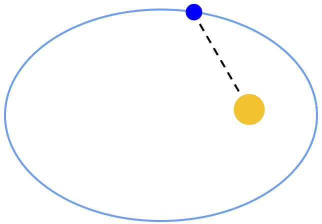
Answer on my last slide

The World is Full of Periodic and Quasi-Periodic Systems

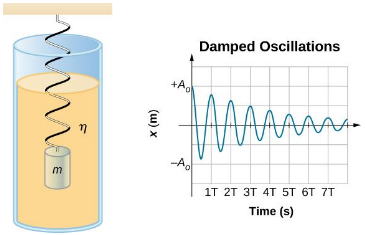
Periodic



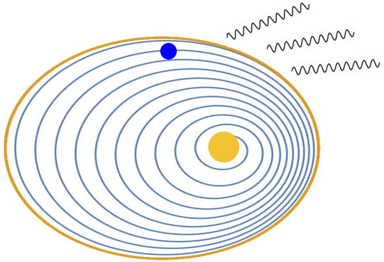
Simple Harmonic Oscillator



Keplerian Motion



Simple Harmonic Oscillator + slow damping

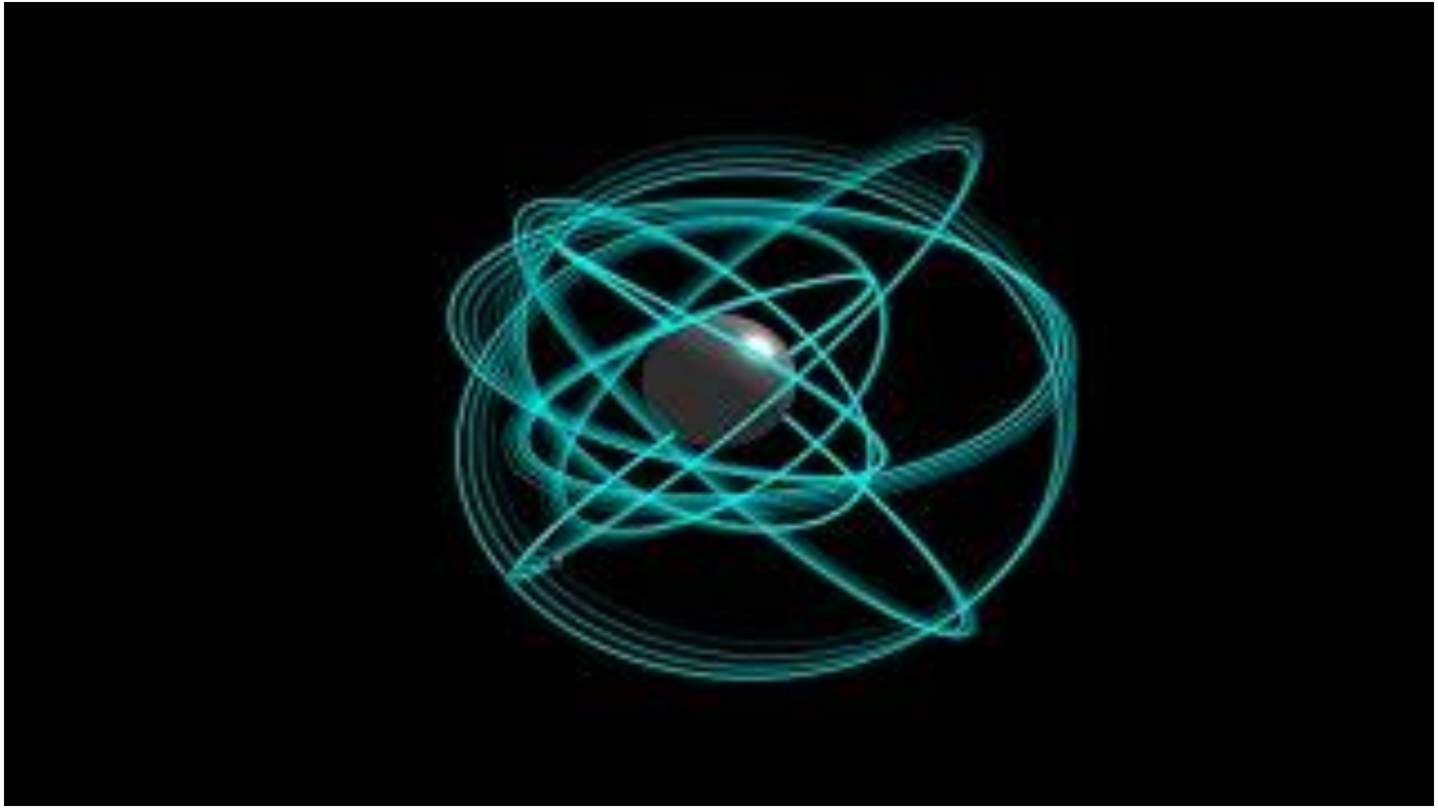


Electromagnetic Inspiral

Quasi- Periodic

$T_{period} \ll T_{loss}$

Extreme-Mass-Ratio Black-Hole Inspirals

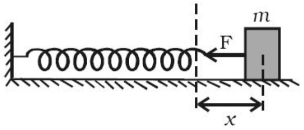


Observables in Periodic Systems: Examples

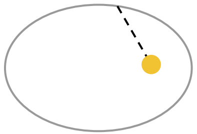
Periodic system with single period T $\longrightarrow$ define angle variable $\alpha = 2\pi \frac{(t - t_0)}{T}$

All variables are Fourier Series $O(\alpha) = \sum_{\Delta n = -\infty}^{\infty} O_{\Delta n}\, e^{-i\Delta n \alpha}$ e.g.

1. Position in the SHO: $x(\alpha) = x_1 e^{-i\alpha} + \text{cc.}$, $x_1 = \frac{1}{2}\left(x_0 + i\frac{T v_0}{2\pi}\right)$



2. Radius in Keplerian motion: $r(\alpha) = \sum_{\Delta n = -\infty}^{\infty} r_{\Delta n}\, e^{-i\Delta n \alpha}$, $r_{\Delta n}(E, L)$ known analytically



Observables in Periodic Systems: Examples

3. The EM field generated by a classical electron in Keplerian motion,

$$A^\mu(x) = \int d^4x \, G^{\mu\nu}(x, x') J_\nu(x')$$

EM Green's function   current from orbiting electron

can be Fourier-expanded in the angle variable of Keplerian motion

$$A^\mu(\vec{x}, \alpha) = \sum_{\Delta n=-\infty}^{\infty} A^\mu_{\Delta n}(\vec{x}) \; e^{-i\Delta n \alpha}$$

The $A^\mu_{\Delta n}(\vec{x})$ are all-order expressions in α/L , known only numerically

Main Question

We saw two examples with known analytical Fourier series,
and one where the coefficients are only known numerically

Is there a universal way to calculate all observables in periodic and quasi-periodic systems *analytically*?

- Periodic and quasi-periodic trajectories
- Emitted EM and gravitational radiation
- EM and gravitational Self-Force (backreaction)
- …

Answer: yes! With the help of Quantum Mechanics

The Quantum Spectral Method

Khalaf, OT '23

Consider the Hamiltonian for a periodic system with one angle variable, and its quantum eigenstates:

$$H\ |n,\sigma\rangle\ =\ E_n\ |n,\sigma\rangle$$

principal quantum number

other quantum numbers

we proved the "Master Equation":

$$\boxed{O_{\Delta n} = \lim_{\hbar\to 0}\sum_{\Delta\sigma}\langle n-\Delta n,\sigma-\Delta\sigma|\,O\,|n,\sigma\rangle}$$

$$n = \frac{N^{classical}}{\hbar} \to \infty$$

$$\sigma = \frac{\Sigma^{classical}}{\hbar} \to \infty$$

"The Δn Fourier coefficient of the *classical observable* O is the classical limit of the Δn transition mediated by the *quantum operator* O"

Khalaf, OT '23

The Master Equation

$$O_{\Delta n} = \lim_{\hbar \to 0} \sum_{\Delta\sigma} \langle n - \Delta n, \sigma - \Delta\sigma | \, O \, | n, \sigma \rangle$$

Master equation

$$n = \frac{N^{classical}}{\hbar} \to \infty$$

$$\sigma = \frac{\Sigma^{classical}}{\hbar} \to \infty$$

- Correspondence principle: quantum numbers go to infinity, their products with ħ are the finite, conserved action variables of the classical system (functions of E, L, etc...)

- Δn does not go to infinity and remains the integer index of the classical Fourier series

- Simple proof using generalized coherent states (next slide)

The Master Equation - Outline of Proof

Generalized coherent states: $|t, N, \Sigma\rangle \equiv \sum_{n,\sigma} e^{-i\frac{E_n}{\hbar}t} f_{n,\sigma}(N, \Sigma)\, |n, \sigma\rangle$

- Time evolution: $e^{-i\frac{H}{\hbar}\delta t}\, |t, N, \Sigma\rangle = |t + \delta t, N, \Sigma\rangle$
- Classical limit: $O(t) = \lim_{\hbar \to 0} \langle t, N, \Sigma|\, O\, |t, N, \Sigma\rangle$

$\longrightarrow$ $f_{n,\sigma}(N, \Sigma)$ has classical saddle point at $(n, \sigma) = \hbar^{-1}(N, \Sigma)$

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The Master Equation - Outline of Proof

$$O(t) = \lim_{\hbar\to 0} \sum_{\Delta n, \Delta\sigma} \sum_{n,\sigma} f^*_{n-\Delta n, \sigma-\Delta\sigma} f_{n,\sigma} \langle n-\Delta n, \sigma-\Delta\sigma | \, O \, | n, \sigma\rangle \times e^{-i\frac{E_n - E_{n-\Delta n}}{\hbar}t}$$

$$\lim_{\hbar\to 0} \frac{E_n - E_{n-\Delta n}}{\hbar} = \frac{2\pi\Delta n}{T}$$

saddle point at $(n,\sigma) = \hbar^{-1}(N,\Sigma)$

the finite, dimensionful action variables of the classical system

$$O(t) = \lim_{\hbar\to 0} \sum_{\Delta n, \Delta\sigma} \langle \hbar^{-1}N - \Delta n, \hbar^{-1}\Sigma - \Delta\sigma \big| \, O \, \big| \hbar^{-1}N, \hbar^{-1}\Sigma\rangle \, e^{-i\frac{2\pi\Delta n}{T}t}$$

(the f factors drop out - can check for O=1)

we also checked the saddle point explicitly for the case of a 1/r potential

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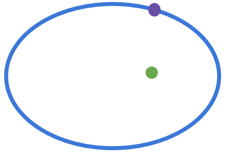
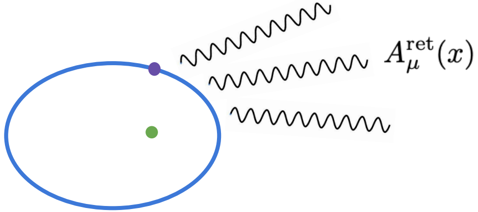
Applications of the QSM so Far

1. Proof-of-principle: time-dependent Keplerian motion

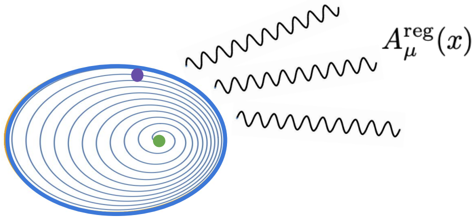


2. First all-multipole analytical result for A^μ from a Keplerian electron



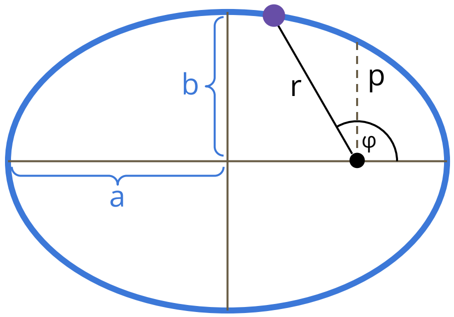
3. First all-multipole analytical result for EM self-force on an inspiralling classical electron +

   inspiralling adiabatic trajectory and EM waveform



First Application: Time-Dependent Keplerian Motion

EOM: $$\mu\ddot{\vec{r}} = -K\frac{\vec{r}}{r^3}$$



Conserved quantities: E - Energy (negative)

L - angular momentum

$$N = K\sqrt{\frac{\mu}{2|E|}} = \frac{1}{2\pi}(I_r + I_\varphi)$$

sum of action variables

$$e = \sqrt{1 - \frac{L^2}{N^2}}$$

eccentricity

$$p = \frac{L^2}{K\mu}$$

semi-latus rectum

$$K = \frac{Qq}{4\pi} \qquad Q = Zq$$

Time-Dependent Keplerian Motion

known classical result: $$r(\alpha) = \frac{p}{1-e^2}\left[1 + \frac{e^2}{2} - 2e\sum_{\Delta n=1}^{\infty} \frac{1}{(\Delta n)^2} \frac{dJ_{\Delta n}(\Delta n\, e)}{de} \cos(\Delta n \alpha)\right]$$

Let's reproduce it with the QSM!

Quantum version: the hydrogen-like atom $H\,|n,l,m\rangle = E_n\,|n,l,m\rangle$ $$E_n = -\frac{\mu K^2}{2\hbar^2 n^2}$$

First, the period $$\lim_{\hbar\to 0} \frac{E_n - E_{n-\Delta n}}{\hbar} = \frac{2\pi \Delta n}{T} \qquad n = \frac{N}{\hbar}$$

$$\downarrow$$

$$T = \frac{\pi N}{|E|}$$ Keplerian period $\longrightarrow$ $$T^2 = \frac{4\pi^2 \mu}{K} a^3$$

Kepler's 3rd law

Time-Dependent Keplerian Motion

Master equation:

$$O_{\Delta n} = \lim_{\hbar\to 0} \sum_{\Delta\sigma} \langle n-\Delta n, \sigma-\Delta\sigma | O | n, \sigma\rangle$$

$$r_{\Delta n} = \lim_{\hbar\to 0} \langle n-\Delta n, l, l | r | n, l, l\rangle \qquad (n,l) = \hbar^{-1}(N,L)$$

Hydrogen-like atom calculation:

(Gordon's integral)

$$\langle n', l | r | n, l\rangle = \frac{\hbar^2}{\mu K} \frac{(-1)^{n'-l}\, 2^{2l+2}\, (nn')^{l+2}\, (n-l-1)}{(2l+1)!\,(n+n')^{2l+4}} \left(\frac{n-n'}{n+n'}\right)^{n-n'-2} \sqrt{\frac{(n+l)!\,(n'+l)!}{(n-l-1)!\,(n'-l-1)!}} \times$$

$$\left[{}_2F_1\left(l-n'+1; n+l; 2l+2; \frac{4nn'}{(n+n')^2}\right) - \frac{n+l+1}{n-l-1}\left(\frac{n-n'}{n+n'}\right)^2 {}_2F_1\left(l-n'+1; n+l+2; 2l+2; \frac{4nn'}{(n+n')^2}\right)\right]$$

we set $(n, l, n', l') = \hbar^{-1}(N, L, N, L) - (0, 0, \Delta n, 0)$

and take the ħ→0 limit

Time-Dependent Keplerian Motion

result:

$$\lim_{\hbar \to 0} \langle n', l | \, r \, | n, l \rangle = -\frac{p}{1-e^2} \frac{e}{\Delta n^2} \frac{d}{de} J_{\Delta n}(e \Delta n) \qquad \Delta n \neq 0$$

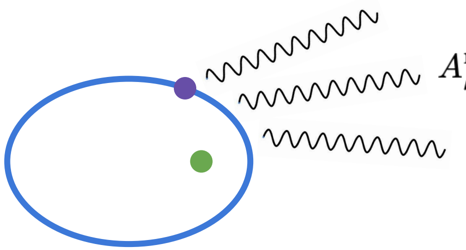
$$\lim_{\hbar \to 0} \langle n, l | \, r \, | n, l \rangle = \frac{p}{1-e^2} \left(1 + \frac{e^2}{2} \right)$$

$$r(\alpha) = \frac{p}{1-e^2} \left[1 + \frac{e^2}{2} - 2e \sum_{\Delta n = 1}^{\infty} \frac{1}{(\Delta n)^2} \frac{d J_{\Delta n}(\Delta n \, e)}{de} \cos(\Delta n \alpha) \right]$$

Time - dependent Keplerian motion

Calculated with the QSM

Second Application: All-Multipole EM Emission



A classical electron in Keplerian motion (no backreaction)

What is the generated electric field at all orders in the multipole expansion?

Current density: $J^\mu(x') = \frac{q}{\mu} p^\mu_{\rm Kep}(t')\,\delta^{(3)}\left[\vec{x}' - \vec{r}_{\rm Kep}(t')\right]$ $\quad p^\mu_{\rm Kep}(t') = (\mu, \mu\,\partial_{t'}\vec{r}_{\rm Kep})$

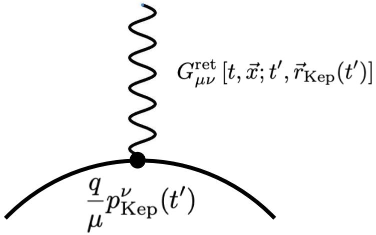
EM Field: $A_\mu^{\rm ret}(x) = \int d^4x\, G^{\rm ret}_{\mu\nu}(x,x') J^\nu(x')$

Retarded EM Green's function

Second Application: All-Multipole EM Emission

By the delta-function support on the orbit:

$$A^{\mu}_{\text{ret}} =$$



$$A^{\mu}_{\text{ret}} = \frac{iq}{\mu} \sum_{l_\gamma=0}^{\infty} \sum_{m_\gamma=-l_\gamma}^{l_\gamma} \sum_{\Delta n} \omega_{\Delta n}\, h^{(1)}_{l_\gamma}(\omega_{\Delta n} r)\ \exp\left[-i \Delta n\, \alpha\right]\, Y^{m_\gamma *}_{l_\gamma}(\theta, \varphi) \sum_{\Delta l, \Delta m} \mathcal{M}^{\mu}_{\Delta, l_\gamma, m_\gamma}(\omega_{\Delta n}, N, L)$$

radial EM wavefunction

Fourier time dependence

angular harmonic

source multipoles

where $\omega_{\Delta n} = \dfrac{2\pi \Delta n}{T}$

where the QSM enters

Second Application: All-Multipole EM Emission

The QSM gives:

$$\mathcal{M}^{\mu}_{\Delta,l_\gamma,m_\gamma}(\omega,N,L) \equiv \lim_{\hbar\to 0} \langle n'l'm'|\, j_{l_\gamma}(\omega\, r)\, Y^{m_\gamma}_{l_\gamma}(\theta,\varphi)\, p^\mu \, |nlm\rangle$$

where we dropped the “Kep” labels to avoid confusion, and (r,θ,φ,p^μ) are quantum operators

I’ll skip the (many) details of the quantum calculation and its classical limit - they appear in the detailed appendices of our 2310.03798, and you’re welcome to ask me later

Second Application: All-Multipole EM Emission

A taste of our analytical calculation:

$$\lim_{\hbar\to 0} \langle l', m'| Y^{m_\gamma}_{l_\gamma, m} (\hat{r}) |l, l\rangle = \delta_{l',m'} \delta_{-\Delta l, m_\gamma} \frac{\cos\left[\frac{\pi(l_\gamma - m_\gamma)}{2}\right]}{2\pi} \sqrt{\frac{(2l_\gamma+1)\,\Gamma\left(\frac{l_\gamma+m_\gamma+1}{2}\right)\Gamma\left(\frac{l_\gamma-m_\gamma+1}{2}\right)}{\Gamma\left(\frac{l_\gamma+m_\gamma}{2}+1\right)\Gamma\left(\frac{l_\gamma-m_\gamma}{2}+1\right)}}$$

$$\lim_{\hbar\to 0} \langle n', l'| j_{l_\gamma} (\omega_{\Delta n}\, r) |n, l\rangle = 2^{l_\gamma} \sum_{j=0}^{\infty} \frac{(-1)^j\,(j+l_\gamma)!}{j!\,(2j+2l_\gamma+1)!}\, \omega_{\Delta n}^{2j+l_\gamma} \lim_{\hbar\to 0} \langle n', l'| r^{2j+l_\gamma} |n, l\rangle$$

$$\lim_{\hbar\to 0} \langle n', l'| r^j |n, l\rangle = \left(\frac{p}{1-e^2}\right)^j (-\eta)^{-\Delta n-\Delta l} \left(\frac{\eta e}{2}\right)^{j+1} \sum_{m=0}^{\infty} L^{j+1-m-\Delta n}_{m+\Delta l+\Delta n}\left(\frac{\eta e \Delta n}{2}\right) L^{j+1-m-\Delta l}_{m}\left(-\frac{\eta e \Delta n}{2}\right) \eta^{-2m}$$

Laguerre polynomials

$$\eta = \sqrt{\frac{N-L}{N+L}} = \frac{1-\sqrt{1-e^2}}{e}$$

All-Multipole EM Emission: Results

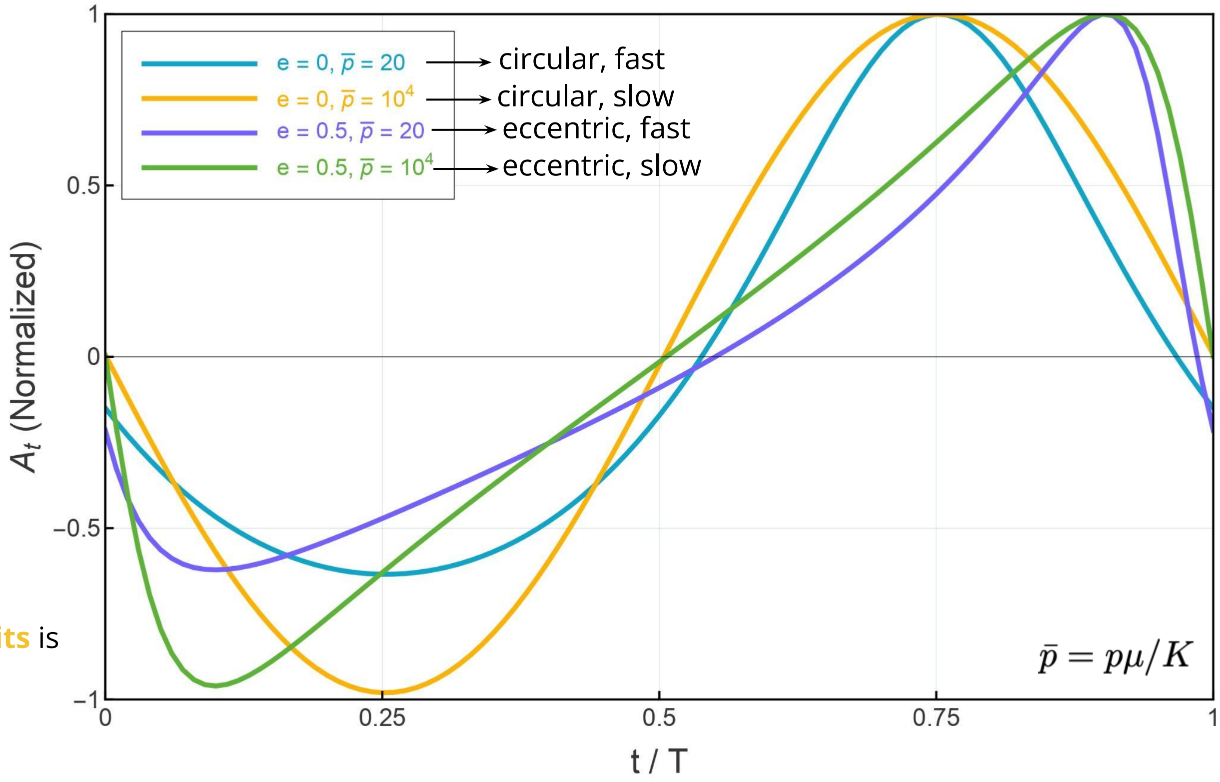
A_t radiated over one period by an electron undergoing Keplerian

The observation point is on the x-axis, far away from the electron's orbit

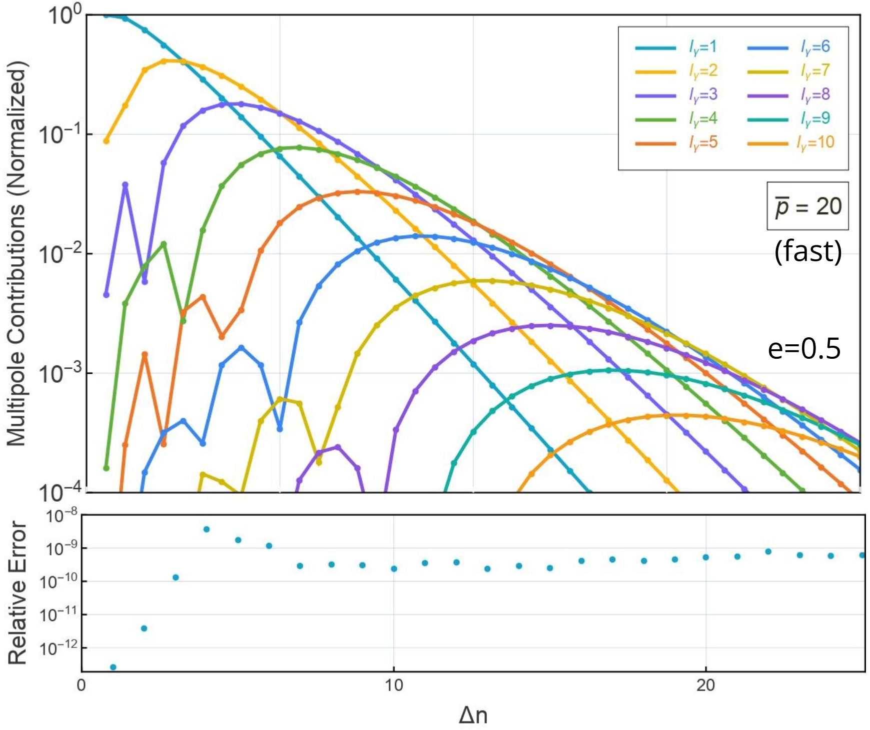
The horizontal and vertical axes are normalized by T and the maximum of the waveform, respectively

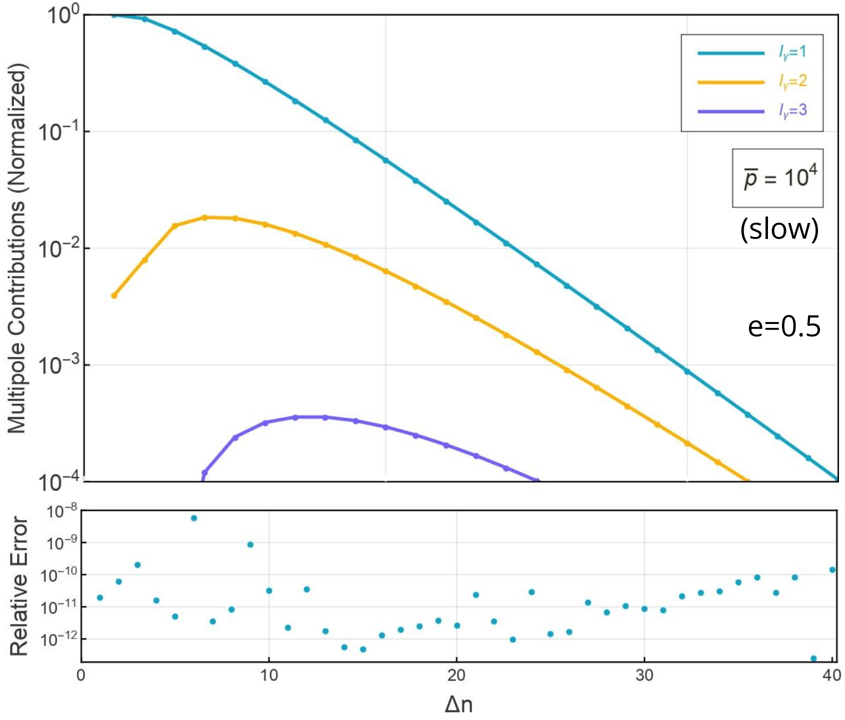
The asymmetry in the **fast orbits** is due to the doppler effect

The sinusoidal shape of the **circular orbits** is due to $\Delta n = m_\gamma$ selection rule



All-Multipole EM Emission: Results





Top: Multipole contributions (without spherical hankel factor)

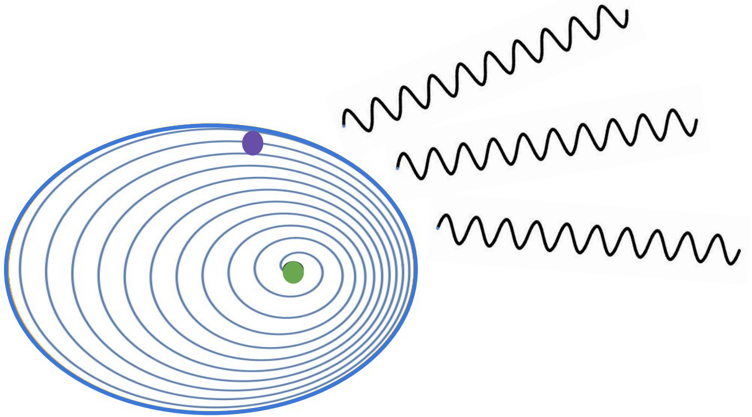
Bottom: Relative error with respect to the (numerical) classical integrals

Third Application: EM Self-Force

The EM field generated by the classical electron carries energy and angular momentum

This leads to a slow time dependence E(t) L(t) so that $\frac{\dot{E}}{E}, \frac{\dot{L}}{L} \ll T^{-1}$

The result is an electromagnetic inspiral



The method of osculating orbits parametrizes the inspiral as keplerian motion with slowly varying “constants of motion” E(t), L(t)

The task is to calculate the loss E(t) , L(t)

from the generated EM field $A_\mu(t)$

This is a (dissipative) self-force calculation

Third Application: EM Self-Force

With the A_μ calculated with the QSM, we get the energy and angular momentum loss:

$$\frac{dE}{dt} = -\lim_{\hbar \to 0} \sum_{\Delta n > 0, \Delta l, \Delta m} (E_n - E_{n'})\, \Gamma_{s.e.}$$

$$\frac{dL}{dt} = -\lim_{\hbar \to 0} \sum_{\Delta n > 0, \Delta l, \Delta m} \hbar\, (l - l')\, \Gamma_{s.e.}$$

where $\Gamma_{s.e.} = -\frac{2q^2 \omega_{\Delta n}}{\hbar \mu^2} \sum_{l_\gamma = 0}^{\infty} \sum_{m_\gamma = -l_\gamma}^{l_\gamma} \mathcal{M}^*_\mu \mathcal{M}^\mu + \mathcal{O}(\hbar^0)$

is the rate for quantum spontaneous emission

In this way we recover the self-force (equivalent to the ALD force) as the classical limit of spontaneous emission

Adiabatic EM Inspiral

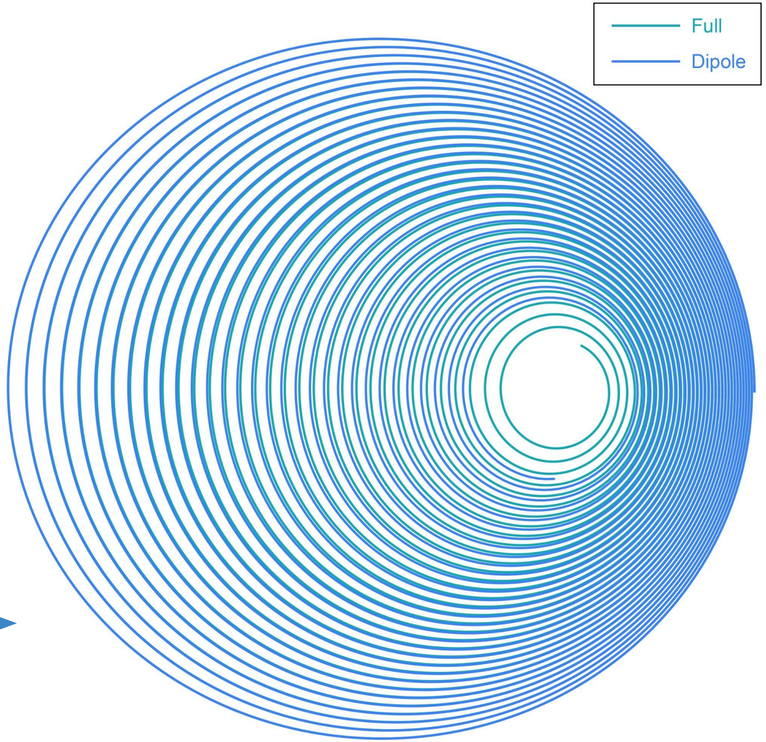
Using our energy and angular momentum loss, we calculate an adiabatic EM inspiral

Our results are $\mathcal{O}\left\{\left(\frac{q}{Q}\right)^1\left(\frac{m}{M}\right)^0\right\}$

$$r(t) = r\left[\alpha(t)\right]|_{E=E(t),L=L(t)}$$

$$\varphi(t) = \varphi\left[\alpha(t)\right]|_{E=E(t),L=L(t)}$$

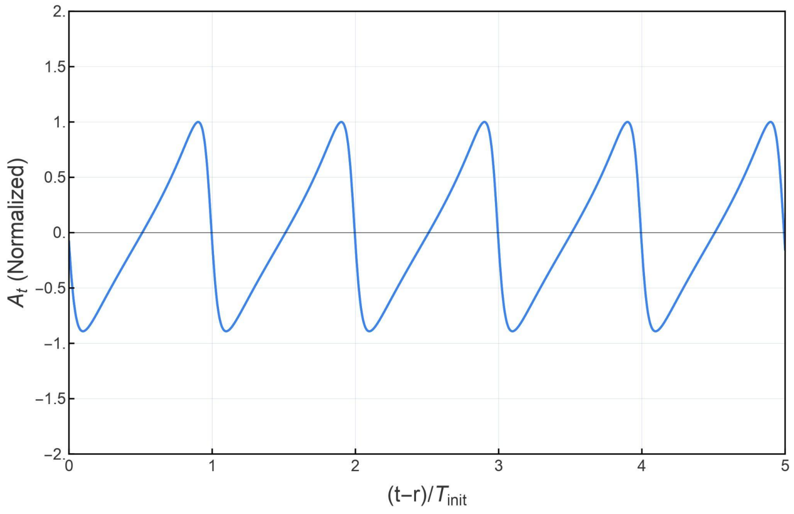
Here we took a very fast inspiral q~Q
just for the visual effect

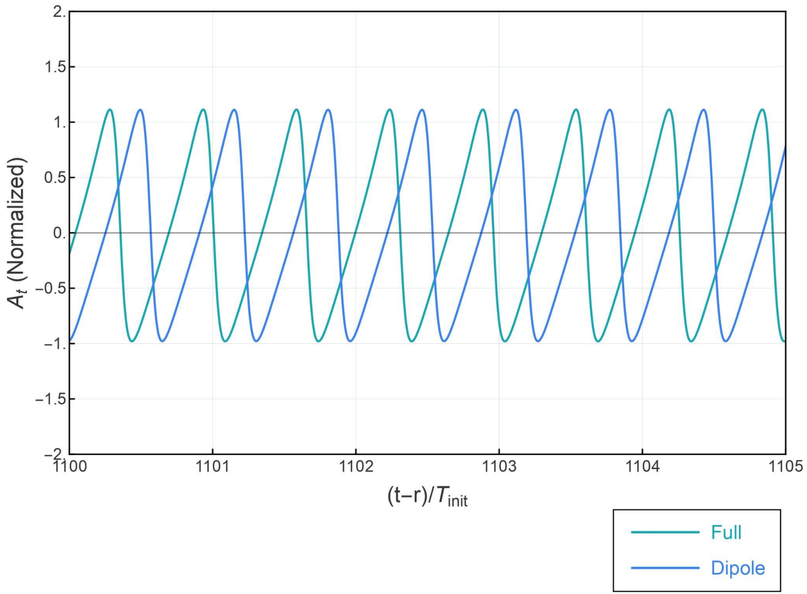


Adiabatic EM Inspiral

In the far field approximation, we also calculate the EM waveform as:

$$A^{\mu}_{\rm ret} = \left\{ \frac{q}{\mu r} \sum_{l_\gamma=0}^{\infty} \sum_{m_\gamma=-l_\gamma}^{l_\gamma} \sum_{\Delta n} e^{-i\Delta n\,\alpha} (-i)^{l_\gamma} Y^{m_\gamma *}_{l_\gamma}(\theta,\varphi) \sum_{\Delta l,\Delta m} \mathcal{M}^{\mu}_{\Delta,l_\gamma,m_\gamma}(\omega_{\Delta n}, N, L) \right\}_{\rm ret}$$

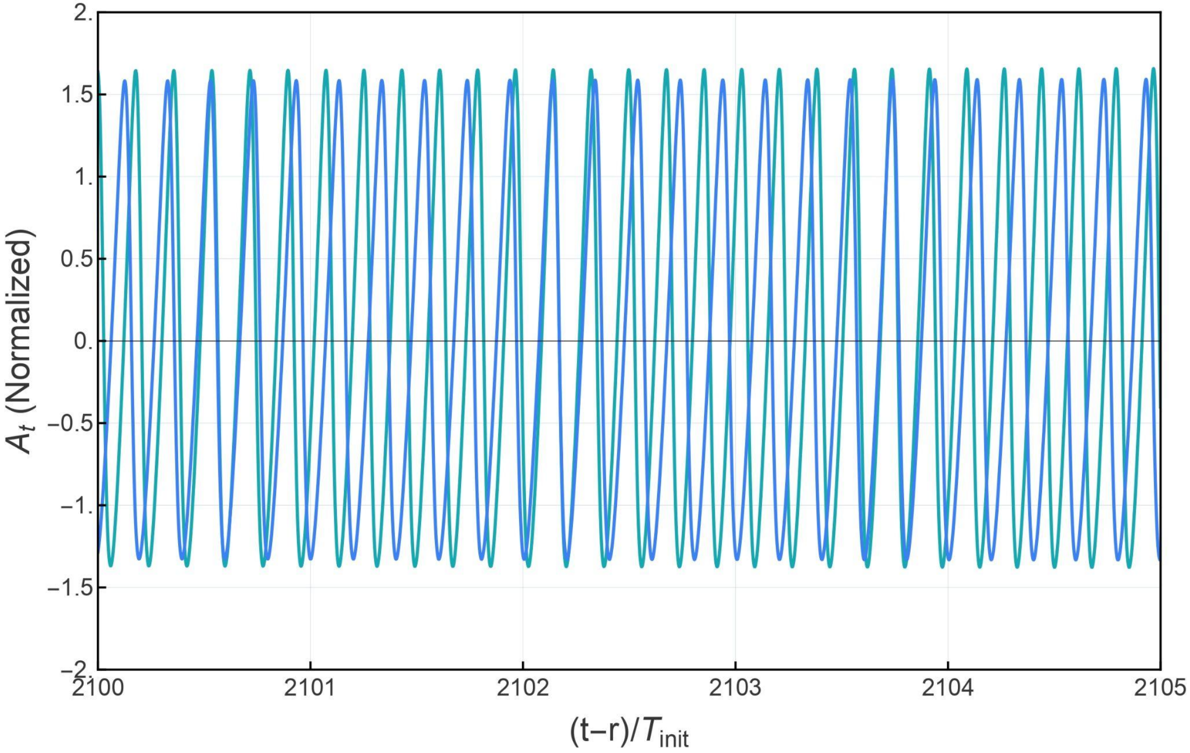




$$Z = Q/q = 4\pi$$

Adiabatic EM Inspiral

In the far field approximation, we also calculate the EM waveform:



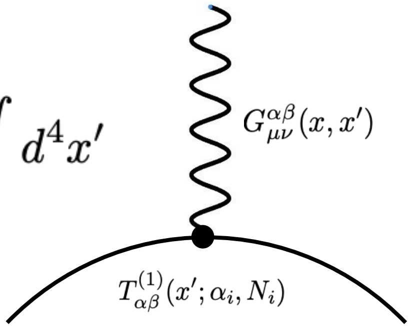
After many cycles, the electron in the full calculation is close to the origin than the one in the dipole

An amplitude difference in addition to phase

Towards Gravitational Self-Force with the QSM

First order gravitational self-force

$$h^{(1)}_{\mu\nu}(x;\alpha_i,N_i) = \int d^4x'$$



The circular line means integrating the Green's function along the worldline of the osculating BH geodesic

Currently this is done numerically, but we are working towards an analytical result with the QSM

We already have the eigenstates $|n,l,m\rangle$ in Schwarzschild/Kerr and can reproduce geodesics

"Wait, Didn't People Use Amplitudes for Inspirals?"

Scattering Amplitudes / NREFT

Perturbative in G or α

All orders in m/M or q/Q

Good for LIGO

Expand in m/M or q/Q
Resum in G or α
→

←
Expand in G or α
Resum in m/M or q/Q

Post Adiabatic / Self-Force

All-orders in G or α

Perturbative in m/M or q/Q

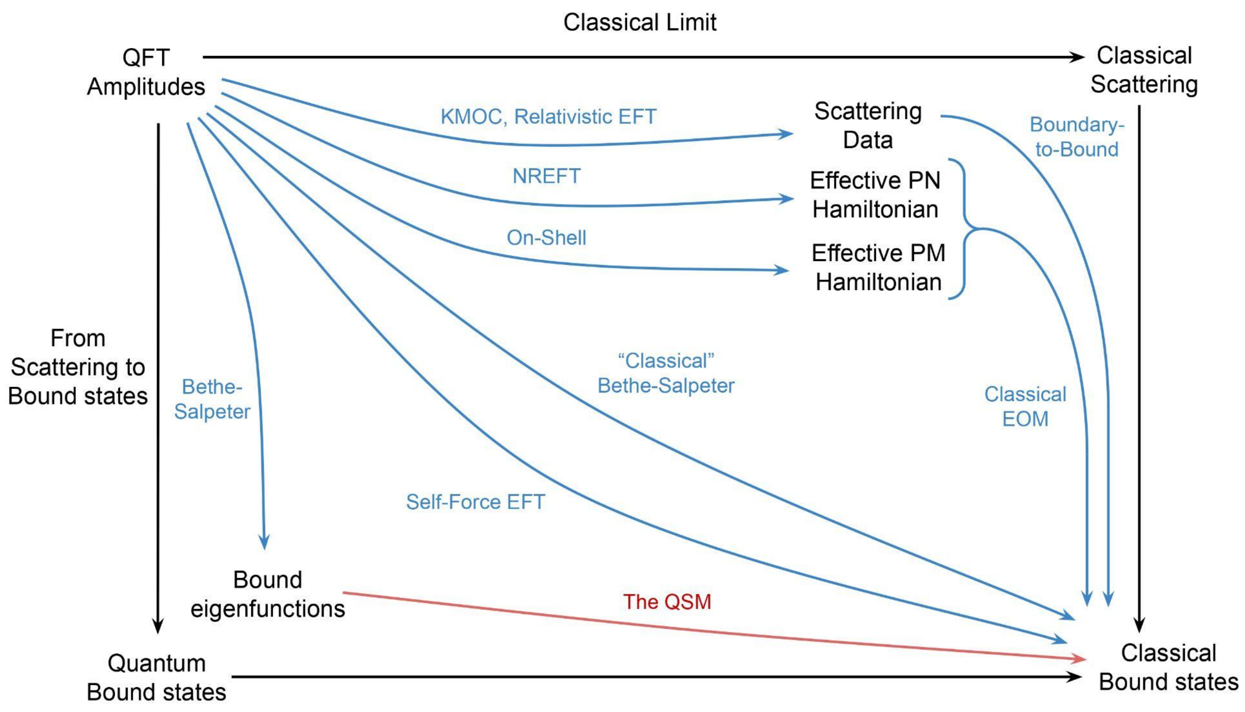
Good for LISA

Goldberger, Rothstein, Porto, Bern, Cheung, Kosower, O'Connell, Huang, Shen……

Poisson, Pound, Barack, Wardell, Warburton, Miller, van de Meent……

We are here - analytically!

The QSM in the Landscape of Quantum-to-Classical Methods



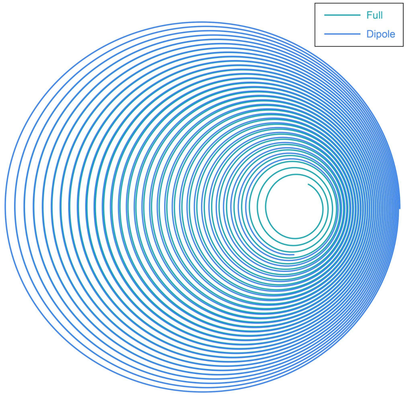
Conclusions and Generalizations

The QSM is a method to obtain the Fourier coefficients of classical observables:

"The Δn Fourier coefficient of the *classical observable* O is the classical limit of the Δn transition mediated by the *quantum operator* O"

We applied it for the analytical calculation of:

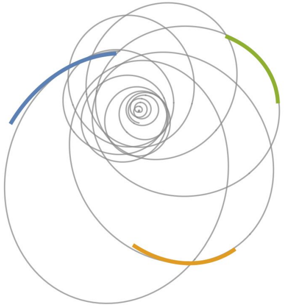
- Time-dependent Keplerian motion
- All-multipole EM radiation from a Keplerian orbit
- EM self-force and an adiabatic EM inspiral



Conclusions and Generalizations

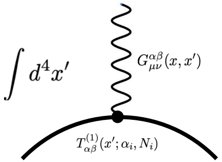
Near future applications:

- All Schwarzschild and Kerr geodesics



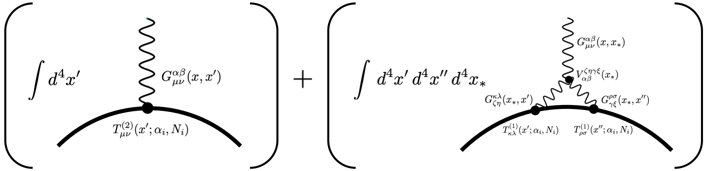
- Gravitational 1SF

$h^{(1)}_{\mu\nu}(x;\alpha_i,N_i) =$



- Gravitational 2SF

$h^{(2)}_{\mu\nu}(x;\alpha_i,N_i) =$



Thank You!

Bill Murray
He's having the day of his life...
over and over again.
Groundhog Day
COLUMBIA PICTURES PRESENTS
A TREVOR ALBERT PRODUCTION A HAROLD RAMIS FILM BILL MURRAY ANDIE MACDOWELL
"GROUNDHOG DAY" CHRIS ELLIOTT MUSIC BY GEORGE FENTON EDITED BY PEMBROKE J. HERRING
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DIRECTED BY HAROLD RAMIS
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Backup

Self-Force

Conclusions and Generalizations

Generalizations:

$$O(\alpha) = \sum_{\Delta n=-\infty}^{\infty} O_{\Delta n}\, e^{-i\Delta n \alpha}$$

bound, single angle variable

$$O(\alpha_1, \ldots, \alpha_k) = \sum_{\Delta n_1=-\infty}^{\infty} \cdots \sum_{\Delta n_k=-\infty}^{\infty} O_{\Delta n_1 \ldots \Delta n_k}\, e^{-i\sum \Delta n_i \alpha_i}$$

multiple angle variables

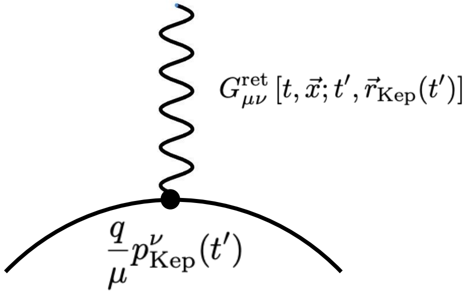
$$O(t) = \int_{-\infty}^{\infty} d\omega\; O_\omega\, e^{-i\omega t}$$

unbound motion

Derivation of All-Multipole EM Emission

Substituting J^μ:

$$A_\mu^{\rm ret}(t,\vec{x}) = \frac{q}{\mu}\int dt'\, G_{\mu\nu}^{\rm ret}\left[t,\vec{x};t',\vec{r}_{\rm Kep}(t')\right]\, p_{\rm Kep}^\nu(t')$$



Multipole expansion of retarded Green's function (see Jackson E&M):

$$G_{\mu\nu}^{\rm ret}(t,\vec{x};t',\vec{x}') = g_{\mu\nu}\frac{\Theta(t-t')}{4\pi R}\,\delta(t-t'-R)$$

$$= g_{\mu\nu}\frac{\Theta(t-t')}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega(t-t')}\left\{i\omega\sum_{l=0}^{\infty} j_l(\omega r_<)\, h_l^{(1)}(\omega r_>)\sum_{m=-l}^{l} Y_l^{m*}(\theta',\varphi')Y_l^m(\theta,\varphi)\right\}$$

(we take the observation point r > r')

Derivation of All-Multipole EM Emission

Substituting multipole expansion:

$$A^{\mu}_{\text{ret}}(t,\vec{x}) = \frac{q}{\mu}\int dt' \, \frac{\Theta(t-t')}{2\pi} \overbrace{\int_{-\infty}^{\infty} d\omega \, e^{-i\omega(t-t')} \left\{ i\omega \sum_{l=0}^{\infty} h_l^{(1)}(\omega r) \sum_{m=-l}^{l} Y_l^m(\theta,\varphi) \right\}}^{\text{independent of source}} \times$$

$$\underbrace{j_l\left[\omega r_{\text{Kep}}(t')\right] \, Y_l^{m*}\left[\theta_{\text{Kep}}(t'), \varphi_{\text{Kep}}(t')\right] \, p^{\nu}_{\text{Kep}}(t')}_{\text{source-dependent}}$$

Derivation of All-Multipole EM Emission

With the QSM we have:

$$A^{\mu}_{\text{ret}}(t,\vec{x}) = \frac{iq}{\mu} \sum_{l_\gamma=0}^{\infty} \sum_{m_\gamma=-l_\gamma}^{l_\gamma} \sum_{\Delta n,\Delta l,\Delta m} Y^{m_\gamma *}_{l_\gamma}(\theta,\varphi) \int_{-\infty}^{\infty} \frac{\Theta(t-t')}{2\pi}\, dt' \int_{-\infty}^{\infty} d\omega\, \omega\, h^{(1)}_{l_\gamma}(\omega r)$$

$$\times \exp\left[-i\Delta n\, \alpha'(t') - i\omega(t-t')\right] \mathcal{M}^{\mu}_{\Delta,l_\gamma,m_\gamma}(\omega,N,L)$$

where: $$\mathcal{M}^{\mu}_{\Delta,l_\gamma,m_\gamma}(\omega,N,L) \equiv \lim_{\hbar\to 0} \langle n'l'm'|\, j_{l_\gamma}(\omega\, r_{\text{Kep}})\, Y^{m_\gamma}_{l_\gamma}(\theta_{\text{Kep}},\varphi_{\text{Kep}})\, p^{\mu}_{\text{Kep}}\, |nlm\rangle$$

A classical limit of a hydrogen atom transition, which we calculate analytically

(r_{Kep},θ_{Kep},φ_{Kep},p^{μ}_{Kep}) are quantum operators

Derivation of EM Self-Force

We work at the adiabatic (0PA) order*. At this order, we can calculate the field $A_\mu(t)$ sourced by an electron on the osculating keplerian orbit defined by E(t) L(t), and averaged over the action-angle α

Van de Meent, Warburton '18

$$\frac{dE}{dt} = q\left\langle \vec{v}\cdot\vec{E}^{\rm reg}\right\rangle_\alpha$$

$$\frac{d\vec{L}}{dt} = q\left\langle \vec{r}\times\left[\vec{E}^{\rm reg}+\vec{v}\times\vec{B}^{\rm reg}\right]\right\rangle_\alpha$$

$$\langle\mathcal{O}(\alpha)\rangle_\alpha \equiv \frac{1}{2\pi}\int_0^{2\pi}\mathcal{O}\,d\alpha$$

E^{reg} and B^{reg} are calculated from A_μ^{reg} . The reg label means the field regular at the source, namely

$$A^\mu_{\rm reg} = \frac{1}{2}\left(A^\mu_{\rm ret} - A^\nu_{\rm adv}\right)$$

Dirac 1938

* Higher post-adiabatic (PA) corrections require the partial derivatives of the field with respect to E , L

Gravitational 1SF

The 1st-order gravitational field generated by a point mass on a Schwarzschild geodesic,

$$h^{\mu\nu}(x) = \int d^4x\,\sqrt{g}\,G^{\mu\nu}{}_{\rho\sigma}(x,x')\,T^{\rho\sigma}(x')$$

Linearized gravitational Green's function in Schwarzschild

Energy-Momentum from mass on geodesic

can be Fourier-expanded in the *two angle variables* of Schwarzschild geodesics

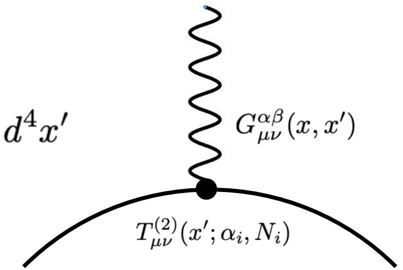
$$h^{\mu\nu} = \sum_{\Delta n_1, \Delta n_2 = -\infty}^{\infty} h^{\mu\nu}_{\Delta n_1, \Delta n_2}(\vec{x})\; e^{-i(\Delta n_1 \alpha_1 + \Delta n_2 \alpha_2)}$$

The $h^{\mu\nu}_{\Delta n_1, \Delta n_2}(\vec{x})$ are all-order expressions in GMm/L , known only numerically

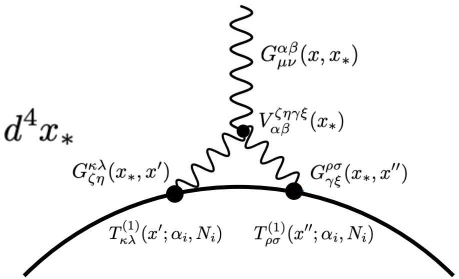
Towards 2nd Order Gravitational Self-Force with the QSM

Future application: second order gravitational self-force

$$h^{(2)}_{\mu\nu}(x;\alpha_i,N_i) = \left(\int d^4x' \right.$$



$$\left. \right) + \left(\int d^4x'\, d^4x''\, d^4x_* \right.$$



$$\left. \right)$$

Using the QSM, we would be able to calculate it analytically

This could open new paths for resummation, with a potential for significant improvements in 2SF efficiency