EFT matching
from analyticity and unitarity

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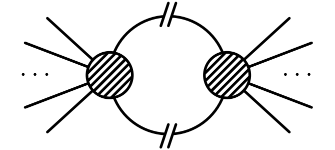
Analyticity and unitarity of on-shell amplitudes

bypass unphysical fields, operators, Lagrangians

avoid gauge and field redefinition redundancies

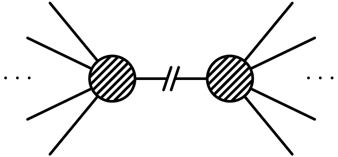
loops cut into lower loops

+ rational terms



trees cut into smaller trees

+ contact terms



recursive construction from the simplest amplitudes

SMEFT applications

- on-shell anomalous dimensions

[Cheung, Shen '15], [Azatov et al. '16], [Bern et al. '19, '20], [Jiang et al. '20], [Elias Miró et al. '20, '21]
[Baratella et al. '20, '20, '21], [Accettulli Huber, De Angelis '21], [Delle Rose et al. '22], [Baratella '22]
[Machado, Renner, Sutherland '22], [Chala '23]

- on-shell operator enumeration

[Shadmi, Weiss '18], [Ma, Shu, Xiao '19], [Falkowski '19], [GD, Machado '19]
[Li, Ren, et al. '20, '20], [Harlander, Kempkens, Schaaf '23]

- on-shell massive amplitudes

[Aoude, Machado '19], [GD, Kitahara, Shadmi, Weiss '19], [GD et al. '20]
[Balkin et al. '21], [Dong, Ma, Shu, Zheng '21, '22], [De Angelis '22]
[Bradshaw, Chang, Chen, Liu, Luty '22, '23], [Liu, Ma, Shadmi, Waterbury '23]

Loop matching from cuts?

[Delle Rose, von Harling, Pomarol '22] examined on-shell the magic zeros found by [Arkani-Hamed, Harigaya '21] in $(g-2)_\mu$

finding that the rational terms of loops do not contribute to the matching

Lessons from positivity

[Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi '06]

- Unitarity $S \cdot S^\dagger = \mathbb{I}$ with $S = 1 + i\mathcal{A}$:

$\mathcal{A}^\dagger(+i\epsilon) \overset{CPT}{=} \mathcal{A}(-i\epsilon)$ sum over intermediate state X

$$(\mathcal{A} - \mathcal{A}^\dagger)/i = \mathcal{A} \cdot \mathcal{A}^\dagger$$

disc. known in 4-point amp.

positive elastic forward elem. $\sim \int \mathrm{d}X \, |\mathcal{A}_{ab\to X}|^2$

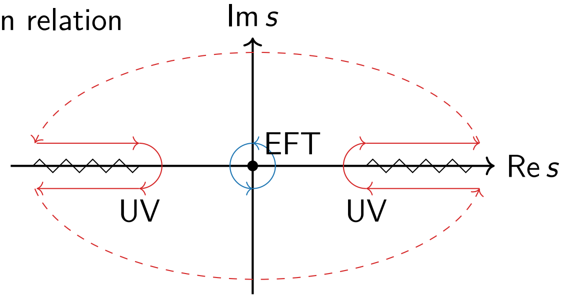
- EFT coefficients

as zero-momentum residues of the *substracted* amplitude

$$c_n = \operatorname*{Res}_{s=0} \frac{\mathcal{A}(s)}{s^{n+1}} \qquad \text{e.g. } \mathcal{A}^{\text{EFT, tree}}_{ab\to ab}(s) = \sum_k c_k \, s^k$$

$$= \frac{1}{2\pi i} \oint_{s=0} \mathrm{d}s \, \frac{\mathcal{A}(s)}{s^{n+1}}$$

- Dispersion relation



vanishes by Froissart for $n \geq 2$

$$c_n = \frac{1}{2\pi} \int_{\Lambda^2}^{\infty} \frac{\mathrm{d}s}{s^{n+1}} \left[\underbrace{\mathrm{Disc}\, \mathcal{A}^{\mathrm{UV}}_{ab\to ab}}_{\geq 0} + (-1)^n \underbrace{\mathrm{Disc}\, \mathcal{A}^{\mathrm{UV}}_{a\bar{b}\to a\bar{b}}}_{\geq 0} \right] + C_\infty$$

≥ 0 for n even ≥ 2

⟶ Low-energy EFT coefficients related to UV discontinuities
at least for four-point (elastic and forward) amplitudes

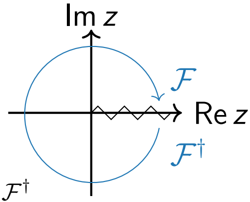
Lessons from anomalous dimensions

- In a massless theory, any $(\log \mu^2)$ comes with a $(-\log s_I)$
- A dilation $z^{D/2}$ with $D \equiv \sum_i p_i^\mu \frac{\partial}{\partial p_i^\mu}$ captures all Mandelstam logs in a single $(-\log z)$ and disregards logs of s_I/s_J ratios
- Form factors $\mathcal{F} \equiv {}_{\text{out}}\langle p_1, \dots p_m | \mathcal{L}(q) | 0 \rangle_{\text{in}}$ have all $s_I \equiv (\sum_{i \in I} p_i^\mu)^2$ Mandelstams positive (momentum influx: q)
- Dilated form factors $\hat{\mathcal{F}}(z) \equiv z^{D/2} \mathcal{F}$ only have singularities at positive z's

at $\sum_k \alpha_k m_k^2 / \sum_I \alpha_I s_I$ in Feynman parameterisation

- Going around the unit circle takes $\mathcal{F}^\dagger$ to $\mathcal{F}$:

$$(e^{-2\pi i})^{D/2}\mathcal{F}^\dagger = \mathcal{F}$$



- Unitarity is $\mathcal{F} = S \cdot \mathcal{F}^\dagger$ or $(\mathcal{F} - \mathcal{F}^\dagger)/i = \mathcal{A} \cdot \mathcal{F}^\dagger$

- So $D \sim -\mu \dfrac{\partial}{\partial \mu}$ is the phase of the S matrix:

$$e^{-\pi i D}\mathcal{F}^\dagger = S \cdot \mathcal{F}^\dagger$$

$\longrightarrow$ The analytic structure in z of m-point form factors is under control

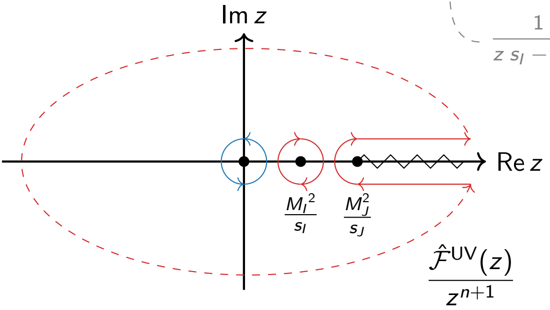
(amplitudes are obtained after crossing and $q \to 0$)

Massless EFT matching

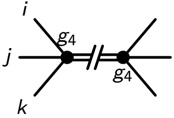
- equate $\mathcal{F}^{\text{EFT}}$ and $\mathcal{F}^{\text{UV}}$ order by order in the zero-momentum expansion
- dilate (with $z^{D/2}$) and enforce $\underset{z=0}{\text{Res}} \dfrac{\hat{\mathcal{F}}^{\text{EFT}}(z)}{z^{n+1}} = \underset{z=0}{\text{Res}} \dfrac{\hat{\mathcal{F}}^{\text{UV}}(z)}{z^{n+1}}$
- EFT: $\underset{z=0}{\text{Res}} \dfrac{\hat{\mathcal{F}}^{\text{EFT}}(s)}{z^{n+1}} = c_n \,\text{poly}_n(s_I)$ with $\mathcal{F}^{\text{EFT}}_{\text{tree}}(s) = \sum_k c_k \,\text{poly}_k(s_I)$
- UV: $\underset{z=0}{\text{Res}} \dfrac{\hat{\mathcal{F}}^{\text{UV}}(z)}{z^{n+1}} = \oint_{z=0} \mathrm{d}z \,\dfrac{\hat{\mathcal{F}}^{\text{UV}}(z)}{z^{n+1}} = \left[\sum \text{Res} + \int \text{Disc} + \int_\infty\right] \dfrac{\hat{\mathcal{F}}^{\text{UV}}(z)}{z^{n+1}}$

$\log(M_J^2 - z\, s_J)$

$\dfrac{1}{z\, s_I - M_I^2}$

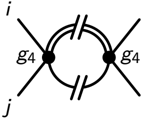


Simplest $\Phi\phi^3$ example



$$: \sum_{ijk \text{ channels}} \underset{z=M^2/s_{ijk}}{\text{Res}} \frac{|\mathcal{A}(\phi\phi\phi \to \Phi)|^2}{z s_{ijk} - M^2} \frac{1}{z^{n+1}}$$

$$= \frac{g_4^2}{M^2} \sum_{\text{channels}} \left(\frac{s_{ijk}}{M^2}\right)^n$$



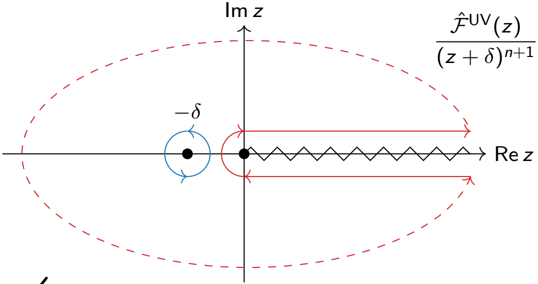
$$: \frac{1}{2\pi} \sum_{\text{channels}} \int_{M^2/s_{ij}}^{\infty} \frac{dz}{z^{n+1}} \int \text{dLIPS} \, |\mathcal{A}(\phi\phi \to \phi\Phi)|^2$$

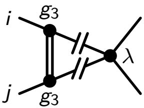
$$= \frac{1}{2\pi} \sum_{\text{channels}} \int_{M^2/s_{ij}}^{\infty} \frac{dz}{z^{n+1}} \frac{1}{8\pi} \left(1 - \frac{M^2}{z s_{ij}}\right) g_4^2$$

$$= \frac{g_4^2}{16\pi^2 n(n+1)} \sum_{\text{channels}} \left(\frac{s_{ij}}{M^2}\right)^n \qquad \text{for } n > 0$$

- use amplitudes instead of form factors in practice
- all EFT orders obtained at once
- nothing to know about, or compute in, the EFT
- fewer legs and loops

Massless cut subtlety





$$: \int_0^\infty \frac{\mathrm{d}z}{z^{n+1}} \int \mathrm{dLIPS}_d \frac{\lambda g_3^2}{(l-p_i)^2 - M^2}$$

$$= \frac{\lambda g_3^2 M^{d-6}}{8(4\pi)^{\frac{d}{2}-2}} \left(\frac{s_{ij}}{M^2}\right)^n \frac{(-1)^{n+1} n! \csc\frac{\pi d}{2}}{\Gamma[\frac{d}{2}+n]}$$

$$= \frac{\lambda g_3^2}{16\pi^2 M^2} \frac{(-1)^n}{n+1} \left(\frac{s_{ij}}{M^2}\right)^n \left(\frac{1}{\bar{\epsilon}} + H_{n+1} + \log\frac{\mu^2}{M^2} + \mathcal{O}(\epsilon)\right)$$

Note phase-space integrals can be complicated,

unlike the hard region expansion ($s_I \ll l^2 \sim M^2$) at low EFT order

Dimreg's power

- Loop contributions in the massless EFT lead to scaleless $\int_0^\infty \mathrm{d}z\, z^{\alpha+\epsilon} = 0$ for suitable ϵ's.

  $\longrightarrow$ The tree-level EFT amplitude is extracted.

- The soft region ($l^2 \sim s_l \ll M^2$) of UV loops is similarly scaleless.

  $\longrightarrow$ The z dispersion extracts the hard region of UV loops.

- Loop contributions at infinity lead to scaleless $\lim_{|z|\to\infty} |z|^{\alpha+\epsilon} = 0$ for a suitable ϵ.

  $\longrightarrow$ The boundary is tree-level exact.

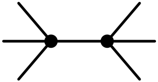
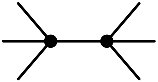
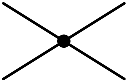
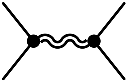
Scarse boundary terms

multiplicity (→ m) · sum of coupling dim. (→ $[c_{\text{EFT}}]$)

- no matching information unless $n \geq \min(4 - m - [c_{\text{EFT}}])/2$
- no boundary term unless $n \leq \min(4 - m - [c_{\text{UV}}])/2$
  $\rightarrow \max[c_{\text{UV}}] \leq 4 - m - 2n \leq \max[c_{\text{EFT}}]$

- but $\max[c_{\text{UV}}] \geq \max[c_{\text{EFT}}]$ by definition
  $\rightarrow$ boundary needed only for $\max[c_{\text{EFT}}] = \max[c_{\text{UV}}]$ $(= 4 - m - 2n)$

e.g.

$n \geq$	EFT	UV
-1		
0		

EFT matching from unitarity and analyticity

provides an alternative to diagrammatic and functional methods

does not require any knowledge about the massless EFT

only uses UV residues and cuts (barring a tree boundary)

extracts all orders of the tree EFT amplitude at once

may lead to new insight and computations