LOCAL COVARIANT PERTURBATION THEORY AT ANY ORDER

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Based on work with K.Santos

2212.07451, 2309.14939

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Covariant Perturbation Theory

- Local: expansion in all IR scales: fields, derivatives, (light) masses
  - EFT matching contribution from integration of heavy d.o.f's
  - RG - Running of local operators
- Covariant: Maintaining manifest gauge covariance at every step (background field method)
- Any Order: Go beyond the usual 1-loop case

Covariant PT: What is Known

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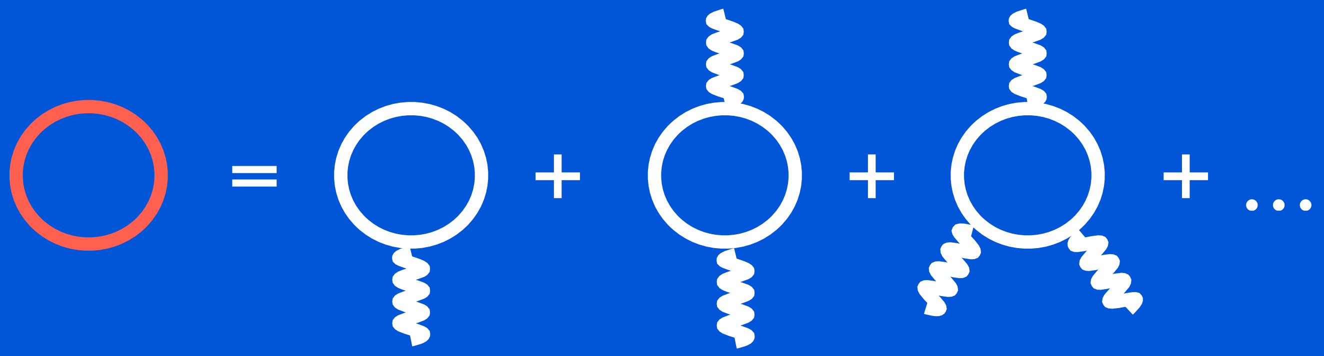
- One-loop determinants of single fields (heat kernel)

Schwinger '51, DeWitt '65, '67, Gilkey '75, Avramidi '90, '91, Fujikawa '79, '80
Review: Vassilevich '03



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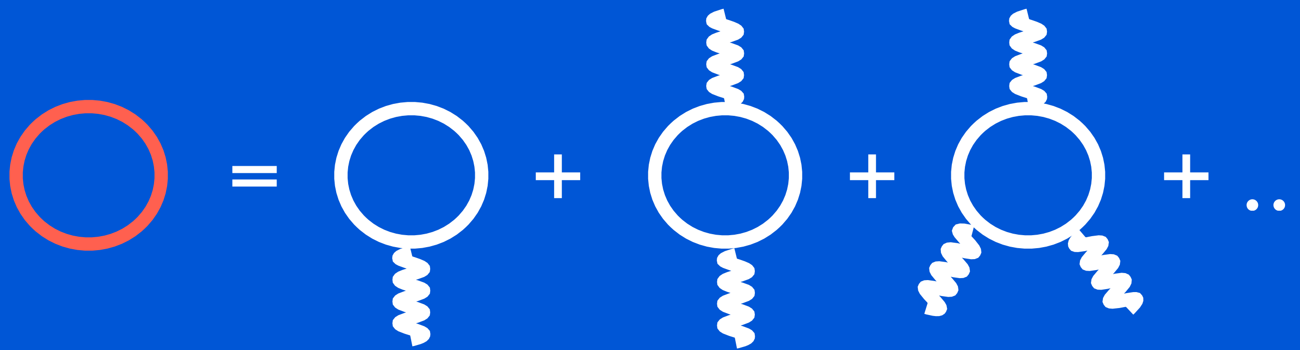
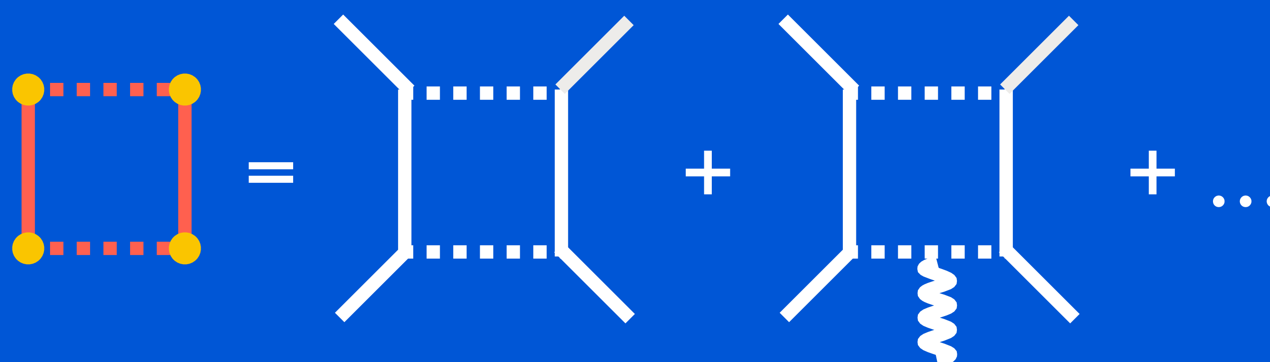


- One loop graphs including mixed fields (mass, spin), derivative couplings

Barvinsky+Vilkovisky '85

Henning et al '14, '16
Drozd et al '15, Aguila et al '16
Zhang '16, Fuentes-Martin et al '16
Ellis et al '17, '20 …



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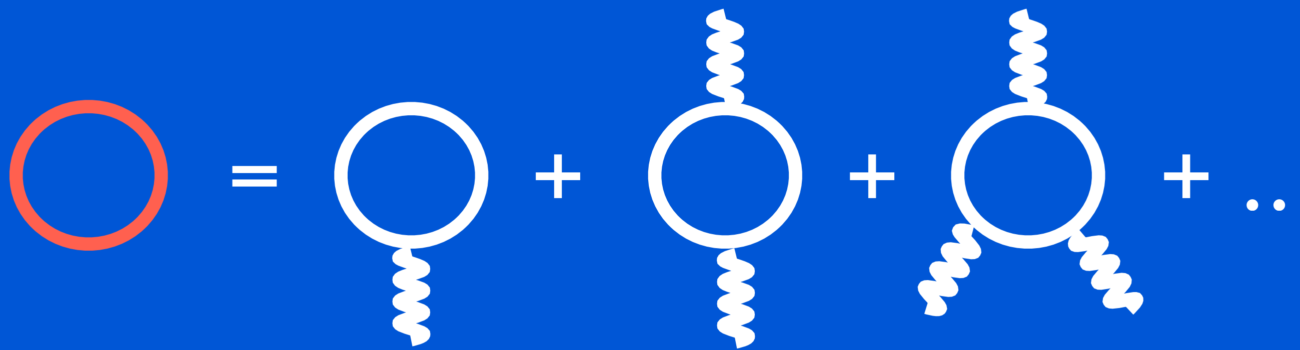

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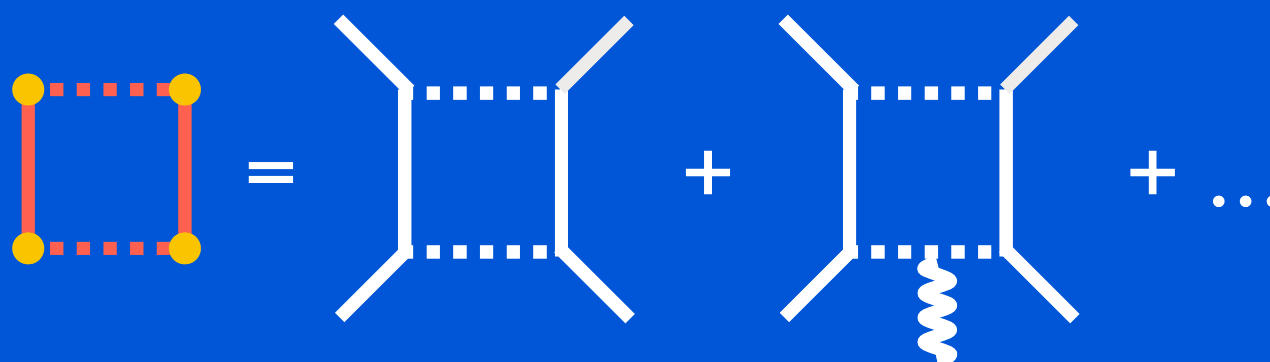
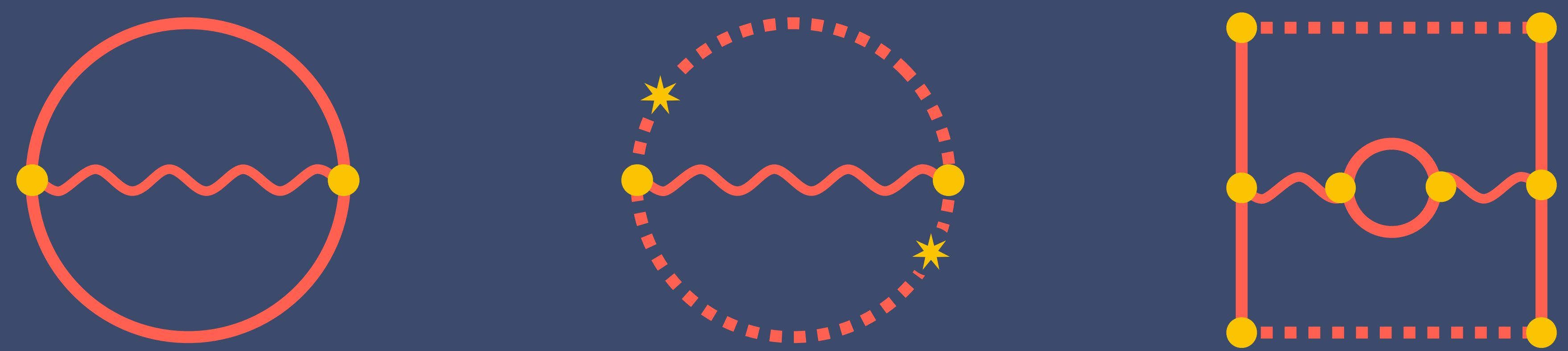


- Some higher-loop special cases (covariantly constant field strength), but no systematic formalism

Goal

- Want to calculate arbitrary, covariant L - loop diagrams and expand in local operators



Background Field Method

- Background field method:

$$\phi(x) = \phi_b(x) + \phi_f(x) \qquad A^\mu(x) = A^\mu_b(x) + A^\mu_f(x) \qquad \text{etc}$$

- Integrating over fluctuations gives effective action $S_{\text{eff}}[\phi_b, A_b, \ldots]$

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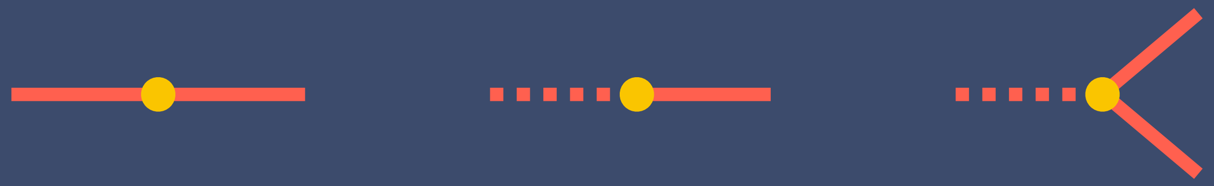
- Background field dependence
  - Couplings, e.g.

$$g\phi\bar{\psi}\psi = \ldots + g\phi_b\,\bar{\psi}_f\psi_f + g\bar{\psi}_b\,\phi_f\psi_f + g\,\phi_f\bar{\psi}_f\psi_f + \ldots$$



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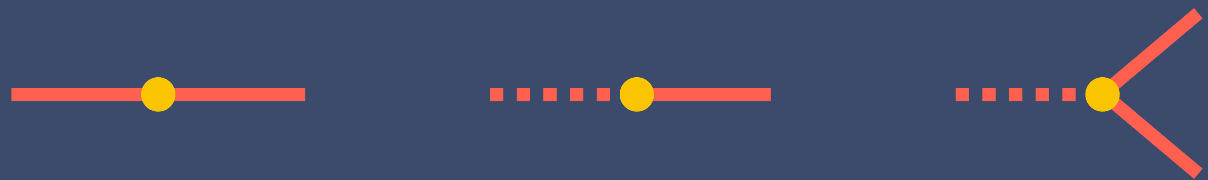

  - Propagators, e.g.

$$\langle\phi_f\bar{\phi}_f\rangle = \frac{-i}{(\partial - iA_b)^2 + m^2} \qquad \langle D\phi_f\bar{\phi}_f\rangle = (\partial - iA_b)\frac{-i}{(\partial - iA_b)^2 + m^2}$$

The Heat Kernel trick

- Represent each propagator by

$$\langle x | \frac{-i}{D^2 + X + m^2} | y \rangle = \int_0^\infty dt \ \langle x | e^{-it(D^2+X+m^2)} | y \rangle$$

- Schwinger parameter t (one for each propagator)

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$$\langle x | \frac{-i}{D^2 + X + m^2} | y \rangle = \int_0^\infty dt \; e^{-itm^2} \int \frac{d^d k}{(2\pi)^d} \, e^{itk^2 - ik(x-y)} \, B(t, X, x, y)$$

- Schwinger parameter t (one for each propagator)

$$B(t, X, x, y) \equiv \frac{\langle x | e^{-it\,(D^2+X)} | y \rangle}{\langle x | e^{-it\,\partial^2} | y \rangle}$$

is analytic in t and has a local, covariant expansion

The Heat Kernel trick

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- Propagators of derivatives of fields

$$\langle x \,|\, D_\mu \frac{-i}{D^2 + X + m^2} \,|\, y \rangle$$

$$= \int_0^\infty dt\ e^{-itm^2} \int \frac{d^d k}{(2\pi)^d}\, e^{itk^2 - ik(x-y)}\, (D_\mu^x - ik_\mu)\, B(t, X, x, y)$$

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- Fermions

$$X = -S^{\mu\nu} F_{\mu\nu}$$

$$\langle x | \frac{i}{i\not{D} - m} | y \rangle$$
$$= \int_0^\infty dt\; e^{-itm^2} \int \frac{d^d k}{(2\pi)^d}\, e^{itk^2 - ik(x-y)}\, (i\not{D}_\mu^x + k_\mu + m)\, B(t, X, x, y)$$

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- Gauge fields, ghosts, etc in a similar way

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- First expand B around $t = 0$

$$B(t;X;x,y) = \sum_n \frac{t^n}{n!} b_{2n}(x,y) \longleftarrow \text{heat kernel coefficients}$$

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- Will need generalized local heat kernel coefficients (LHKC)

$$\lim_{y \to x} (\dots D^x_\mu)(\dots D^y_\nu) b_{2n}(x, y) \equiv [b_{2n;\mu\dots;\nu\dots}](x)$$

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- A Mathematica notebook for calculation of the LHKC's is available with GG+Santos, arXiv 2212.07451

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- "Standard" local Coefficients $[b_{2n}]$

```
In[∘]:= b[2, {}, {}]
        b[4, {}, {}]
```

Out[∘]= $\{\{X, -i\}\}$

Out[∘]= $\left\{\left\{F_{a_1 a_2} F_{a_1 a_2}, \frac{1}{6}\right\}, \left\{X_{;a_1 a_1}, -\frac{1}{3}\right\}, \{XX, -1\}\right\}$

EXAMPLES OF LOCAL COEFFICIENTS

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- Generalized local coefficients (e.g., $[b_{0;\mu;\nu}]$ or $[b_{2;;\mu\nu}]$)

```
In[∘]:= b[0, {"μ"}, {"ν"}]
        b[2, {}, {"μ", "ν"}]
```

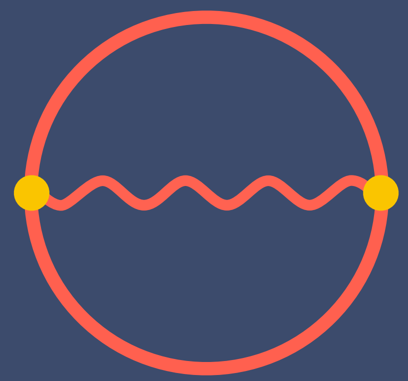
Out[∘]= $\left\{\left\{F_{\mu\nu}, -\frac{i}{2}\right\}\right\}$

Out[∘]= $\left\{\left\{X_{;\nu\mu}, -\frac{5\,i}{6}\right\}, \left\{F_{a_1\mu;a_1\nu}, -\frac{1}{12}\right\}, \left\{X_{;\mu\nu}, \frac{i}{2}\right\}, \left\{F_{a_1\nu;a_1\mu}, -\frac{1}{12}\right\}, \left\{F_{\nu\mu}X, \frac{1}{3}\right\}, \left\{F_{a_1\mu}F_{a_1\nu}, \frac{i}{12}\right\}, \left\{F_{a_1\nu}F_{a_1\mu}, \frac{i}{12}\right\}, \left\{XF_{\nu\mu}, \frac{1}{6}\right\}\right\}$

THE MASTER FORMULA

GG + Santos ’23
GG ’23

Feynman Graph G



$$\Gamma(t_i, x_n, k_i) \equiv \Pi_i\,(\ldots)B_i\,\Pi_n\,C_n$$

$$I(t_i, x_n, k_i) \equiv e^{-it_ik_i^2 - ix_n\mathbb{B}_{ni}k_i}$$

Gauge and Lorentz invariant!

$\mathbb{B}$ is the incidence matrix of the graph

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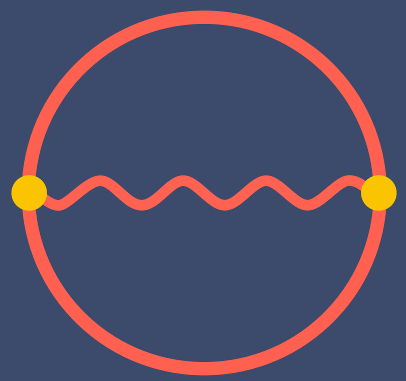

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Gauge and Lorentz invariant!

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$\mathbb{B}$ is the incidence matrix of the graph

$$S^G_{\text{eff}} = \int_{t_i} e^{-it_i m_i^2} \int_{x_n} \int_{k_i} I(t_i, x_n, k_i) \, \Gamma(t_i, x_n, k_i)$$

The master formula

GG + Santos '23
GG '23

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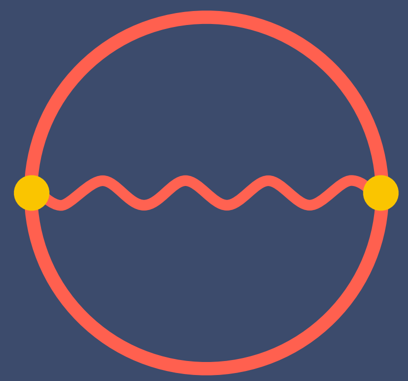

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$$S^G_{\text{eff}} = \int_{t_i} e^{-it_i m_i^2} \int_x \tilde{I}(t_i, i\partial_{x_n}, z_i)\, \Gamma(t_i, x_n, -i\overleftarrow{\partial}_{z_i})\Big|_{z_i=0,\ x_n=x}$$

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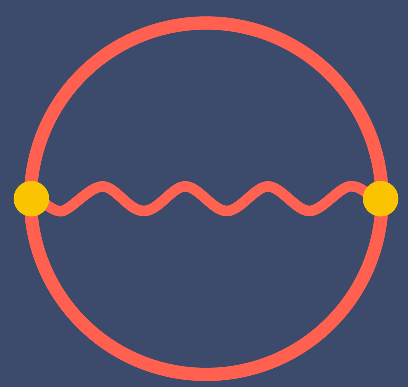

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$$S_{\rm eff}^G = \int_{t_i} e^{-it_im_i^2} \int_x \tilde{I}(t_i, i\partial_{x_n}, z_i)\,\Gamma(t_i, x_n, \text{-}\,i\overleftarrow{\partial}_{z_i})\Big|_{z_i=0,\ x_n=x}$$

- Only LHKC's appear
- $\tilde{I}$ is universal (depends only on topology of graph)
- $I \to \tilde{I}$ equivalent to loop momentum integration

The universal momentum integral

$$\tilde{I}(t_i, p_n, z_i) = (4\pi)^{-\frac{dL}{2}} \Delta^{-\frac{d}{2}} \exp\left(\frac{1}{4} z_i \mathbb{Q}_{ij} z_j - i z_i \mathbb{R}_{in} p_n - p_m \mathbb{U}_{mn} p_n\right)$$

Incidence matrix: $$\mathbb{B}_{ni} = \begin{cases} +1 & \text{if edge } i \text{ enters vertex } n \\ -1 & \text{if edge } i \text{ leaves vertex } n \\ 0 & \text{else} \end{cases}$$

$$\longrightarrow \quad \mathbb{T} \equiv \begin{pmatrix} it_1 & & \\ & it_2 & \\ & & \ddots \end{pmatrix} + \epsilon^{-1} \mathbb{B}^{\mathrm{T}} \mathbb{B}$$

- Δ is the leading coefficient of $\det \mathbb{T}$ as $\epsilon \to 0$
- $\mathbb{T}^{-1} = \mathbb{Q} + \epsilon\, \mathbb{Q}' + \epsilon^2 \mathbb{Q}'' + \ldots$, $\mathbb{R} = \mathbb{Q}' \mathbb{B}^{\mathrm{T}}$, $\mathbb{U} = \mathbb{B} \mathbb{Q}'' \mathbb{B}^{\mathrm{T}}$

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- Δ, $\Delta\mathbb{Q}_{ij}$, $\Delta\mathbb{R}_{in}$, and $\Delta\mathbb{U}_{mn}$ are polynomials in the t_i
- Δ and $\Delta\mathbb{U}_{mn}$ $\rightarrow$ first and second Szymanzik polynomials
- $\Delta\mathbb{Q}_{ij}$ and $\Delta\mathbb{R}_{in}$ are new graph polynomials

Golz '17, GG '23

- All four polynomials can be written as sums over subgraphs (all vertices, fewer edges)
- Gives rise to many interesting relations...

The universal momentum integral

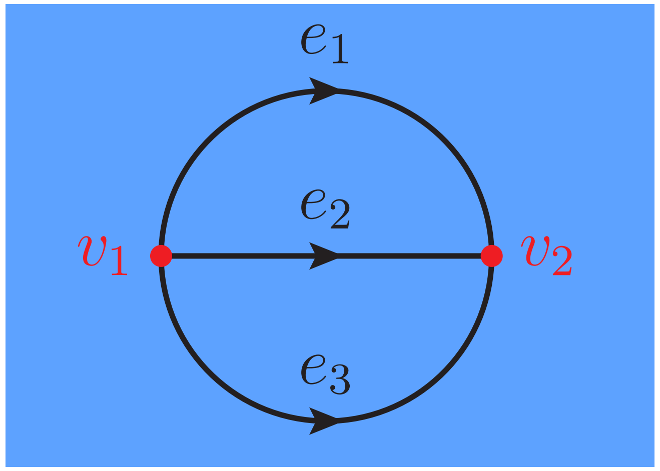
$$\tilde{I}(t_i, p_n, z_i) = (4\pi)^{-\frac{dL}{2}} \Delta^{-\frac{d}{2}} \exp\left(\frac{1}{4} z_i \, \mathbb{Q}_{ij} \, z_j - i z_i \, \mathbb{R}_{in} \, p_n - p_m \, \mathbb{U}_{mn} \, p_n \right)$$

$$\Delta = \tau_1\tau_2 + \tau_2\tau_3 + \tau_1\tau_3$$

$$\mathbb{Q} = \frac{1}{\Delta} \begin{pmatrix} \tau_2 + \tau_3 & -\tau_3 & -\tau_2 \\ -\tau_3 & \tau_1 + \tau_3 & -\tau_1 \\ -\tau_2 & -\tau_1 & \tau_1 + \tau_2 \end{pmatrix}$$

$$\mathbb{R} = \frac{1}{\Delta} \begin{pmatrix} -\tau_2\tau_3 & \tau_2\tau_3 \\ -\tau_1\tau_3 & \tau_1\tau_3 \\ -\tau_1\tau_2 & \tau_1\tau_2 \end{pmatrix}$$

$$\mathbb{U} = \frac{\tau_1\tau_2\tau_3}{\Delta} \begin{pmatrix} -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix}$$



$(\; \tau_i = i t_i \;)$

Example

- Integrating out massive fermion at one loop

$\mathscr{L}_{\rm int} \supset g\bar{\chi}_b \phi_f \psi_f \qquad C \equiv g\chi_b$

$$\mathcal{L}_{\rm eff}^{D=6} = \frac{1}{16\pi^2}\frac{1}{m^2}\left[-\frac{1}{2}\,\mathrm{tr}\left(\bar{C}_{;\mu}\gamma^\mu \left[b_2^f\right] C + \bar{C}\gamma^\mu \left[b^f_{2;\mu}\right] C\right) + \left(\frac{3}{2} - \log\frac{m^2}{\mu^2}\right)\mathrm{tr}\left[b^s_{2;;\mu}\right]\bar{C}\gamma^\mu C \right.$$
$$\left. + \frac{i}{3}\,\mathrm{tr}\,\bar{C}_{;(\mu\nu\rho)}\gamma^\mu g^{\nu\rho} C + \mathrm{tr}\,\bar{C}\gamma^\mu\left[b^f_{2;\mu}\right]C - \frac{i}{2}g^{\nu\rho}\,\mathrm{tr}\left(2\bar{C}_{;\nu}\gamma^\mu\left[b^f_{0;\mu\rho}\right]C + \bar{C}\gamma^\mu\left[b^f_{0;\mu\nu\rho}\right]C\right)\right],$$

↓

$$\mathcal{L}_{\rm eff}^{D=6} = \frac{1}{16\pi^2}\frac{1}{m^2}\left[\frac{1}{8}\,\mathrm{tr}\,\bar{C}_{;\mu}\gamma^\mu(\not{F}^f)C - \frac{1}{8}\,\mathrm{tr}\,C\not{F}^f\gamma^\mu C_{;\mu} + \frac{i}{3}\,\mathrm{tr}\,\bar{C}_{;(\mu\nu\rho)}\gamma^\mu g^{\nu\rho}C \right.$$
$$\left. + \left(\frac{1}{4} - \frac{1}{6}\log\frac{m^2}{\mu^2}\right)\mathrm{tr}\,F^s_{\mu\nu;\nu}\bar{C}\gamma^\mu C + \frac{1}{4}\,\mathrm{tr}\,\bar{C}\gamma^\mu F^f_{\mu\nu;\nu}C\right].$$

Summary

- Developed general formalism to compute local covariant n-loop Feynman graphs (matching, running)
  - Generalized HK expansion (automated)
  - Expressed n-loop momentum integral in terms of graph polynomials (analytic expressions in terms of $\mathbb{B}$)

- Future directions:
  - Include gravitational background
  - Tackle Schwinger-Parameter integrals (in EFT, one mass, no momenta)
  - Automation ?