

LOCAL COVARIANT PERTURBATION THEORY AT ANY ORDER

Gero von Gersdorff

Recontres de Vietnam, 10/01/2024

Based on work with K.Santos

2212.07451, 2309.14939



Departamento
de Física

COVARIANT PERTURBATION THEORY

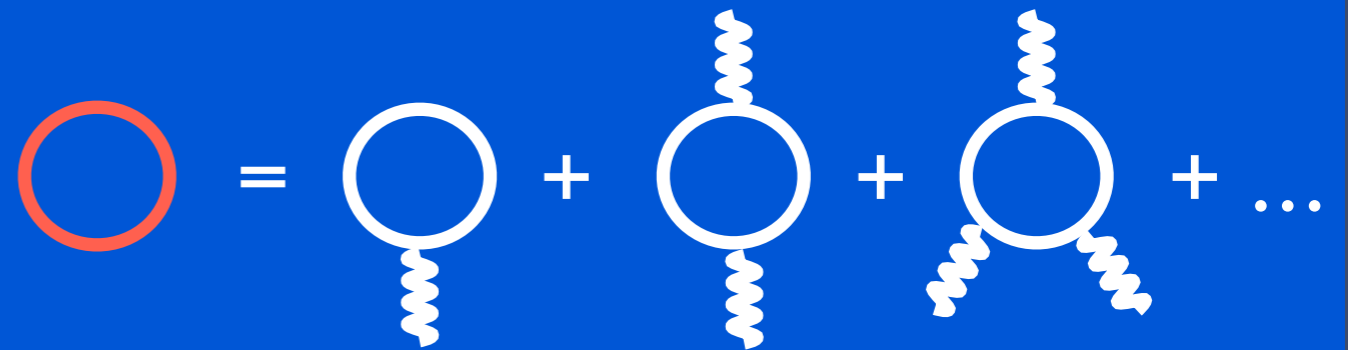
- ▶ **Local**: expansion in all IR scales: fields, derivatives, (light) masses
 - ▶ EFT matching contribution from integration of heavy d.o.f's
 - ▶ RG - Running of local operators
- ▶ **Covariant**: Maintaining manifest gauge covariance at every step (background field method)
- ▶ **Any Order**: Go beyond the usual 1-loop case

COVARIANT PT: WHAT IS KNOWN

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► One-loop determinants of single fields (heat kernel)

Schwinger '51, DeWitt '65, '67,
Gilkey '75, Avramidi '90, '91,
Fujikawa '79, '80
Review: Vassilevich '03



The diagram shows an equation where a red circle is equal to a sum of three white circles with wavy lines, followed by an ellipsis. The first white circle has a wavy line at the bottom. The second white circle has wavy lines at the top and bottom. The third white circle has wavy lines at the top, bottom, and two diagonal sides. This represents the expansion of a one-loop determinant into a series of diagrams with increasing numbers of external legs.

$$\text{Red Circle} = \text{White Circle with bottom wavy line} + \text{White Circle with top and bottom wavy lines} + \text{White Circle with top, bottom, and two diagonal wavy lines} + \dots$$

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A diagrammatic equation showing a red circle on the left, followed by an equals sign, and then a series of terms: a white circle with a wavy line at the bottom, plus a white circle with wavy lines at the top and bottom, plus a white circle with wavy lines at the top, bottom, and right, plus an ellipsis.

▶ One loop graphs including mixed fields (mass, spin), derivative couplings

Barvinsky+Vilkovisky '85

Henning et al '14, '16
Drozd et al '15, Aguila et al '16
Zhang '16, Fuentes-Martin et al '16
Ellis et al '17, '20 ...

A diagrammatic equation showing a red square with yellow dots at its corners on the left, followed by an equals sign, and then a series of terms: a white square with wavy lines on the top and bottom edges, plus a white square with wavy lines on the top, bottom, and right edges, plus an ellipsis.

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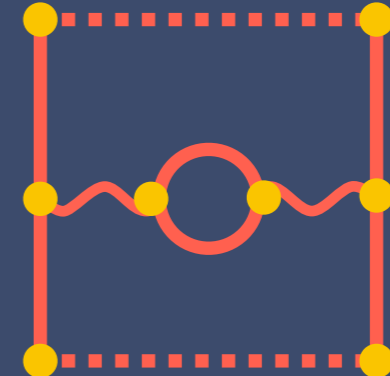
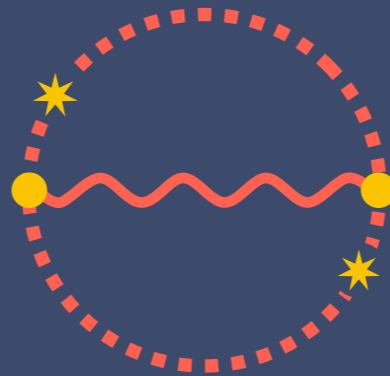
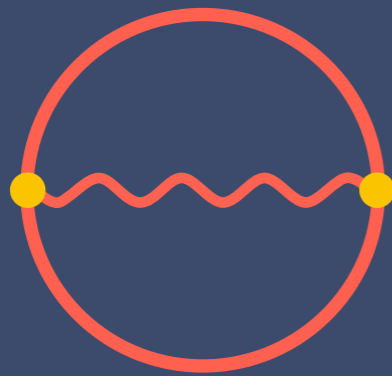
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A diagrammatic equation showing a red square with yellow dots at its corners on the left, followed by an equals sign, then a series of terms separated by plus signs. The first term is a white square with four external lines extending outwards from its corners. The second term is a white square with four external lines extending outwards from its corners and a wavy line extending downwards from the bottom edge. This sequence is followed by a plus sign and an ellipsis.

▶ Some higher-loop special cases (covariantly constant field strength), but no systematic formalism

GOAL

- ▶ Want to calculate arbitrary, covariant \mathcal{L} - loop diagrams and expand in local operators



BACKGROUND FIELD METHOD

▶ Background field method:

$$\phi(x) = \phi_b(x) + \phi_f(x) \quad A^\mu(x) = A_b^\mu(x) + A_f^\mu(x) \quad \text{etc}$$

▶ Integrating over fluctuations gives effective action $S_{\text{eff}}[\phi_b, A_b, \dots]$

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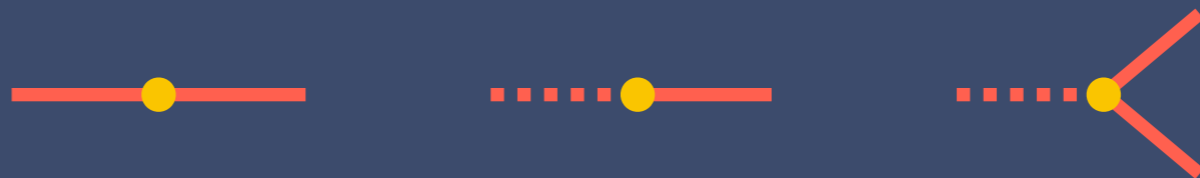
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- ▶ Background field dependence

- ▶ **Couplings**, e.g.

$$g\phi\bar{\psi}\psi = \dots + g\phi_b \bar{\psi}_f\psi_f + g\bar{\psi}_b \phi_f\psi_f + g\phi_f \bar{\psi}_f\psi_f + \dots$$



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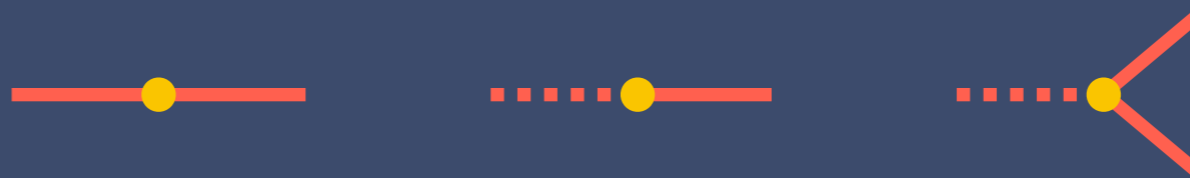
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- ▶ **Propagators**, e.g.

$$\langle \phi_f \bar{\phi}_f \rangle = \frac{-i}{(\partial - iA_b)^2 + m^2} \quad \langle D\phi_f \bar{\phi}_f \rangle = (\partial - iA_b) \frac{-i}{(\partial - iA_b)^2 + m^2}$$

THE HEAT KERNEL TRICK

- ▶ Represent each propagator by

$$\langle x | \frac{-i}{D^2 + X + m^2} | y \rangle = \int_0^\infty dt \langle x | e^{-it(D^2 + X + m^2)} | y \rangle$$

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$$\langle x | \frac{-i}{D^2 + X + m^2} | y \rangle = \int_0^\infty dt e^{-itm^2} \int \frac{d^d k}{(2\pi)^d} e^{itk^2 - ik(x-y)} B(t, X, x, y)$$

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$$B(t, X, x, y) \equiv \frac{\langle x | e^{-it(D^2+X)} | y \rangle}{\langle x | e^{-it\partial^2} | y \rangle}$$

is analytic in t and has a local, covariant **expansion**

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► Propagators of **derivatives** of fields

$$\begin{aligned} & \langle x | D_{\mu} \frac{-i}{D^2 + X + m^2} | y \rangle \\ &= \int_0^{\infty} dt e^{-itm^2} \int \frac{d^d k}{(2\pi)^d} e^{itk^2 - ik(x-y)} (D_{\mu}^x - ik_{\mu}) B(t, X, x, y) \end{aligned}$$

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▶ Fermions

$$\begin{aligned} & \langle x | \frac{i}{i\mathcal{D} - m} | y \rangle \\ &= \int_0^\infty dt e^{-itm^2} \int \frac{d^d k}{(2\pi)^d} e^{itk^2 - ik(x-y)} (i\mathcal{D}_\mu^x + k_\mu + m) B(t, X, x, y) \end{aligned}$$

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↓

- ▶ Gauge fields, ghosts, etc in a similar way

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- ▶ A Mathematica notebook for calculation of the **LHKC's** is available with [GG+Santos, arXiv 2212.07451](#)

EXAMPLES OF LOCAL COEFFICIENTS

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▶ “Standard” local Coefficients $[b_{2n}]$

```
In[*]:= b[2, {}, {}]
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Out[*]= {{X, -i}}
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Out[*]= {{{Fa1a2Fa1a2,  $\frac{1}{6}$ }, {X;a1a1,  $-\frac{1}{3}$ }, {XX, -1}}
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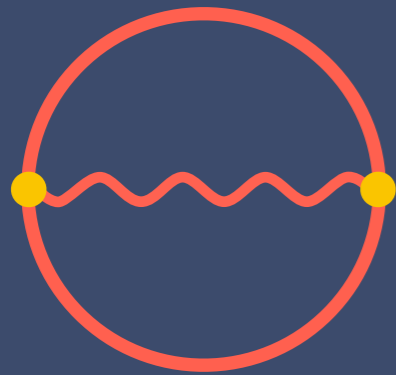
▶ Generalized local coefficients (e.g., $[b_{0;\mu;\nu}]$ or $[b_{2;;\mu\nu}]$)

```
In[*]:= b[0, {"μ"}, {"ν"}]  
        b[2, {}, {"μ"}, {"ν"}]  
  
Out[*]= {{Fμν,  $-\frac{i}{2}$ }}  
  
Out[*]= {{X;νμ,  $-\frac{5i}{6}$ }, {Fa1μ;a1ν,  $-\frac{1}{12}$ }, {X;μν,  $\frac{i}{2}$ },  
        {Fa1ν;a1μ,  $-\frac{1}{12}$ }, {FνμX,  $\frac{1}{3}$ }, {Fa1μFa1ν,  $\frac{i}{12}$ }, {Fa1νFa1μ,  $\frac{i}{12}$ }, {XFνμ,  $\frac{1}{6}$ }}
```


THE MASTER FORMULA

GG + Santos '23
GG '23

Feynman Graph G



$$\Gamma(t_i, x_n, k_i) \equiv \prod_i (\dots) B_i \prod_n C_n$$

$$I(t_i, x_n, k_i) \equiv e^{-it_i k_i^2 - ix_n \mathbb{B}_{ni} k_i}$$

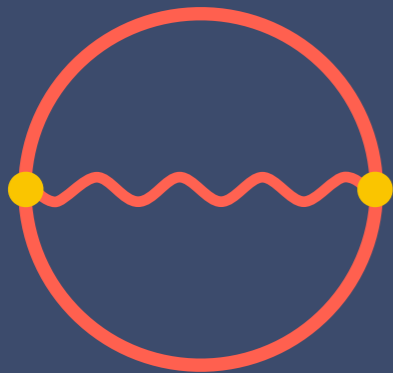
Gauge and
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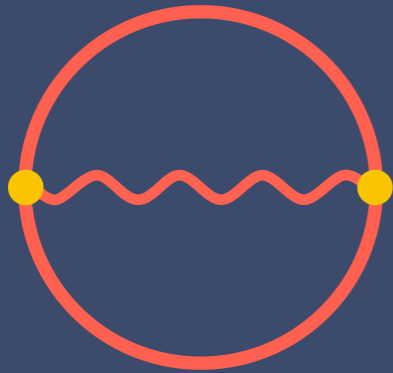
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$$S_{\text{eff}}^G = \int_{t_i} e^{-it_i m_i^2} \int_{x_n} \int_{k_i} I(t_i, x_n, k_i) \Gamma(t_i, x_n, k_i)$$

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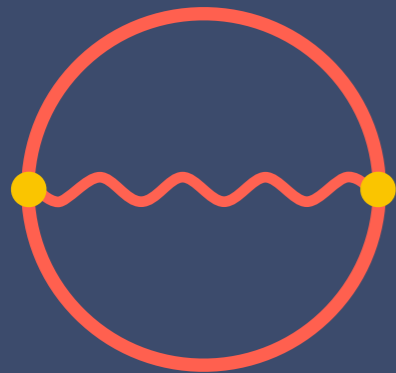
$$I(t_i, x_n, k_i) \equiv e^{-it_i k_i^2 - ix_n \mathbb{B}_{ni} k_i} \rightarrow \tilde{I}(t_i, p_n, z_i)$$

$$S_{\text{eff}}^G = \int_{t_i} e^{-it_i m_i^2} \int_x \tilde{I}(t_i, i\partial_{x_n}, z_i) \Gamma(t_i, x_n, -i\overleftarrow{\partial}_{z_i}) \Big|_{z_i=0, x_n=x}$$

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- ▶ Only **LHKC's** appear
- ▶ \tilde{I} is **universal** (depends only on **topology** of graph)
- ▶ $I \rightarrow \tilde{I}$ equivalent to loop momentum integration

THE UNIVERSAL MOMENTUM INTEGRAL

$$\tilde{I}(t_i, p_n, z_i) = (4\pi)^{-\frac{dL}{2}} \Delta^{-\frac{d}{2}} \exp \left(\frac{1}{4} z_i \mathbb{Q}_{ij} z_j - iz_i \mathbb{R}_{in} p_n - p_m \mathbb{U}_{mn} p_n \right)$$

Incidence matrix: $\mathbb{B}_{ni} = \begin{cases} +1 & \text{if edge } i \text{ enters vertex } n \\ -1 & \text{if edge } i \text{ leaves vertex } n \\ 0 & \text{else} \end{cases}$

$$\longrightarrow \mathbb{T} \equiv \begin{pmatrix} it_1 & & & \\ & it_2 & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix} + \epsilon^{-1} \mathbb{B}^T \mathbb{B}$$

▶ Δ is the leading coefficient of $\det \mathbb{T}$ as $\epsilon \rightarrow 0$

▶ $\mathbb{T}^{-1} = \mathbb{Q} + \epsilon \mathbb{Q}' + \epsilon^2 \mathbb{Q}'' + \dots$, $\mathbb{R} = \mathbb{Q}' \mathbb{B}^T$, $\mathbb{U} = \mathbb{B} \mathbb{Q}'' \mathbb{B}^T$

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- ▶ Δ , $\Delta \mathbb{Q}_{ij}$, $\Delta \mathbb{R}_{in}$, and $\Delta \mathbb{U}_{mn}$ are **polynomials** in the t_i
- ▶ Δ and $\Delta \mathbb{U}_{mn} \rightarrow$ first and second **Szymanzik** polynomials
- ▶ $\Delta \mathbb{Q}_{ij}$ and $\Delta \mathbb{R}_{in}$ are new **graph polynomials** Golz '17, GG '23

- ▶ All four polynomials can be written as sums over subgraphs (all vertices, fewer edges)
- ▶ Gives rise to many interesting relations...

THE UNIVERSAL MOMENTUM INTEGRAL

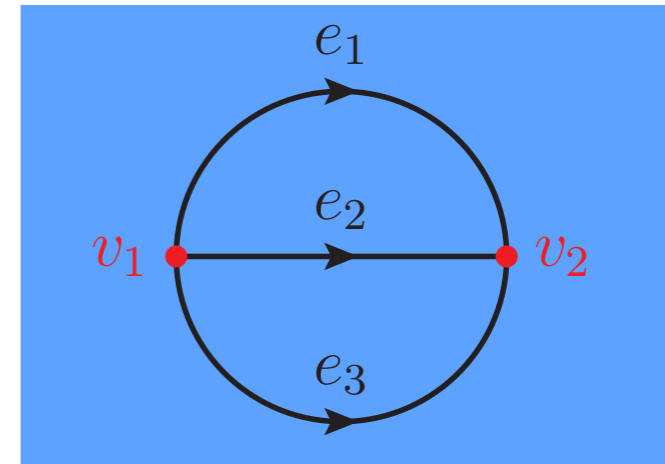
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$$\Delta = \tau_1 \tau_2 + \tau_2 \tau_3 + \tau_1 \tau_3$$

$$\mathbb{Q} = \frac{1}{\Delta} \begin{pmatrix} \tau_2 + \tau_3 & -\tau_3 & -\tau_2 \\ -\tau_3 & \tau_1 + \tau_3 & -\tau_1 \\ -\tau_2 & -\tau_1 & \tau_1 + \tau_2 \end{pmatrix}$$

$$\mathbb{R} = \frac{1}{\Delta} \begin{pmatrix} -\tau_2 \tau_3 & \tau_2 \tau_3 \\ -\tau_1 \tau_3 & \tau_1 \tau_3 \\ -\tau_1 \tau_2 & \tau_1 \tau_2 \end{pmatrix}$$

$$\mathbb{U} = \frac{\tau_1 \tau_2 \tau_3}{\Delta} \begin{pmatrix} -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix}$$



$$(\tau_i = it_i)$$

EXAMPLE

► Integrating out massive fermion at one loop

$$\mathcal{L}_{\text{int}} \supset g \bar{\chi}_b \phi_f \psi_f \quad C \equiv g \chi_b$$



$$\mathcal{L}_{\text{eff}}^{D=6} = \frac{1}{16\pi^2} \frac{1}{m^2} \left[-\frac{1}{2} \text{tr} \left(\bar{C}_{;\mu} \gamma^\mu [b_2^f] C + \bar{C} \gamma^\mu [b_{2;\mu}^f] C \right) + \left(\frac{3}{2} - \log \frac{m^2}{\mu^2} \right) \text{tr} [b_{2;;\mu}^s] \bar{C} \gamma^\mu C \right. \\ \left. + \frac{i}{3} \text{tr} \bar{C}_{;(\mu\nu\rho)} \gamma^\mu g^{\nu\rho} C + \text{tr} \bar{C} \gamma^\mu [b_{2;\mu}^f] C - \frac{i}{2} g^{\nu\rho} \text{tr} \left(2\bar{C}_{;\nu} \gamma^\mu [b_{0;\mu\rho}^f] C + \bar{C} \gamma^\mu [b_{0;\mu\nu\rho}^f] C \right) \right],$$



$$\mathcal{L}_{\text{eff}}^{D=6} = \frac{1}{16\pi^2} \frac{1}{m^2} \left[\frac{1}{8} \text{tr} \bar{C}_{;\mu} \gamma^\mu (\not{F}^f) C - \frac{1}{8} \text{tr} C \not{F}^f \gamma^\mu C_{;\mu} + \frac{i}{3} \text{tr} \bar{C}_{;(\mu\nu\rho)} \gamma^\mu g^{\nu\rho} C \right. \\ \left. + \left(\frac{1}{4} - \frac{1}{6} \log \frac{m^2}{\mu^2} \right) \text{tr} F_{\mu\nu;\nu}^s \bar{C} \gamma^\mu C + \frac{1}{4} \text{tr} \bar{C} \gamma^\mu F_{\mu\nu;\nu}^f C \right].$$

SUMMARY

- ▶ Developed general formalism to compute local covariant n -loop Feynman graphs (matching, running)
 - ▶ Generalized HK expansion (automated)
 - ▶ Expressed n -loop momentum integral in terms of graph polynomials (analytic expressions in terms of \mathbb{B})
- ▶ Future directions:
 - ▶ Include gravitational background
 - ▶ Tackle Schwinger-Parameter integrals (in EFT, one mass, no momenta)
 - ▶ Automation ?