NESTED SOFT-COLLINEAR SUBTRACTION, REINVENTED

Based on: JHEP 02 (2024) 016, arXiv:2310.17598

In collaboration with: Kirill Melnikov, Raoul Röntsch, Chiara Signorile-Signorile, Davide Maria Tagliabue

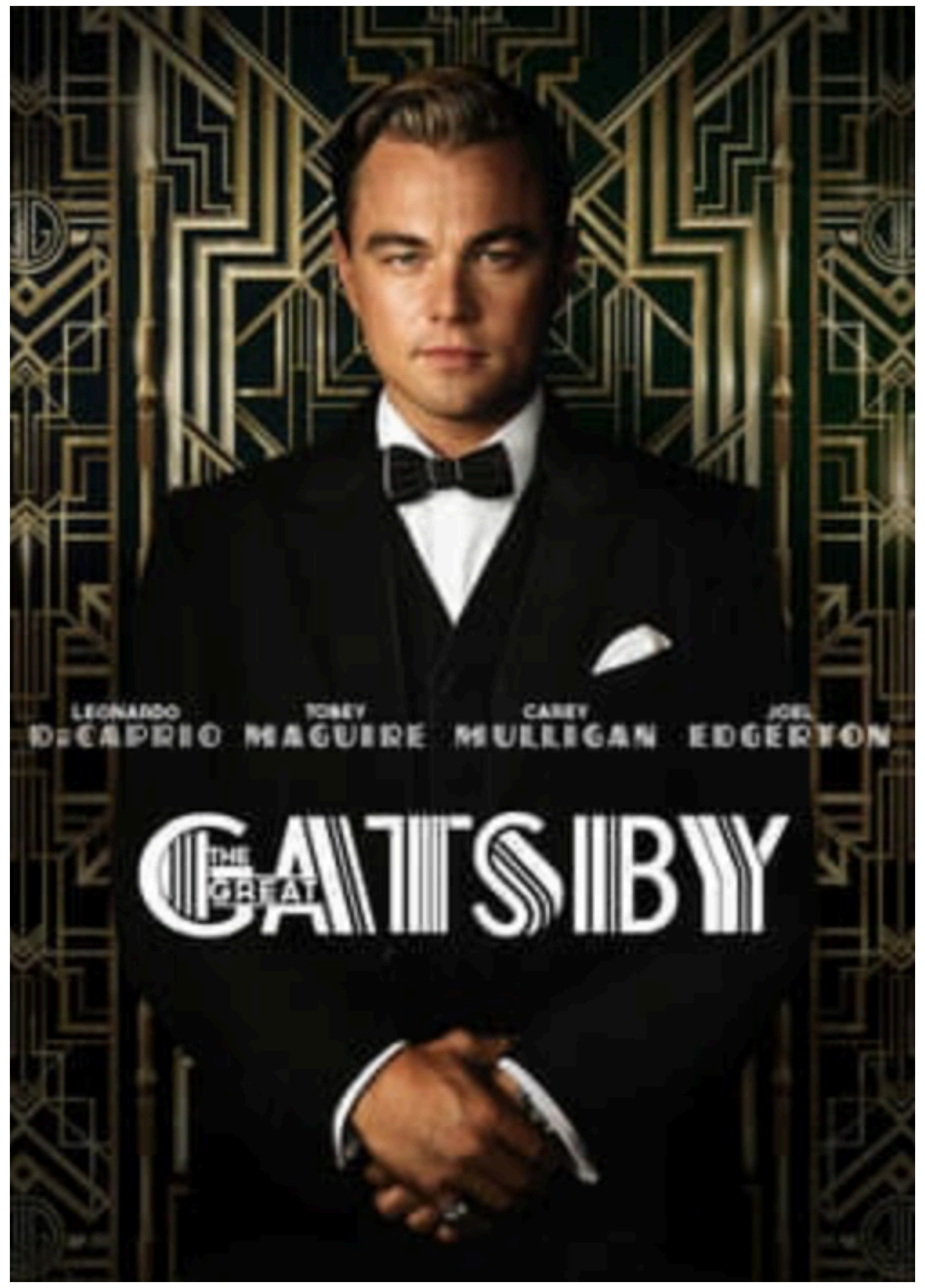
Federica Devoto

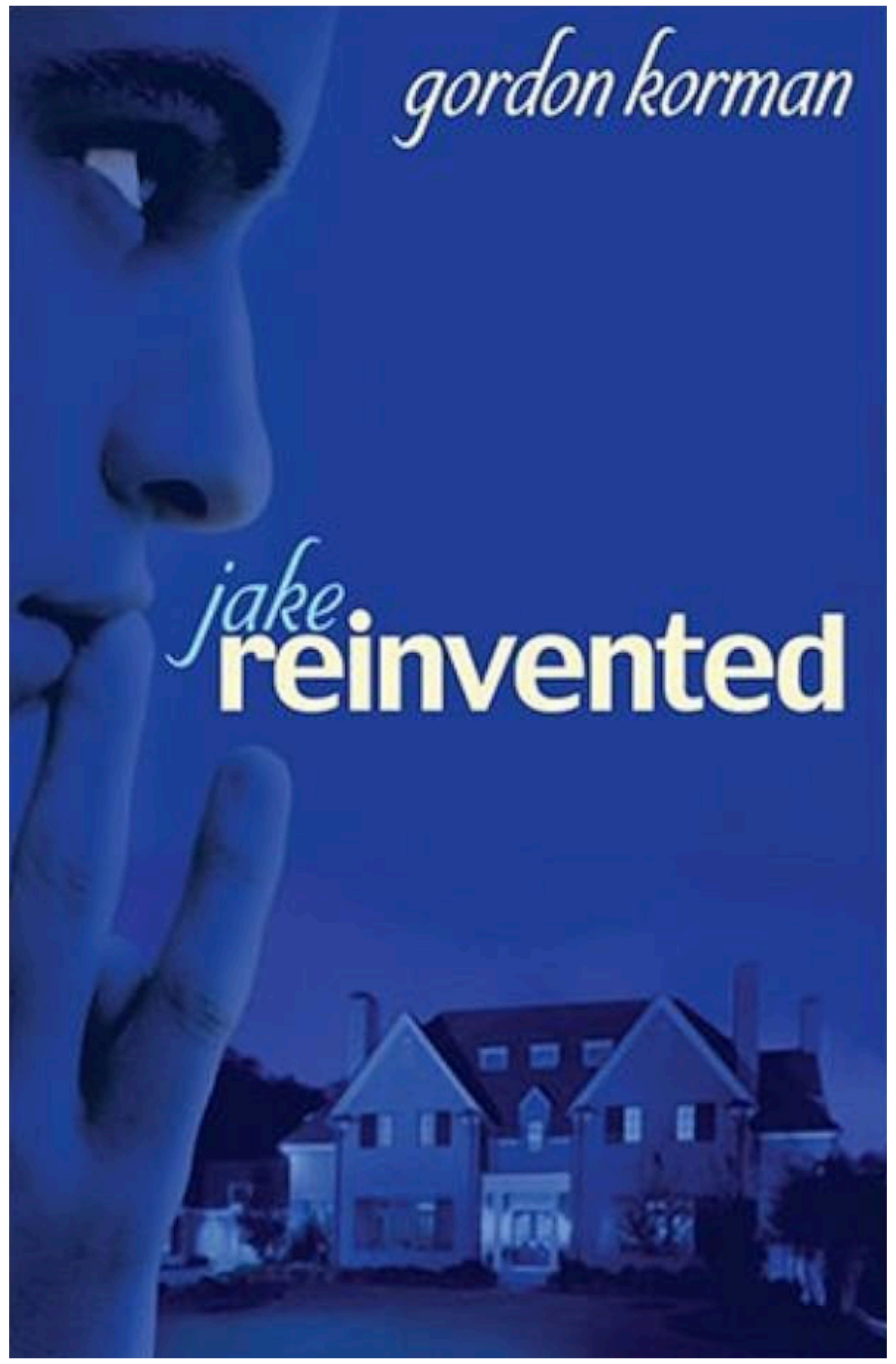
CERN QCD seminar, 25/03/2024

Nested soft-collinear subtractions in NNLO QCD computations

Fabrizio Caola[1,2], Kirill Melnikov[3], Raoul Röntsch[3].

'17





A fresh look at the nested soft-collinear subtraction scheme: NNLO QCD corrections to N-gluon final states in $q\bar{q}$ annihilation

Federica Devoto,[a] Kirill Melnikov,[b] Raoul Röntsch,[c] Chiara Signorile-Signorile[b,d,e]
Davide Maria Tagliabue[c]

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OUTLINE

- Introduction & Motivation: what is a subtraction scheme? Why do we need it? Warm-up at NLO
- Overview: difficulties of subtraction at NNLO. Focus on **nested soft-collinear subtraction scheme**, strengths and limitations. Does it work "in real life"?
- What's new: **pushing NSC towards processes with high-multiplicity of colored particles**
- Conclusions & outlooks: what's next?

INTRODUCTION

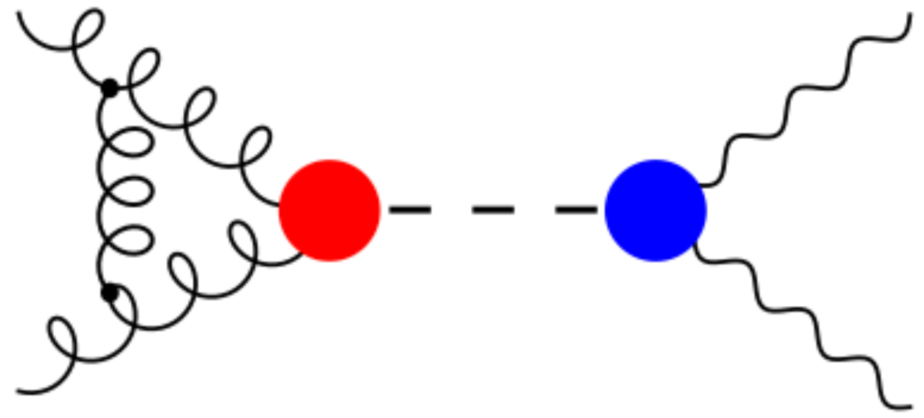
$$\mathrm{d}\sigma = \sum_{i_1,i_2} \int \mathrm{d}x_1 \mathrm{d}x_2 f_{i_1}(x_1) f_{i_2}(x_2) \mathrm{d}\sigma_{i_1 i_2}(x_1, x_2) F_J \left(1 + \mathcal{O}(\Lambda^n_{\mathrm{QCD}}/Q^n)\right), \quad n \geq 1.$$

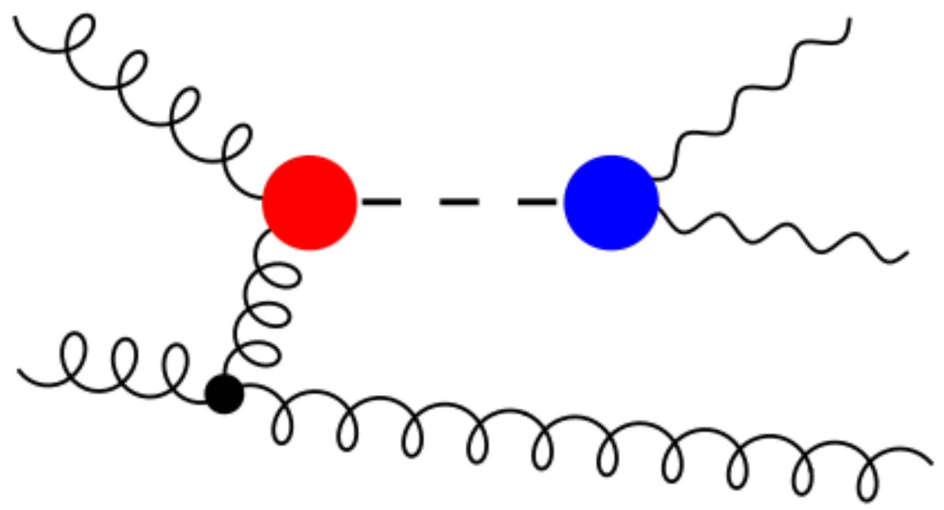
Parton distribution functions

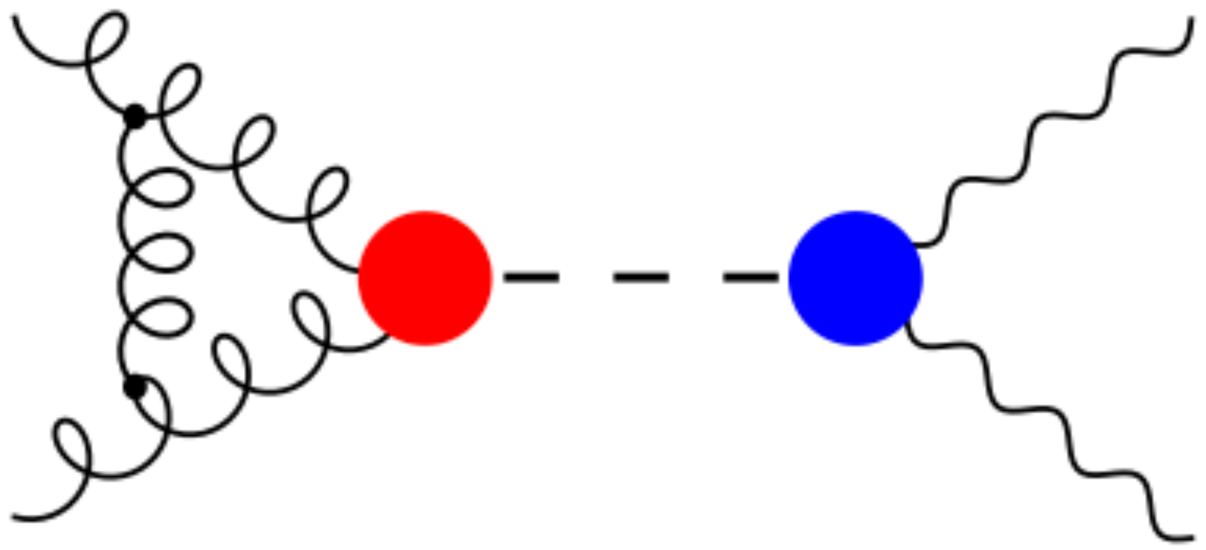
Partonic cross section

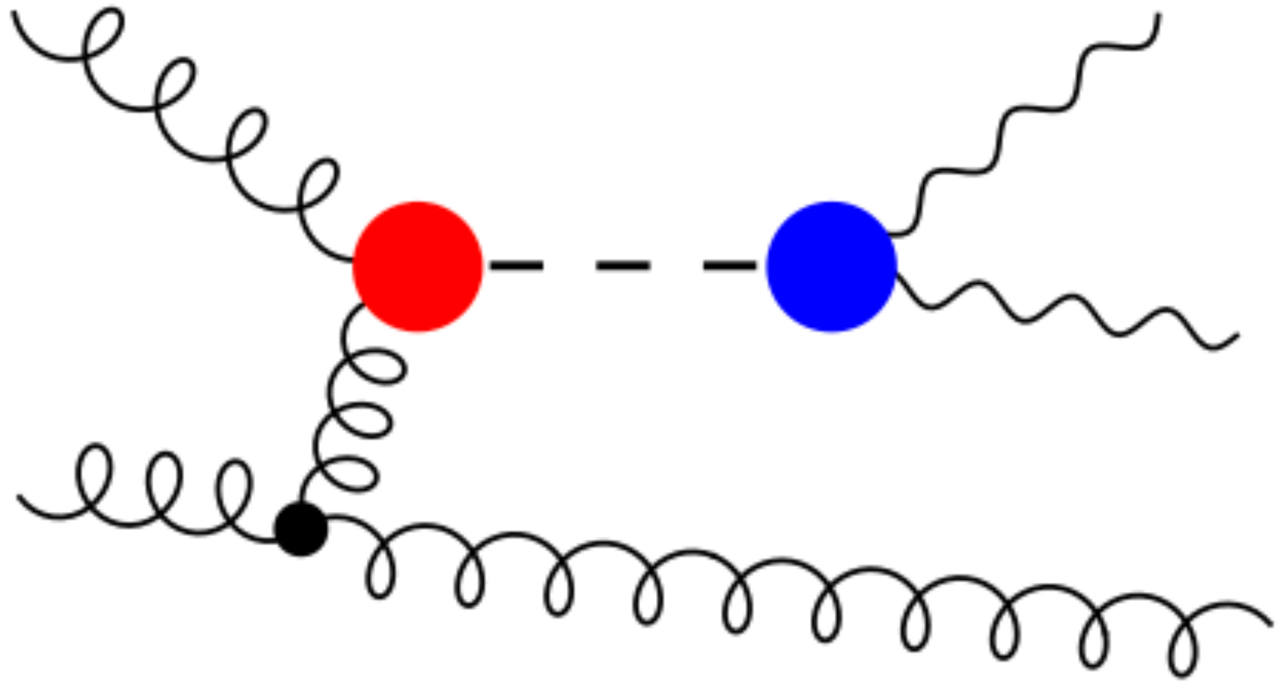
Encodes non-perturbative effects

$$\mathrm{d}\sigma_{ij} = \mathrm{d}\sigma_{ij,\,\mathrm{LO}}\left(1 + \alpha_s\,\Delta^{QCD}_{ij,\,\mathrm{NLO}} + \alpha_{ew}\,\Delta^{EW}_{ij,\,\mathrm{NLO}} + \alpha_s^2\,\Delta^{QCD}_{ij,\,\mathrm{NNLO}} + \alpha_s\,\alpha_{ew}\,\Delta^{QCD\otimes EW}_{ij,\,\mathrm{NNLO}} + \ldots\right)$$









In general, UV & IR divergent

Understood

Needs to be combined with legs

Divergences are **explicit**

IR divergent when integrated over radiation PS

Divergences are **implicit,** i.e. appear after integrating

How do we deal with these divergences?

= finite!

∞ ∞

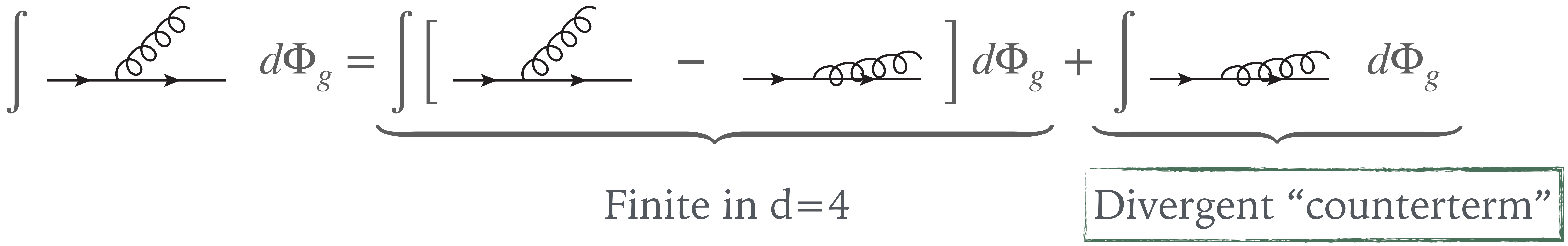
Inclusive: not a problem, KLN ensures cancellation. Relatively interesting

Differential: more interesting, but need proper subtraction of divergences

Structure of divergences get more and more involved at higher orders, important to understand and organise the cancellation

"Subtraction scheme"

Main idea: add and subtract the divergent configurations



The choice of the counterterm defines a given subtraction scheme

Identikit of suitable counterterm:

- Approximate full matrix element in all singular limits
- Easy to integrate
- Other optional (?) features: locality, Lorentz invariance, limit number of spurious singularities etc.

STATUS OF SUBTRACTION

NLO

Catani-Seymour (CS)
[9602277]

Frixione-Kunst-Signer (FKS)
[9512328]

Currently implemented in full generality in fast and efficient NLO generators *[Gleisberg, Krauss '07, Frederix, Gehrmann, Greiner '08, Hasegawa, Moch, Uwer '09, Frederix, Frixione, Maltoni, Stelzer '09, Alioli, Nason, Oleari, Re '10, Reuter et al. '16*

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SOLVED

STATUS OF SUBTRACTION

NNLO

Extraction of real-emission singularities was the main bottleneck for NNLO predictions.

Example: di-jet two-loop amplitudes $\sim$ 20 years ago *[Glover, Oleari, Tejeda-Yeomans '01]*,
di-jet production at NNLO $\sim$ 7 ago *[Currie, De Ridder, Gehrmann, Glover, Huss, Pires '17]*

Many schemes are available:

Antenna *[Gehermann-De Ridder et al. 0505111]*

ColorfullNNLO *[Del Duca et al. 1603.08927]*

Nested-soft-collinear subtraction *[Caola et al. 1702.01352]*

Residue subtraction *[Czakon 1005.0274]*

New strategies have been explored:

Analytic Sector Subtraction *[Magnea et al. 1806.09570]*

Geometric IR subtraction *[Herzog 1804.07949]*

Unsubtraction *[Sborlini et al. 1608.01584]*

FDR *[Pittau, 1208.5457]*

Universal Factorisation *[Sterman et al.2008.12293]*

1) **Physical transparency**
2) **Generality**
3) **Locality**
4) **Analyticity**
5) **Efficiency**

Hard to combine these 5 criteria in one scheme!

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3) **Locality**
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Ongoing efforts towards generality + efficiency!

Hard to combine these 5 criteria in one scheme!

WARM-UP: NSC@NLO

Simple (yet instructive) example: Z+j production in proton collision

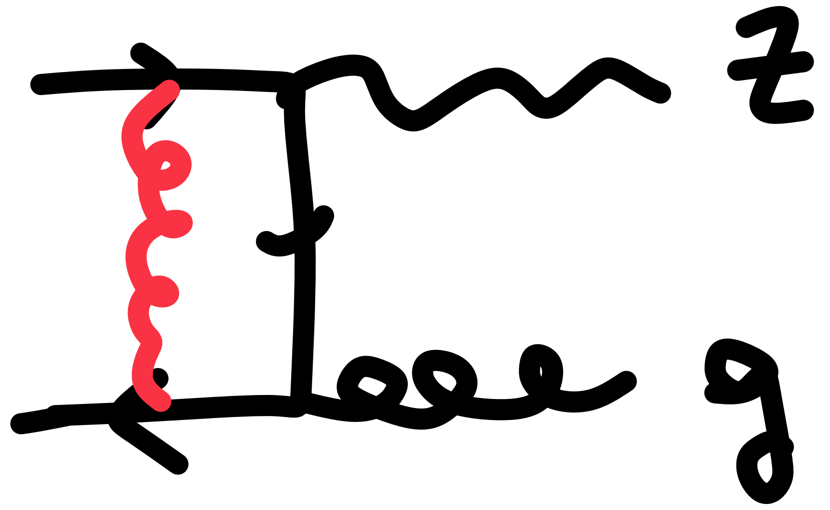
$pp \to Z + j \sim \alpha_s^1$

At **NLO QCD**: include contributions $\sim \alpha_s^2$

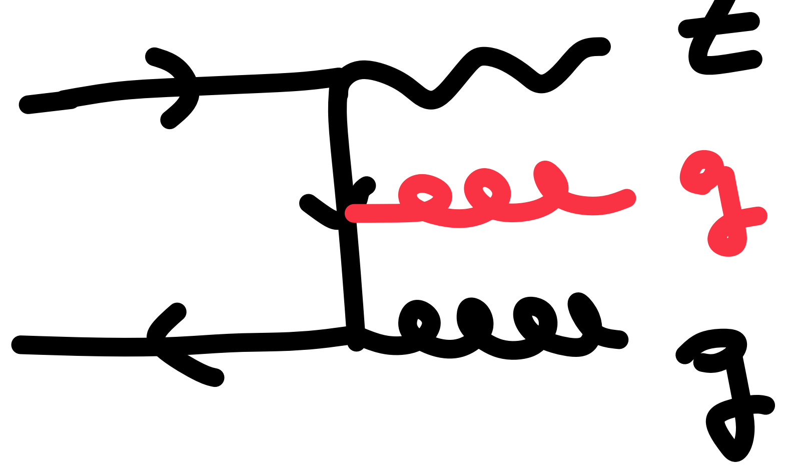
Let's procede step-by-step:

- Identify singular configurations
- Make singularities explicit
- Combine them to get finite result

Loop corrections



Real emissions



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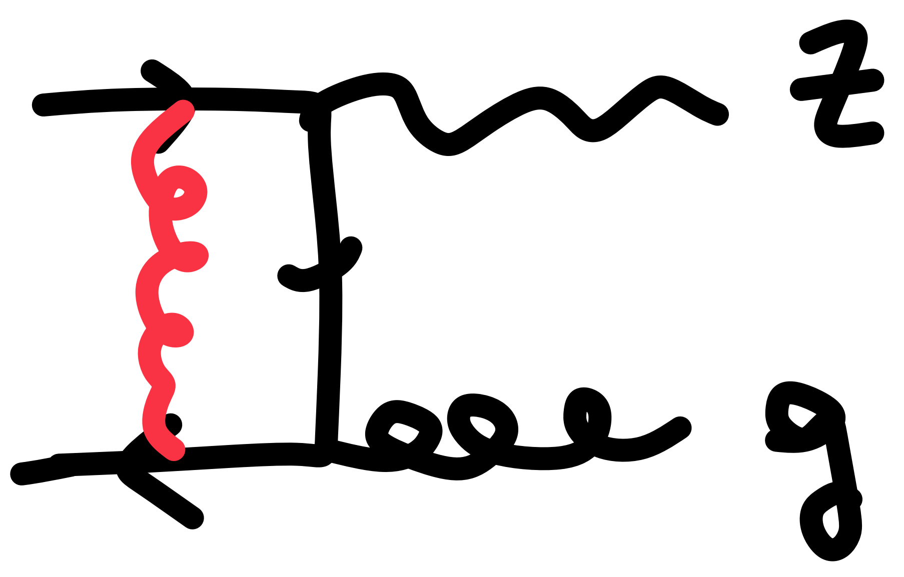

Divergent when loop momentum becomes soft/collinear

Real emissions

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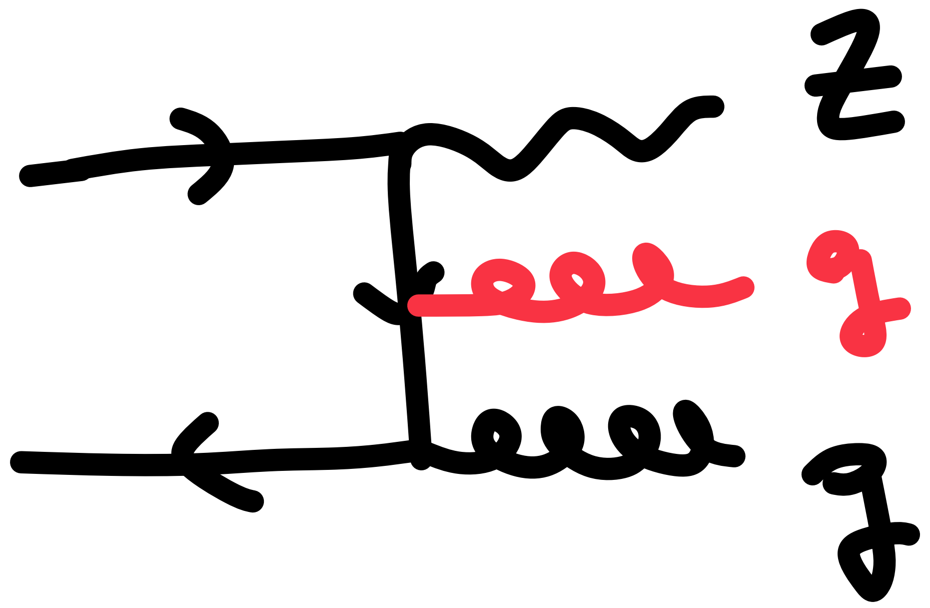
Real emissions

Integration over gluon
phase space divergent
in soft/collinear regions



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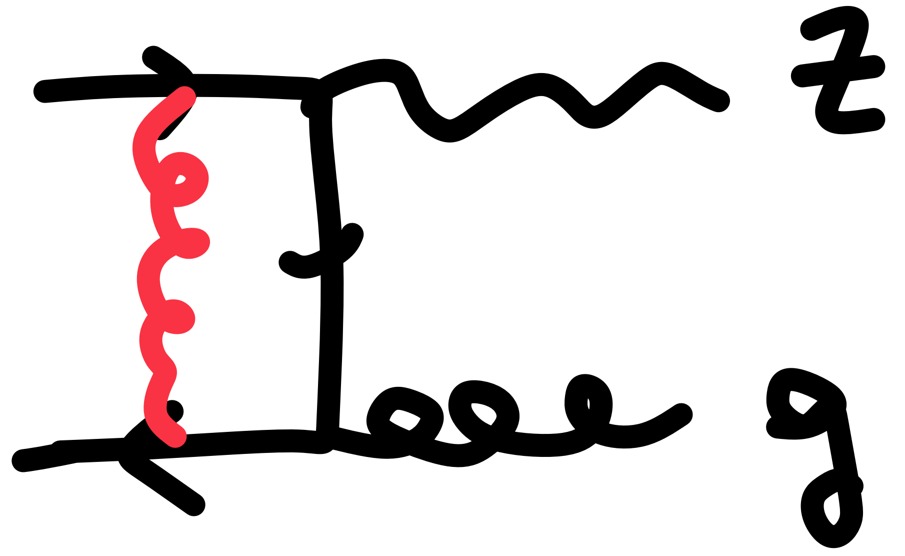

$$\langle F_{\rm LV}(1\ldots n)\rangle = \frac{\alpha_s}{2\pi}\Big\langle 2\Re\big(\mathcal{I}_1(\epsilon)\big)\, F_{\rm LM}\Big\rangle$$

$$\mathcal{I}_1(\epsilon) = \frac{1}{2}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\sum_i \frac{1}{\mathbf{T}_i^2}\left(\mathbf{T}_i^2\frac{1}{\epsilon^2}+\gamma_i\frac{1}{\epsilon}\right)\sum_{j\neq i}\mathbf{T}_i\cdot\mathbf{T}_j\left(\frac{\mu^2}{2p_i\cdot p_j}\right)^{\epsilon} e^{i\lambda_{ij}\pi\epsilon}$$

Catani, 1998

Color correlations

$$\langle F_{\rm LV}(1_q,2_{\bar q};3_g)\rangle = [\alpha_s]\Big\{(C_A-2C_F)\left[\frac{1}{\epsilon^2}+\frac{3}{2\epsilon}\right](s_{12})^{-\epsilon}\cos(\pi\epsilon) - \left[\frac{C_A}{\epsilon^2}+\frac{3C_A+2\beta_0}{4\epsilon}\right]\left((s_{13})^{-\epsilon}+(s_{23})^{-\epsilon}\right)\Big\}\langle F_{\rm LM}(1_q,2_{\bar q};3_g)\rangle + \langle F_{\rm LV}^{\rm fin}(1_q,2_{\bar q};3_g)\rangle$$

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Let's procede step-by-step:

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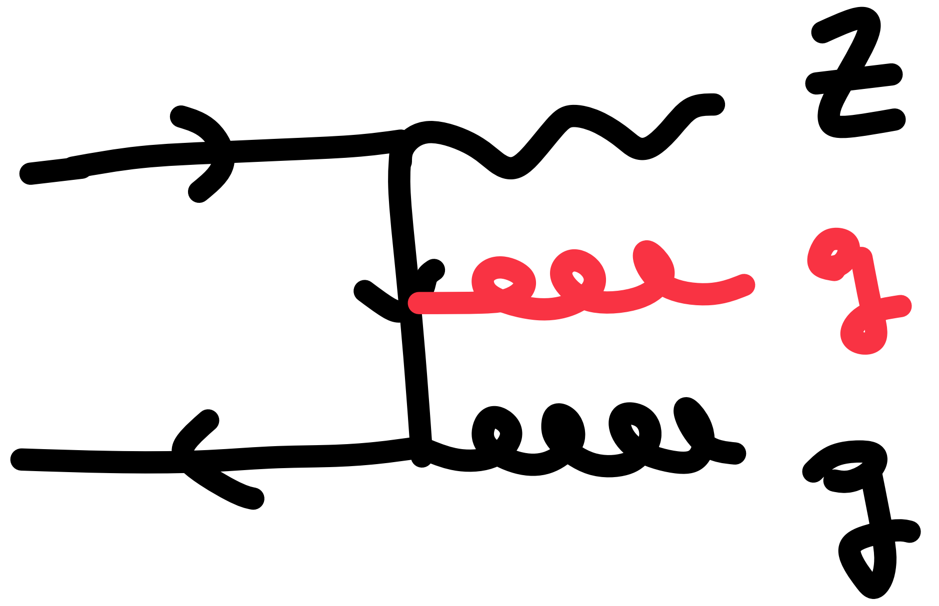
Integration over gluon phase space divergent in soft/collinear regions



- **Make singularities explicit**

NSC philosophy: subtract singularities in a nested way, i.e. regulate soft first, then collinear

$$< F_{LM}(1,2,3,4) > = < (I - S_4) F_{LM} \Delta^{(4)}(1,2,3,4) > + < S_4 \Delta^{(4)} F_{LM}(1,2,3,4) >$$

$$= < (I - S_4)(I - C_{4i}) \Delta^{(4)} F_{LM}(1,2,3,4) > + \sum_i < (I - S_4) C_{4i} \Delta^{(4)} F_{LM}(1,2,3,4) > + < S_4 \Delta^{(4)} F_{LM}(1,2,3,4) >$$

Fully-regulated — Hard-collinear counterterm — Soft counterterm

- **Make singularities explicit**

"Old-fashioned" strategy: evaluate each counterterm explicitly

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Parametrised in energy&angles

Explicit breaking of Lorentz invariance

$$< S_4 F_{LM}(1,2,3,4) > \sim \int d\Phi_4 \sum_{i,j} \frac{s_{ij}}{s_{i4}\, s_{j4}} < F_{LM}^{ij}(1,2,3) > \sim \frac{E_{max}^{-2\epsilon}}{\epsilon^2} \sum_{i,j} T_i \cdot T_j \eta_{ij}^{-\epsilon} K(i,j) < F_{LM}(1,2,3) >$$

Soft factorisation

Color correlation

- **Make singularities explicit**

"Old-fashioned" strategy: evaluate each counterterm explicitly

$$< S_4 F_{LM}(1,2,3,4) > \sim \int d\Phi_4 \sum_{i,j} \frac{s_{ij}}{s_{i4}\, s_{j4}} < F_{LM}^{ij}(1,2,3) > \sim \frac{E_{max}^{-2\epsilon}}{\epsilon^2} \sum_{i,j} T_i \cdot T_j \eta_{ij}^{-\epsilon} K(i,j) < F_{LM}(1,2,3) >$$

Soft factorisation

Color correlation

$$< (I - S_4) C_{4i} F_{LM}(1,2,3,4) >$$

2 IS limits, 1 FS limit.. Need to introduce partitions to deal with 1 collinear singularity at a time!

$$1 = \omega_{41} + \omega_{42} + \omega_{43}$$

$$C_{4j}\,\omega_{4i} = \delta_{ji} \qquad \omega_{4i} = \frac{1/\rho_{4i}}{\sum_j 1/\rho_{4_j}}$$

NB: Soft limits "see" the color charge of all colored particles, collinear limits only see the color charge of the emitter

IS

$$\sum_{i=1}^{2} \langle (I - S_4)\, C_{i4}\, 2\,\Delta_{\perp,34}^{(3)}\, \omega^{i4}\, F_{\rm LM}(1_q, 2_{\bar q}; 3_g, 4_g) \rangle$$

$$= -\frac{[\alpha_s]}{\epsilon} C_F \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \sum_{i=1}^{2} \Big\langle (2E_i)^{-2\epsilon} \int_0^1 dz\, P_{qq}^{\rm NLO}(z, L_i)\, F_{\rm LM}^{(i)}(1_q, 2_{\bar q}; 3_g|z) \Big\rangle$$

FS

$$\frac{[\alpha_s]}{\epsilon} C_A (2E_3)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} P_{gg}^{\rm NLO}(L_3) \langle F_{\rm LM}(1_q, 2_{\bar q}; 3_g) \rangle$$

[I also cheated a bit… initial state collinear limits also need PDF renormalization to be fully regulated]

- Combine them to get finite result

```
sigmaNLO =
  - braαs/ϵ^2 (4 Emax^2/mu2)^-ϵ (CA (eta[1, 2]^-ϵ KK[1, 2] - eta[1, 3]^-ϵ KK[1, 3] - eta[2, 3]^-ϵ KK[2, 3]) - 2 CF eta[1, 2]^-ϵ KK[1, 2] )
    FLM[p1_q, p2_q̄, p3_g] + braαs/ϵ CA (4 E3^2/mu2)^-ϵ Gamma[1 - ϵ]^2/Gamma[1 - 2 ϵ] PggNLOFS[L3] × FLM[p1_q, p2_q̄, p3_g] +
   braαs/ϵ TR Nf (4 E3^2/mu2)^-ϵ Gamma[1 - ϵ]^2/Gamma[1 - 2 ϵ] γzgqq22 2 FLM[p1_q, p2_q̄, p3_g] +
   braαs ((CA - 2 CF) (1/ϵ^2 + 3/(2 ϵ)) (s12/mu2)^-ϵ Cos[π ϵ] - (CA/ϵ^2 + (3 CA + 2 β0)/(4 ϵ)) ((s13/mu2)^-ϵ + (s23/mu2)^-ϵ)) FLM[p1_q, p2_q̄, p3_g] +
   asontwopi FLVfin[p1_q, p2_q̄, p3_g] +
   braαs CF 1/ϵ Gamma[1 - ϵ]^2/Gamma[1 - 2 ϵ] ((4 E1^2/mu2)^-ϵ (3/2 + 1/ϵ (1 - (Emax^2/E1^2)^-ϵ)) + (4 E2^2/mu2)^-ϵ (3/2 + 1/ϵ (1 - (Emax^2/E2^2)^-ϵ))) FLM[p1_q, p2_q̄, p3_g] +
   asontwopi ONLO[2 Δ34 FLM[p1_q, p2_q̄, p3_g, p4_g]] +
   asontwopi ONLO[Δ34 (FLM[p1_q, p2_q̄, p3_qp, p4_q̄p] + FLM[p1_q, p2_q̄, p3_q̄p, p4_qp])] +
   braαs CF (Pqqfin[z, E1] × FLM[z p1_q, p2_q̄, p3_g, z ] + Pqqfin[z, E2] × FLM[p1_q, z p2_q̄, p3_g, z]);
```

Check analytic pole cancellation

```
Normal[Series[sigmaNLO //. softfunctions //. splittings /. replaceSij /. L3 → Log[Emax/E3] /. β0 → 11/6 CA - 2/3 Nf TR,
    {ϵ, 0, -1}]] // PowerExpand // Simplify

0
```

∘ Combine them to get finite result

```
Coefficient[%, ϵ, 0]

asontwopi FLVfin[p1q, p2q, p3g] +
 braαs CA FLM[p1q, p2q, p3g] (67/9 - 2 π^2/3 - 2 (Log[E3] - Log[Emax])^2 + (11/6 - 2 Log[E3] + 2 Log[Emax]) (-2 Log[E3] + Log[mu2/4])) +
 1/9 braαs Nf TR FLM[p1q, p2q, p3g] (-23 + Log[4096] + 12 Log[E3] - 6 Log[mu2]) +
 braαs CF FLM[p1q, p2q, p3g] (-6 Log[2] + 2 Log[E1]^2 + 2 Log[E2]^2 - 8 Log[2] Log[Emax] - 4 Log[Emax]^2 +
    Log[E1] (-3 + Log[16] - 2 Log[mu2]) + Log[E2] (-3 + Log[16] - 2 Log[mu2]) + 3 Log[mu2] + 4 Log[Emax] Log[mu2]) +
 braαs CF (FLM[z p1q, p2q, p3g, z] (1 - z + 4 D1[z] - 1/2 (2 + 2 z - 4 D0[z] - 3 delta[1 - z]) (Log[4] + 2 Log[E1] - Log[mu2]) -
        2 (1 + z) Log[1 - z]) + FLM[p1q, z p2q, p3g, z]
      (1 - z + 4 D1[z] - 1/2 (2 + 2 z - 4 D0[z] - 3 delta[1 - z]) (Log[4] + 2 Log[E2] - Log[mu2]) - 2 (1 + z) Log[1 - z])) +
 1/6 braαs FLM[p1q, p2q, p3g] (-3 (CA - 2 CF) (π^2 - 4 Log[2]^2 + Log[64] + 3 (Log[E1] + Log[E2] - Log[mu2] + Log[eta[1, 2]]) -
        4 Log[2] (Log[E1] + Log[E2] - Log[mu2] + Log[eta[1, 2]]) - (Log[E1] + Log[E2] - Log[mu2] + Log[eta[1, 2]])^2) +
    2 (5 CA - Nf TR) (Log[E1] + Log[E2] + 2 Log[E3] - 2 Log[mu2] + Log[16 eta[1, 3]] + Log[eta[2, 3]]) - 3 CA (8 Log[2]^2 +
        4 Log[2] (Log[E1] + Log[E3] - Log[mu2] + Log[eta[1, 3]]) + (Log[E1] + Log[E3] - Log[mu2] + Log[eta[1, 3]])^2 +
        4 Log[2] (Log[E2] + Log[E3] - Log[mu2] + Log[eta[2, 3]]) + (Log[E2] + Log[E3] - Log[mu2] + Log[eta[2, 3]])^2)) +
 asontwopi ONLO[2 Δ34 FLM[p1q, p2q, p3g, p4g]] + asontwopi ONLO[Δ34 (FLM[p1q, p2q, p3qp, p4qp] + FLM[p1q, p2q, p3qp, p4qp])] +
 braαs FLM[p1q, p2q, p3g]
  (-CF π^2/3 + 1/2 (CA + 2 CF) (Log[4] + 2 Log[Emax] - Log[mu2])^2 + CF Log[eta[1, 2]]^2 -
    (-2 Log[Emax] + Log[mu2/4]) (-((CA - 2 CF) Log[eta[1, 2]]) + CA (Log[eta[1, 3]] + Log[eta[2, 3]]))) +
```

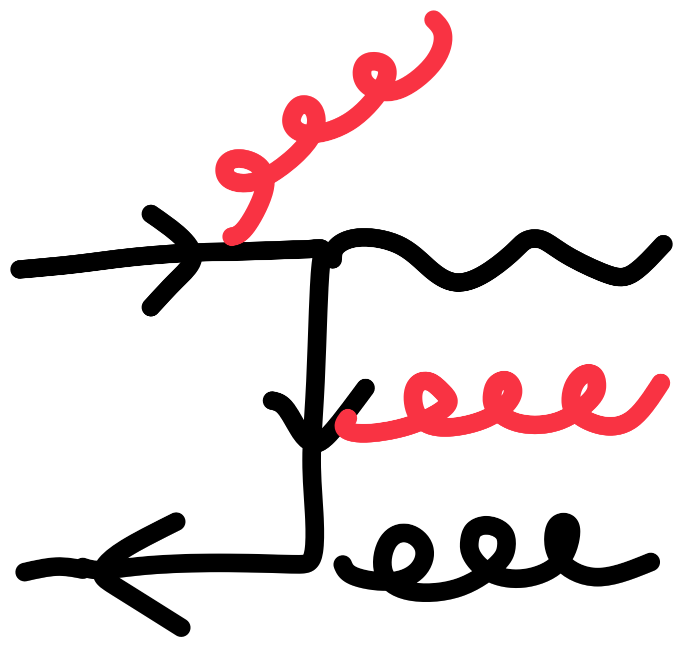
And then try to find recurring structures that make final result nice and cute

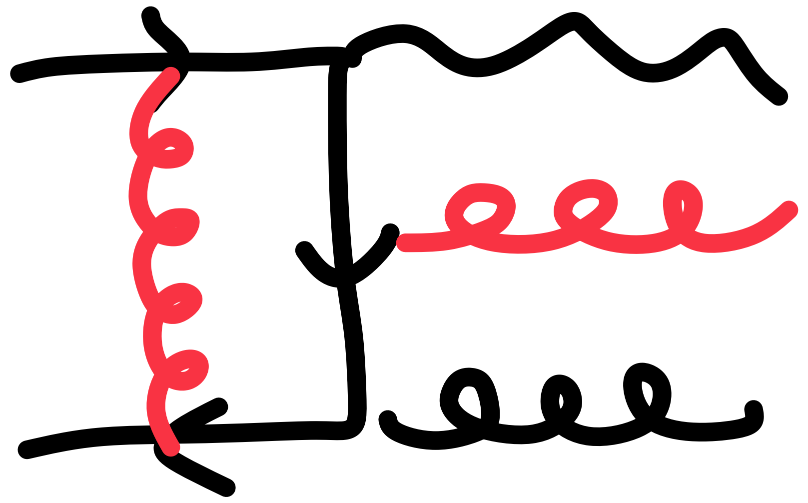
$$
\begin{aligned}
\mathrm{d}\sigma^{qq}_{\mathrm{NLO}} = {} & n_f \left\langle \mathcal{O}^{(4)}_{\mathrm{nlo}} F_{\mathrm{LM}}(1_q, 2_{\bar{q}}; 3_q, 4_{\bar{q}}) \right\rangle + \left\langle \mathcal{O}^{(4)}_{\mathrm{nlo}}\, 2\,\Delta^{(3)}_{\perp,34}\, F_{\mathrm{LM}}(1_q, 2_{\bar{q}}; 3_g, 4_g) \right\rangle \\
& + \left\langle F^{\mathrm{fin}}_{\mathrm{LV}}(1_q, 2_{\bar{q}}; 3_q) \right\rangle \\
& + [\alpha_s]\, C_F \sum_{i=1}^{2} \int_0^1 dz \left\langle \tilde{P}^{\mathrm{NLO}}_{qq}(z, E_c)\, F^{(i)}_{\mathrm{LM}}(1_q, 2_{\bar{q}}; 3_g | z) \right\rangle \\
& + [\alpha_s] \left\langle F_{\mathrm{LM}}(1_q, 2_{\bar{q}}; 3_q) \right\rangle \Bigg[T_R\, n_f \left(-\frac{23}{9} + \frac{2}{3} \log\left(\frac{E_3}{E_c}\right) - \frac{1}{3} \log\left(\eta_{13}\, \eta_{23}\right) \right) \\
& \qquad + C_F \left(\frac{2\pi^2}{3} + 6 \log\left(\frac{2E_c}{\mu}\right) \right) + C_A \left(\frac{67}{9} - \frac{4\pi^2}{3} + \frac{1}{3} \log\left(\frac{E_c}{E_3}\right) + \log^2\left(\frac{E_c}{E_3}\right) \right. \\
& \qquad \left. \frac{1}{3} \left(5 + 3 \log\left(\frac{E_c}{E_3}\right) \right) \log\left(\eta_{13}\, \eta_{23}\right) \right) + \mathrm{Li}_2(1 - \eta_{13}) + \mathrm{Li}_2(1 - \eta_{23}) \Bigg],
\end{aligned}
$$

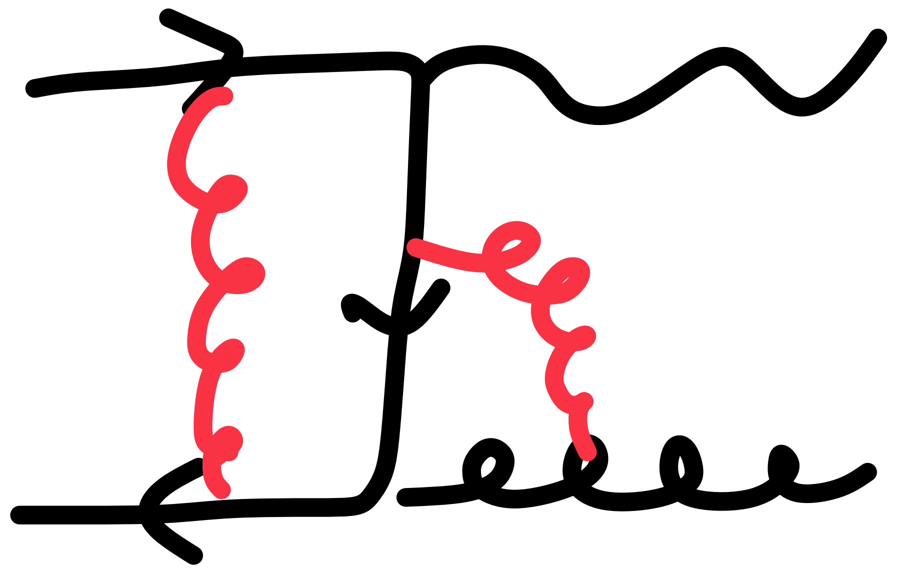
NNLO SUBTRACTION

NNLO COMPLEXITY

$$d\sigma_{NNLO} = d\sigma_{RR} + d\sigma_{RV} + d\sigma_{VV} + d\sigma_{PDF}$$







Catani, 1998

Many singular configurations…

VV is "simple", RV and RR much more intricate

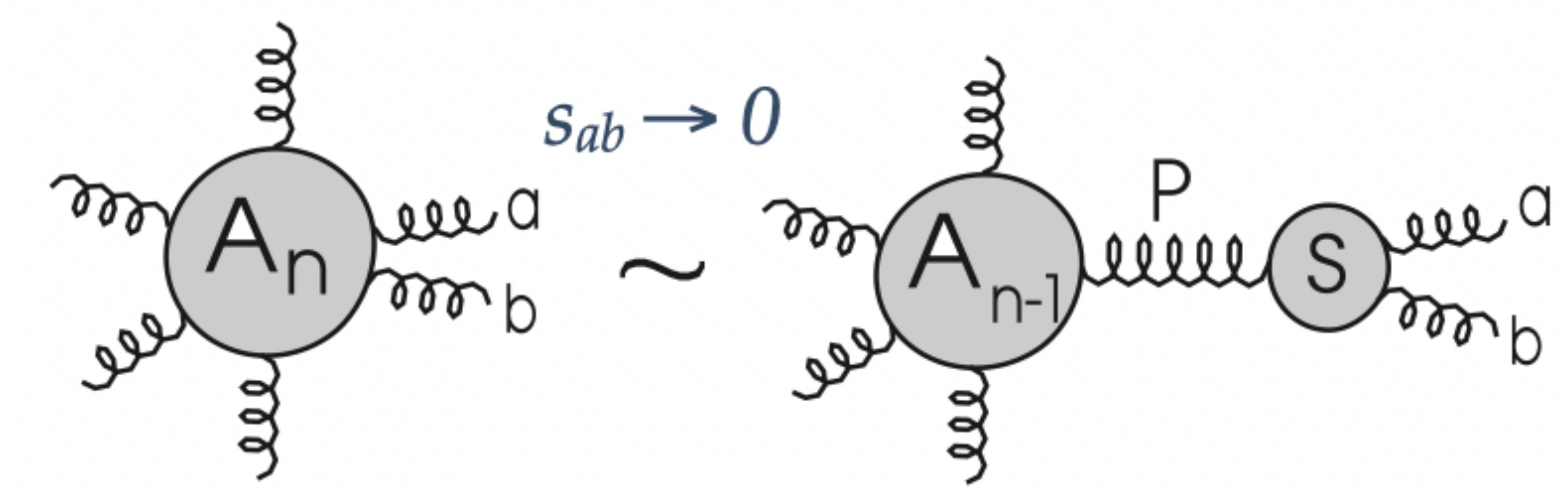
- Overlapping singularities
- Interplay soft/collinear limits

} ?!

IR singularities at 2-loops encoded in Catani's 2-loop operator

$$\begin{aligned} \boldsymbol{I}^{(2)}_{\mathrm{RS}}(\epsilon,\mu^2;\{p\}) &= -\frac{1}{2}\boldsymbol{I}^{(1)}(\epsilon,\mu^2;\{p\})\left(\boldsymbol{I}^{(1)}(\epsilon,\mu^2;\{p\}) + 4\pi\beta_0\frac{1}{\epsilon}\right) \\ &+ \frac{e^{+\epsilon\psi(1)}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\left(2\pi\beta_0\frac{1}{\epsilon}+K\right)\boldsymbol{I}^{(1)}(2\epsilon,\mu^2;\{p\}) \\ &+ \boldsymbol{H}^{(2)}_{\mathrm{RS}}(\epsilon,\mu^2;\{p\}) \ , \end{aligned}$$

Under IR limits, RR factorises into **universal kernels** x **lower multiplicity matrix elements**



Arises as a consequence of **locality** + **unitarity** in QFT

Under IR limits, RR factorises into **universal kernels** x **lower multiplicity matrix elements**

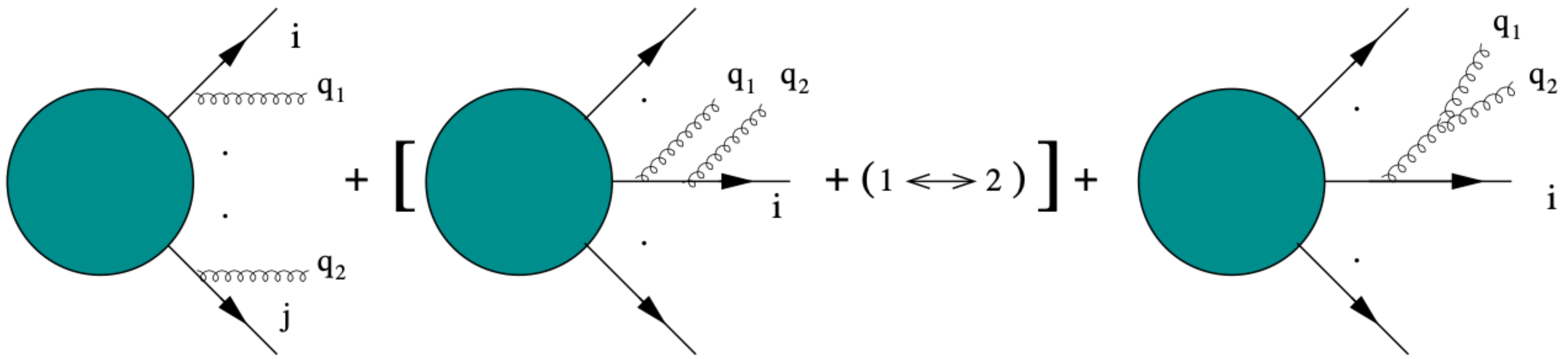
- Double soft *[Catani, Grazzini 9908523]*

$$|\mathcal{M}_{g,g,a_1,\ldots,a_n}(q_1,q_2,p_1,\ldots,p_n)|^2 \simeq \left(4\pi\alpha_S\mu^{2\epsilon}\right)^2$$
$$\cdot\left[\frac{1}{2}\sum_{i,j,k,l=1}^{n} \mathcal{S}_{ij}(q_1)\,\mathcal{S}_{kl}(q_2)\,|\mathcal{M}^{(i,j)(k,l)}_{a_1,\ldots,a_n}(p_1,\ldots,p_n)|^2 - C_A\sum_{i,j=1}^{n}\mathcal{S}_{ij}(q_1,q_2)|\mathcal{M}^{(i,j)}_{a_1,\ldots,a_n}|^2\right]$$

~ NLO^2

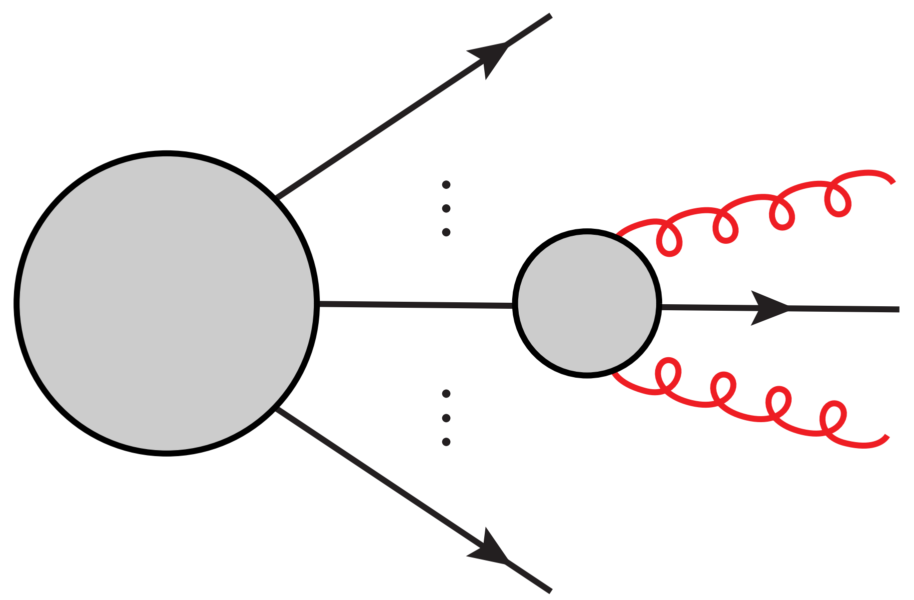
Pure NNLO

Purely non abelian



Under IR limits, RR factorises into **universal kernels** x **lower multiplicity matrix elements**

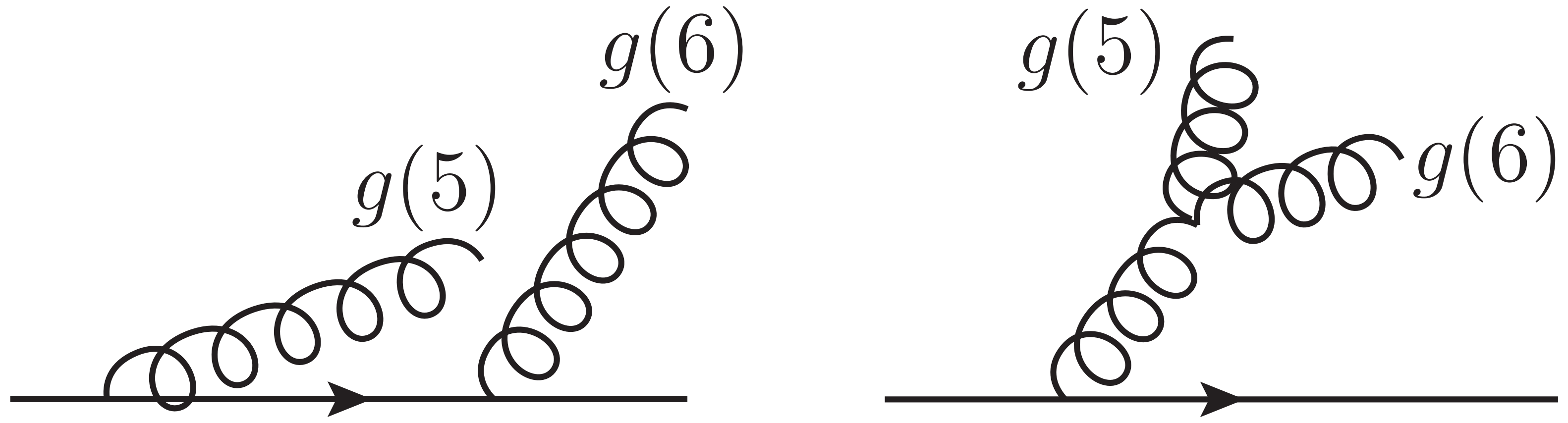
- Double soft *[Catani, Grazzini 9908523]*
- Triple collinear limit *[Catani, Grazzini 9810389]*



$$|\mathcal{M}_{a_1,a_2,a_3,\ldots}(p_1,p_2,p_3,\ldots)|^2 \simeq \frac{4}{s_{123}^2}(4\pi\mu^{2\epsilon}\alpha_S)^2\,\mathcal{T}^{ss'}_{a,\ldots}(p,\ldots)\,\hat{P}^{ss'}_{a_1a_2a_3}$$

$$\mathcal{T}^{s_1s_1'}_{a_1,\ldots}(p_1,\ldots) \equiv \sum_{\text{spins}\neq s_1,s_1'}\sum_{\text{colours}} \mathcal{M}^{c_1,c_2,\ldots;s_1,s_2,\ldots}_{a_1,a_2,\ldots}(p_1,p_2,\ldots)\left[\mathcal{M}^{c_1,c_2,\ldots;s_1',s_2,\ldots}_{a_1,a_2,\ldots}(p_1,p_2,\ldots)\right]^\dagger$$

Different triple collinear topologies to disentangle



$$1 = \theta\left(\eta_{61} < \frac{\eta_{51}}{2}\right) + \theta\left(\frac{\eta_{51}}{2} < \eta_{61} < \eta_{51}\right) + \theta\left(\eta_{51} < \frac{\eta_{61}}{2}\right) + \theta\left(\frac{\eta_{61}}{2} < \eta_{51} < \eta_{61}\right)$$

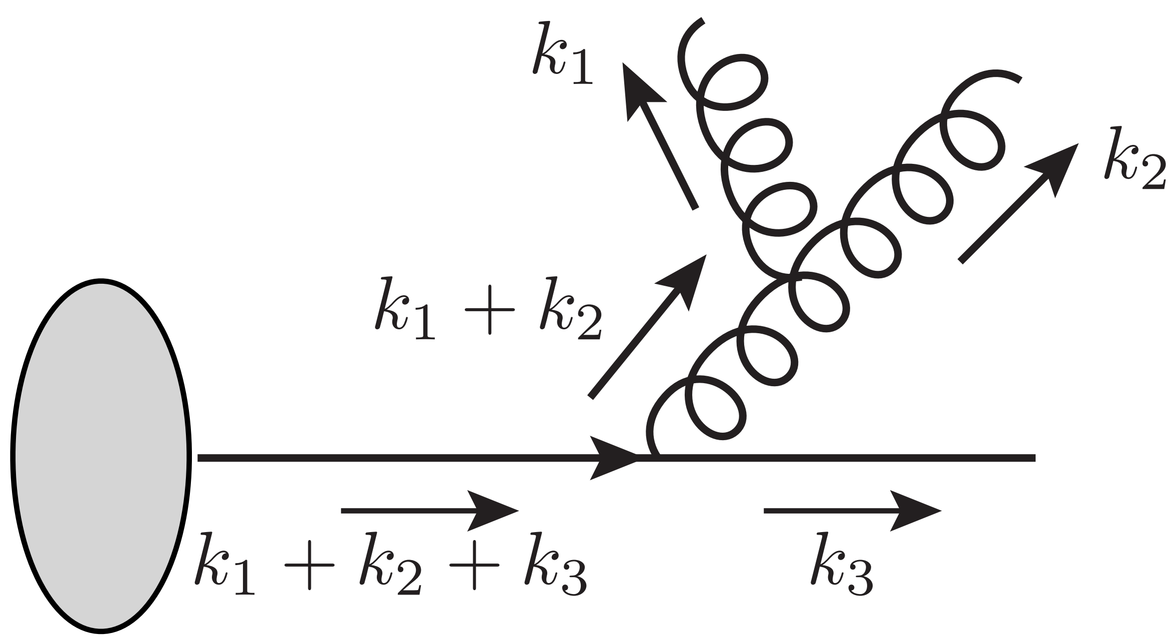
$$= \theta^{(a)} + \theta^{(b)} + \theta^{(c)} + \theta^{(d)}$$

Under IR limits, RR factorises into **universal kernels** x **lower multiplicity matrix elements**

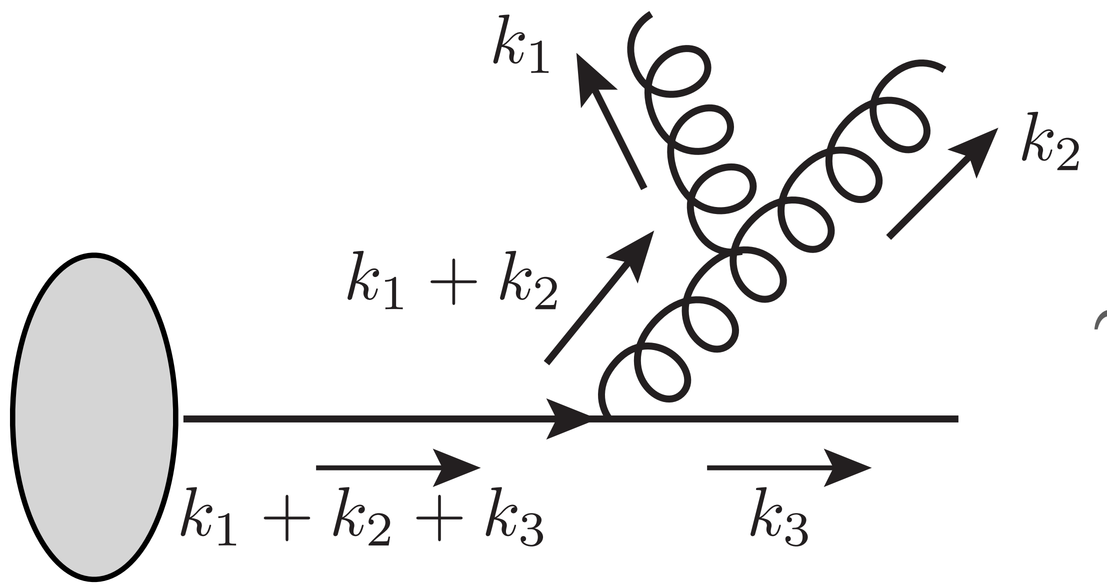
- Double soft *[Catani, Grazzini 9908523]*
- Triple collinear *[Catani, Grazzini 9810389]*
- One loop single soft *[Catani, Grazzini 0007142]*
- One loop single collinear *[Kosower 9901201, Bern, Del Duca, Kilgore, Schmidt 9903516]*

NLO-like kernels are not problematic. We need to understand the pure NNLO structures for generic processes

e.g.



OVERLAPPING SINGULARITIES – SOFT/COLLINEAR INTERPLAY

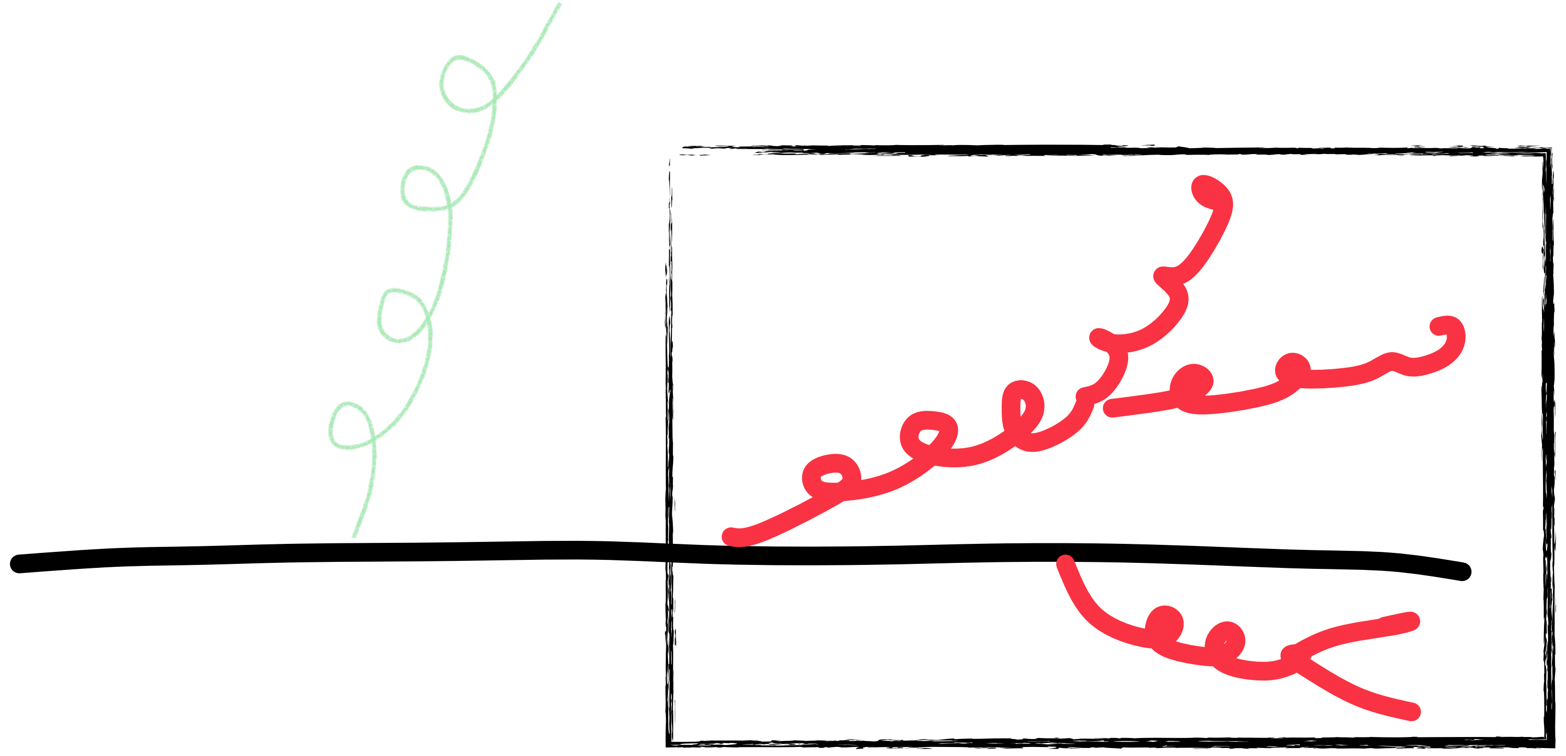


$$\sim \frac{1}{(k_1+k_2)^2}\frac{1}{(k_1+k_2+k_3)^2} = \frac{1}{2k_1\cdot k_2}\frac{1}{2k_1\cdot k_2 + 2k_1\cdot k_3 + 2k_2\cdot k_3} \iff k_1 \to 0 \text{ and } k_2 \parallel k_3$$

Overlapping energy/angle (e.g. soft/collinear) singularity!

Turns out to be an artifact of individual Feynman diagrams. **On-shell scattering amplitudes are free from entangled singularities**

Color coherence

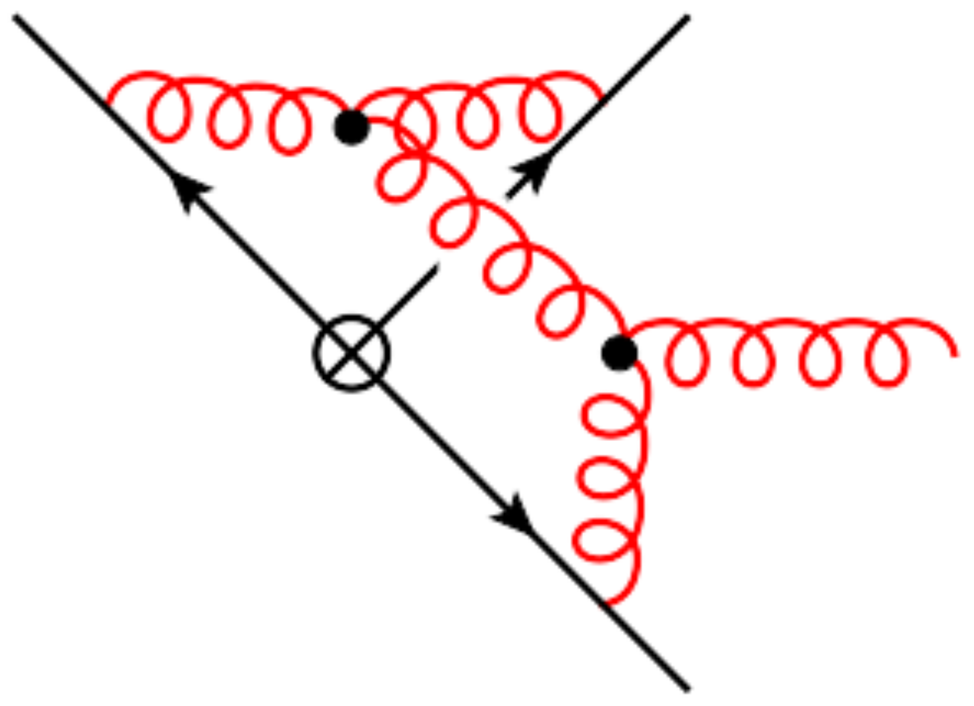


Soft gluon only sensitive to color charge of collinear subsystem, no S/C interplay!

INTERMEZZO: S/C INTERPLAY AT HIGHER ORDERS

Soft gluon only sensitive to color charge of collinear subsystem, no S/C interplay!

Is this true beyond NNLO?



[Dixon, Hermann, Yan, Zhu 2019]

Non-planar contributions to the two-loop soft gluon current contain triple color-correlated contributions

$$\boldsymbol{S}^{+,(2)}_{a,ikj} = \left(V^q_{ij}\right)^2 f^{aa_kb} f^{ba_ia_j} \boldsymbol{T}^{a_i}_i \boldsymbol{T}^{a_j}_j \boldsymbol{T}^{a_k}_k \left[\frac{\langle ik\rangle}{\langle iq\rangle\langle qk\rangle} F(z^{ij}_k, \epsilon) - \frac{\langle jk\rangle}{\langle jq\rangle\langle qk\rangle} F(z^{ji}_k, \epsilon) \right]$$

These terms **break strict collinear factorization** in space-like collinear limits

It would be interesting to see what happens at the level of the cross section… dijet@N3LO

Out of sight for now… but dream big!

BACK TO NNLO

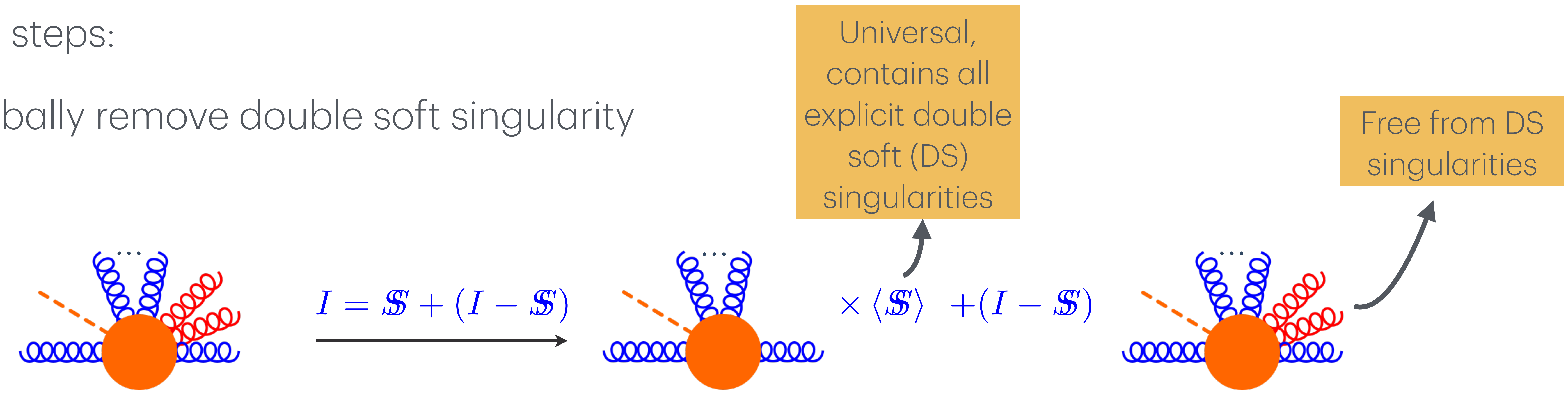
Use of color coherence is what distinguishes original **sector decomposition (Czakon, 2010)** from the **nested soft-collinear subtraction scheme (Caola, Melnikov, Roentsch 2017)**

No interplay between soft and collinear → subtract soft limits first, then collinear

"Nested approach"

Three steps:

- Globally remove double soft singularity



BACK TO NNLO

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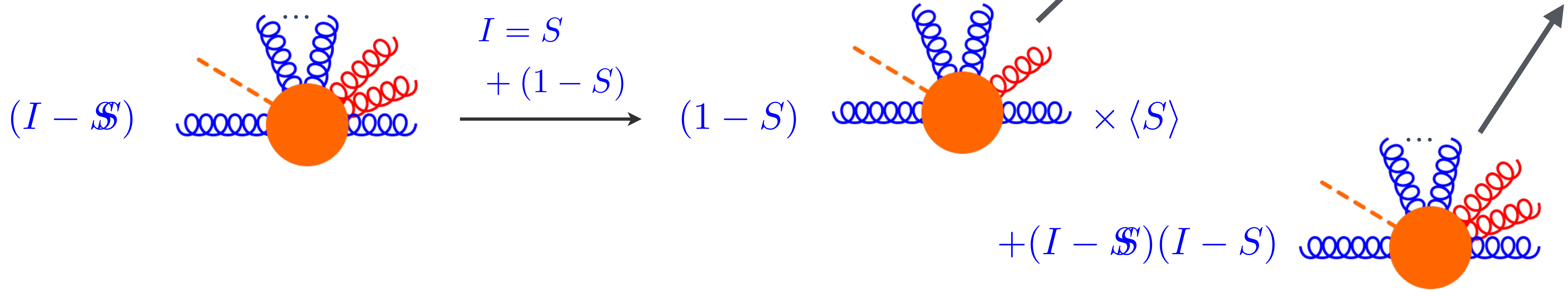
Three steps:

- Globally remove double soft singularity
- Globally remove single soft singularity

~NLO

Only contains collinear singularities



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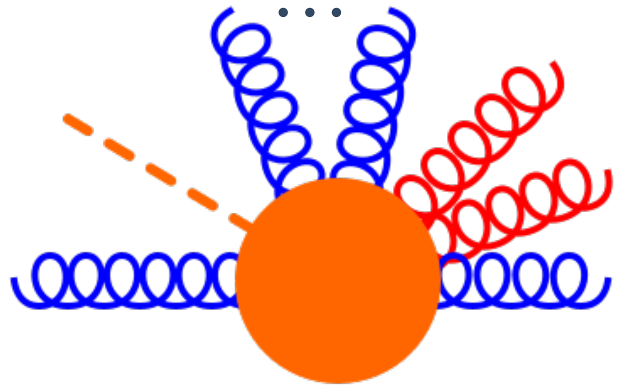
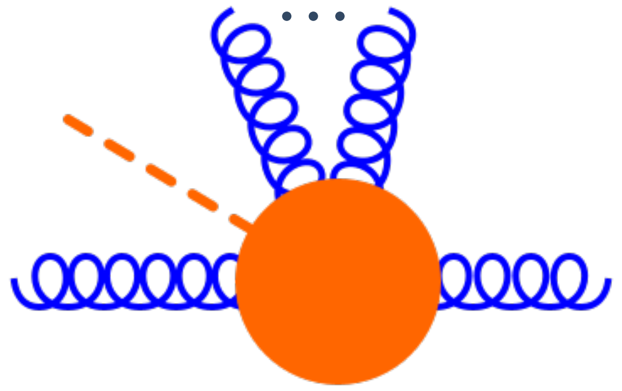
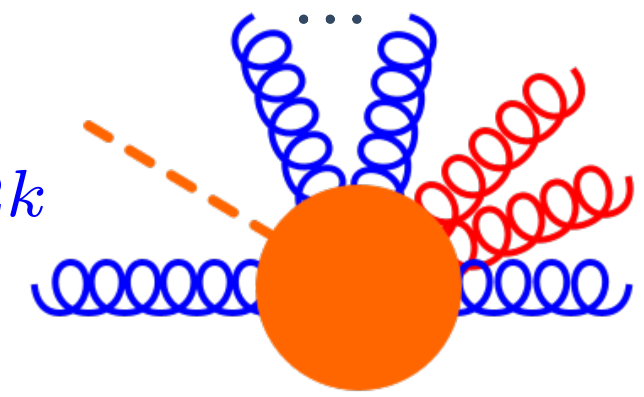
Three steps:

"Nested approach"

- Globally remove double soft singularity
- Globally remove single soft singularity
- FKS partition and treat one collinear singularity at a time

$(I-\mathbb{S})(I-S)$  $\xrightarrow{I=\mathbb{C}+(I-\mathbb{C})}$ $\sum_k$  $\langle P_{ijk}\rangle$

$+\sum_k (I-\mathbb{S})(I-S)(I-\mathbb{C}_{ijk})\Delta^{1k,2k}$ 

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Use of color coherence is what distinguishes original **sector decomposition (Czakon, 2010)** from the **nested soft-collinear subtraction scheme (Caola, Melnikov, Roentsch 2017)**

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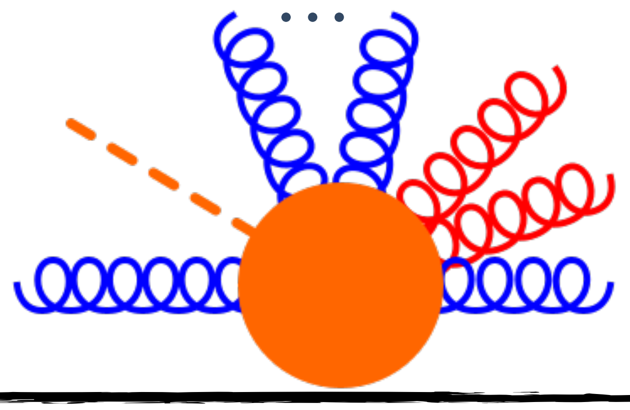
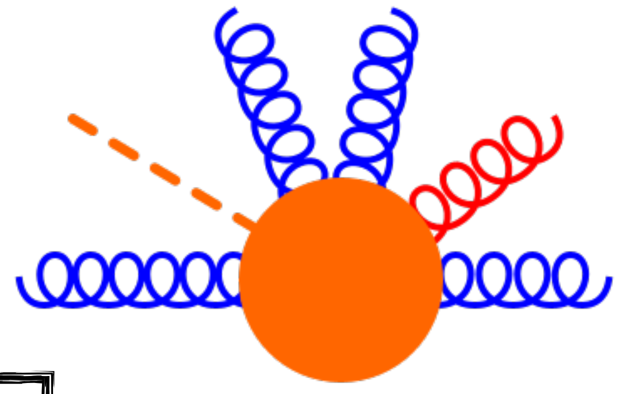
$$+\sum_k (I-\mathbb{S})(I-S)(I-\mathbb{C}_{ijk})\Delta^{1k,2k}$$



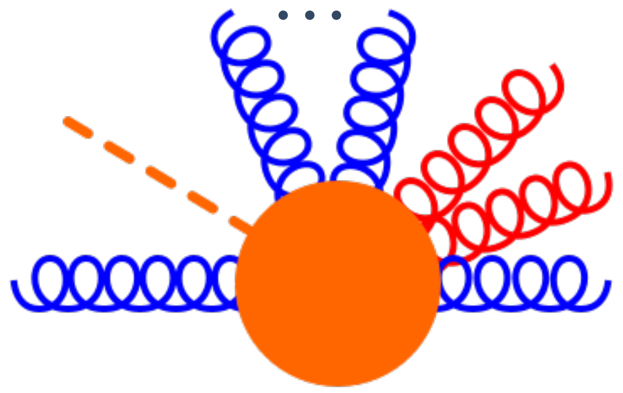
$$\xrightarrow{I = C + (1-C)} \quad (I-S)(I-C)$$



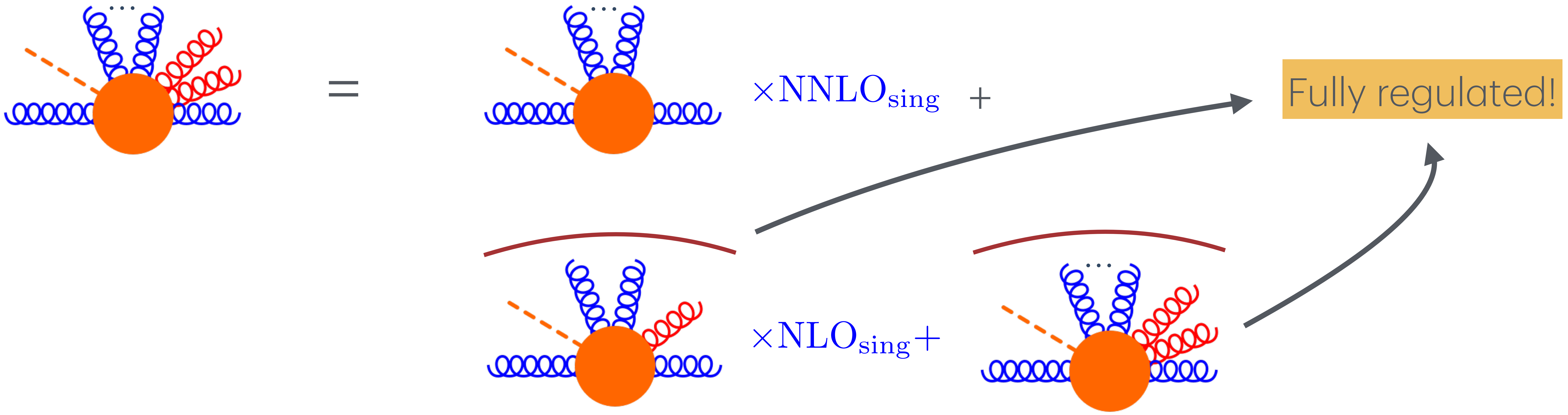
$$\times \langle P \rangle$$

Fully regulated!

$$+\sum_{k,a} (I-\mathbb{S})(I-S)(I-\mathbb{C}_{ijk})\theta^{(a)}(I-C_a)$$



FINAL RESULT



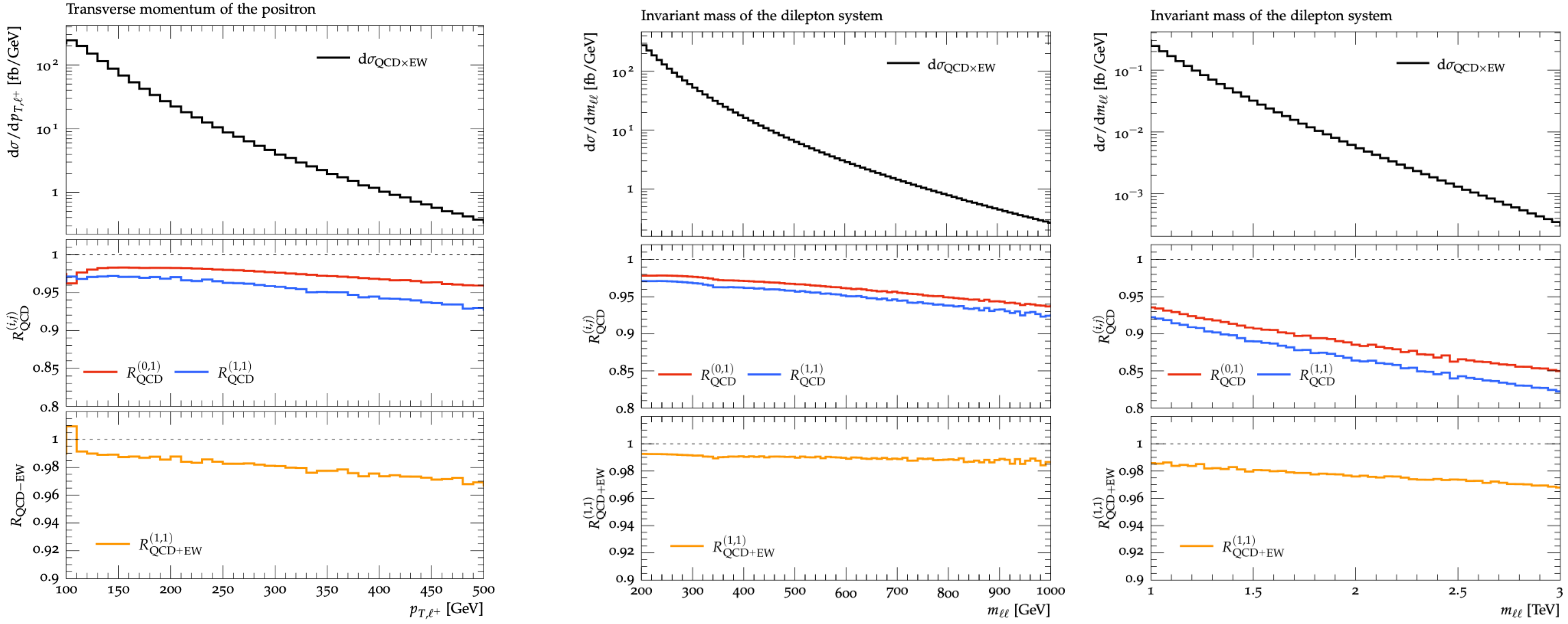
- Singularities are explicitly extracted in the counterterms, constructed from universal structures
- Counterterms are integrated analytically *"once and for all"*
- Minimal approach: only *inequivalent* physical limits are subtracted

Local in phase space & Analytic

[Caola, Delto, Frellesvig, Melnikov, 2018+19]

DOES IT WORK?

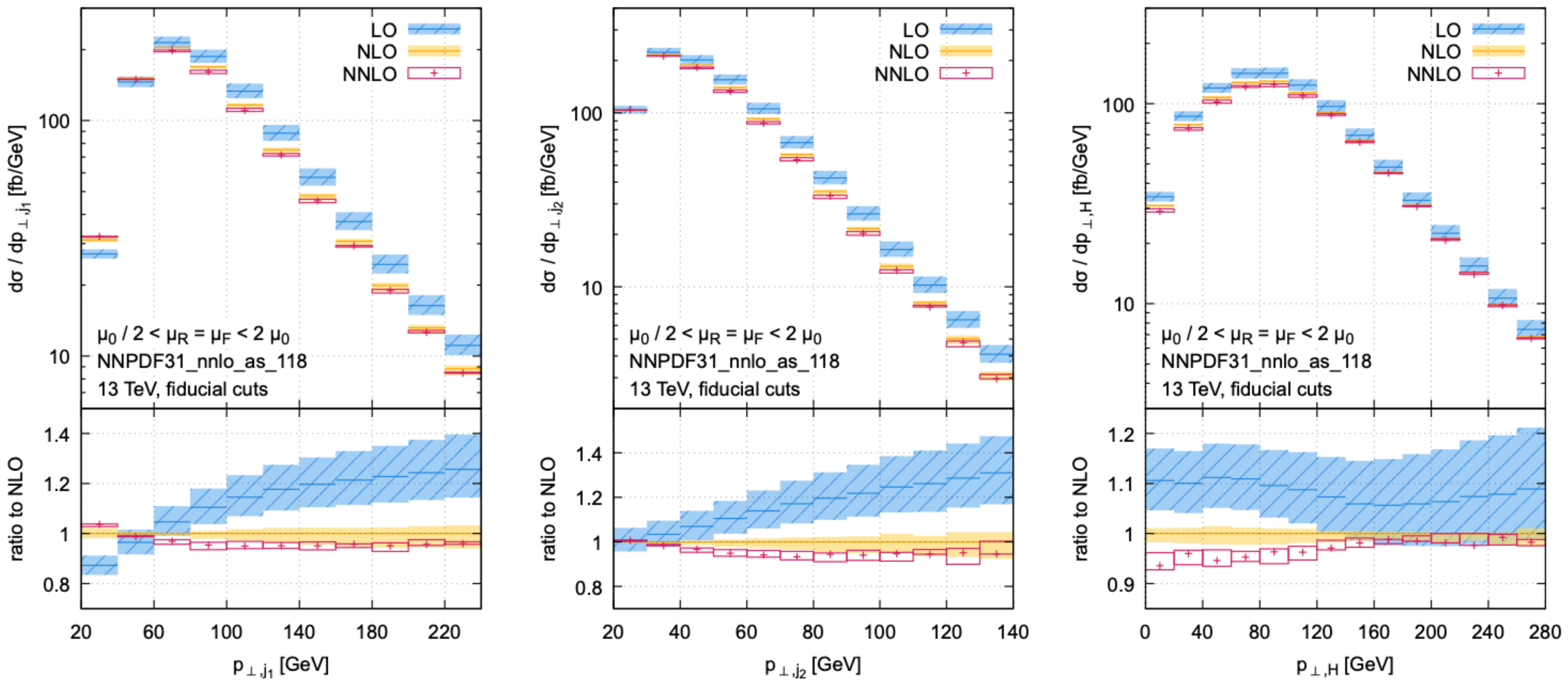
Mixed QCDxEW corrections to neutral current DY



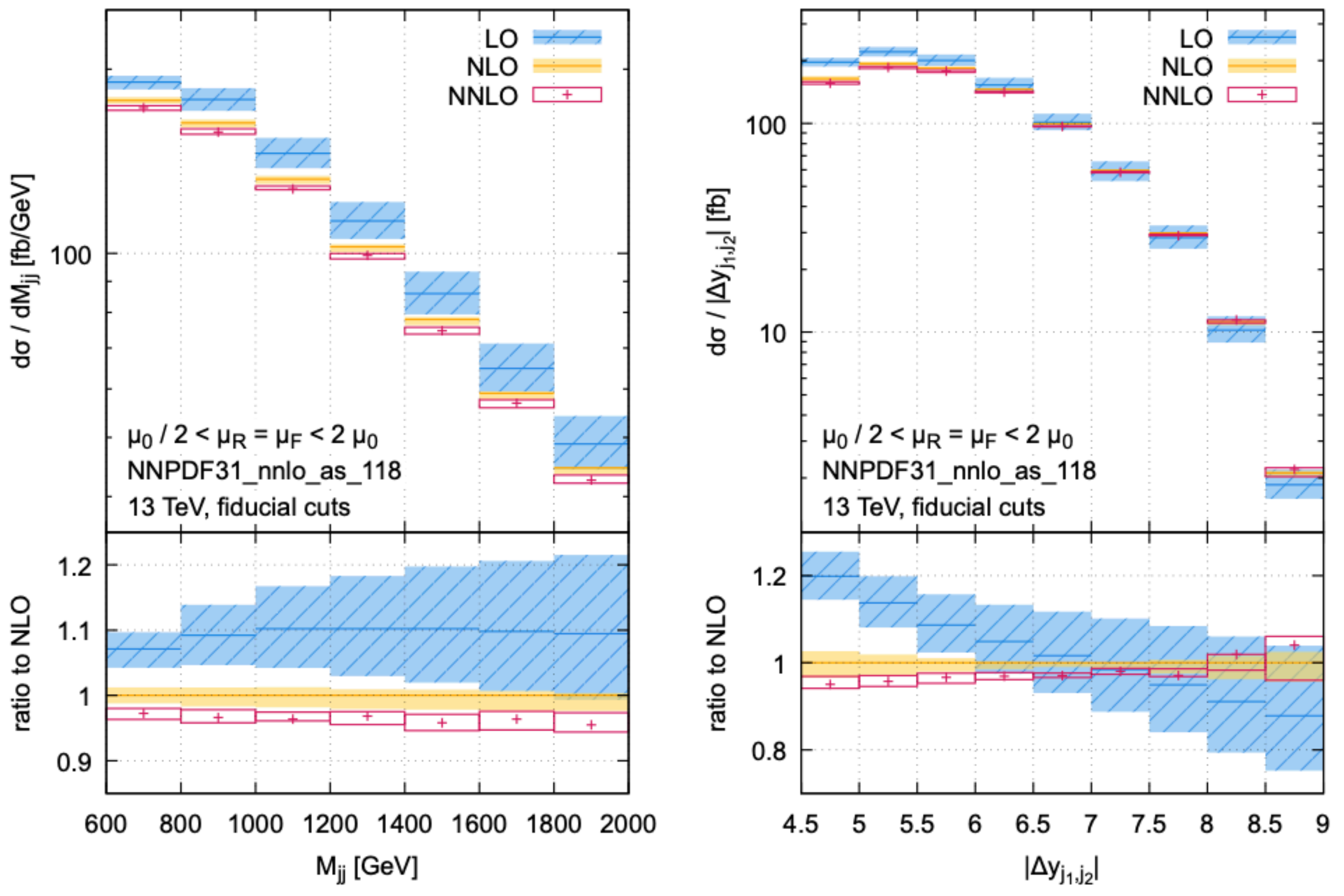
[Buccioni, Caola,Chawdhry, **FD**,Heller, von Manteuffel, Melnikov, Röntsch,Signorile-Signorile, 2022]

DOES IT WORK?

WBF Higgs production + H->bb/ZZ decay



[Asteriadis, Caola, Melnikov, Röntsch, 2022]



TOWARDS COLORFUL FINAL STATES

Back to our original example: Z+j @NNLO

$$
\begin{aligned}
\frac{1}{3!}\langle F_{\rm LM}(1_q,2_{\bar q};3_g,4_g,5_g)\rangle &= \langle S_{45}\,\Delta^{(45)}F_{\rm LM}^{4>5}\rangle + \langle (I-S_4)S_5\,\Delta^{(45)}F_{\rm LM}^{4>5}\rangle \\
&+ \Big\langle (I-S_{45})(I-S_5)\Big\{\sum_{i\in{\rm TC}}\Big[\Theta^{(a)}C_{45,i}(I-C_{5i}) + \Theta^{(b)}C_{45,i}(I-C_{45}) \\
&\quad + \Theta^{(c)}C_{45,i}(I-C_{4i}) + \Theta^{(d)}C_{45,i}(I-C_{45})\Big]\omega_{4i5i}\Big\}\,\Delta^{(45)}F_{\rm LM}^{4>5}\Big\rangle \\
&- \Big\langle (I-S_{45})(I-S_5)\sum_{(ij)\in{\rm DC}} C_{4i}C_{5j}\,\omega_{4i5j}\,\Delta^{(45)}F_{\rm LM}^{4>5}\Big\rangle \\
&+ \Big\langle (I-S_{45})(I-S_5)\Big\{\sum_{i\in{\rm TC}}\big[\Theta^{(a)}C_{5i} + \Theta^{(b)}C_{45} + \Theta^{(c)}C_{4i} + \Theta^{(d)}C_{45}\big]\,\omega_{4i5i} \\
&\quad + \sum_{(ij)\in{\rm DC}}\big[C_{4i}+C_{5j}\big]\,\omega_{4i5j}\Big\}\,\Delta^{(45)}F_{\rm LM}^{4>5}\Big\rangle \\
&+ \Big\langle (I-S_{45})(I-S_5)\Big\{\sum_{i\in{\rm TC}}\Big[\Theta^{(a)}(I-C_{45,i})(I-C_{5i}) + \Theta^{(b)}(I-C_{45,i})(I-C_{45}) \\
&\quad + \Theta^{(c)}(I-C_{45,i})(I-C_{4i}) + \Theta^{(d)}(I-C_{45,i})(I-C_{45})\Big]\omega_{4i5i} \\
&\quad + \sum_{(ij)\in{\rm DC}}(I-C_{4i})(I-C_{5j})\,\omega_{4i5j}\Big\}\,\Delta^{(45)}F_{\rm LM}^{4>5}\Big\rangle
\end{aligned}
$$

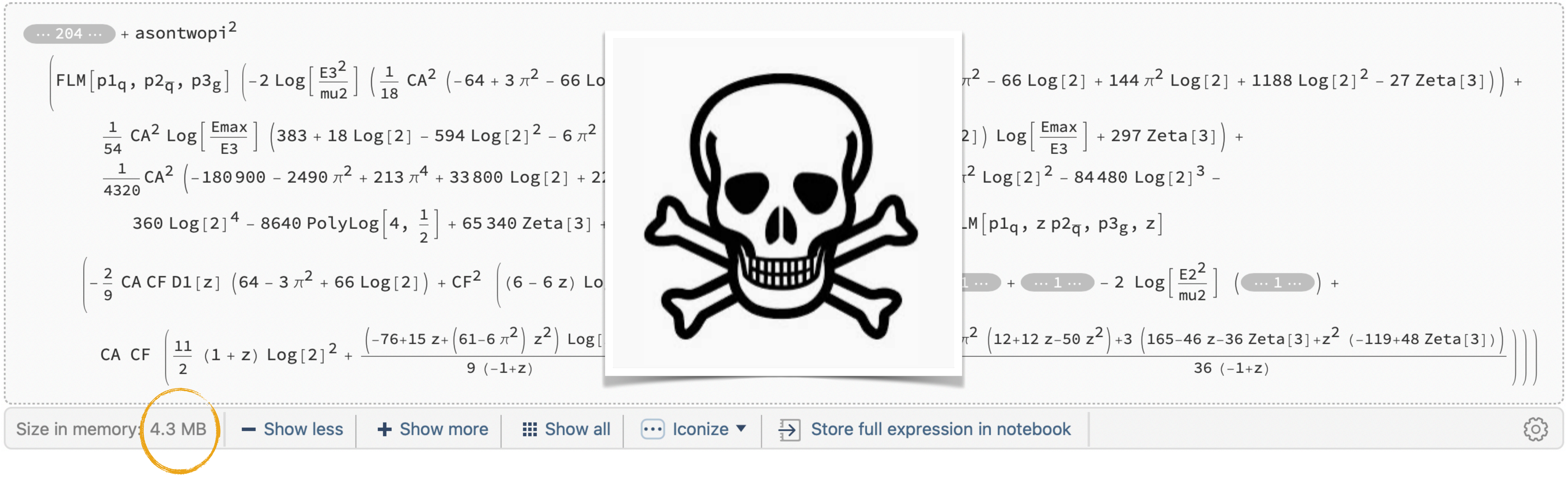
In principle, everything is known to deal with a completely generic process

In practice, several issues encountered:

- Bookkeeping increases dramatically
- Color correlations become crucial, SU(Nc) algebra does not close for n>=4

TOWARDS COLORFUL FINAL STATES

Back to our original example: Z+j @NNLO

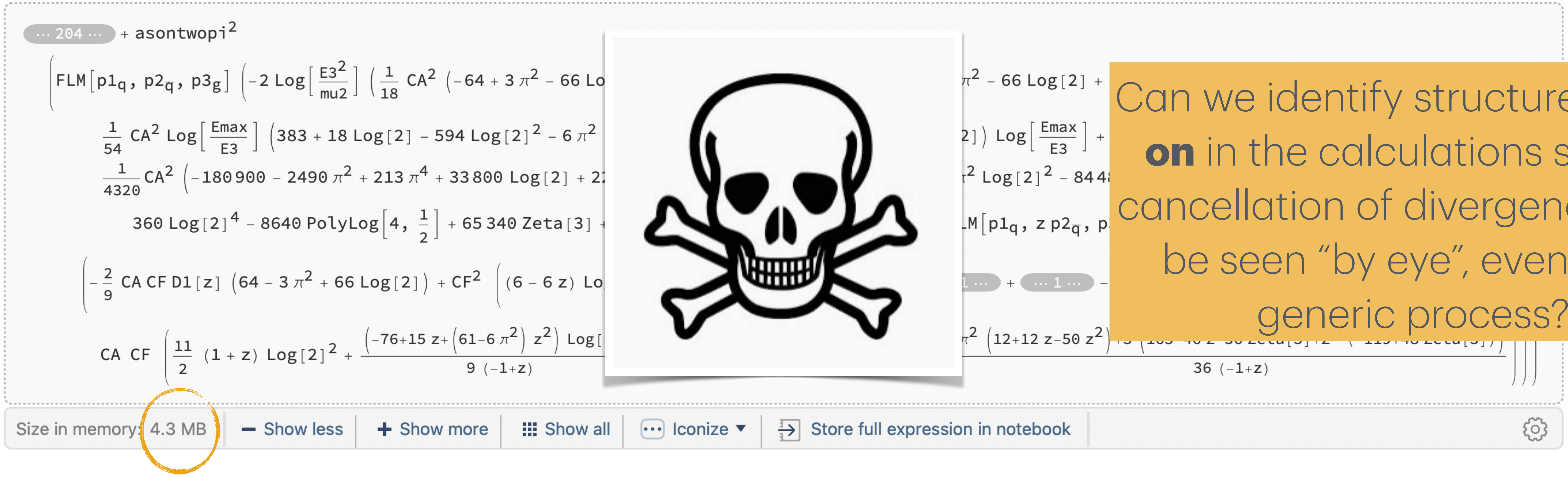


Evaluating each subtraction term explicitly hides structures & simplifications

"Asymmetry": VV very simple pole structure, RR structure obscured by energy ordering, partitioning, etc..

TOWARDS COLORFUL FINAL STATES

Back to our original example: Z+j @NNLO



Can we identify structures **early on** in the calculations so that cancellation of divergences can be seen "by eye", even for a generic process?

Evaluating each subtraction term explicitly hides structures & simplifications

"Asymmetry": VV very simple pole structure, RR structure obscured by energy ordering, partitioning, etc..

Can we identify structures **early on** in the calculations so that cancellation of divergences can be seen “by eye”, even for a generic process?

A fresh look at the nested soft-collinear subtraction scheme: NNLO QCD corrections to N-gluon final states in $q\bar{q}$ annihilation

Federica Devoto,[a] **Kirill Melnikov,**[b] **Raoul Röntsch,**[c] **Chiara Signorile-Signorile**[b,d,e] **Davide Maria Tagliabue**[c]

(2024)

Main idea: look at the pole structure of the virtuals to infer similar structures for the reals

Case of study: $q\bar{q} \to X + Ng$

Similar efforts were recently made also in the context of antenna subtraction, see e.g. arXiv:2310.19757

Main idea: look at the pole structure of the virtuals to infer similar operators for the reals

Warm-up @ NLO

- Virtuals: $I_V(\epsilon) = \bar{I}_1(\epsilon) + \bar{I}_1^{\dagger}(\epsilon)$ $\quad \bar{I}_1(\epsilon) = \frac{1}{2}\sum_{i\neq j}^{Np} \frac{\mathcal{V}_i^{\text{sing}}(\epsilon)}{\boldsymbol{T}_i^2}(\boldsymbol{T}_i \cdot \boldsymbol{T}_j)\left(\frac{\mu^2}{2p_i \cdot p_j}\right)^{\epsilon} e^{i\pi\lambda_{ij}\epsilon}$ $\quad \mathcal{V}_i^{\text{sing}}(\epsilon) = \frac{\boldsymbol{T}_i^2}{\epsilon^2} + \frac{\gamma_i}{\epsilon}$, $\; N_p = N+2$

- Reals: $I_S(\epsilon) = -\frac{(2E_{\max}/\mu)^{-2\epsilon}}{\epsilon^2}\sum_{i\neq j}^{N_p} \eta_{ij}^{-\epsilon} K_{ij}(\boldsymbol{T}_i \cdot \boldsymbol{T}_j)$ $\quad I_C(\epsilon) = \sum_{i=1}^{N_p} \frac{\boxed{\Gamma_{i,f_i}}}{\epsilon}$

$I_V(\epsilon) + I_S(\epsilon)$: Highest pole trivially cancel; Color correlations cancel } Remnant single pole canceled by $I_C(\epsilon)$

"Generalised anomalous dimensions" (Γ_{i,f_i})

$I_T(\epsilon) = I_V(\epsilon) + I_S(\epsilon) + I_C(\epsilon)$ **Finite!**

$$d\hat{\sigma}^{\text{NLO}} = [\alpha_s]\langle I_T(\epsilon) \cdot F_{\text{LM}}\rangle + [\alpha_s]\left[\langle P_{aa}^{\text{NLO}} \otimes F_{\text{LM}}\rangle + \langle F_{\text{LM}} \otimes P_{aa}^{\text{NLO}}\rangle\right] + \langle F_{\text{LV}}^{\text{fin}}\rangle + \langle \mathcal{O}_{\text{NLO}}\Delta^{(\mathfrak{m})} F_{\text{LM}}(\mathfrak{m})\rangle$$

NEW APPROACH AT NNLO

Think about **structures** arising in VV and look for their friends in RV and RR

Ideally the result will be ~NLO^2 as much as possible

$$\boldsymbol{I}^{(2)}_{\rm R.S.}(\epsilon,\mu^2;\{p\}) = -\frac{1}{2}\boldsymbol{I}^{(1)}(\epsilon,\mu^2;\{p\})\left(\boldsymbol{I}^{(1)}(\epsilon,\mu^2;\{p\})+4\pi\beta_0\frac{1}{\epsilon}\right) + \frac{e^{+\epsilon\psi(1)}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\left(2\pi\beta_0\frac{1}{\epsilon}+K\right)\boldsymbol{I}^{(1)}(2\epsilon,\mu^2;\{p\}) + \boldsymbol{H}^{(2)}_{\rm R.S.}(\epsilon,\mu^2;\{p\}) \,,$$

Catani, 1998

$$\langle F_{\rm VV}\rangle = [\alpha_s]^2\left\langle\left[\frac{1}{2}I_{\rm V}^2(\epsilon) - \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_E}}\left(\frac{\beta_0}{\epsilon}I_{\rm V}(\epsilon) - \left(\frac{\beta_0}{\epsilon}+K\right)I_{\rm V}(2\epsilon)\right)\right]\cdot F_{\rm LM}\right\rangle$$
$$+ [\alpha_s]^2\left\langle\left[-\frac{1}{2}\left[\bar{I}_1(\epsilon),\bar{I}_1^\dagger(\epsilon)\right] + \mathcal{H}_{2,\rm tc} + \mathcal{H}_{2,\rm tc}^\dagger + \mathcal{H}_{2,\rm cd} + \mathcal{H}_{2,\rm cd}^\dagger\right]\cdot F_{\rm LM}\right\rangle$$
$$+ [\alpha_s]\left\langle I_{\rm V}(\epsilon)\cdot F_{\rm LV}^{\rm fin}\right\rangle + \left\langle F_{\rm LV^2}^{\rm fin}\right\rangle + \left\langle F_{\rm VV}^{\rm fin}\right\rangle .$$

Color correlated contributions:

$$\begin{cases} \sim T_i\cdot T_j \\ \sim T_i\cdot T_j\cdot T_k \\ \sim (T_i\cdot T_j)\cdot(T_k\cdot T_l)\end{cases}$$

Different patterns of cancellations!

COLOR CORRELATIONS AND WHERE TO FIND THEM

We know they can only arise from soft real emissions and loop amplitudes

Double soft → $\sim T_i \cdot T_j$

$$\langle S_{\mathfrak{mn}} \Theta_{\mathfrak{mn}} F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle_{T^2}$$
$$= g_{s,b}^4 \sum_{i<j}^{N_p} \int [dp_{\mathfrak{m}}][dp_{\mathfrak{n}}]\Theta(E_{\mathfrak{m}}-E_{\mathfrak{n}})\langle \widetilde{S}_{ij}(p_{\mathfrak{m}},p_{\mathfrak{n}})\ (\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\cdot F_{\rm LM}\rangle$$
$$= [\alpha_s]^2\left[\frac{C_A}{\epsilon^2}c_1(\epsilon)+\frac{\beta_0}{\epsilon}c_2(\epsilon)+\beta_0\, c_3(\epsilon)\right]\langle \widetilde{I}_{\rm S}(2\epsilon)\cdot F_{\rm LM}\rangle + \langle S_{\mathfrak{mn}}\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle_{T^2}^{\rm fin}$$

Pole content identical to $I_S(2\epsilon)$!

Double soft → $\sim (T_i \cdot T_j)\cdot(T_k \cdot T_l)$

"Factorised contribution"

$$\langle S_{\mathfrak{mn}}\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle_{T^4} = 2g_{s,b}^4\sum_{(ij),(kl)}^{N_p}\Big\langle\int[dp_{\mathfrak{m}}][dp_{\mathfrak{n}}]\Theta(E_{\mathfrak{m}}-E_{\mathfrak{n}})S_{ij}(p_{\mathfrak{m}})S_{kl}(p_{\mathfrak{n}})$$
$$\times\{\boldsymbol{T}_i\cdot\boldsymbol{T}_j,\boldsymbol{T}_k\cdot\boldsymbol{T}_l\}\cdot F_{\rm LM}\Big\rangle$$
$$=[\alpha_s]^2\frac{1}{2}\langle I_{\rm S}^2(\epsilon)\cdot F_{\rm LM}\rangle\ .$$

$$\langle S_{\mathfrak{mn}}\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle$$
$$=[\alpha_s]^2\left\langle\left[\frac{1}{2}I_{\rm S}^2(\epsilon)+\left(\frac{C_A}{\epsilon^2}c_1(\epsilon)+\frac{\beta_0}{\epsilon}c_2(\epsilon)+\beta_0\,c_3(\epsilon)\right)\widetilde{I}_{\rm S}(2\epsilon)\right]\cdot F_{\rm LM}\right\rangle$$
$$+\langle S_{\mathfrak{mn}}\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle_{T^2}^{\rm fin}\ .$$

$I_S^2(\epsilon) + I_V^2(\epsilon)$ takes care of "quartic" color-correlated poles

COLOR CORRELATIONS AND WHERE TO FIND THEM

We know they can only arise from soft real emissions and loop amplitudes

Soft real-virtual

$$S_{\mathfrak{m}} F_{\rm RV}(\mathfrak{m}) = -g_{s,b}^2 \sum_{(ij)}^{N_p} \Big\{ 2\, S_{ij}(p_{\mathfrak{m}})\, (\boldsymbol{T}_i \cdot \boldsymbol{T}_j) \cdot F_{\rm LV} - \frac{\alpha_s(\mu)}{2\pi} \frac{\beta_0}{\epsilon}\, 2\, S_{ij}(p_{\mathfrak{m}})\, (\boldsymbol{T}_i \cdot \boldsymbol{T}_j) \cdot F_{\rm LM}$$
$$- 2 \frac{[\alpha_s]}{\epsilon^2} C_A A_K(\epsilon) \left(S_{ij}(p_{\mathfrak{m}})\right)^{1+\epsilon} (\boldsymbol{T}_i \cdot \boldsymbol{T}_j) \cdot F_{\rm LM}$$
$$- [\alpha_s] \frac{4\pi\, \Gamma(1+\epsilon)\Gamma^3(1-\epsilon)}{\epsilon\, \Gamma(1-2\epsilon)} \sum_{\substack{k=1\\k\neq i,j}}^{N_p} \kappa_{ij}\, S_{ki}(p_{\mathfrak{m}}) \left(S_{ij}(p_{\mathfrak{m}})\right)^{\epsilon} f_{abc}\, T_k^a\, T_i^b\, T_j^c\, F_{\rm LM} \Big\}$$

Triple color correlators

Only contributes in processes with 2 colored particles in the initial state and for processes with Np >=4

Non-trivial phase space integral

$$\langle S_{\mathfrak{m}} F_{\rm RV}(\mathfrak{m}) \rangle = [\alpha_s]^2 \left\langle \frac{1}{2} \Big[I_{\rm S}(\epsilon) \cdot I_{\rm V}(\epsilon) + I_{\rm V}(\epsilon) \cdot I_{\rm S}(\epsilon) \Big] \cdot F_{\rm LM} \right\rangle$$
$$+ [\alpha_s] \left\langle I_{\rm S}(\epsilon) \cdot F_{\rm LV}^{\rm fin} \right\rangle - [\alpha_s]^2 \frac{\Gamma(1-\epsilon)}{e^{\epsilon \gamma_E}} \frac{\beta_0}{\epsilon} \left\langle I_{\rm S}(\epsilon) F_{\rm LM} \right\rangle$$
$$- \frac{[\alpha_s]^2}{\epsilon^2} C_A A_K(\epsilon) \left\langle \widetilde{I}_{\rm S}(2\epsilon) \cdot F_{\rm LM} \right\rangle$$
$$+ [\alpha_s]^2 \left\langle \left(\frac{1}{2} \Big[I_{\rm S}(\epsilon)\,,\, \overline{I}_1(\epsilon) - \overline{I}_1^{\dagger}(\epsilon) \Big] + I_{\rm tri}^{\rm RV}(\epsilon) \right) \cdot F_{\rm LM} \right\rangle$$

The subtraction term can be almost fully written in terms of our NLO Catani-like operators

And so on.......(hard-collinear RV, single soft RR etc.)

CANCELLATION OF DOUBLE COLOR-CORRELATED POLES

Recall: $I_T = I_V + I_S + I_C$ = finite! $= I_T^2 - I_C^2$ ➡ No color correlated poles!

$$\Sigma_N^{(V+S),\mathrm{el}} = [\alpha_s]^2 \frac{1}{2}\left\langle \left[I_V^2 + I_V I_S + I_S I_V + I_S^2 + 2I_C I_V + 2 I_C I_S\right] \cdot F_{LM}\right\rangle$$
$$+ [\alpha_s]^2 \frac{\beta_0}{\epsilon}\frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_E}}\left\langle\left[-\left[I_S(\epsilon) + I_V(\epsilon)\right] + I_V(2\epsilon) + \tilde{c}(\epsilon)\,\widetilde{I}_S(2\epsilon)\right]\cdot F_{LM}\right\rangle$$
$$+ [\alpha_s]^2\left\langle\left[K\frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_E}}\, I_V(2\epsilon) + C_A\left(\frac{c_1(\epsilon)}{\epsilon^2} - \frac{A_K(\epsilon)}{\epsilon^2} - 2^{2+2\epsilon}\delta_g^{C_A}(\epsilon)\right)\right.\right.$$
$$\left.\left.\times\, \widetilde{I}_S(2\epsilon)\right]\cdot F_{LM}\right\rangle + [\alpha_s]\left\langle\left[I_V(\epsilon) + I_S(\epsilon)\right]\cdot F_{LV}^{\mathrm{fin}}\right\rangle,$$

No color correlated poles!

$$\sim -I_{V+S}(\epsilon) + I_{V+S}(2\epsilon) + (\tilde{c}(\epsilon) - 1)\tilde{I}_S(2\epsilon) + \tilde{I}_S(2\epsilon) - I_S(2\epsilon)$$

$\mathcal{O}(\epsilon)$ $\mathcal{O}(\epsilon^2)$ $\mathcal{O}(\epsilon)$

With similar arguments one can show that all terms are free of color correlated poles

CANCELLATION OF TRIPLE COLOR-CORRELATED POLES

Double origin: **explicit** or **commutators** of I operators

From VV ($\mathscr{H}_2$) and soft RV (I_{tri}^{RV})

Again present in VV and soft RV

Computed explicitly up to $\mathcal{O}(\epsilon^0)$

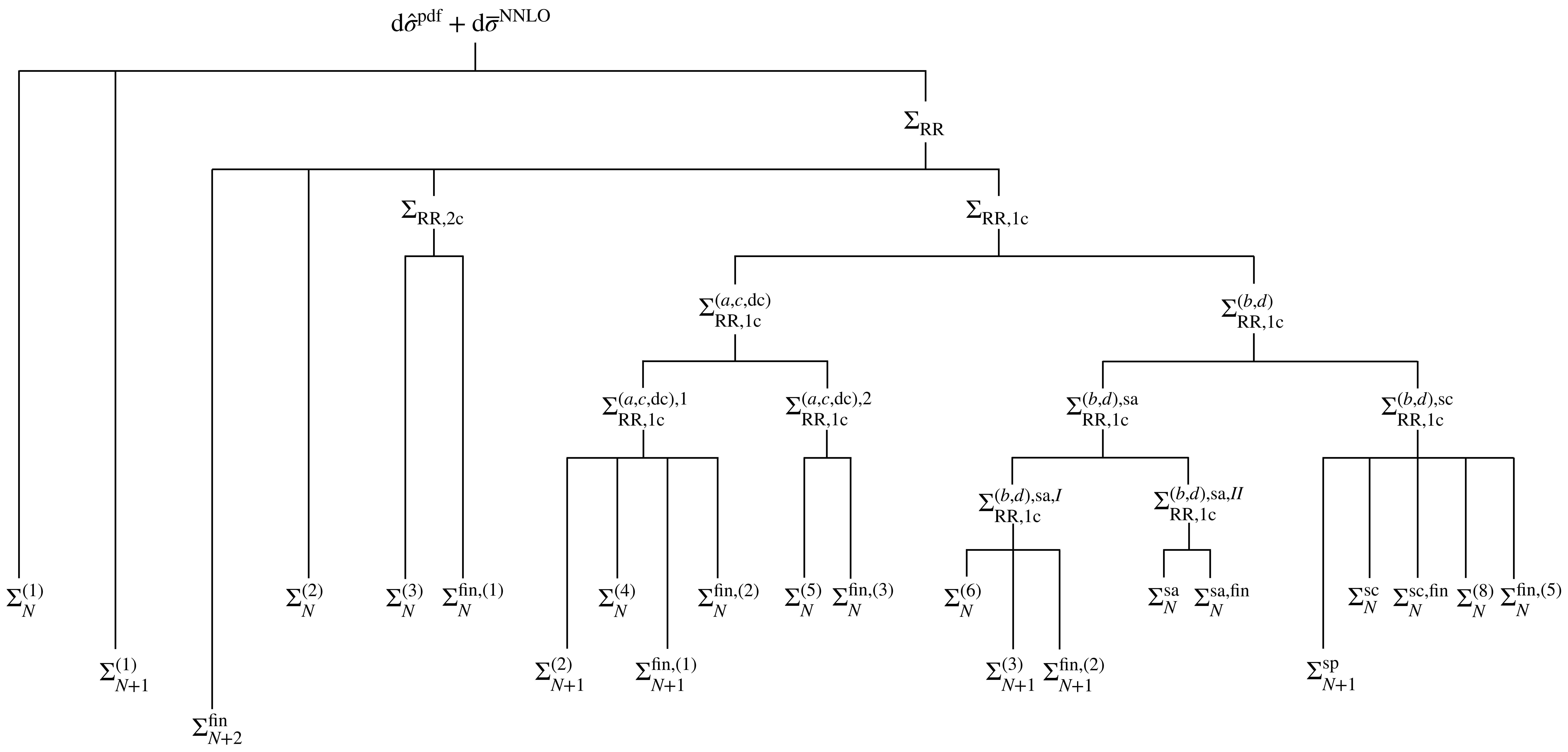
$$\Sigma_N^{\text{tri}} = [\alpha_s]^2 \left\langle \left(\frac{1}{2} \left[I_{\text{S}}(\epsilon) \, , \, \overline{I}_1(\epsilon) - \overline{I}_1^\dagger(\epsilon) \right] + I_{\text{tri}}^{\text{RV}}(\epsilon) \right) \cdot F_{\text{LM}} \right\rangle ,$$

$$+ [\alpha_s]^2 \left\langle \left(-\frac{1}{2} \left[\overline{I}_1(\epsilon), \overline{I}_1^\dagger(\epsilon) \right] + \mathcal{H}_{2,\text{tc}} + \mathcal{H}_{2,\text{tc}}^\dagger \right) \cdot F_{\text{LM}} \right\rangle$$

$$I_{\text{tri}}^{(\text{cc})} = -\left[I_+, I_- \right] + \left[2I_+ + I_{\text{S}}, I_- \right] + \mathcal{H}_{2,\text{tc}} + \mathcal{H}_{2,\text{tc}}^\dagger$$

By computing these commutators one can see that the poles exactly cancel!

$$\mathcal{H}_{2,\text{tc}} = \frac{1}{2\epsilon} [\Gamma, C] \qquad \overline{I}_1^{(\text{cc})} = \frac{\Gamma}{\epsilon} + C + \mathcal{O}(\epsilon)$$


$d\hat{\sigma}^{\rm pdf} + d\overline{\sigma}^{\rm NNLO}$
$\Sigma_{\rm RR}$
$\Sigma_{\rm RR,2c}$
$\Sigma_{\rm RR,1c}$
$\Sigma^{(a,c,{\rm dc})}_{\rm RR,1c}$
$\Sigma^{(b,d)}_{\rm RR,1c}$
$\Sigma^{(a,c,{\rm dc}),1}_{\rm RR,1c}$
$\Sigma^{(a,c,{\rm dc}),2}_{\rm RR,1c}$
$\Sigma^{(b,d),{\rm sa}}_{\rm RR,1c}$
$\Sigma^{(b,d),{\rm sc}}_{\rm RR,1c}$
$\Sigma^{(b,d),{\rm sa},I}_{\rm RR,1c}$
$\Sigma^{(b,d),{\rm sa},II}_{\rm RR,1c}$
$\Sigma^{(1)}_{N}$
$\Sigma^{(1)}_{N+1}$
$\Sigma^{\rm fin}_{N+2}$
$\Sigma^{(2)}_{N}$
$\Sigma^{(3)}_{N}$
$\Sigma^{{\rm fin},(1)}_{N}$
$\Sigma^{(2)}_{N+1}$
$\Sigma^{(4)}_{N}$
$\Sigma^{{\rm fin},(1)}_{N+1}$
$\Sigma^{{\rm fin},(2)}_{N}$
$\Sigma^{(5)}_{N}$
$\Sigma^{{\rm fin},(3)}_{N}$
$\Sigma^{(6)}_{N}$
$\Sigma^{(3)}_{N+1}$
$\Sigma^{{\rm fin},(2)}_{N+1}$
$\Sigma^{\rm sa}_{N}$
$\Sigma^{\rm sa,fin}_{N}$
$\Sigma^{\rm sp}_{N+1}$
$\Sigma^{\rm sc}_{N}$
$\Sigma^{\rm sc,fin}_{N}$
$\Sigma^{(8)}_{N}$
$\Sigma^{{\rm fin},(5)}_{N}$

FINAL RESULT

Finite remainders for the generic process $q\bar{q} \to X + Ng$

$$2s\, \mathrm{d}\hat{\sigma}_{\mathrm{db}}^{\mathrm{NNLO}} = \left[\frac{\alpha_s(\mu)}{2\pi}\right]^2 \left\langle \mathcal{P}_{qq}^{\mathrm{NLO}} \otimes F_{\mathrm{LM}} \otimes \mathcal{P}_{qq}^{\mathrm{NLO}} \right\rangle$$

$$\begin{aligned}2s\, \mathrm{d}\hat{\sigma}_{\mathrm{sb}}^{\mathrm{NNLO}} = \left[\frac{\alpha_s(\mu)}{2\pi}\right]^2 \Big\{ &\left\langle \mathcal{P}_{qq}^{\mathrm{NLO}} \otimes \left[I_{\mathrm{T}}^{(0)} \cdot F_{\mathrm{LM}}\right]\right\rangle + \left\langle \left[I_{\mathrm{T}}^{(0)} \cdot F_{\mathrm{LM}}\right] \otimes \mathcal{P}_{qq}^{\mathrm{NLO}} \right\rangle \\ &+ \left\langle \mathcal{P}_{qq}^{\mathcal{W}} \otimes \left[\mathcal{W}_1^{1\|\mathfrak{n},\mathrm{fin}} \cdot F_{\mathrm{LM}}\right]\right\rangle + \left\langle \left[\mathcal{W}_2^{2\|\mathfrak{n},\mathrm{fin}} \cdot F_{\mathrm{LM}}\right] \otimes \mathcal{P}_{qq}^{\mathcal{W}} \right\rangle \\ &+ \left\langle \mathcal{P}_{qq}^{\mathrm{NNLO}} \otimes F_{\mathrm{LM}} \right\rangle + \left\langle F_{\mathrm{LM}} \otimes \mathcal{P}_{qq}^{\mathrm{NNLO}} \right\rangle \\ &+ \left\langle \mathcal{P}_{qq}^{\mathrm{NLO}} \otimes F_{\mathrm{LV}}^{\mathrm{fin}} \right\rangle + \left\langle F_{\mathrm{LV}}^{\mathrm{fin}} \otimes \mathcal{P}_{qq}^{\mathrm{NLO}} \right\rangle \Big\},\end{aligned}$$

$$\begin{aligned}2s\, \mathrm{d}\hat{\sigma}_{\mathrm{el}}^{\mathrm{NNLO}} = \left[\frac{\alpha_s(\mu)}{2\pi}\right]^2 \Big\{ &\left\langle \left[I_{\mathrm{cc}}^{\mathrm{fin}} + I_{\mathrm{tri}}^{\mathrm{fin}} + I_{\mathrm{unc}}^{\mathrm{fin}}\right] \cdot F_{\mathrm{LM}} \right\rangle \\ &+ \sum_{i=1}^{N_p} \left\langle \left[\gamma^{\mathcal{W}}(L_i)\, \theta_{i2}\, \mathcal{W}_i^{i\|\mathfrak{n},\mathrm{fin}} + \delta_g^{(0)}\, \mathcal{W}_i^{\mathfrak{m}\|\mathfrak{n},\mathrm{fin}} + \delta_g^{\perp} \mathcal{W}_r^{(i)}\right] \cdot F_{\mathrm{LM}} \right\rangle \Big\} \\ &+ \left[\frac{\alpha_s(\mu)}{2\pi}\right] \left\langle I_{\mathrm{T}}^{(0)} \cdot F_{\mathrm{LV}}^{\mathrm{fin}} \right\rangle + \left\langle S_{\mathfrak{m}\mathfrak{n}} \Theta_{\mathfrak{m}\mathfrak{n}} F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n}) \right\rangle_{T^2}^{\mathrm{fin}} + \left\langle F_{\mathrm{LV}^2}^{\mathrm{fin}} \right\rangle + \left\langle F_{\mathrm{VV}}^{\mathrm{fin}} \right\rangle\end{aligned}$$

- Ready to be implemented in a numerical code
- Trivial dependence on number of partons
- Analytic proof of pole cancellation for generic process at NNLO!!

CONCLUSIONS AND OUTLOOKS

- Higher order calculations are important for LHC precision physics program; they involve infrared singularities that need to be regulated
- There is a freedom in how to regulate such divergences -> subtraction scheme. I presented generalities and recent advances in the context of the nested soft-collinear subtraction scheme, goal is to deal with **high-multiplicity final states** (beyond 2->2)
- With the new approach, we are able to **analytically prove** the cancellation of IR singularities for a generic NNLO process (only gluons for now)
- What's next: generalization of the NSC "new" approach to off-diagonal partonic channels, include quarks in the final state, etc., numerical studies, phenomenological applications WIP


Thank you for your attention!

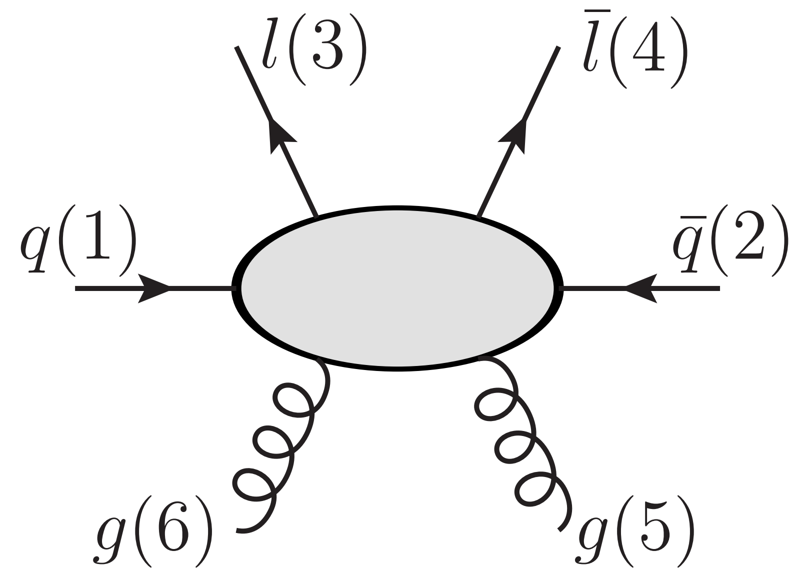
PHASE SPACE PARTITIONS

Efficient way to simplify the problem: introduce **partition functions** (following FKS philosophy):

- **Unitary partition**
- Select a **minimum number of singularities** in each sector
- Do **not affect** the **analytic integration** of the counterterms

Definition of partition functions benefits from remarkable degree of **freedom**: different approaches can be implemented

Examples: Nested soft-collinear subtraction $q\bar{q} \to Z \to e^- e^+ g\, g$ *[Caola, Melnikov, Röntsch 1702.01352]*



$$1 = \omega^{51,61} + \omega^{52,62} + \omega^{51,62} + \omega^{52,61}$$

$$\omega^{51,61} = \frac{\rho_{25}\,\rho_{26}}{d_5\, d_6}\left(1 + \frac{\rho_{15}}{d_{5621}} + \frac{\rho_{16}}{d_{5612}}\right) \qquad \omega^{51,62} = \frac{\rho_{25}\,\rho_{16}\,\rho_{56}}{d_5\, d_6\, d_{5612}}$$

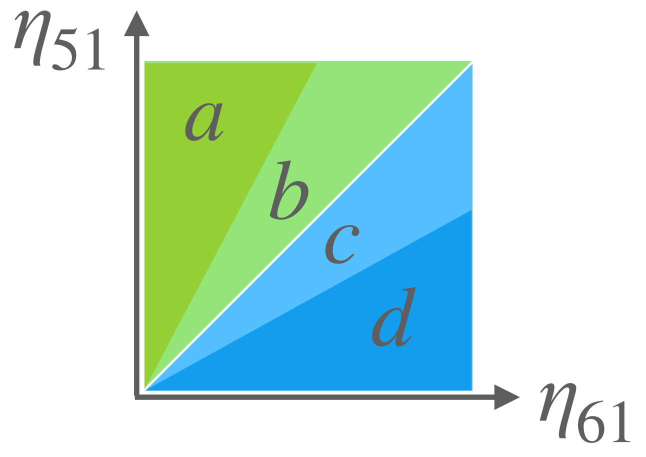
$$\omega^{52,62} = \frac{\rho_{15}\,\rho_{16}}{d_5\, d_6}\left(1 + \frac{\rho_{25}}{d_{5621}} + \frac{\rho_{26}}{d_{5612}}\right) \qquad \omega^{52,61} = \frac{\rho_{15}\,\rho_{26}\,\rho_{56}}{d_5\, d_6\, d_{5621}}$$

$$\rho_{ab} = 1 - \cos\vartheta_{ab}\,,\ \eta_{ab} = \rho_{ab}/2$$

$$d_{i=5,6} = \rho_{1i} + \rho_{2i} = 2$$

$$d_{5621} = \rho_{56} + \rho_{52} + \rho_{61}$$

$$d_{5612} = \rho_{56} + \rho_{51} + \rho_{62}$$



$$1 = \theta\left(\eta_{61} < \frac{\eta_{51}}{2}\right) + \theta\left(\frac{\eta_{51}}{2} < \eta_{61} < \eta_{51}\right) + \theta\left(\eta_{51} < \frac{\eta_{61}}{2}\right) + \theta\left(\frac{\eta_{61}}{2} < \eta_{51} < \eta_{61}\right)$$

$$= \theta^{(a)} + \theta^{(b)} + \theta^{(c)} + \theta^{(d)}$$