ALEGRO Workshop 21/03/2024

Beam quality preservation using multi-staged rectangular waveguide

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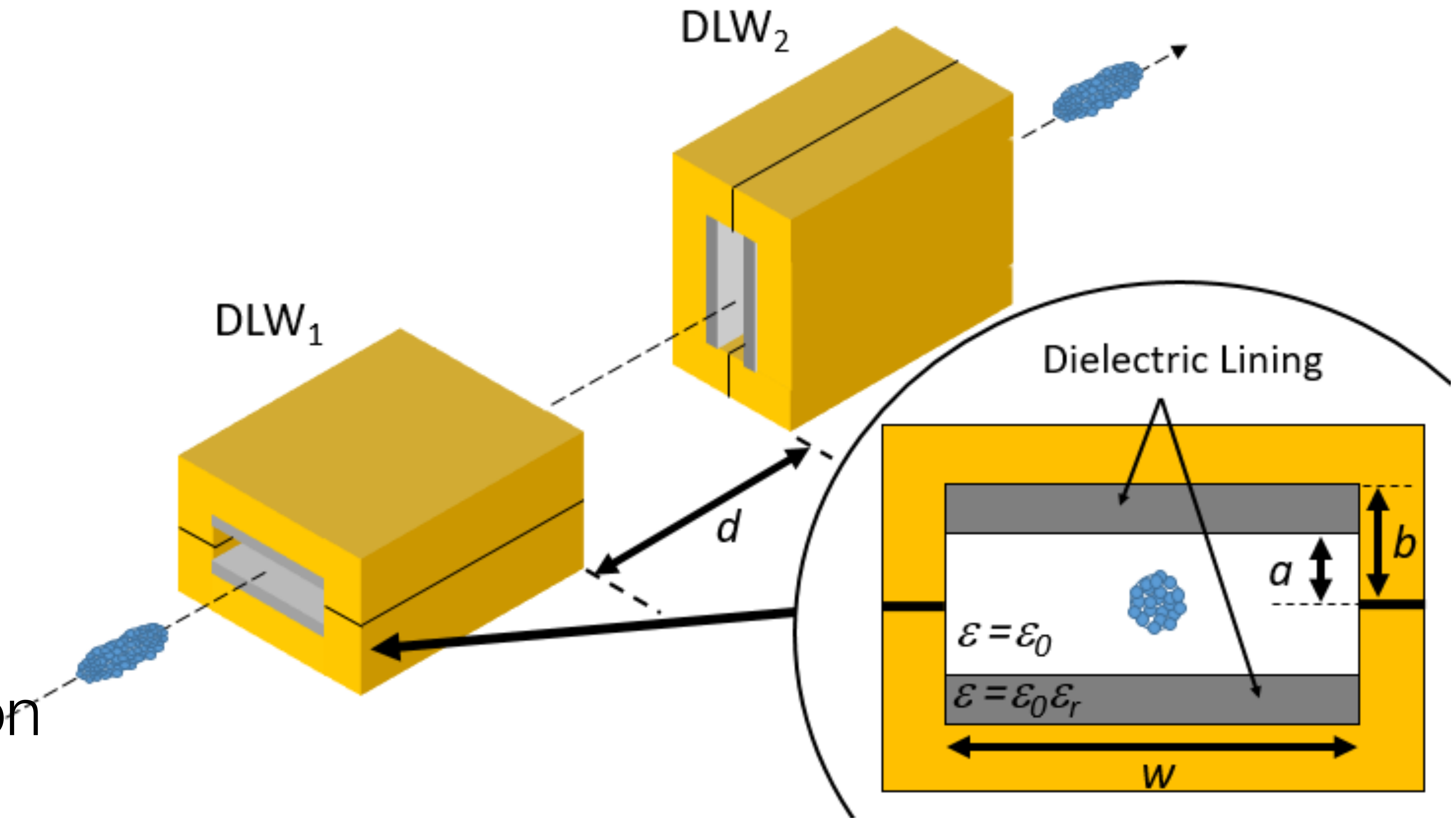
Outline

- Fourier decomposition of the longitudinal THz electric field into multipole components (monopole, dipole, quadrupole, sextupole, octupole, etc.)
- Quadrupolar field and correlated transverse energy distribution
- Orthogonal-multistaging configuration and matching to preserve the 6D phase space
- Conclusions and outlook



Six-dimensional phase space preservation in a terahertz-driven multistage dielectric-lined rectangular waveguide accelerator

Öznur Apsimon, Graeme Burt, Robert B. Appleby, Robert J. Apsimon, Darren M. Graham, and Steven P. Jamison

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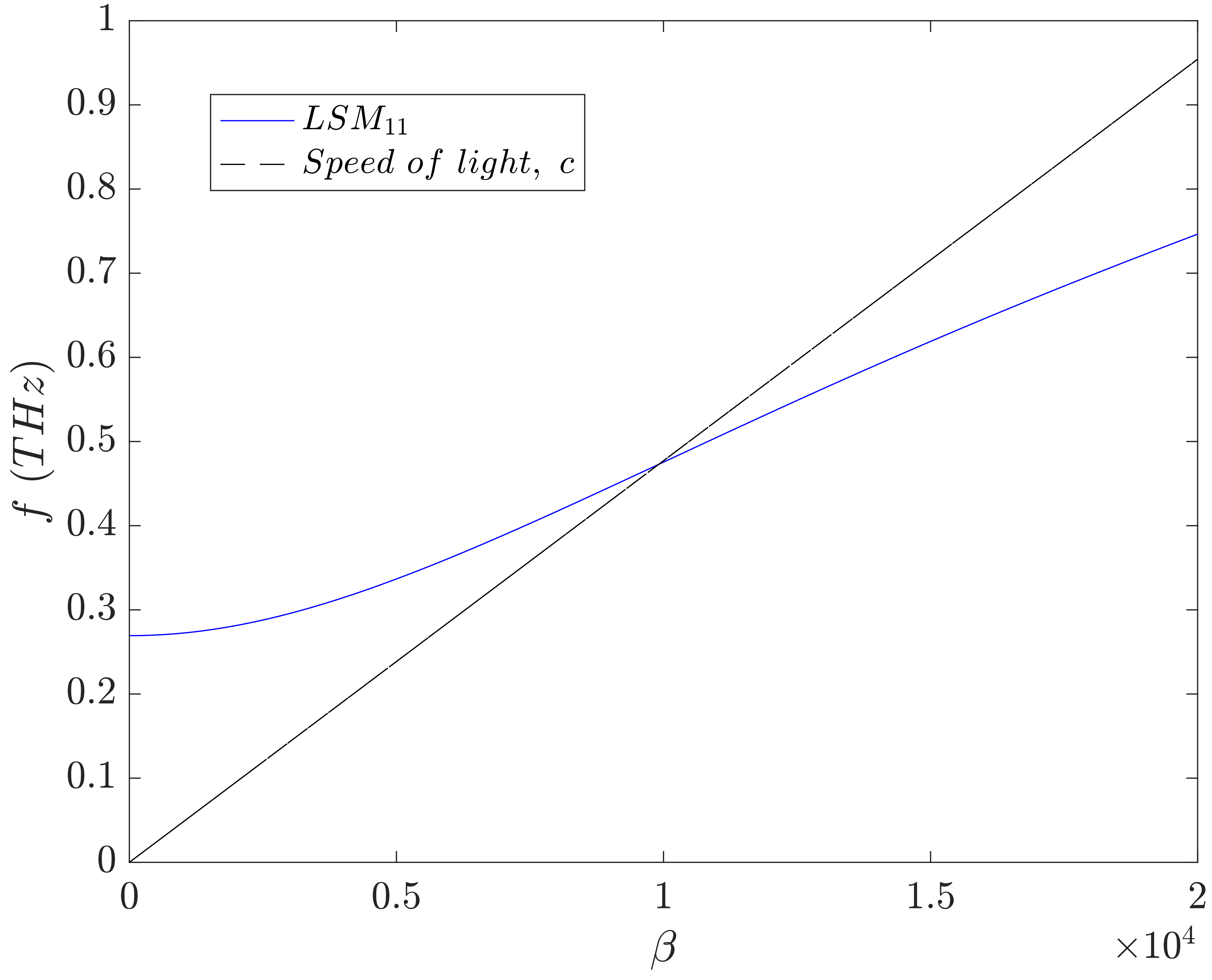
TM$_{11}$ for acceleration

$$E_z(x,y,z,\omega) = A(-ik_z)k_y^a \sin(\frac{\pi}{w}(x+\frac{w}{2}))\cos(k_y^a y)e^{i\omega\frac{z}{c}-k_z+\phi}$$

Synchronisation with a relativistic beam



- Find the roots of the dispersion relation to find $\beta = k_0$ using Newton-Raphson method. *Alisa Healy, Thesis, Lancaster*

$$k_y^a \sin(k_y^b(b-a))\sin(k_0 a) = \epsilon_r k_0 \cos(k_0 a)\cos(k_y^b(b-a))$$

- A single frequency of 0.465 THz synchronises to the relativistic beam, $v_p \sim c$.

$$\beta = k_z = k_0 \simeq 9746$$

- The propagation constant in vacuum is defined as,

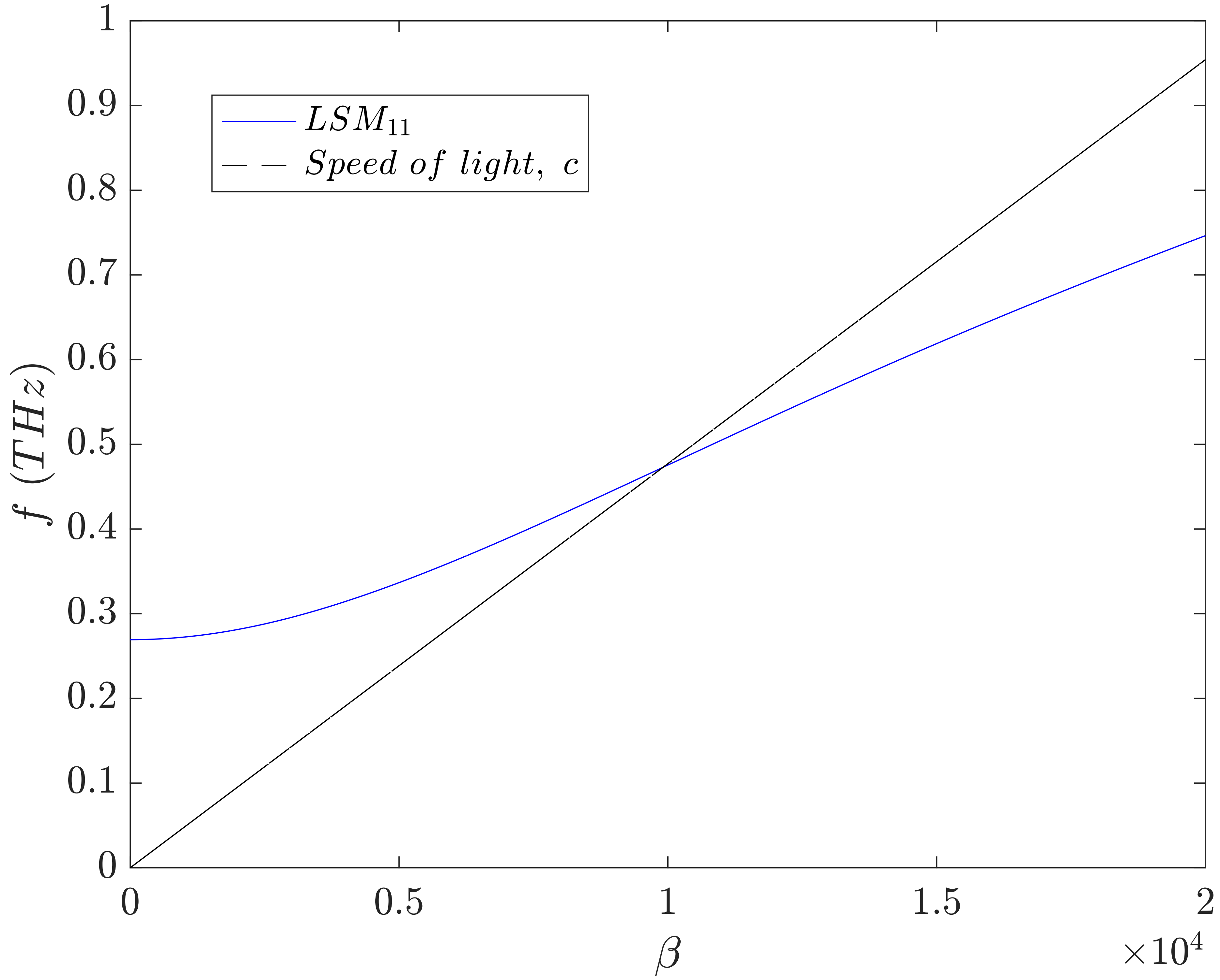
$$k_{y,mn}^a = \sqrt{k_0^2 - (\frac{m\pi}{w})^2 - \beta_{mn}^2} \qquad k_{y,11}^a = i2.62\times10^3 = i\frac{\pi}{w}$$

TM$_{11}$ for acceleration

drop time dependency for single frequency and v_p=c

$$E_z(x,y,z,\omega) = A(-ik_z)k_y^a \sin(\frac{\pi}{w}(x+\frac{w}{2}))\cos(k_y^a y)e^{i\omega\frac{z}{c}-k_z z+\phi}$$

Synchronisation with a relativistic beam



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- A single frequency of 0.465 THz synchronises to the relativistic beam, v_p ~ c.

$$\beta = k_z = k_0 \simeq 9746$$

- The propagation constant in vacuum is defined as,

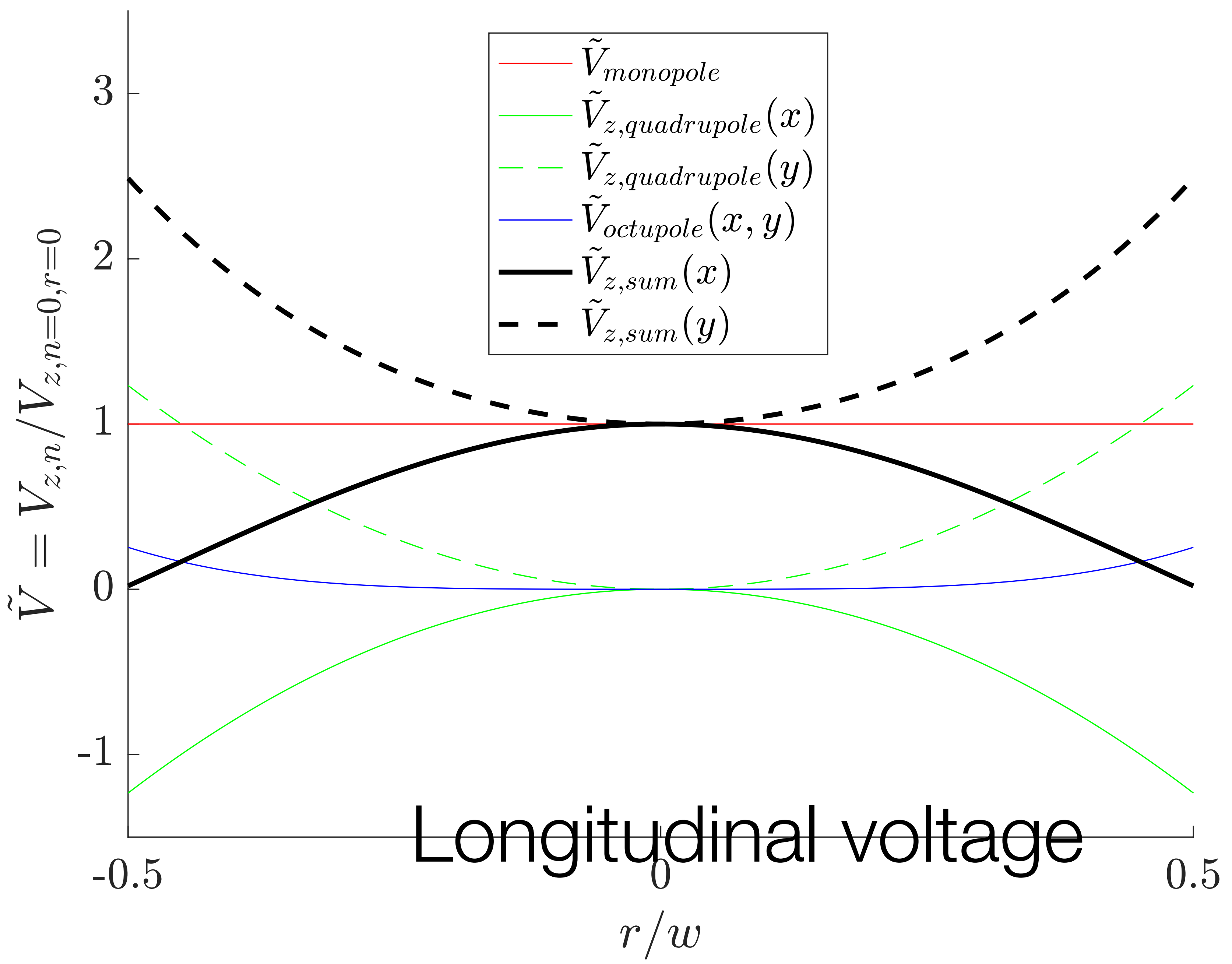
$$k_{y,mn}^a = \sqrt{k_0^2 - (\frac{m\pi}{w})^2 - \beta_{mn}^2} \qquad k_{y,11}^a = i2.62\times 10^3 = i\frac{\pi}{w}$$

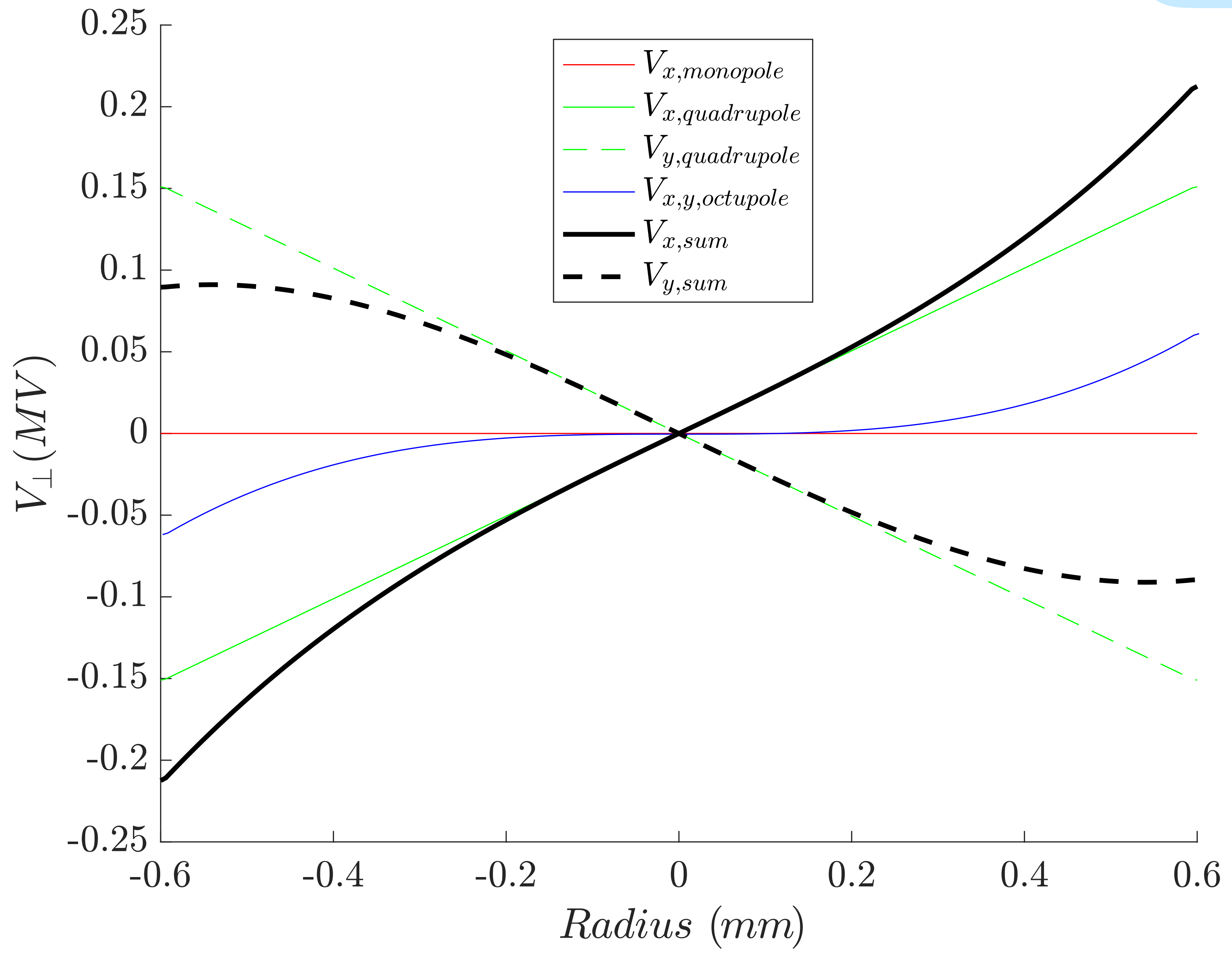
Fourier decomposition of TM_{11}

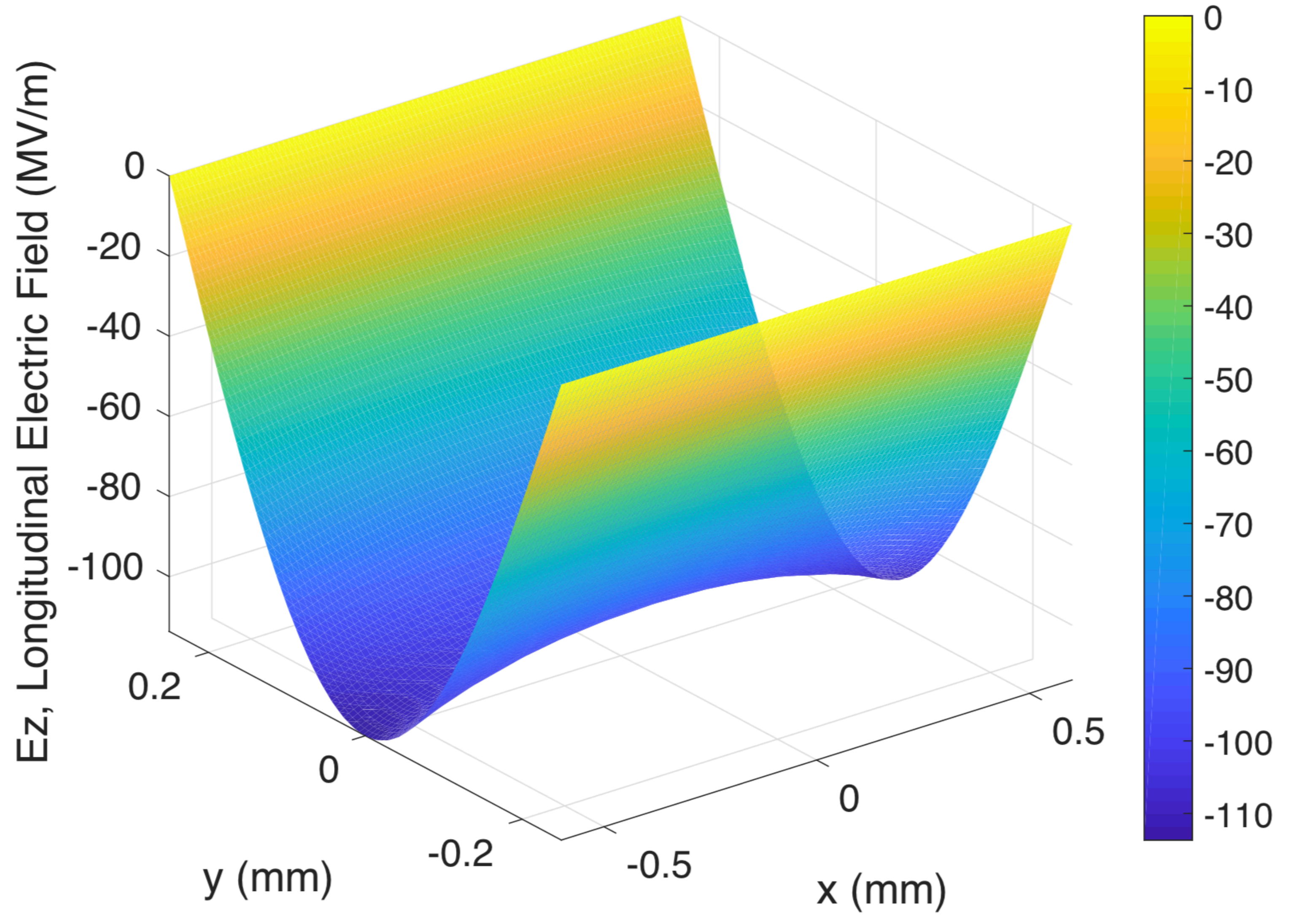
$$V_z(r,\theta) = -iA\beta L k_y^a \cos\left(\frac{\pi}{w} r\cos(\theta)\right)\cos(k_y^a r\sin(\theta))$$

Voltage induced on longitudinal axis due to TM_{11} in polar coordinates.

$$V_z(x,y) = \int_0^L E_z(x,y)dz = E_z(x,y)L$$







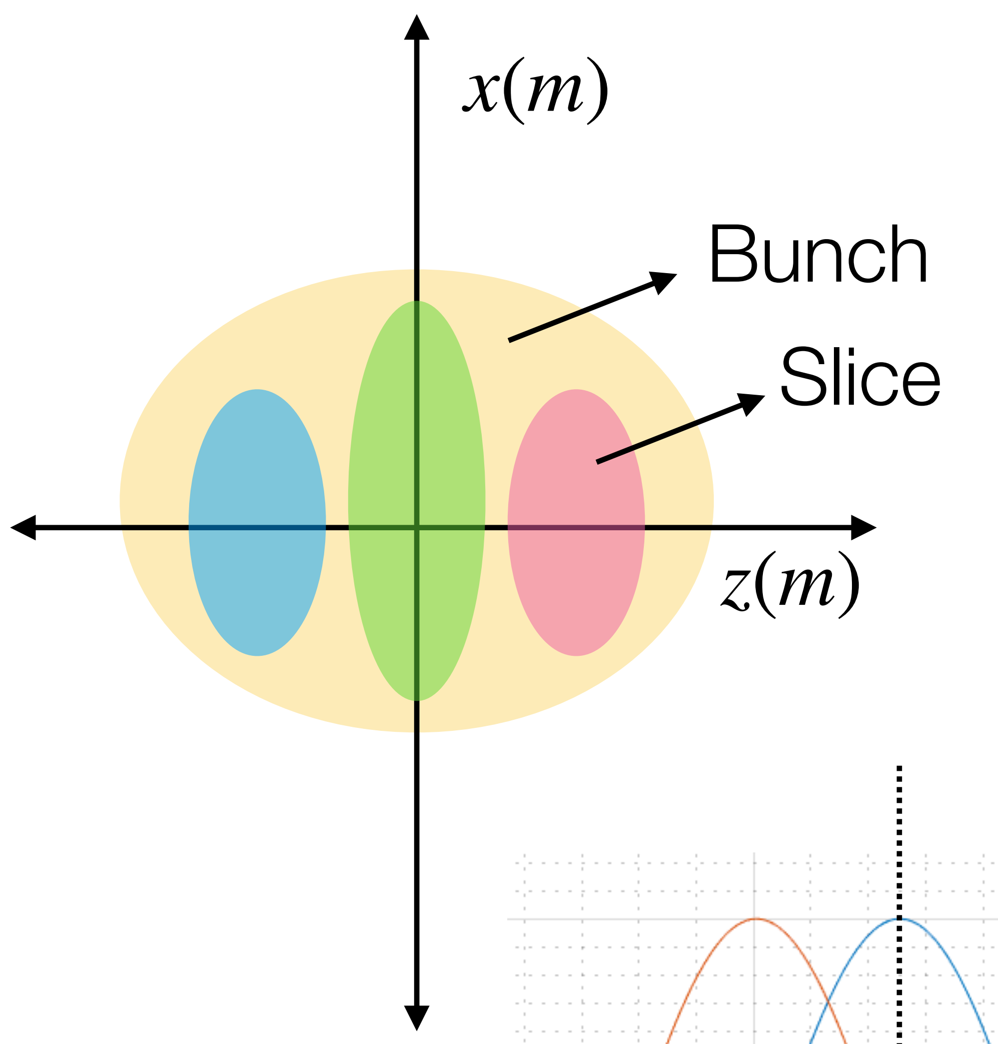
- Multipole components = (Fourier components) / V_0 (axial monopole field),
- Transverse voltage induced according to Panofsky-Wenzel theorem.

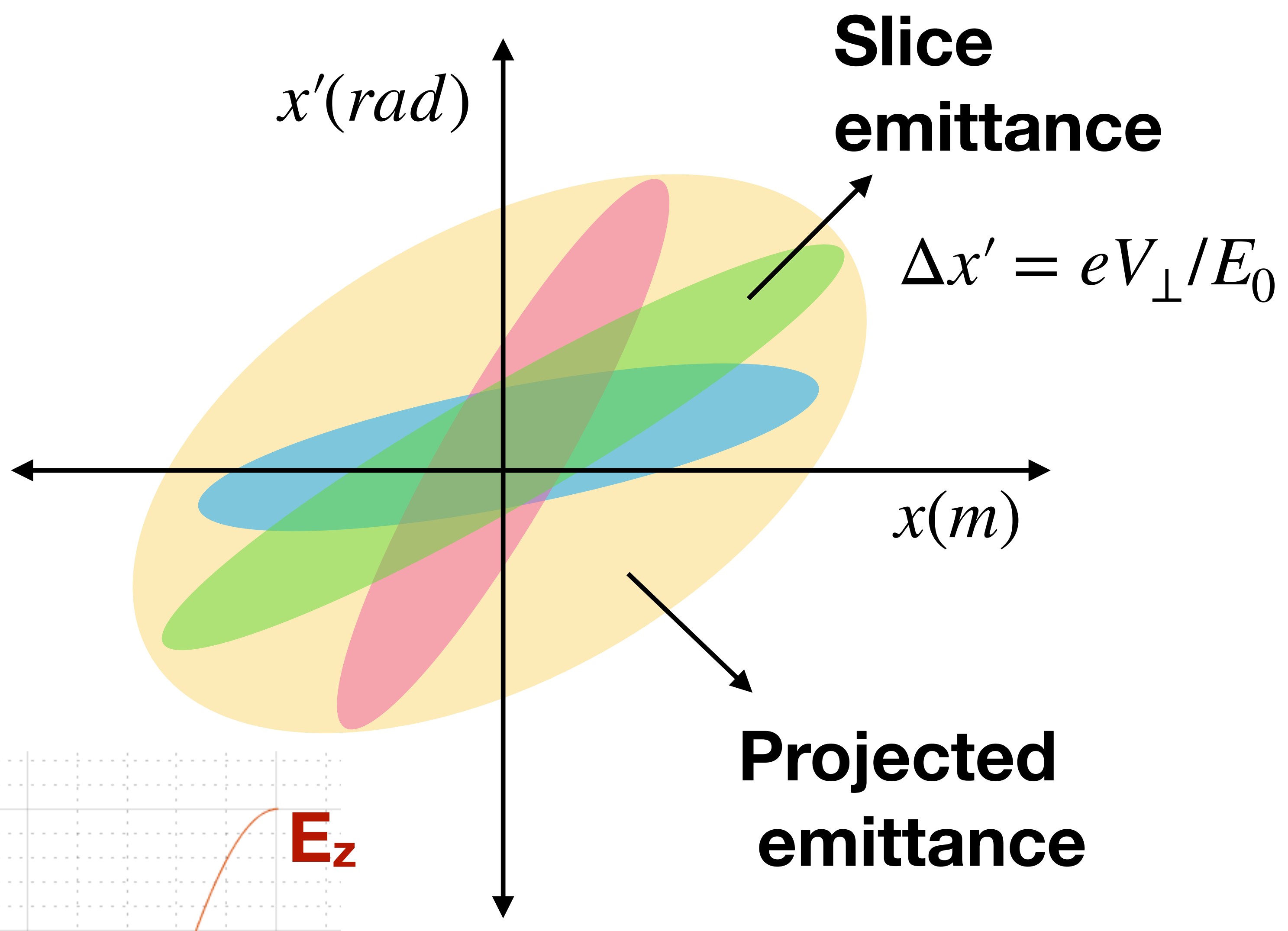
$$V_\perp = -\frac{ic}{\omega}\int_0^L dz\, \nabla_\perp E_z(z, z/c)$$

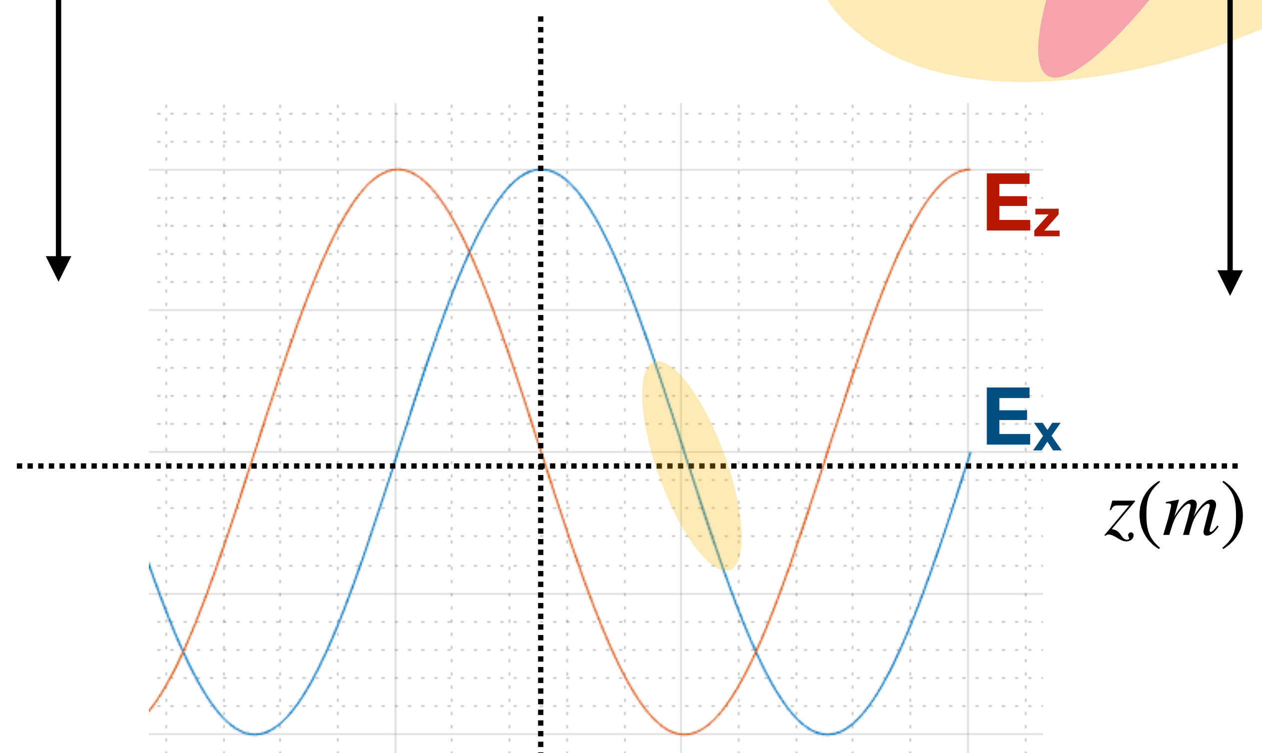
Transverse Voltage Seen by Particles

- Amplitude of the deflection voltage (DC) is related to slice emittance (ignore time dependence).
- No DC contribution from monopole components.
- Quadrupole: linear effect, no slice emittance growth (only phase space rotation) but small projected emittance growth.
- DC octupole component prominent for larger radii.
- Time dependent voltage (AC) introduces projected emittance increase.

$$V_{\perp}(x,y) sin(\omega \frac{z}{v_z})$$

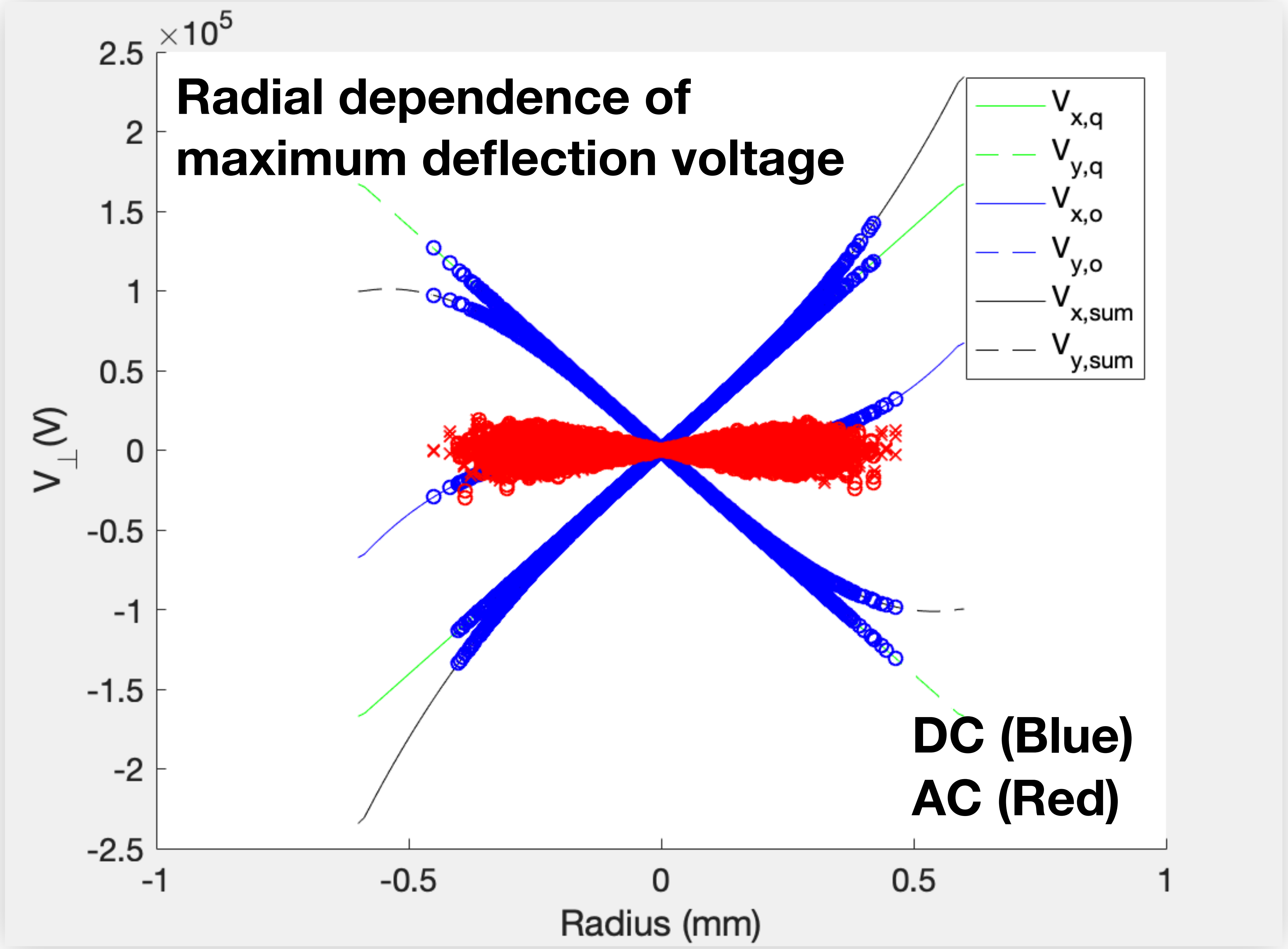






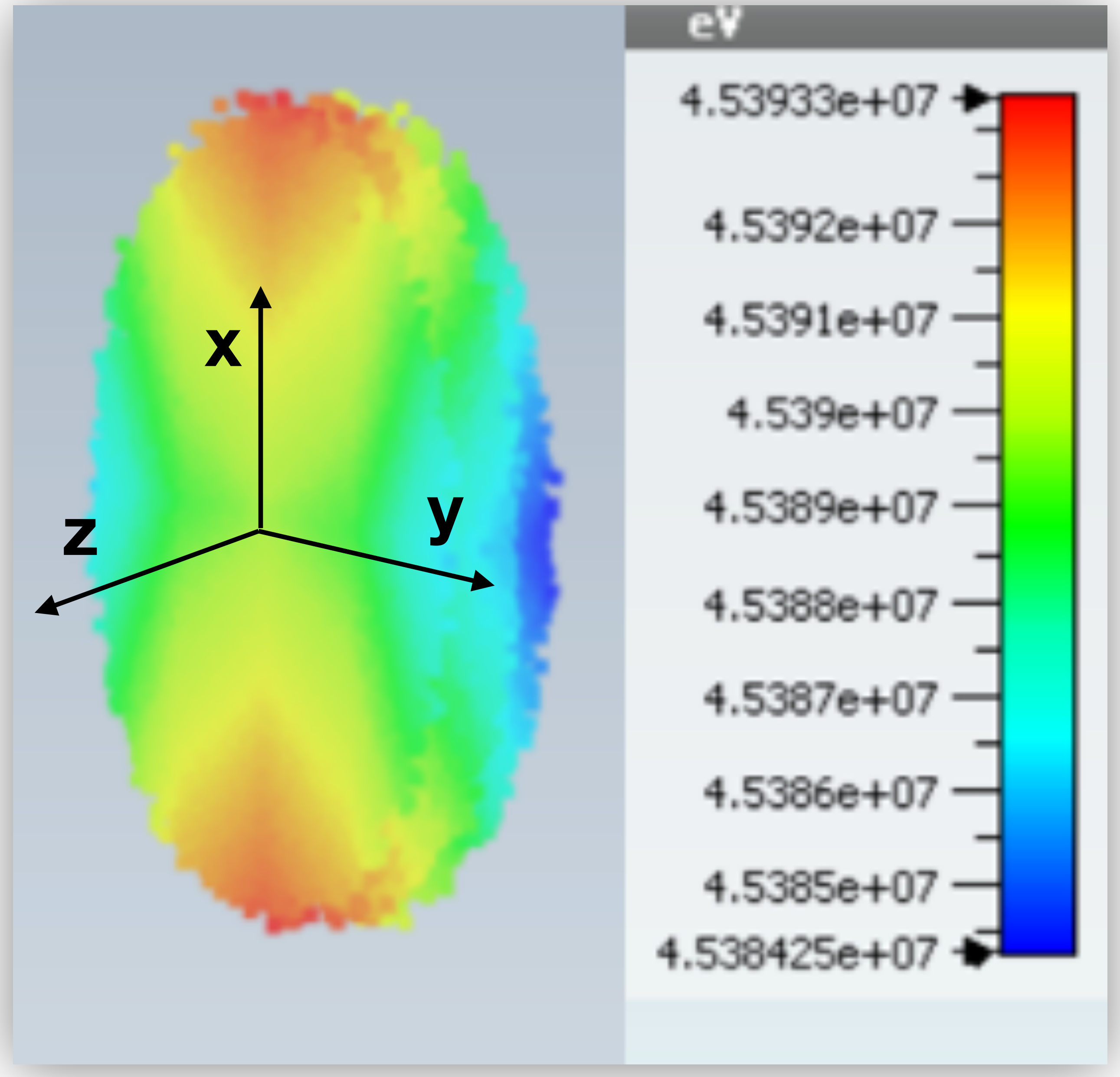
Max. deflection at min. acceleration

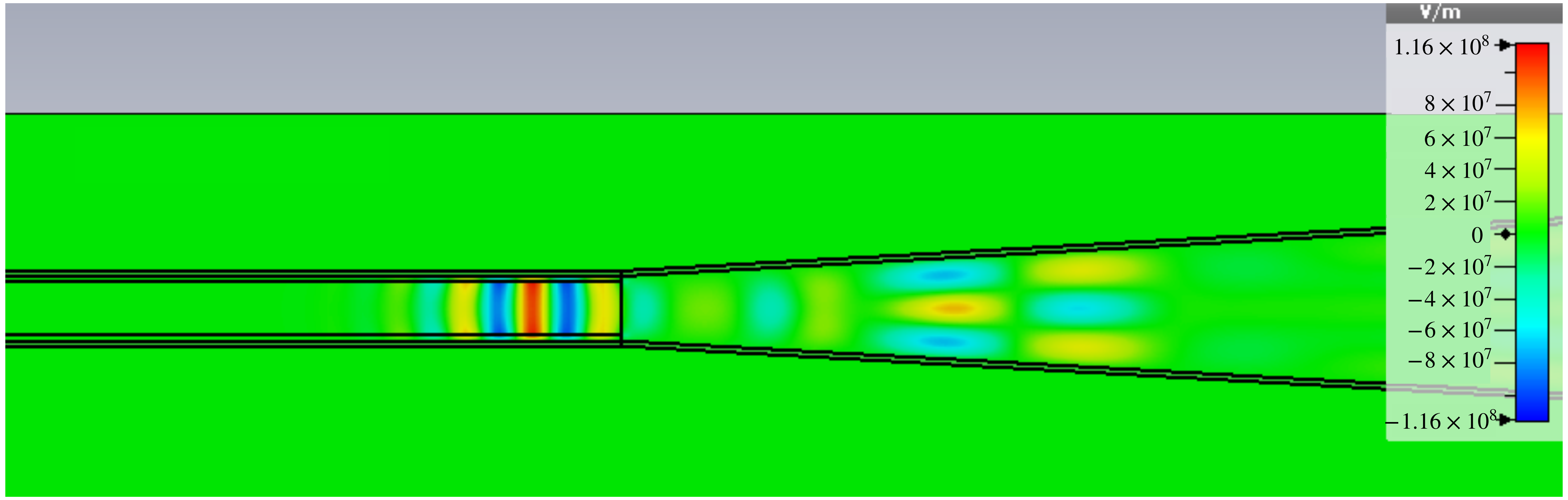
Particles see different deflection with respect to their location in the bunch.



Correlated Energy Spread and Correction

CST - Beam Real Space



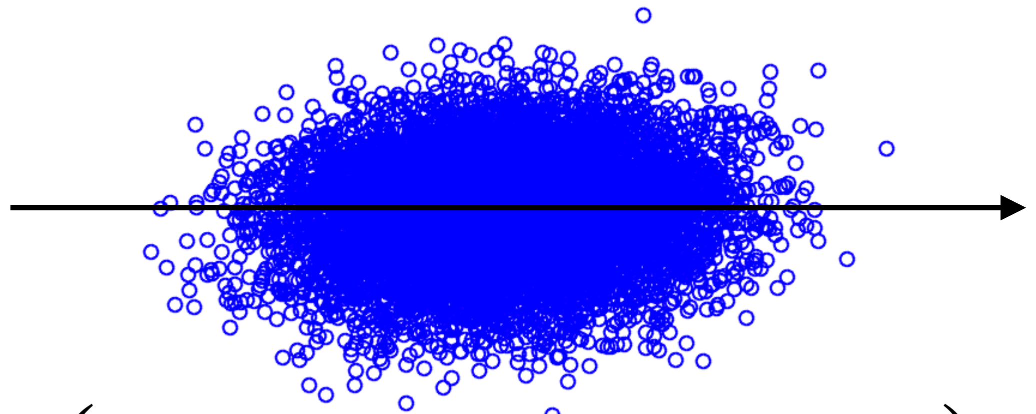


Initial parameter:
10um, 1mrad, 25fs, 45MeV
Energy gain:
390keV, spread 1.6keV

- Beam dynamics with CST, not ideal.
- Computationally heavy for systematic studies.
- Custom thin lens tracking code using multipole components calculated.

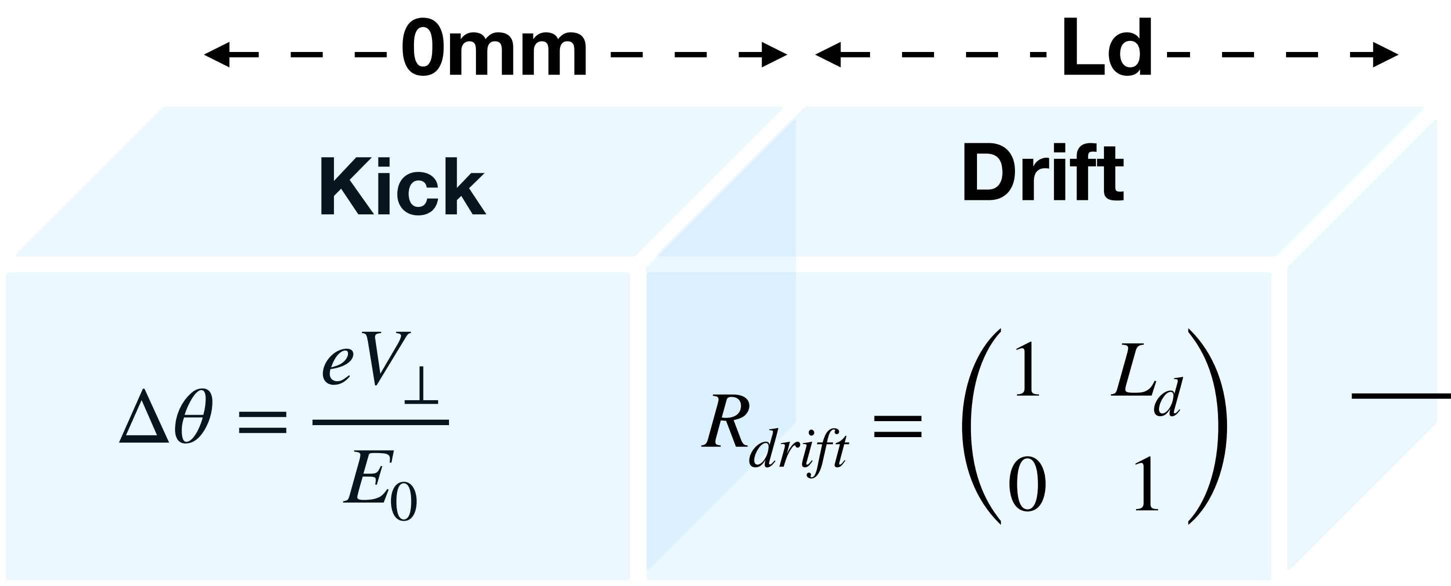
Thin Lens Tracking: Single DLW

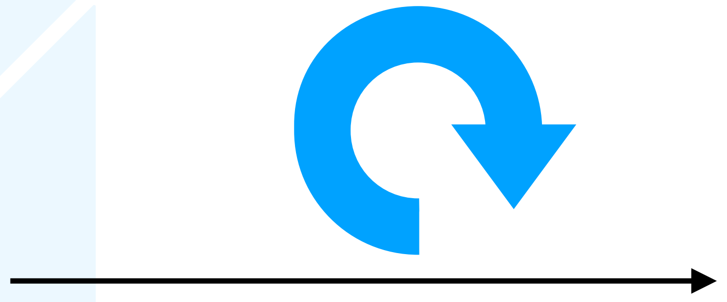
Incident beam



$(x_0, y_0, z_0, p_{x_0}, p_{y_0}, p_{z_0})$

Total interaction length, 4mm





Multiple slices of kick-drift

Apply deflection due to $V_\perp$

$\theta_{kick} = \theta_0 + \Delta\theta$

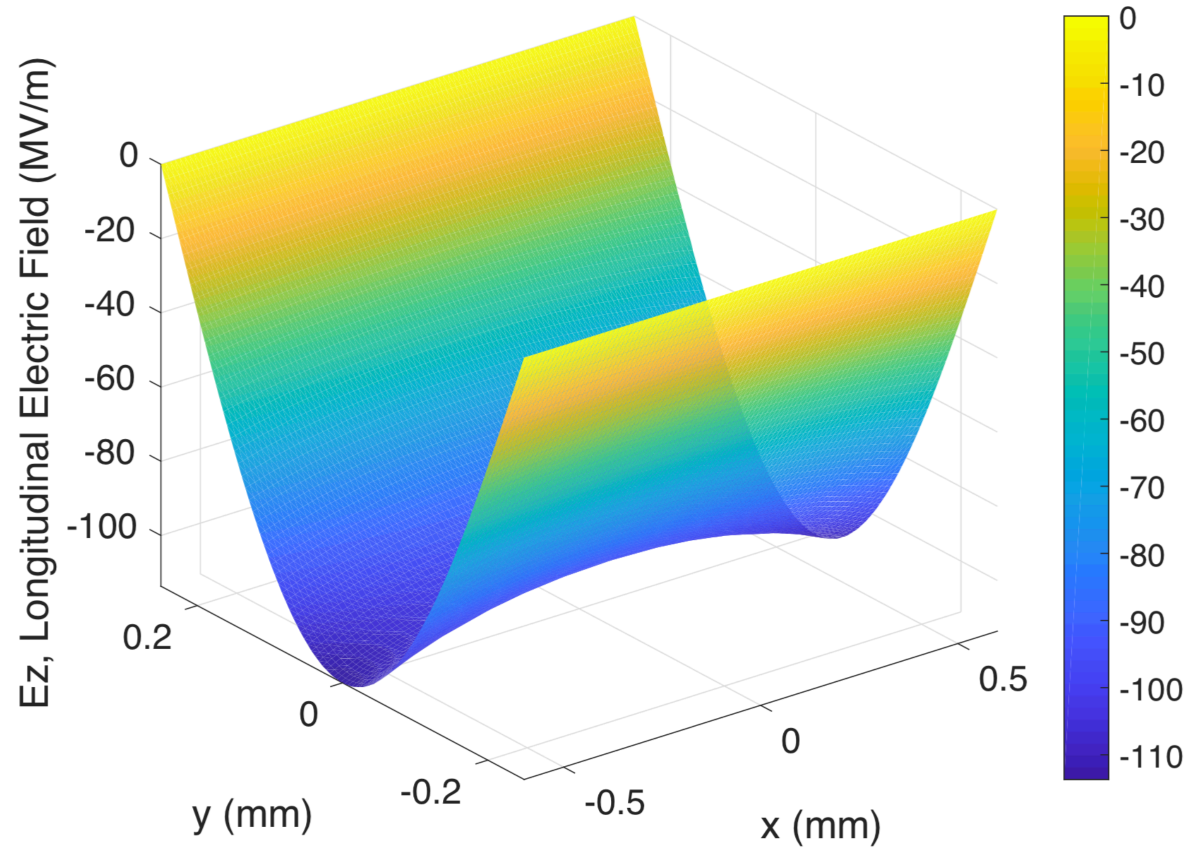
$\Delta\theta = eV_\perp / E_0$

Update positions

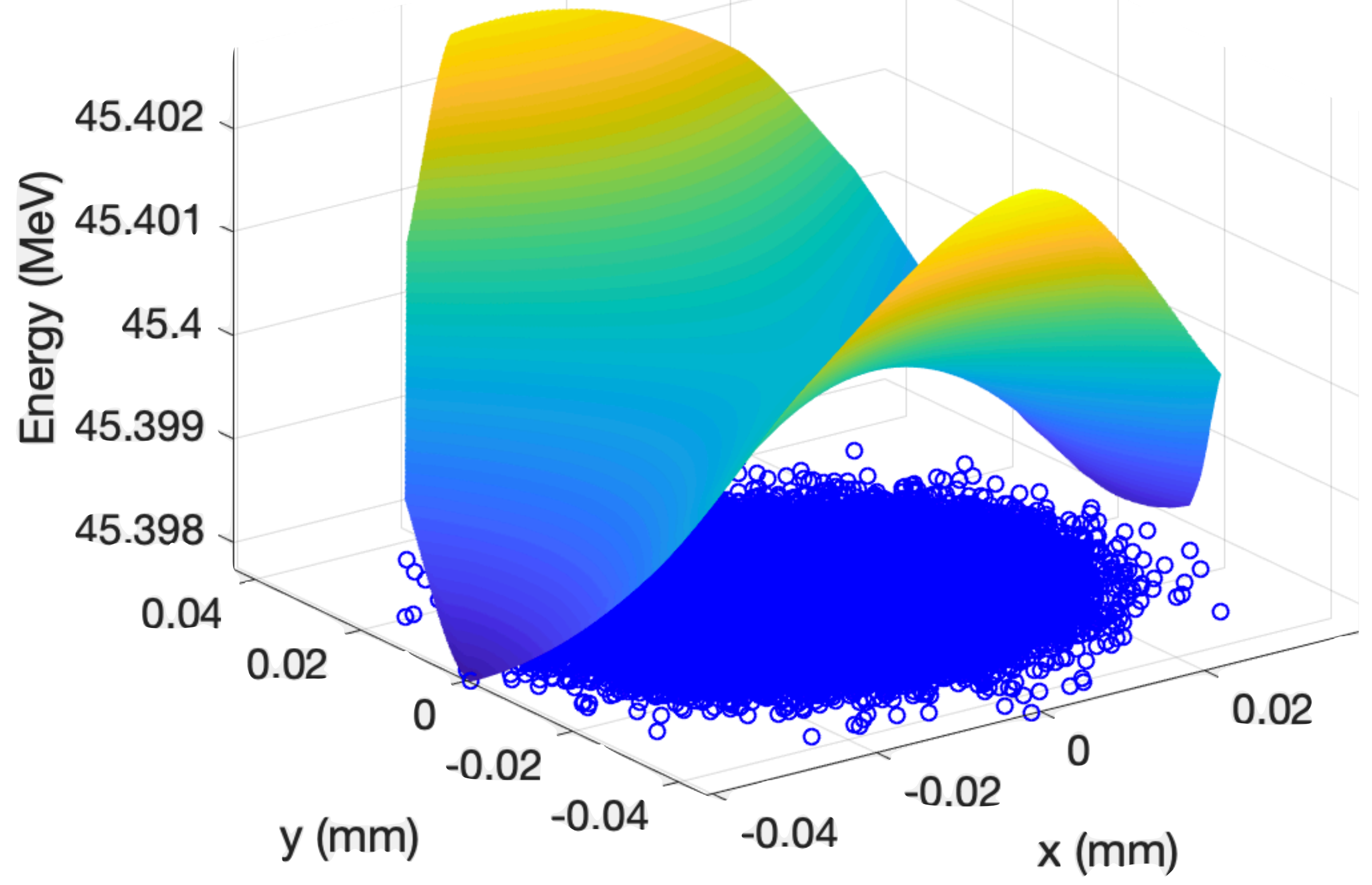
$x_{drift,2} = x_{drift} + \theta_{kick} L_d$

$\theta_{drift} = \theta_{kick}$

Particles at x-y plane



Energy distribution at x-y plane

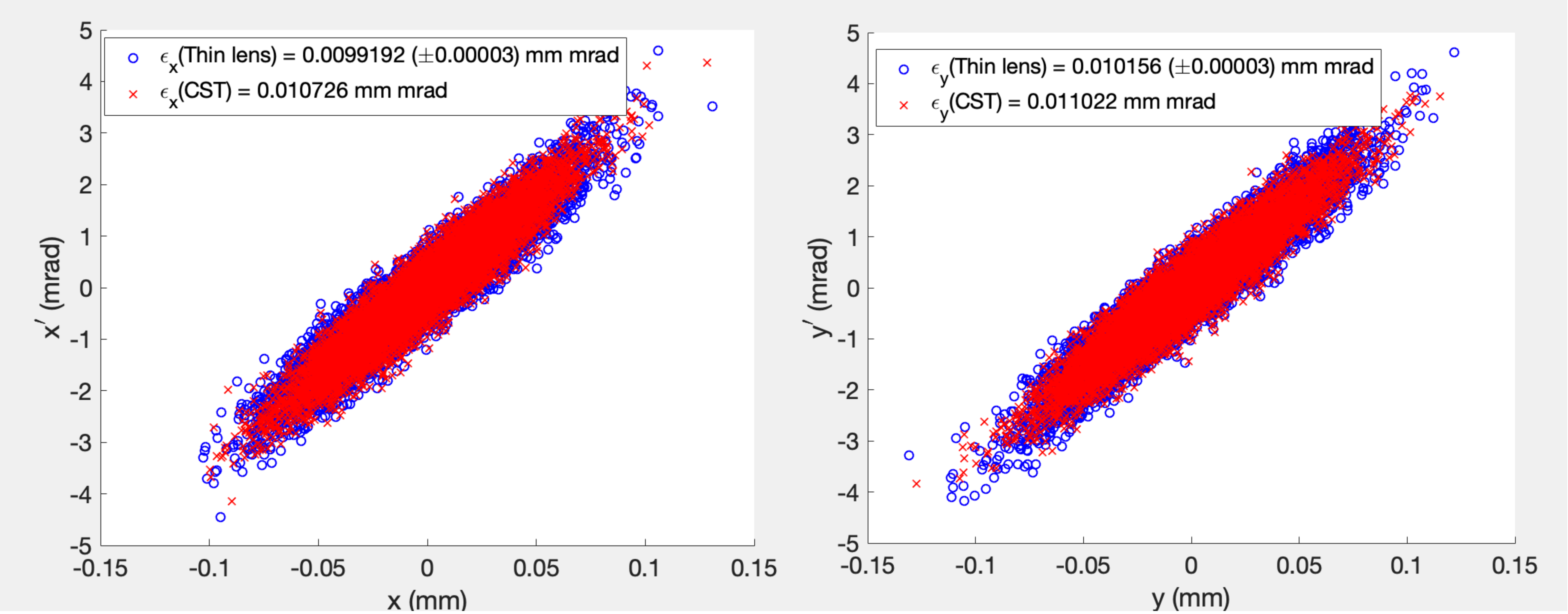


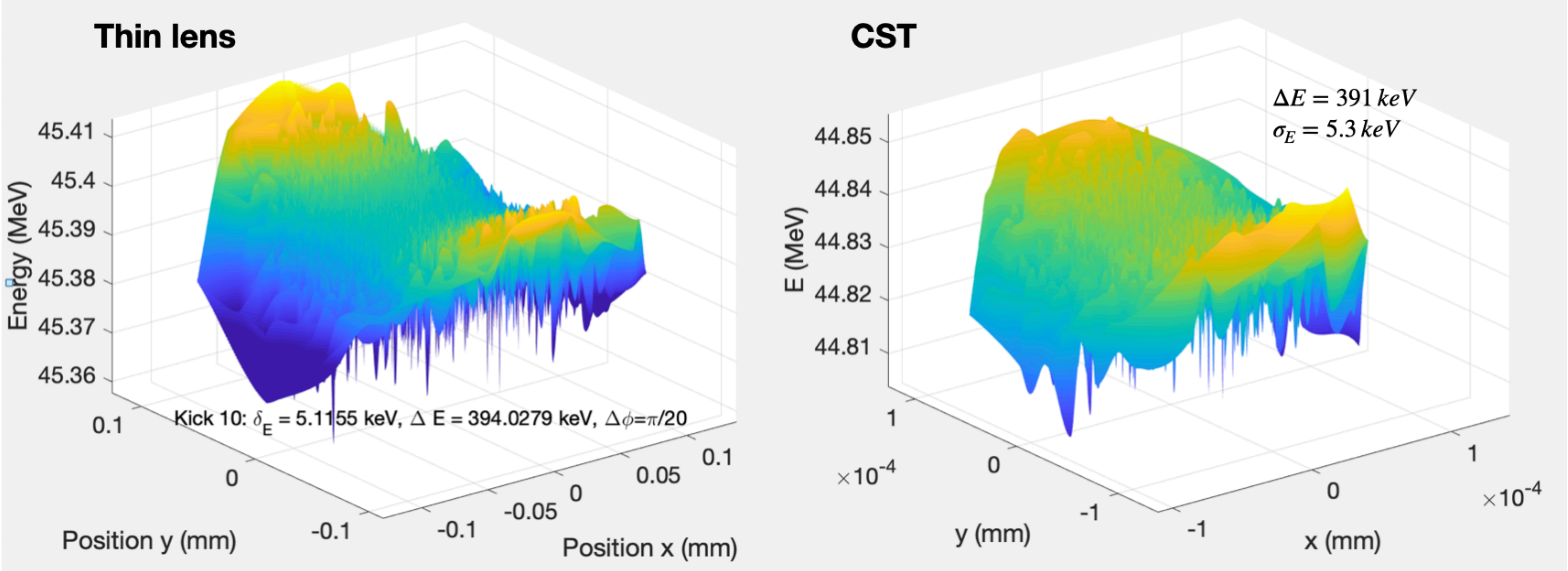
E_0 (MeV)	45
Trans. emittance (mm mrad)	0.01
Beam Size (1σ, mm)	10
Divergence (1σ, mrad)	1
Bunch length (fs)	25
Accelerating Field (MV/m)	100
Max. Energy Gain (keV)	400

- **Interaction length is assumed to be 4 mm.**
- Multi step tracking:
  - Thin lens kick (zero-length) due to $V_\perp$ applied,
  - Phase space tracked for the slice length.
  - Same repeated N time with **linearly decreased kick amplitude**.
- Quadrupolar transverse energy distribution after single DLW

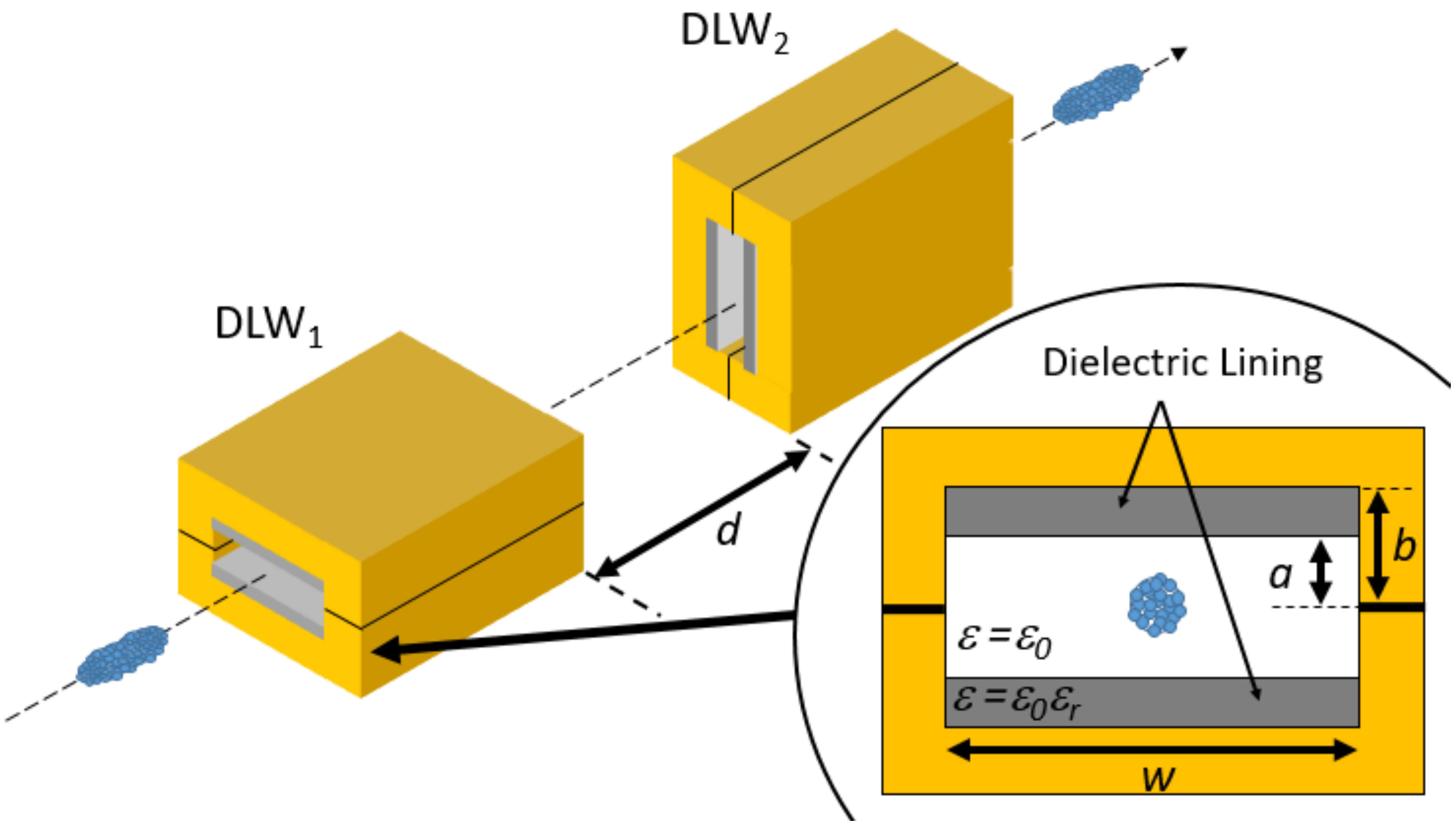
Benchmarking against CST

- Energy gain: 400 keV, 10um, 1mrad and 45 MeV beam total energy.
- **Identical distribution** imported both in CST and custom thin lens tracking.
- In both case beam drifted across the horn until interaction (4 mm co-propagation).
- CST presents kinetic energy rather than total energy.
- Remnant fields (~5kV/m) induced in the horn causing minor energy errors.
- **Transverse phase space is benchmarked** within statistical tracking errors (0.3% for 100k particles.)
- **Energy distribution in the transverse plane is benchmarked** against CST: 5.2 keV energy spread and ~400keV energy gain.



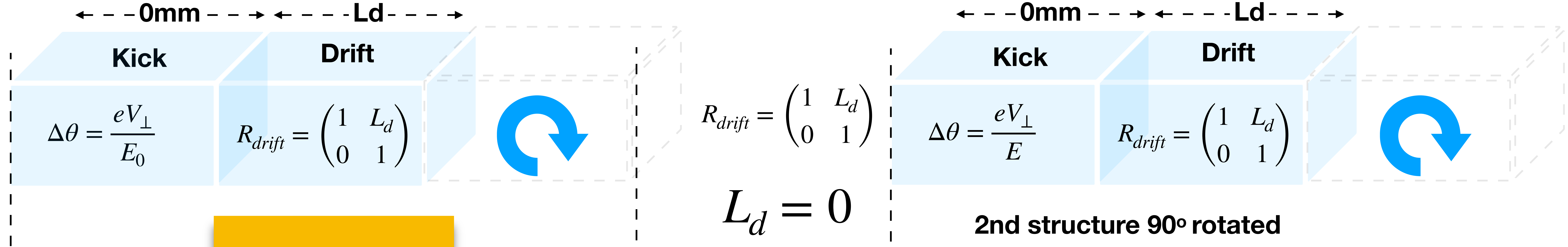


Can we correct the correlated energy distribution using orthogonally staged DLWs?



Multi-Stage Case 1: Longitudinal field cancellation with zero drift

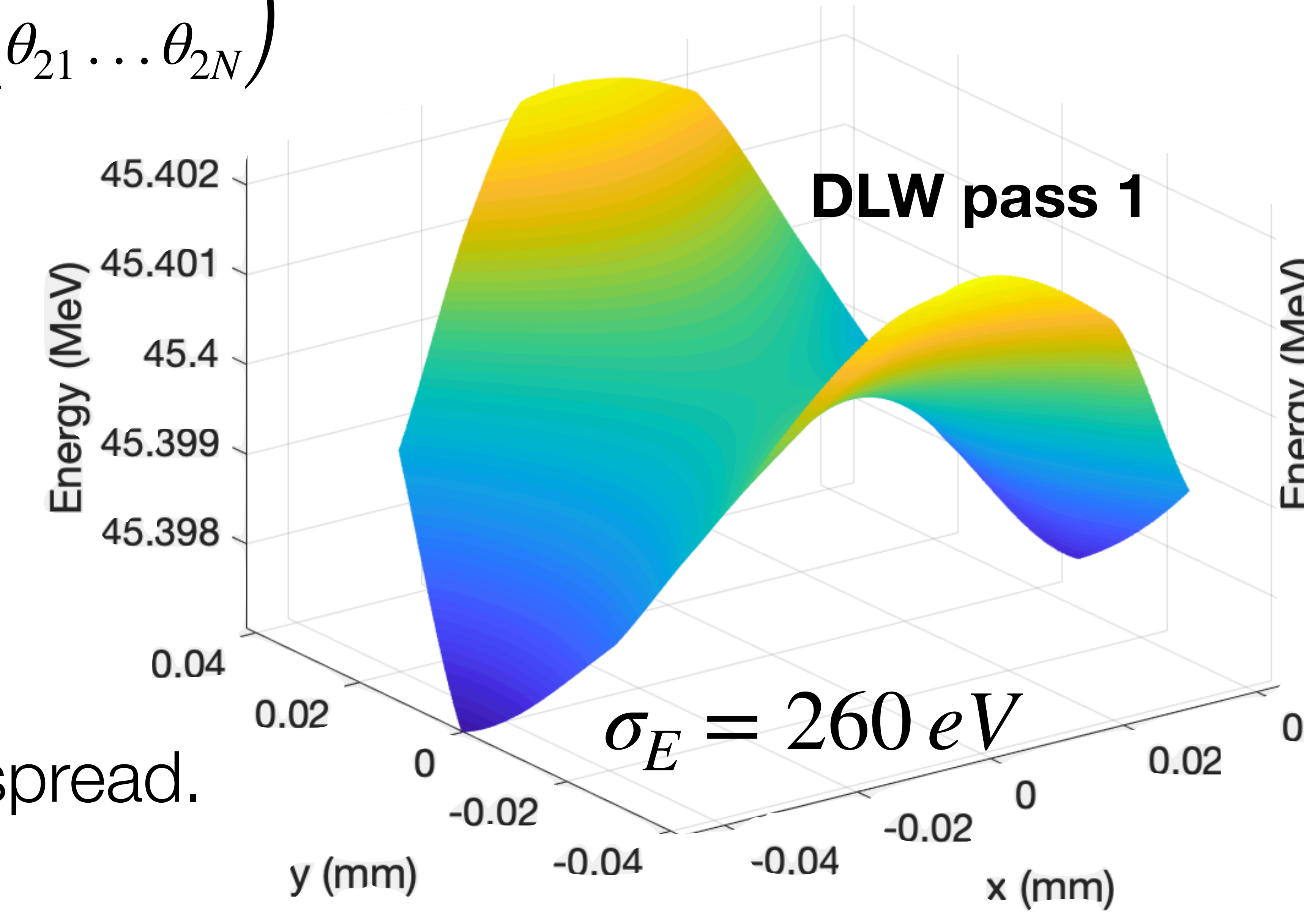
The First Stage / Drift Section / The Second Stage

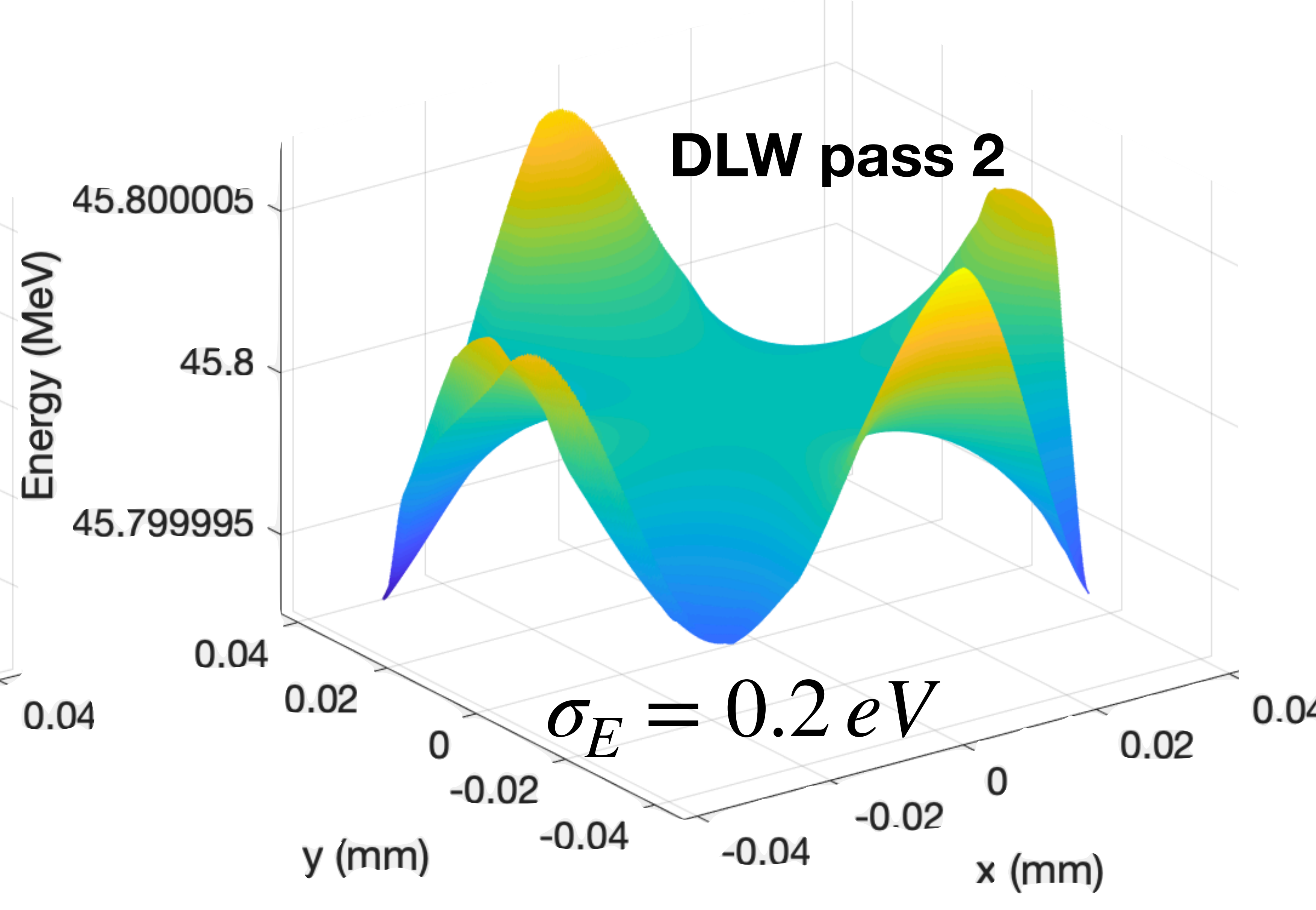


2nd structure 90º rotated

$\begin{pmatrix} x_{11} \ldots x_{1N} \\ \theta_{11} \ldots \theta_{1N} \end{pmatrix}$ $\begin{pmatrix} x_{21} \ldots x_{2N} \\ \theta_{21} \ldots \theta_{2N} \end{pmatrix}$

ΔE_z = 400 keV
$\sigma_x = 10\mu m$
$\sigma'_x = 1mrad$
$\sigma_z = 25fs$
$N = 100k$
$\sigma_{E,uncorr} = 0$



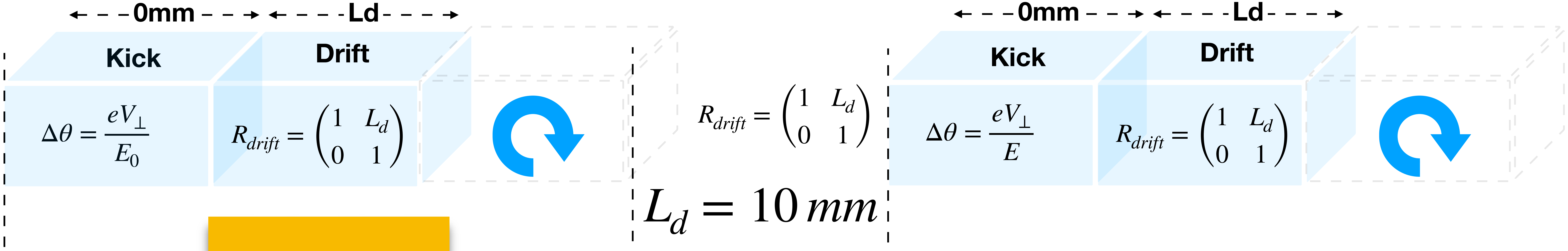


- Double pass scheme
- with **no uncorrelated** initial energy spread.
- Quadrupole terms cancel out.
- Very weak higher order effect (octupole).

For infinitesimal bunch length.

Multi-Stage Case 2: introduce a drift between stages

The First Stage | **Drift Section** | **The Second Stage**



$$\begin{pmatrix} x_{11} \dots x_{1N} \\ \theta_{11} \dots \theta_{1N} \end{pmatrix} \qquad \begin{pmatrix} x_{21} \dots x_{2N} \\ \theta_{21} \dots \theta_{2N} \end{pmatrix}$$

ΔE_z = 400 keV
$\sigma_x = 10\mu m$
$\sigma'_x = 1mrad$
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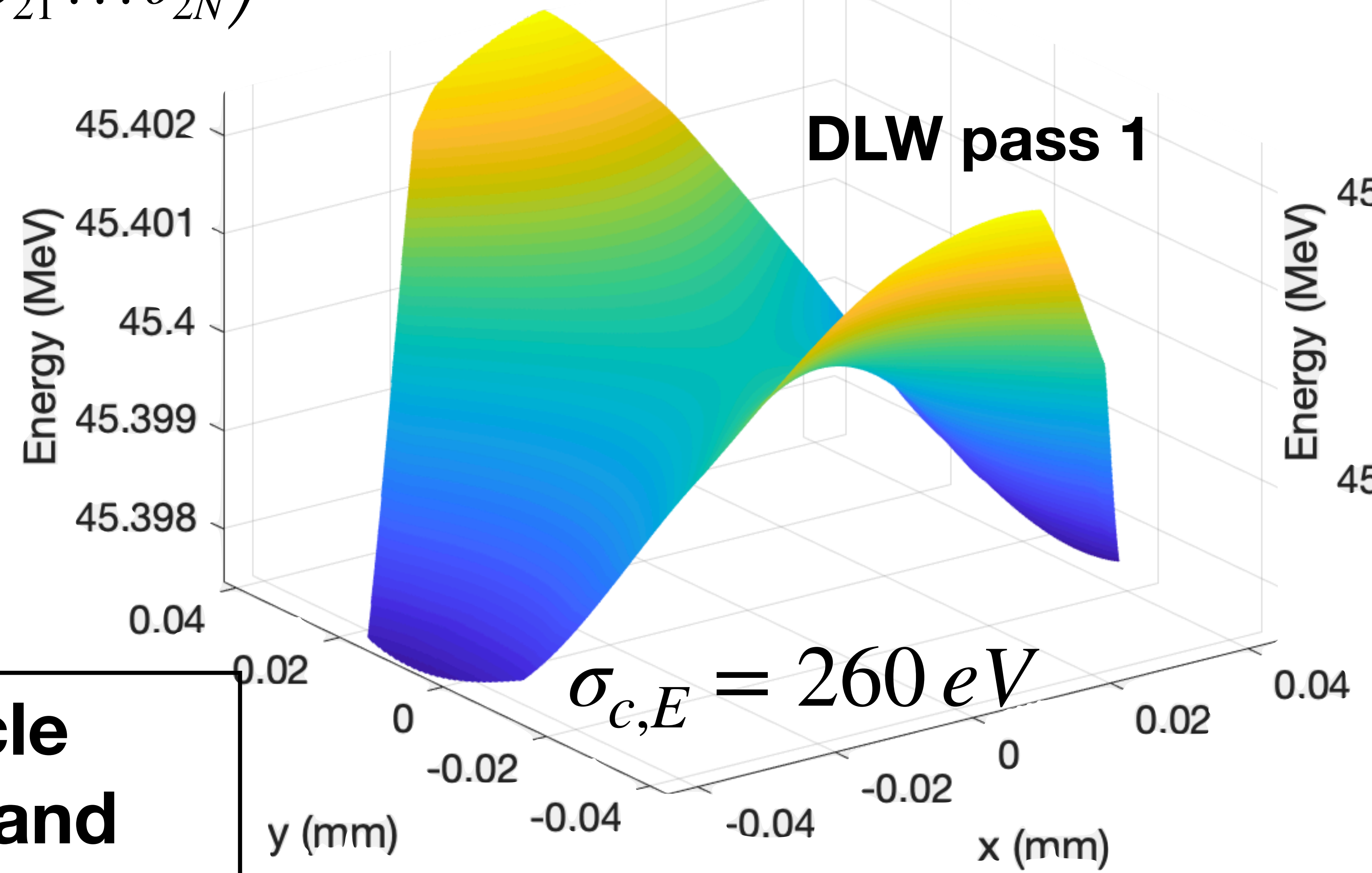
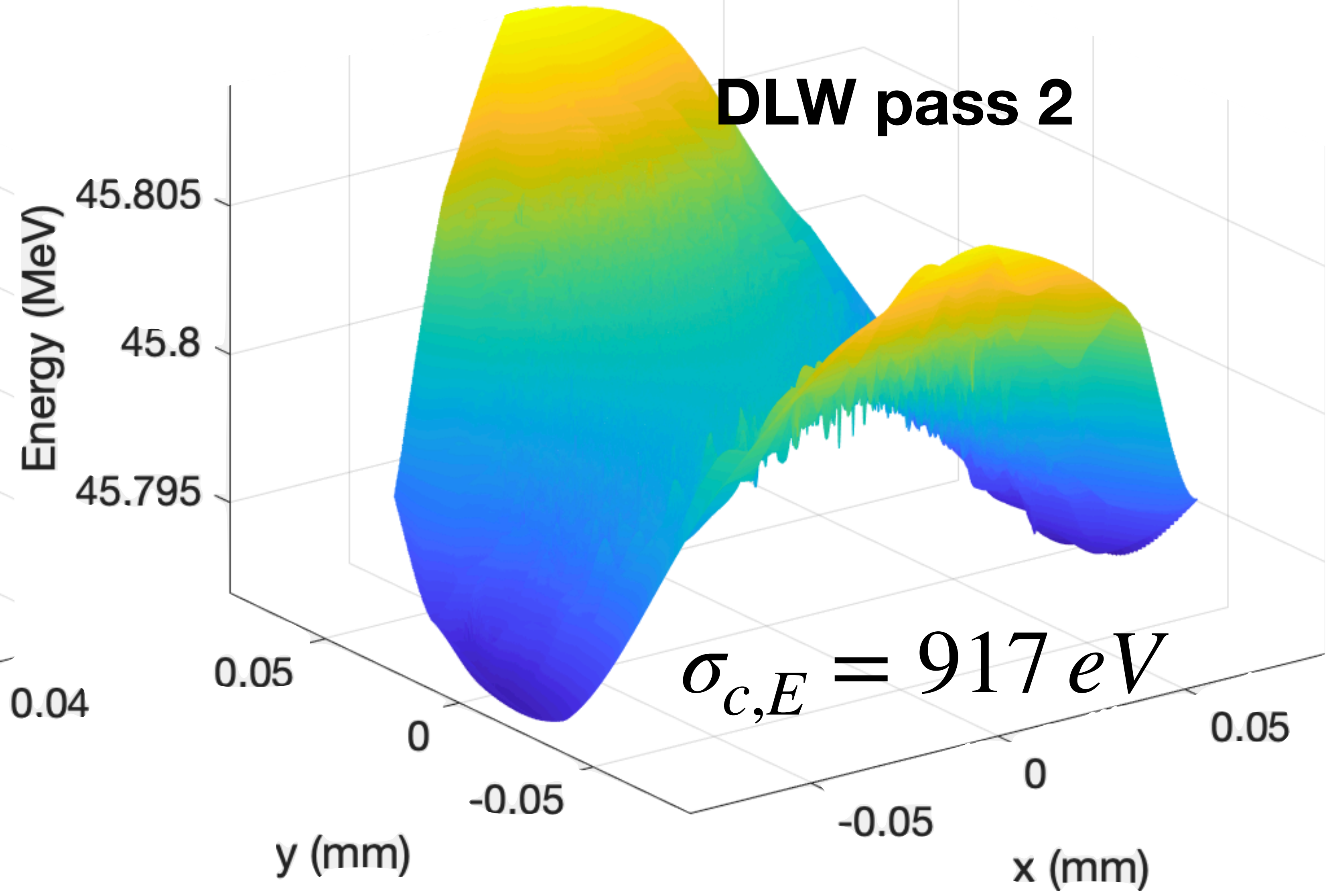
$N = 100k$
$\sigma_{E,uncorr} = 0$





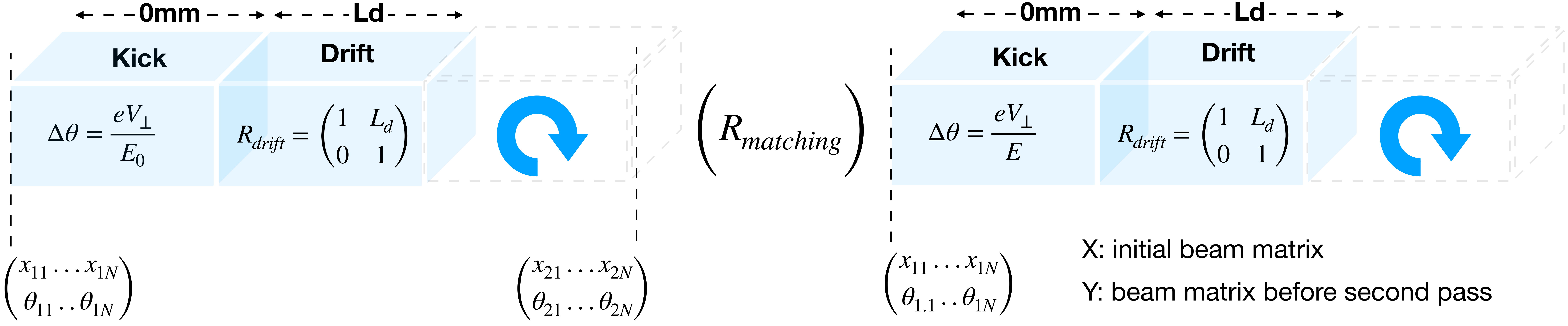
Energy spread induced as particle positions diverge during the drift and experience larger off-axis quadrupole field.

For infinitesimal bunch length.

Solution: One needs to match the positions of particles before second pass to their initial positions before the first pass.

Multi-Stage Case 3: Matching

The First Stage | **Drift Section** | **The Second Stage**



The least square fit to a system of linear equations.

$$\begin{pmatrix} x_{21} \dots x_{2N} \\ \theta_{21} \dots \theta_{2N} \end{pmatrix} = R_{DLW} \begin{pmatrix} x_{11} \dots x_{1N} \\ \theta_{1.1} \dots \theta_{1N} \end{pmatrix}$$

$$R_{matching} = R_{DLW}^{-1}$$

Calculate matching matrix

$$Y = R_{DLW} X$$

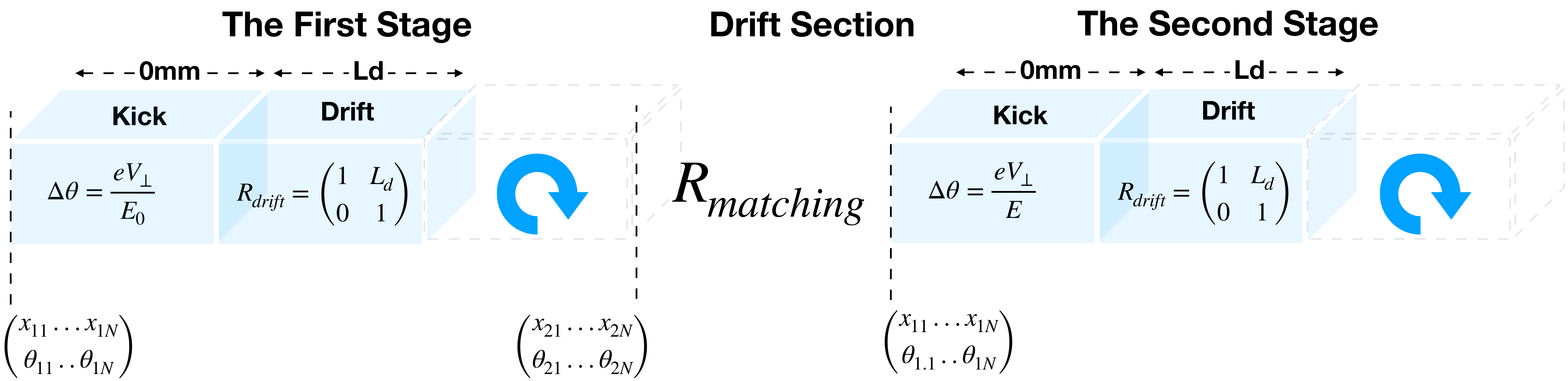
$$YX^T = R_{DLW} XX^T$$

$$YX^T(XX^T) = R_{DLW} XX^T(XX^T)$$

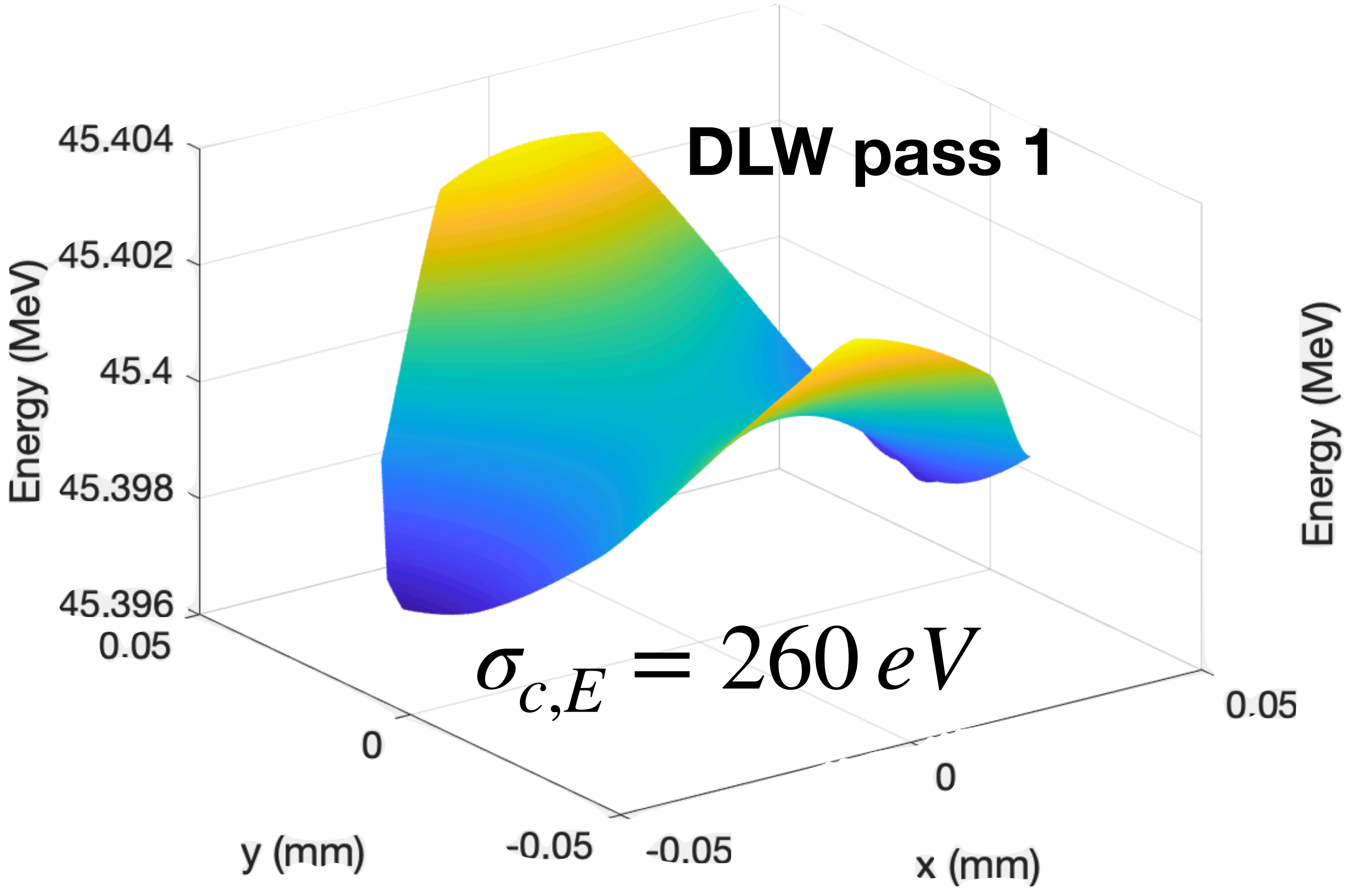
$$YX^+ = R_{DLW}$$

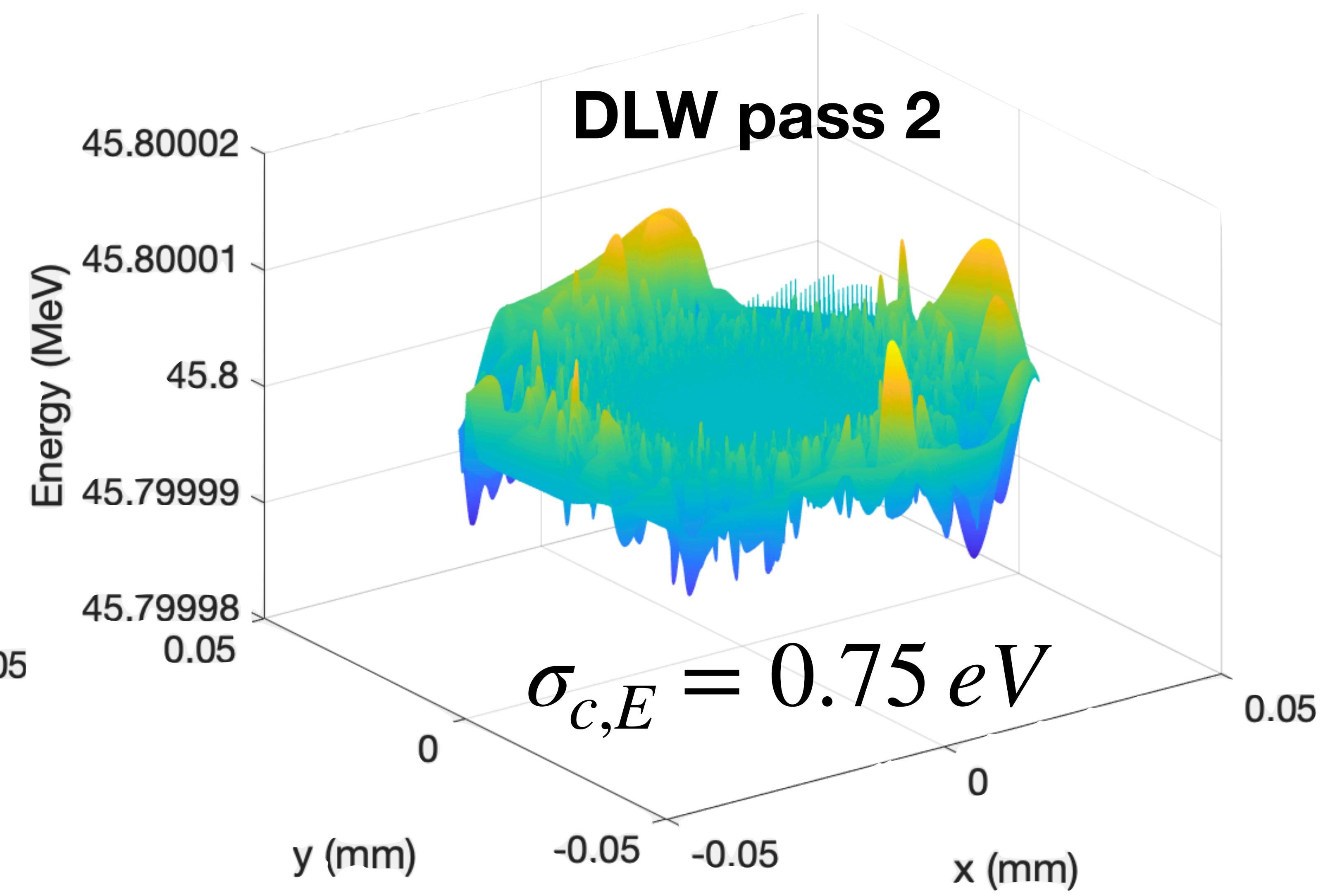
Moore-Penrose pseudo-inverse

Multi-Stage Case 3: Matching



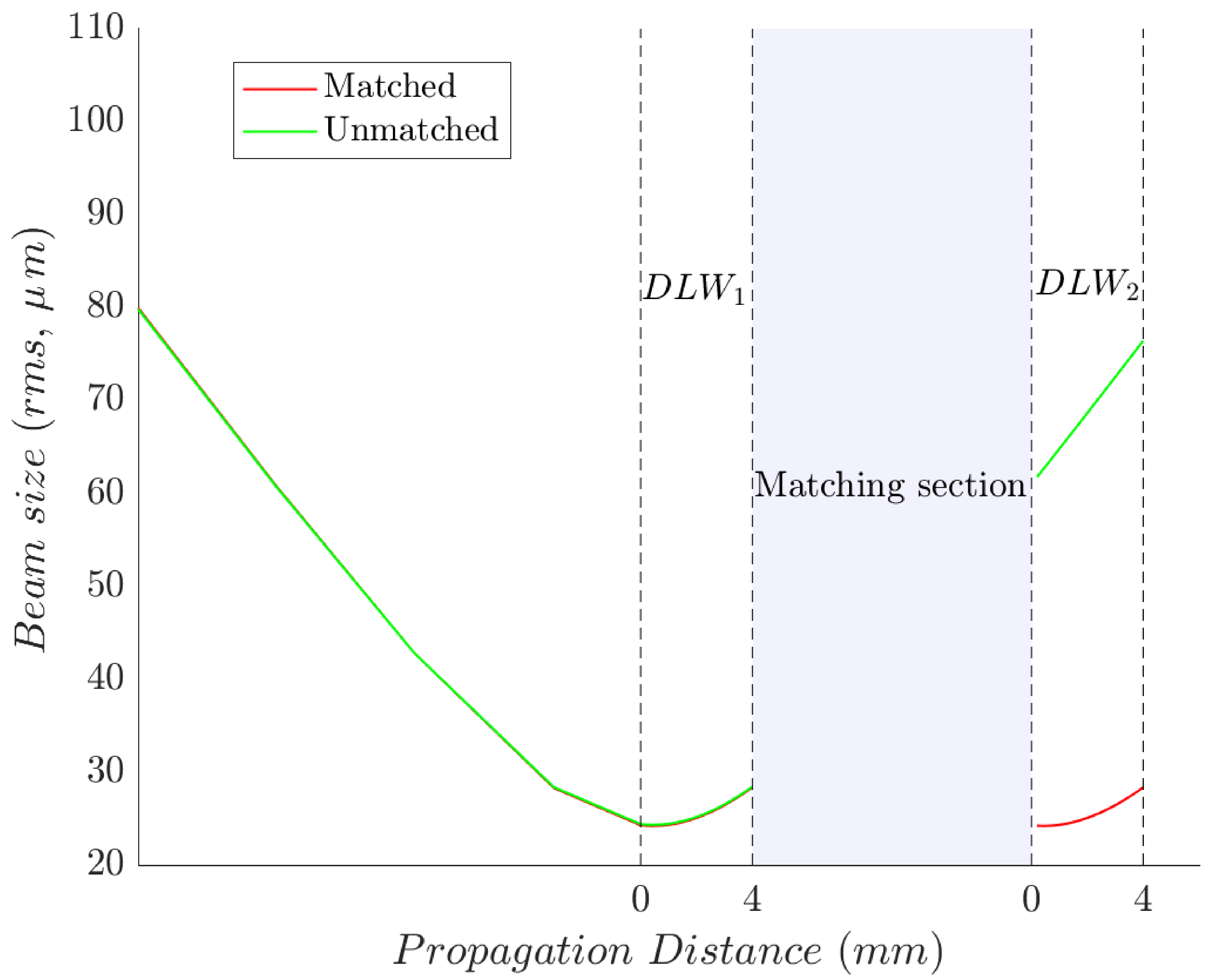
- Double pass scheme with **no uncorrelated** initial energy spread.
- Point to point optics, reconstructs initial particle positions at the DLW2 entrance.
- Correlated energy spread induced by transverse effects of DLWs are compensated with matching optics.

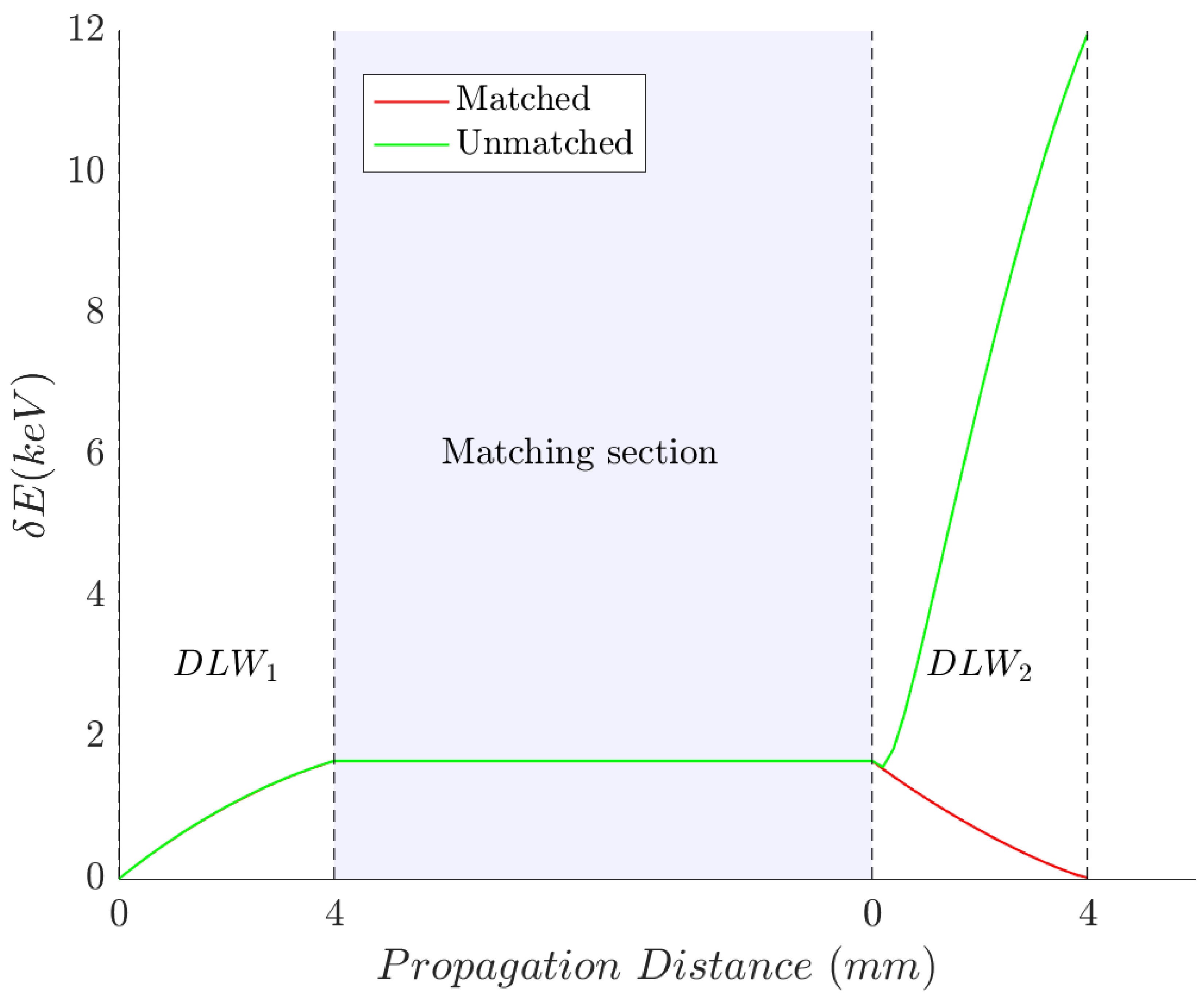




For infinitesimal bunch length.

Impact of matching section on energy spread





- Envelope evolution for the matched and unmatched beam.
- The correlated energy spread can be fully corrected.

Is this too good to be true?

Is suggested matching transfer matrix physical?

A physical solution: non-periodic transfer matrix

Hill's equation defines the particle trajectories under periodic focusing. Solution to Hill's equation and its derivative represents the rms position and angle of the beam. This can be arranged into a non-periodic matrix that can be computed between arbitrary points of "0" and "s".

$$M = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}(cos\psi_s + \alpha_0 sin\psi_s) & \sqrt{\beta_s\beta_0}sin\psi_s \\ \frac{1}{\sqrt{\beta_s\beta_0}}\Big((\alpha_0 - \alpha_s)cos\psi_s - (1+\alpha_0\alpha_s)sin\psi_s\Big) & \sqrt{\frac{\beta_0}{\beta_s}}(cos\psi_s - \alpha_s sin\psi_s) \end{pmatrix}$$

$(\alpha_0, \beta_0, \gamma_0)$ ⟶ Twiss parameters that define **initial** beam distribution in (x, θ) space (phase space).

$(\alpha_s, \beta_s, \gamma_s)$ ⟶ Twiss parameters that define **final** beam distribution in (x, θ) space (phase space).

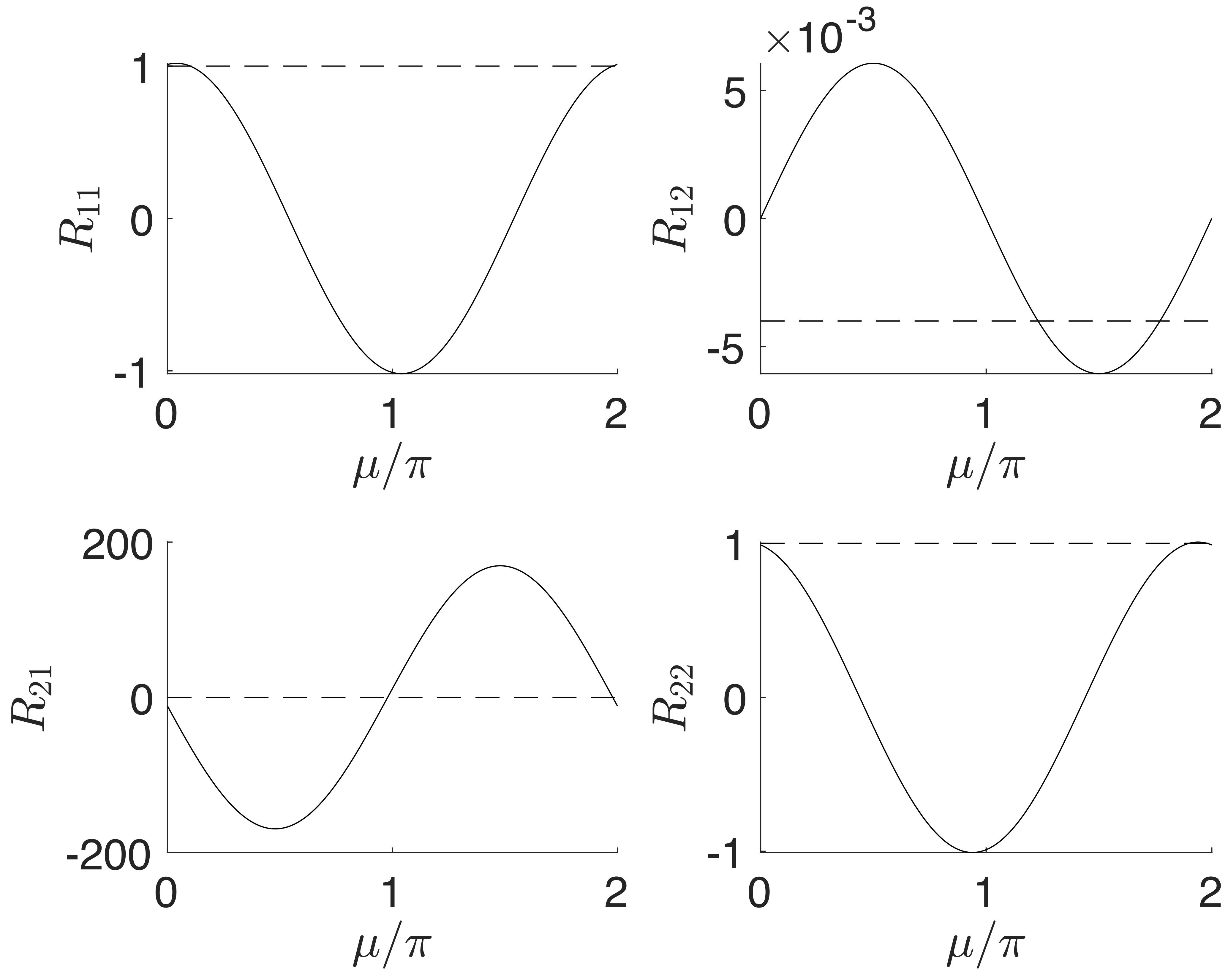
ψ_s ⟶ Rms phase rotation that beam undergone during the propagation through a certain segment.

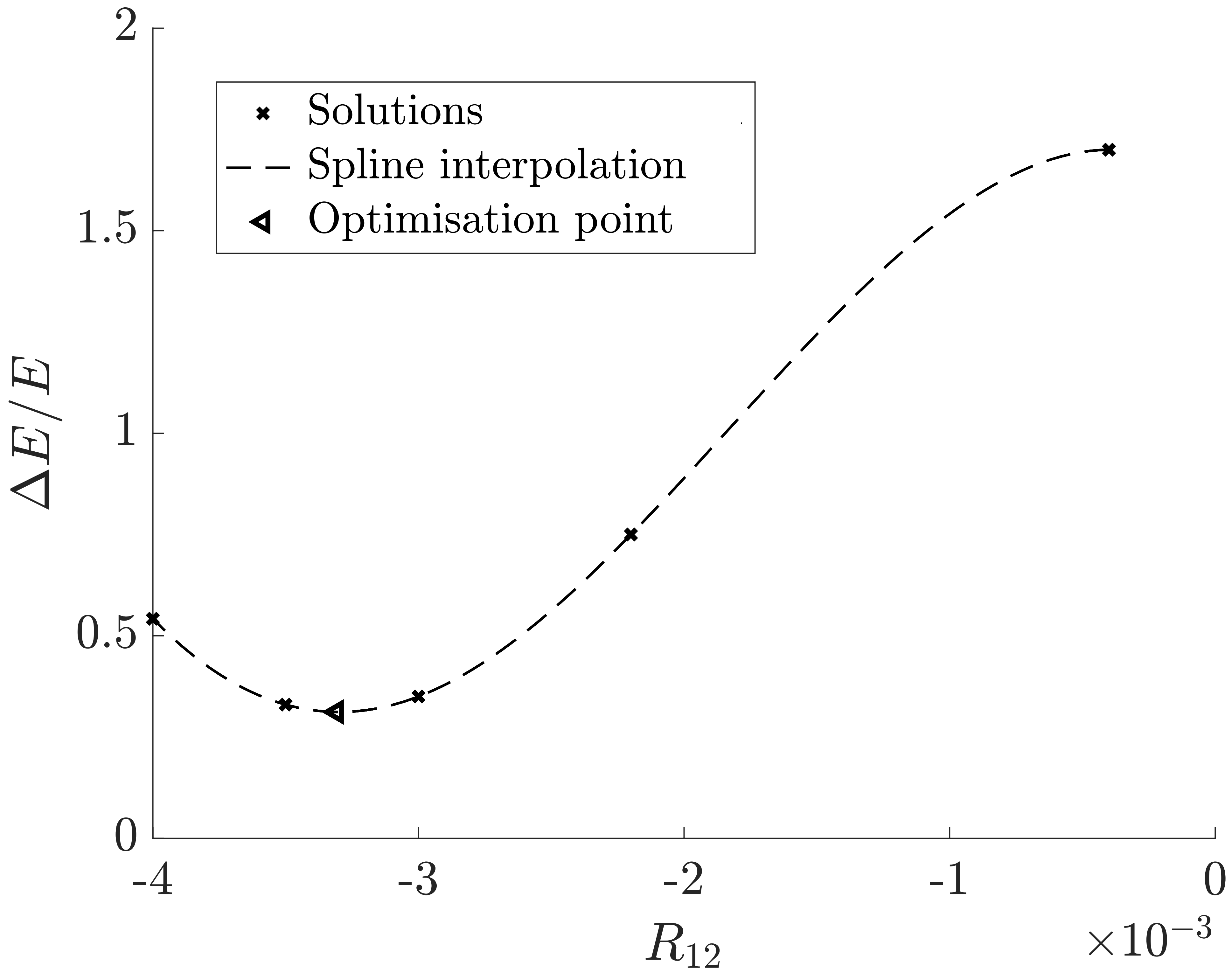
Any physical transformation should satisfy the non-periodic transfer matrix.

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle x'x \rangle & \langle x'^2 \rangle \end{pmatrix} = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} \epsilon$$

Twiss parameters can be computed using rms values given in beam matrix.

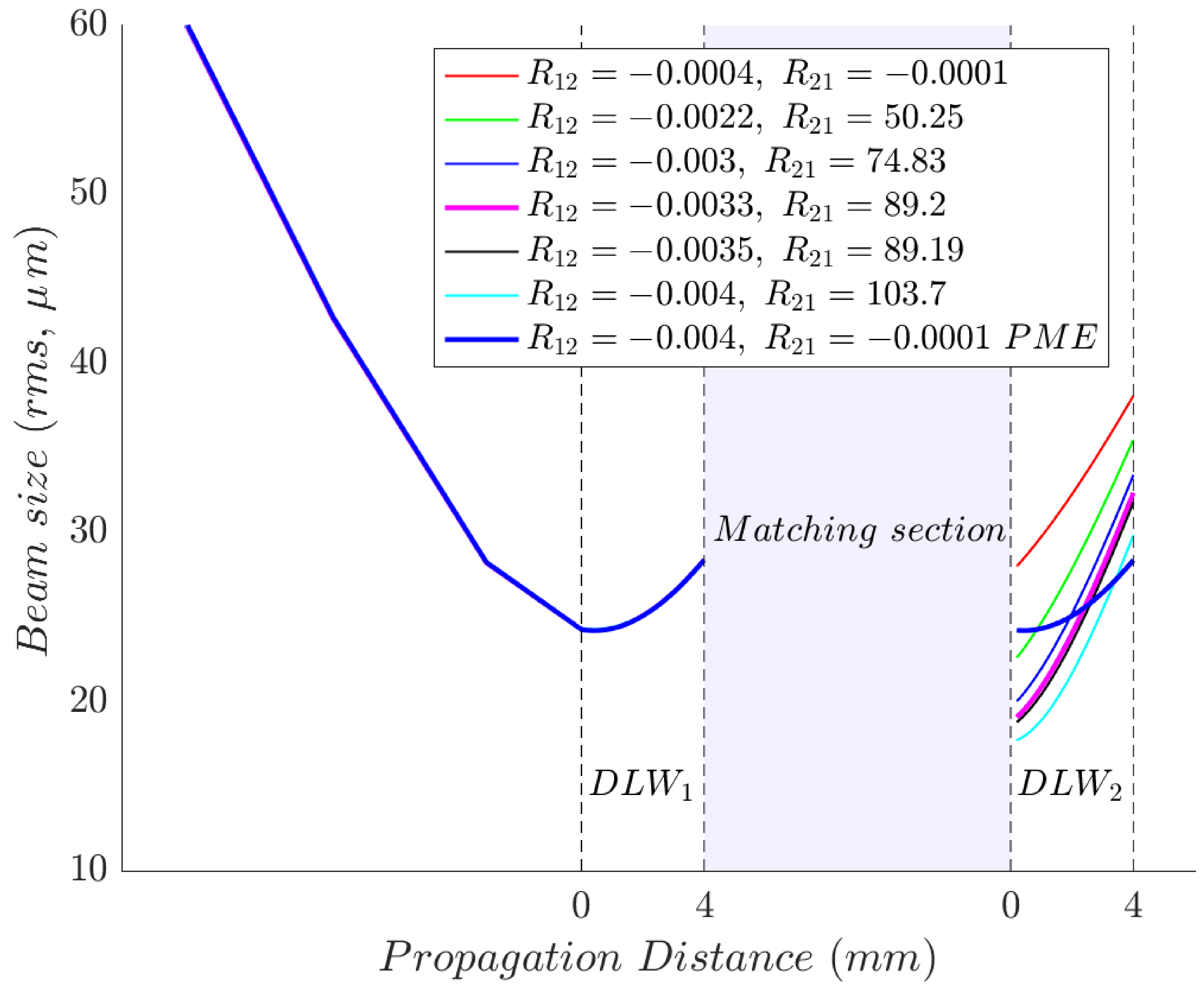
A physical solution





- No simultaneous solutions at a single phase advance value.
- Energy spread is sensitive to R_{12} the most.

A physical transformation



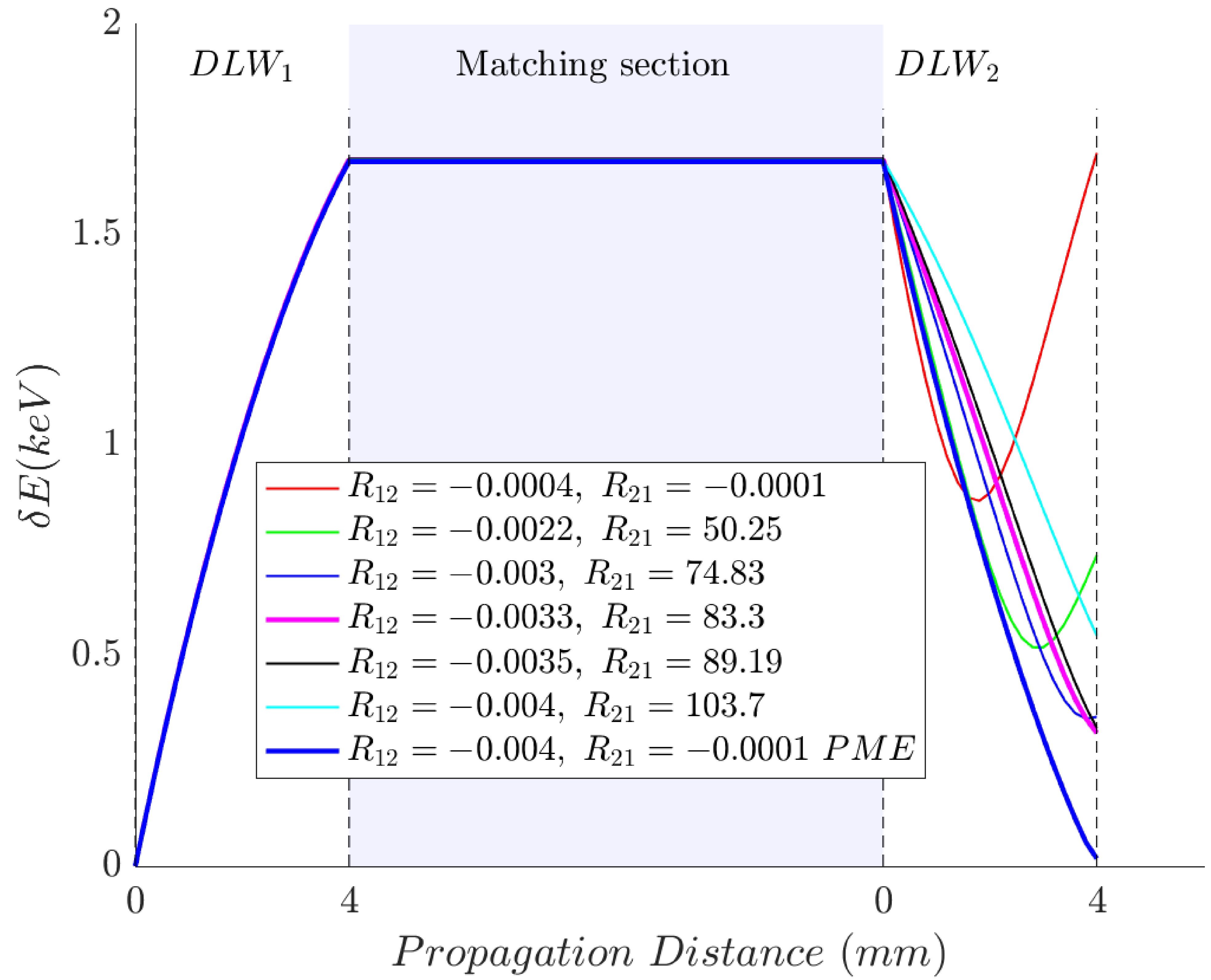


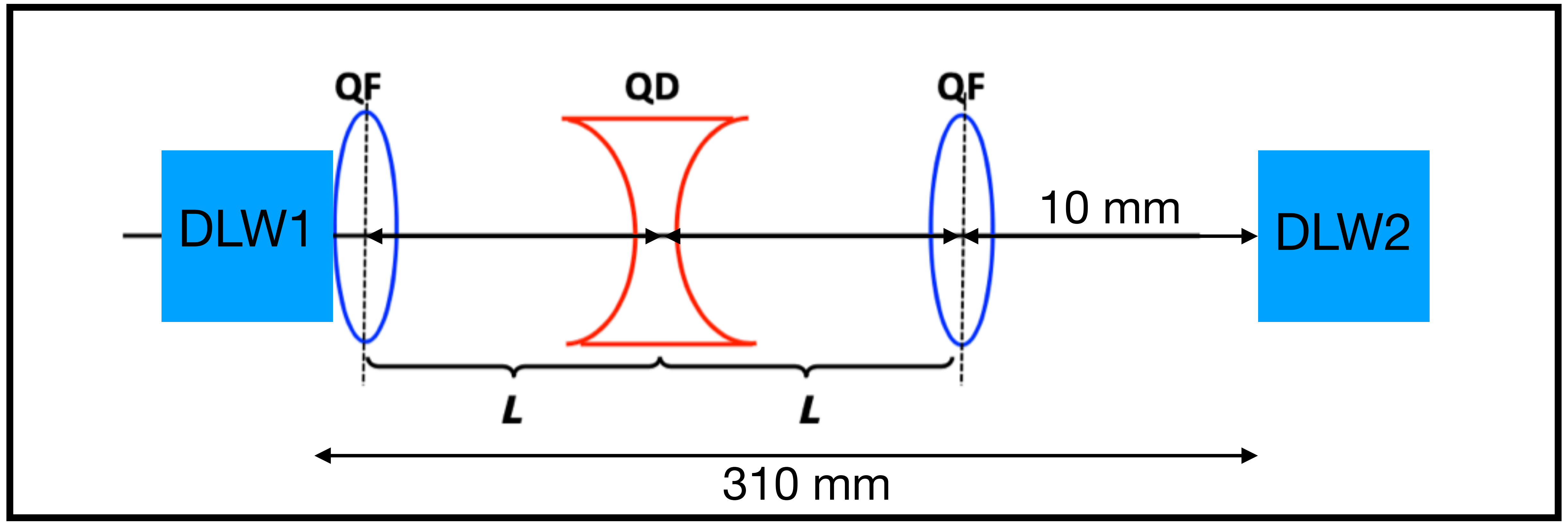
TABLE I. The solutions of nonperiodic transfer matrix for the matching section at given phase advances and resulting final energy spread values for each solution. The residual energy spread is minimized for the solution at $\mu = 1.816$.

μ (π)	R_{11}	R_{12}	R_{21}	R_{22}	$\Delta E/E$ (keV)
1.98	1.001	−0.0004	−0.0001	0.98	1.7
1.883	0.898	−0.0022	50.25	0.99	0.75
1.834	0.81	−0.003	74.83	0.96	0.35
1.816	0.77	−0.0033	83.3	0.94	0.31
1.803	0.75	−0.0035	89.19	0.92	0.33
1.77	0.672	−0.004	103.7	0.8711	0.5422

- Correction using Twiss matrix helps converging to a physical solution.
- But… Does a realistic matching lattice exist?

Does a realistic lattice exist?

- Transfer matrix elements for each lattice calculated using thick elements via multi objective fitting to the matrix elements deduced from pseudoinverse matching and non-periodic matrix correction steps.



Design	R_{11}	R_{12}	R_{21}	R_{22}
No drift	0.6897	−0.0033	13.1928	0.9039
$Q100D_3 10$	0.7845	−0.0033	51.5138	0.9738
$Q50D_3 10$	0.7629	−0.0033	97.2900	0.9003
$Q100D_1 150D_2 150$	0.8118	−0.0035	129.0208	0.9870
$Q100D_1 200D_2 200$	0.9157	−0.0037	139.9490	1.1135
$Q100D_1 200D_2 200D_3 200$	0.7461	−0.0033	46.8733	0.9202

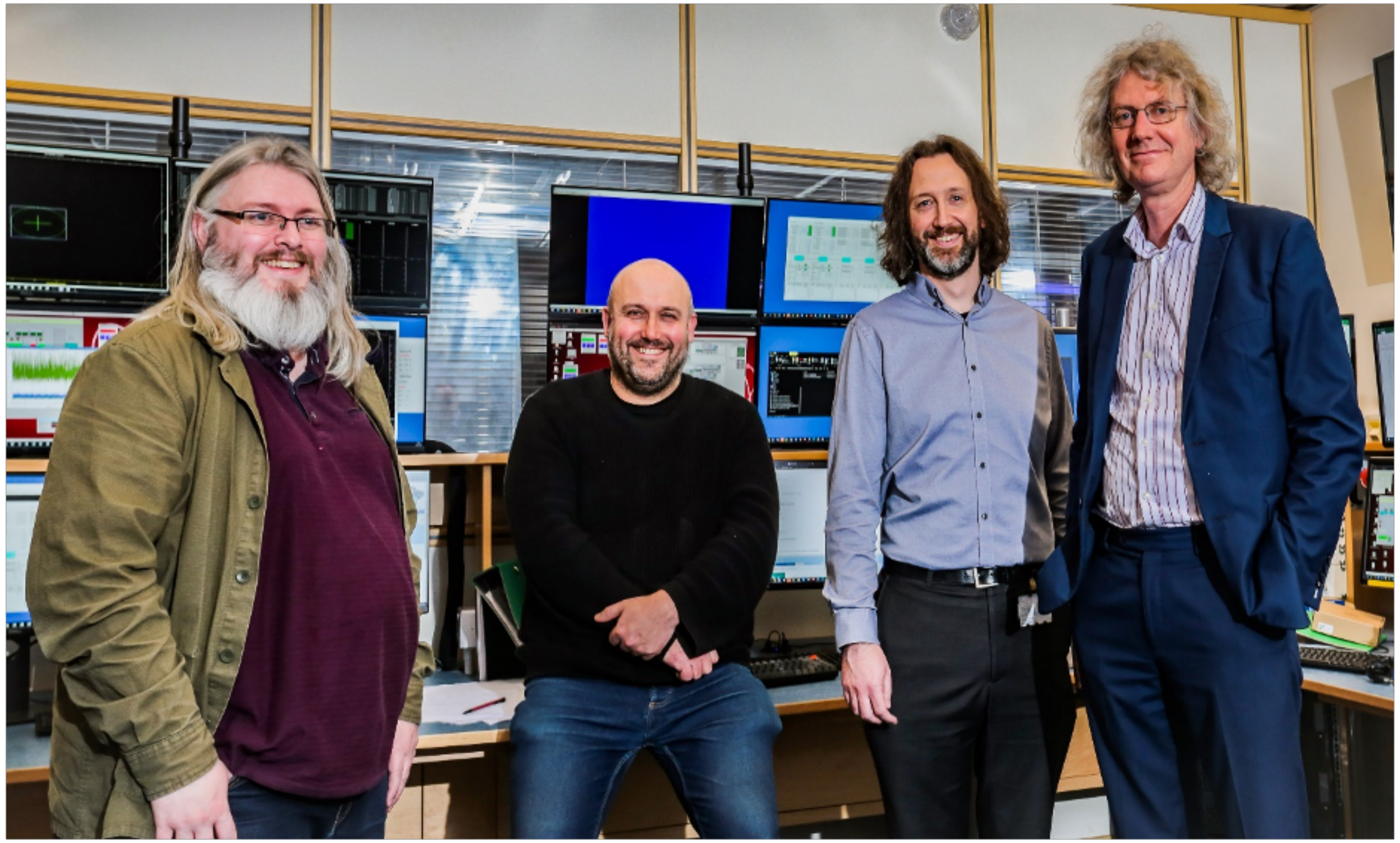
Design	Matching section length (cm)	g (T/m)	r (mm)	ΔE (keV)
No drift	30	20.4/111.5/7.3	50/9/138	0.92
$Q100D_3 10$	31	75.3/0.5/66.2	13.3/201/15.1	0.54
$Q50D_3 10$	16	310.2/−0.18/219.2	3.2/567/4.6	0.46
$Q100D_1 150D_2 150$	40	48.8/−1.8/50.2	20/55.8/20	0.49
$Q100D_1 200D_2 200$	50	46.5/−3.7/47.9	21.5/268/21	0.86
$Q100D_1 200D_2 200D_3 200$	65	61/1.2/10.5	16.4/807/95.4	0.5

Conclusions

- Emittance
  - Unlike all symmetric THz and RF structures we find the monopole term has zero transverse variation at the synchronous point, meaning **no monopolar defocussing term**
  - The only strong transverse field term is a quadrupole that if DC would not cause any emittance growth, we get zero slice emittance growth
  - As we have a time varying quadrupole, **a small projected emittance growth of 0.4%** of a 50 fs, 100um beam with a 400 keV acceleration.
- Energy spread
  - PI only - 0.75 eV
  - PI+NPTM - 0.31 keV
  - PI+NPTM+MOO - 0.54 keV over 800 keV energy gain.
- Demonstrating the 6D phase space preservation.

Cockcroft Institute: Terahertz Acceleration Group

THz Group Academic Leads



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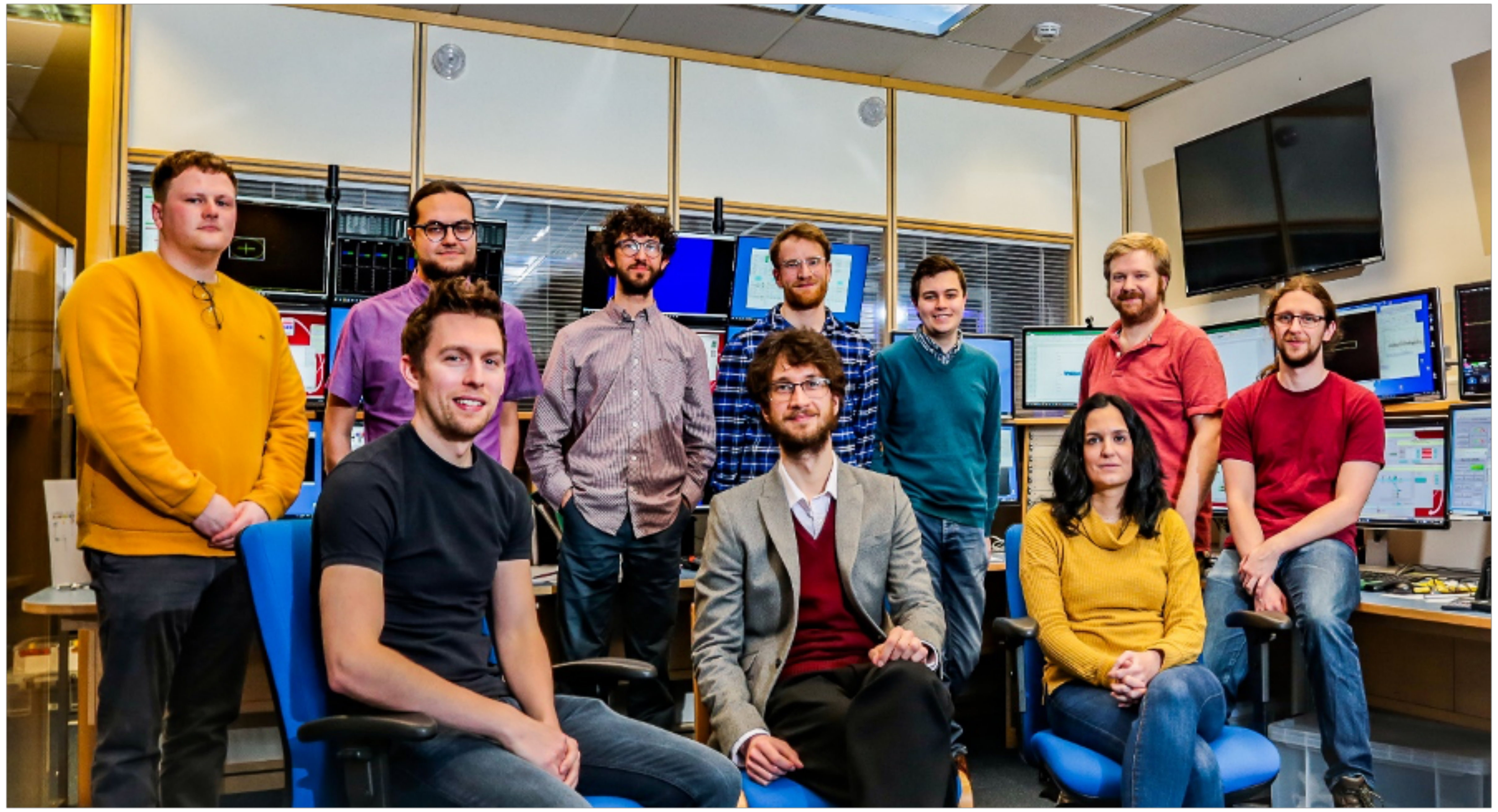
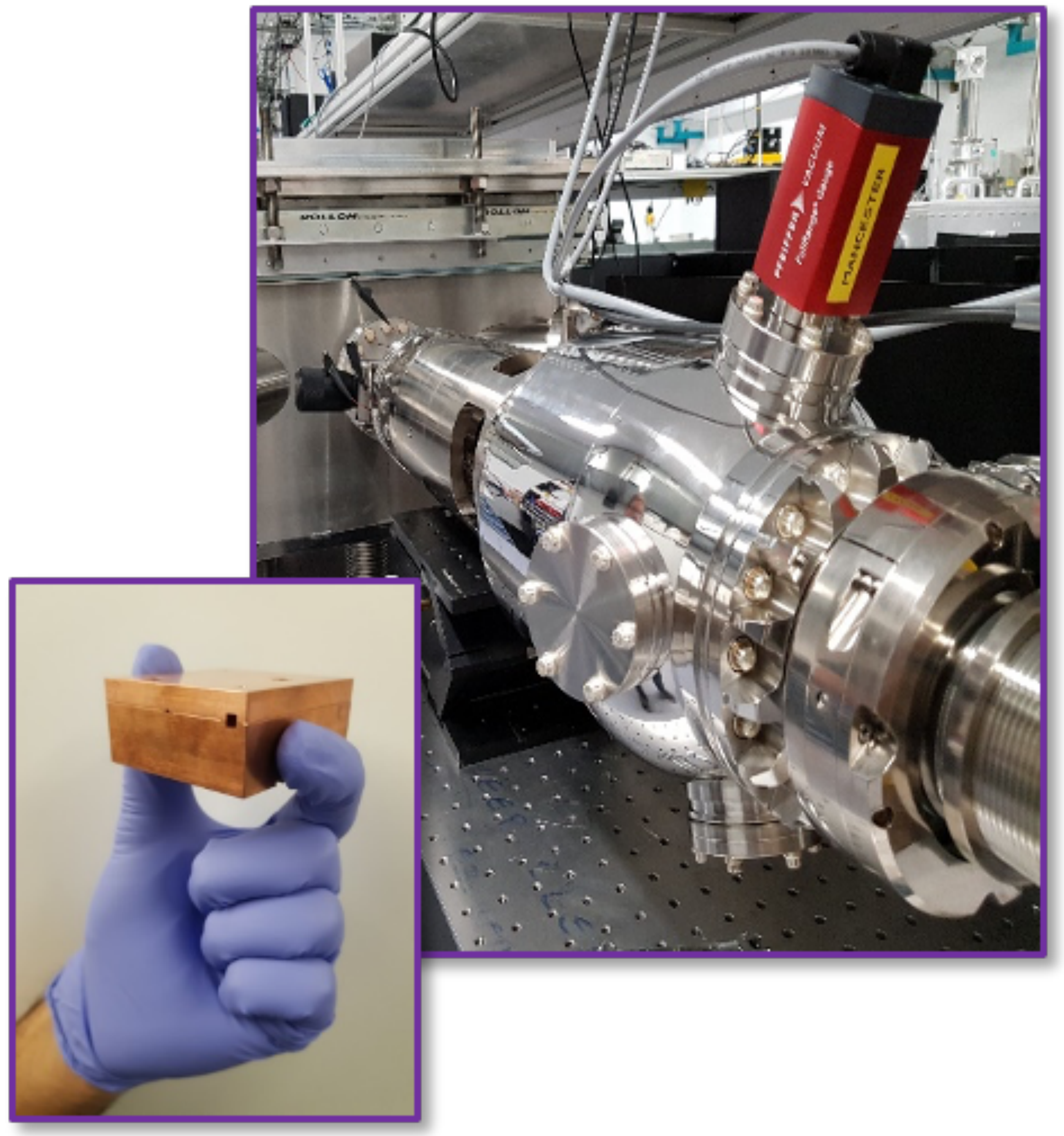
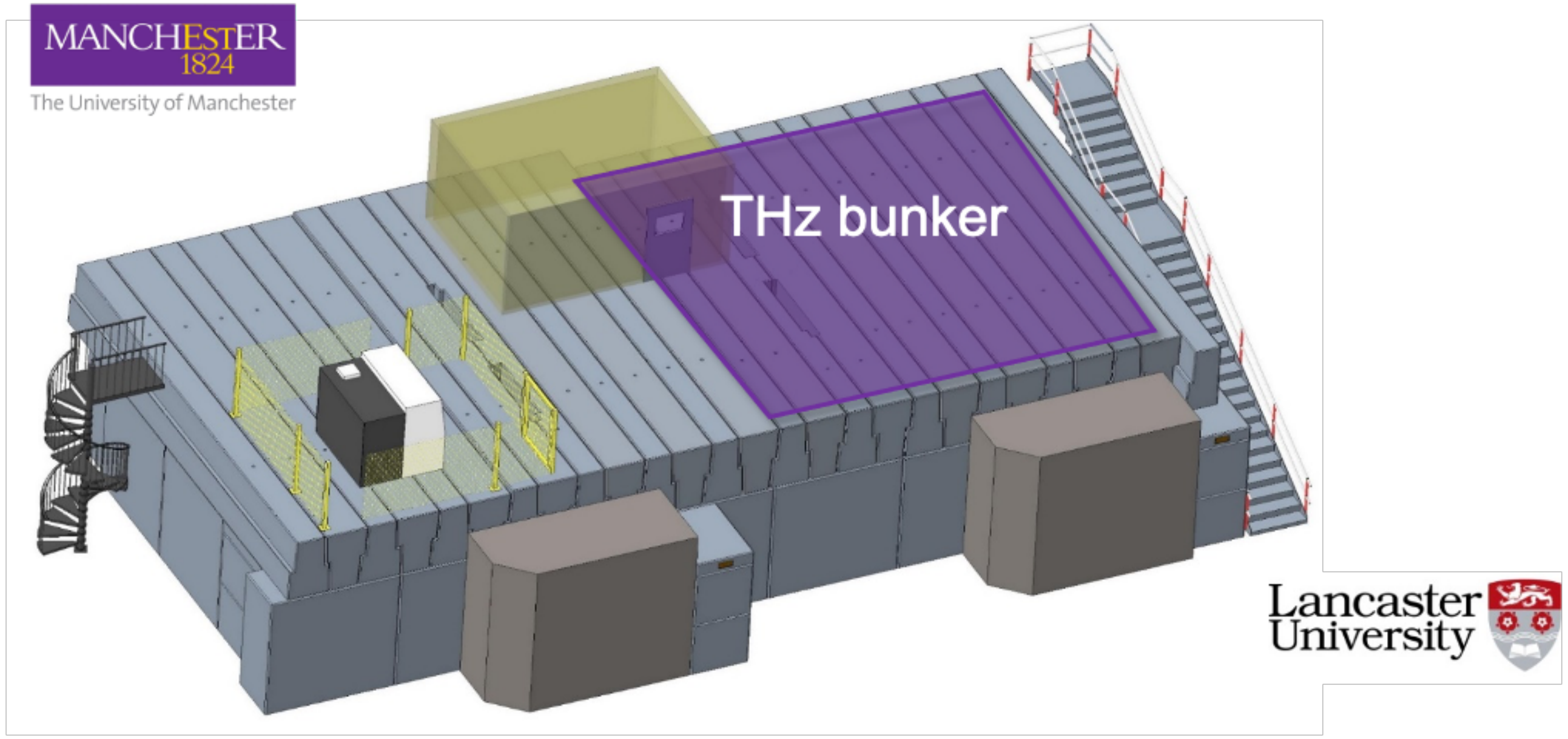
PhD students
Christopher Shaw
Joseph Bradbury
Ryan McGuigan
Patrick Dalton
Filip Peczek
Aras Amini

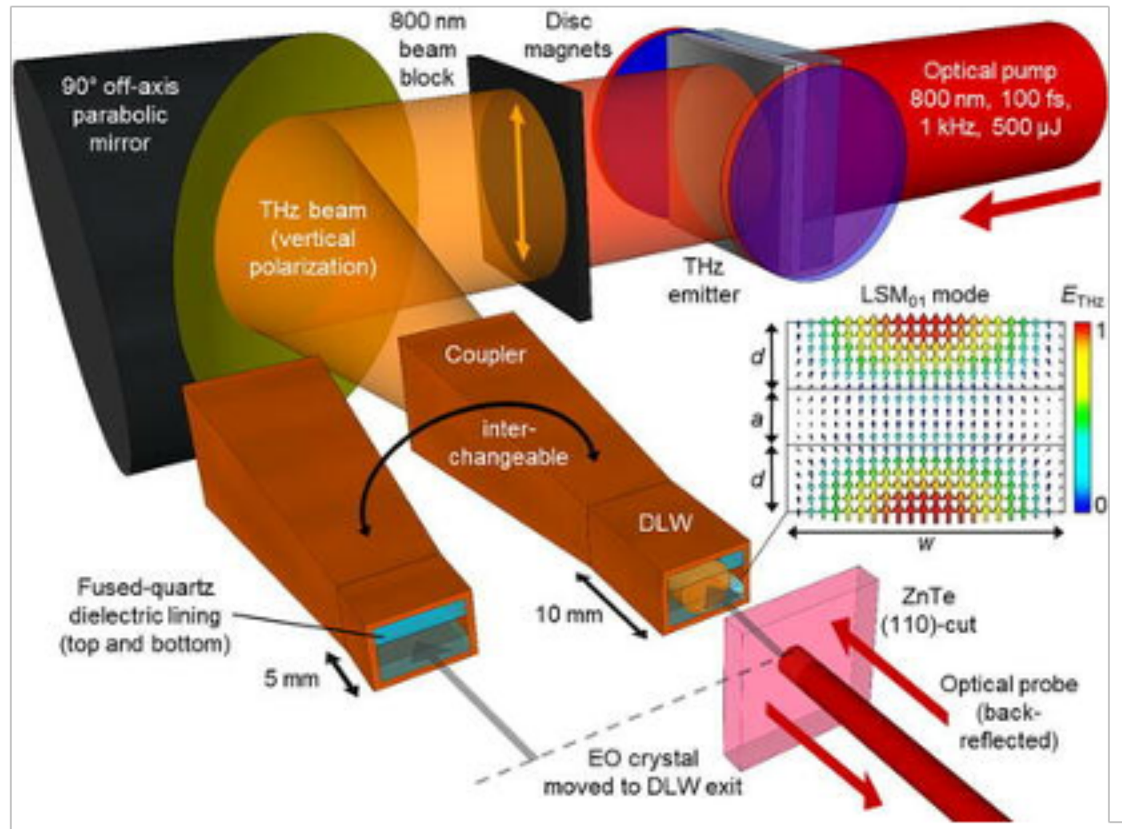


THz bunker





Waveguide characterisation



V. Georgiadis *et al. Appl. Phys. Lett.* **118,** 144102 (2021)

THz-driven injector development

- 100 keV photo-electrons boost to > 1 MeV
- Compression, acceleration, diagnostics
- Test concepts for scaling to relativistic energies

Backup Slides

Decomposing Longitudinal THz Field to Multipole Components

k_y^a, y-directed propagation constant in vacuum.

$\beta = k_z$, the propagation constant inside the waveguide.

w, horizontal width of the rectangular structure.

- Longitudinal field propagation in vacuum:

$$k^2 = k_x^2 + k_y^2 + k_z^2$$

$$\left(\frac{\omega}{c}\right)^2 = \left(\frac{\omega}{\beta c}\right)^2 + k_x^2 + k_y^2$$

- At ultra relativistic limit ($\beta = 1$) and for very narrow band (single frequency)

$$k_x = ik_y$$

- Fourier components of longitudinal voltage.

$$V_z(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n cos(n\theta) + \sum_{n=1}^{\infty} b_n sin(n\theta)$$

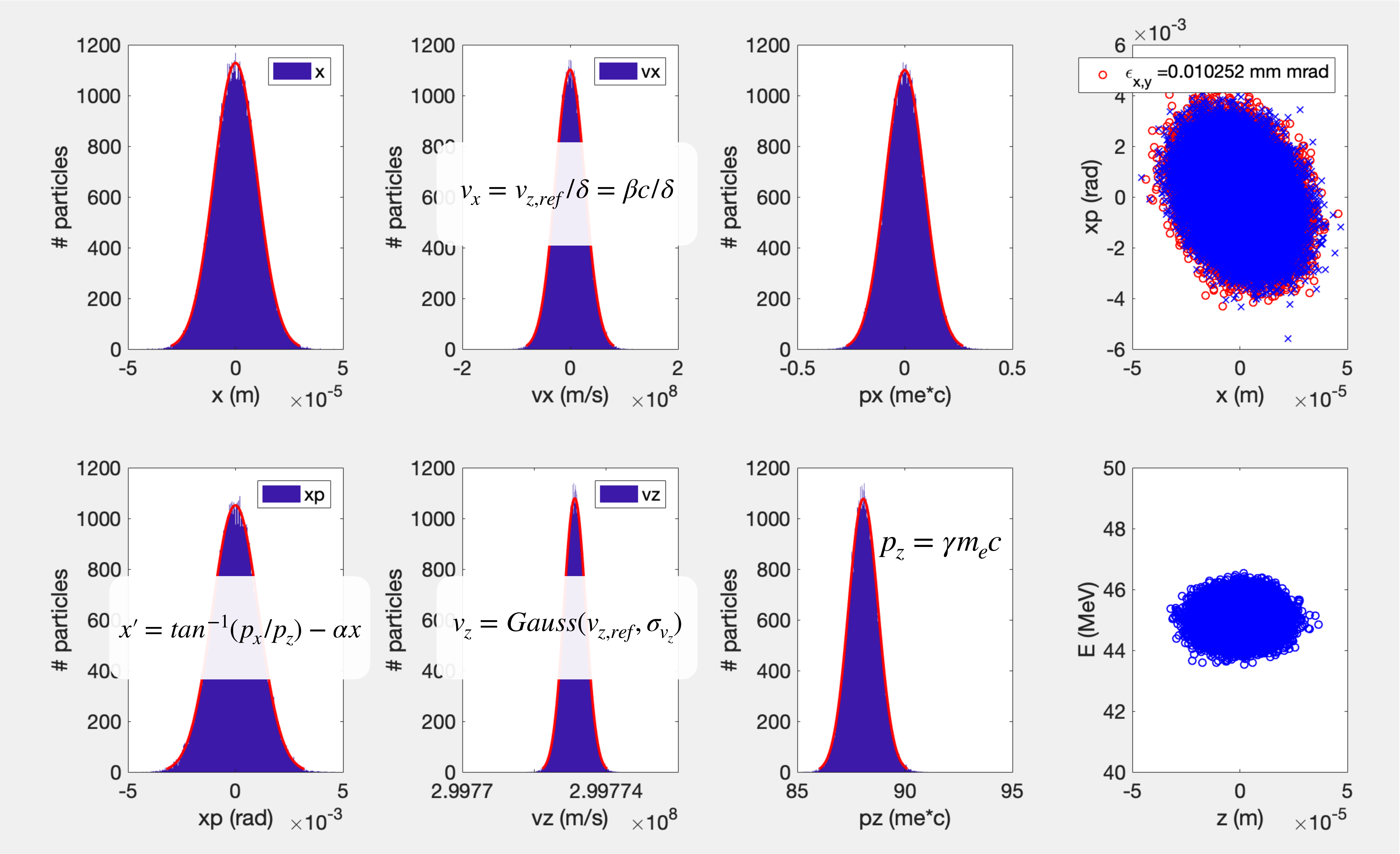
$$a_0 = \frac{\alpha}{\pi}\int_{-\pi}^{\pi} \cos\big((\pi/w)r\cos\theta\big)\cos\big(k_y^a r\sin\theta\big)d\theta$$

$$a_n = \frac{\alpha}{\pi}\int_{-\pi}^{\pi} cos(\frac{\pi}{w}rcos\theta)cos(k_y^a rsin\theta)cos(n\theta)d\theta$$

$$b_n = \frac{\alpha}{\pi}\int_{-\pi}^{\pi} cos(\frac{\pi}{w}rcos\theta)cos(k_y^a rsin\theta)sin(n\theta)d\theta$$

Gaussian Particle Generation for Tracking

- Capability to import custom beams into CST PIC.
- Custom thin lens tracking for speed.



Induce initial beam divergence

Non-relativistic transverse velocity

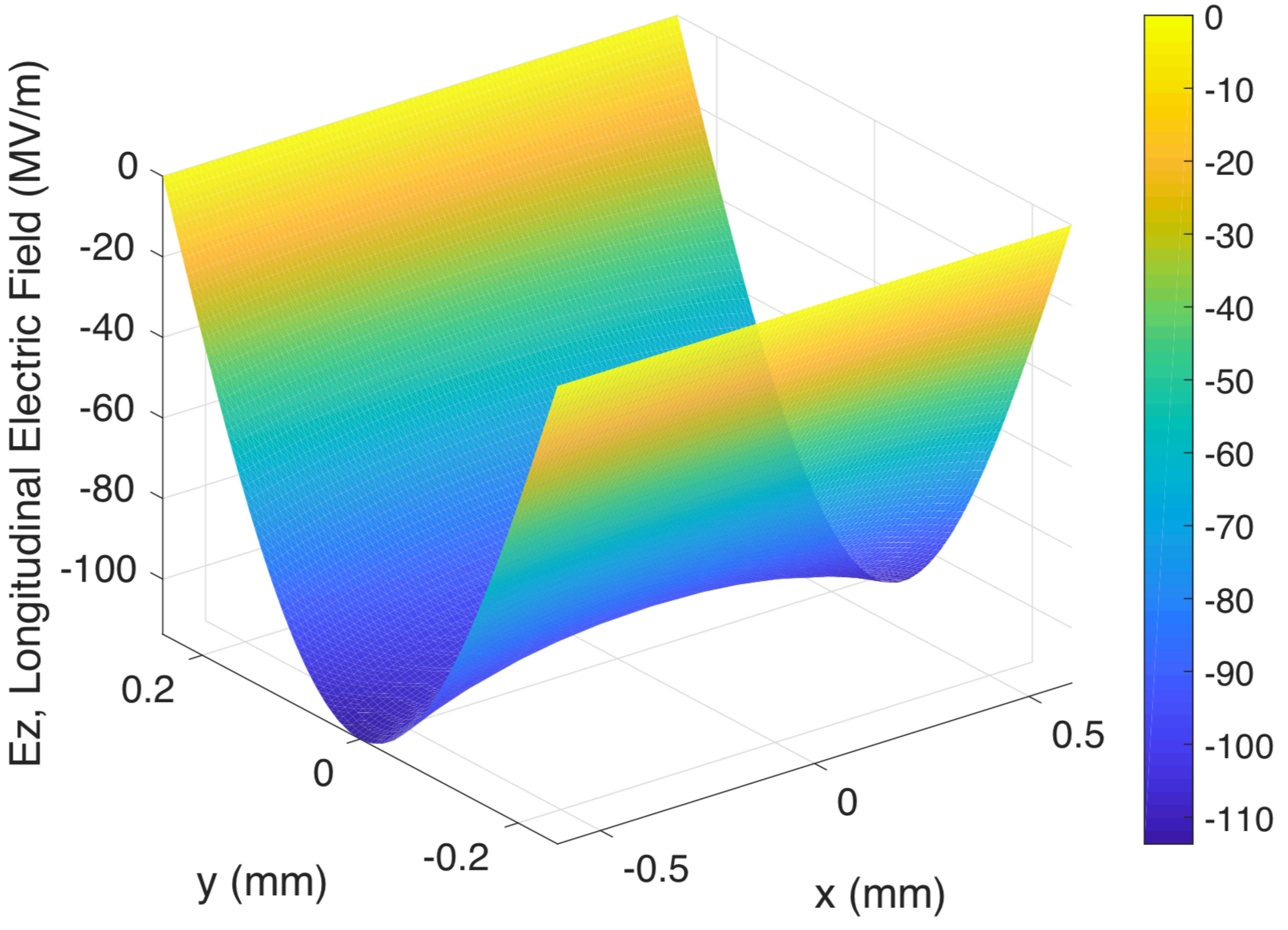
$$x' = tan^{-1}(p_x/p_z) - \alpha x$$

Induce initial uncorrelated energy spread

Spread particle velocities around the reference particle as a Gaussian distribution.

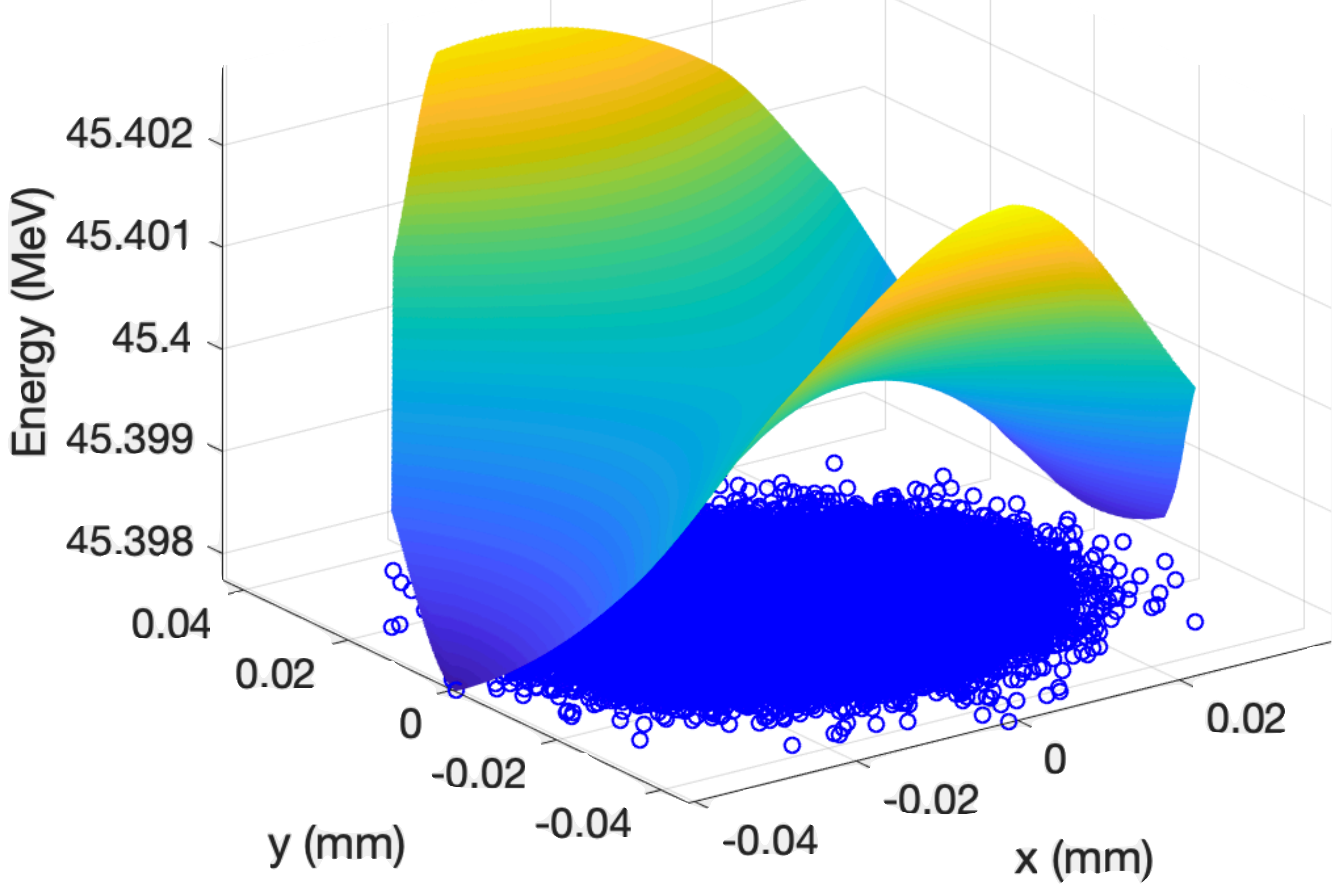
Correlated Energy Spread and Correction

Analytical field distribution for 400 keV gain



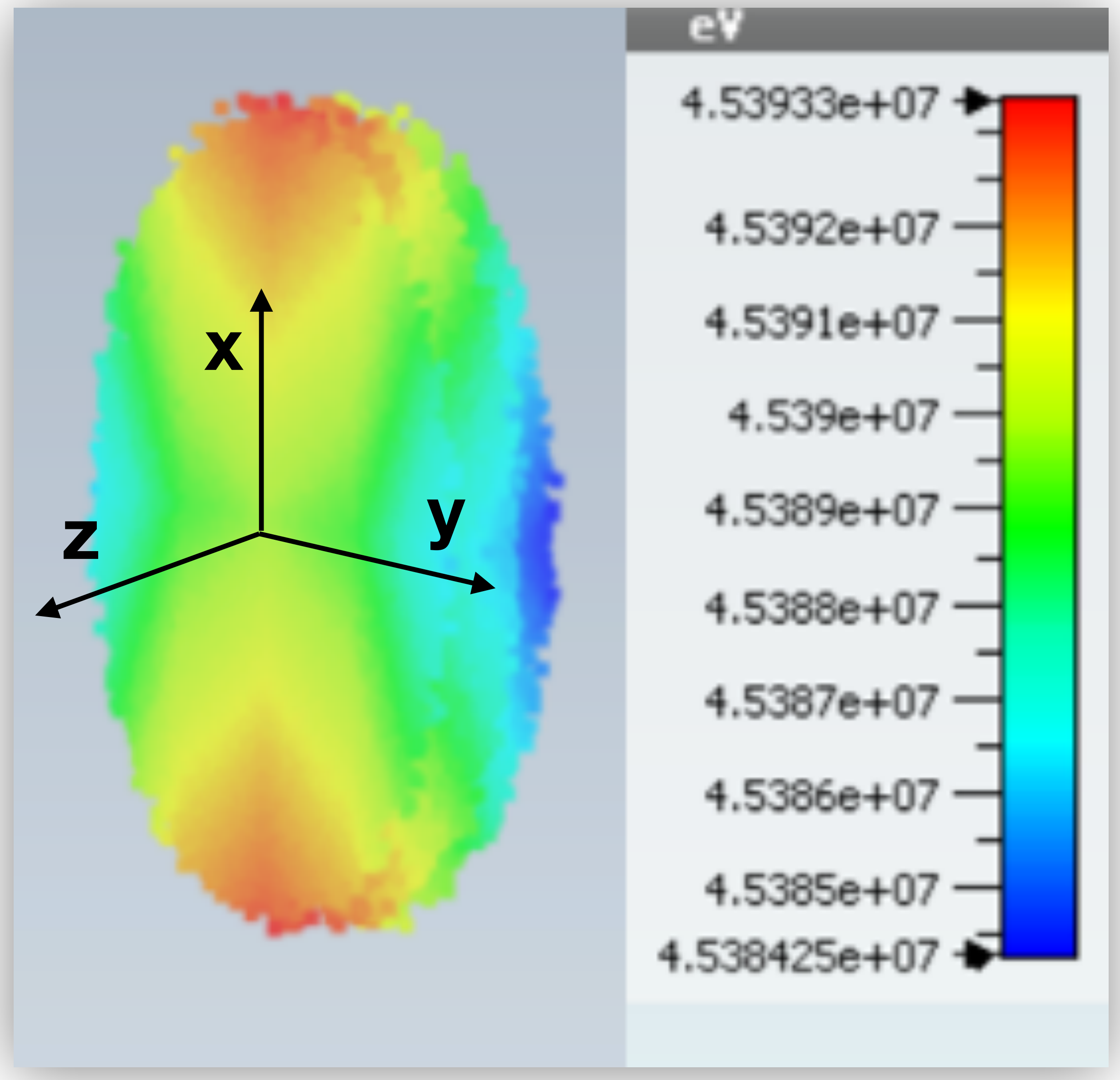
Longitudinal field distribution on transverse plane

Thin lens tracking: Single bunch slice



Field pattern maps itself onto the beam energy distribution. Energy spread 0.3keV.

CST - Beam Real Space



Initial parameter:
10um, 1mrad, 25fs, 45MeV
Energy gain:
390keV, spread 1.6keV

Can we correct the correlated energy distribution with a 90^{o} rotated DLW?