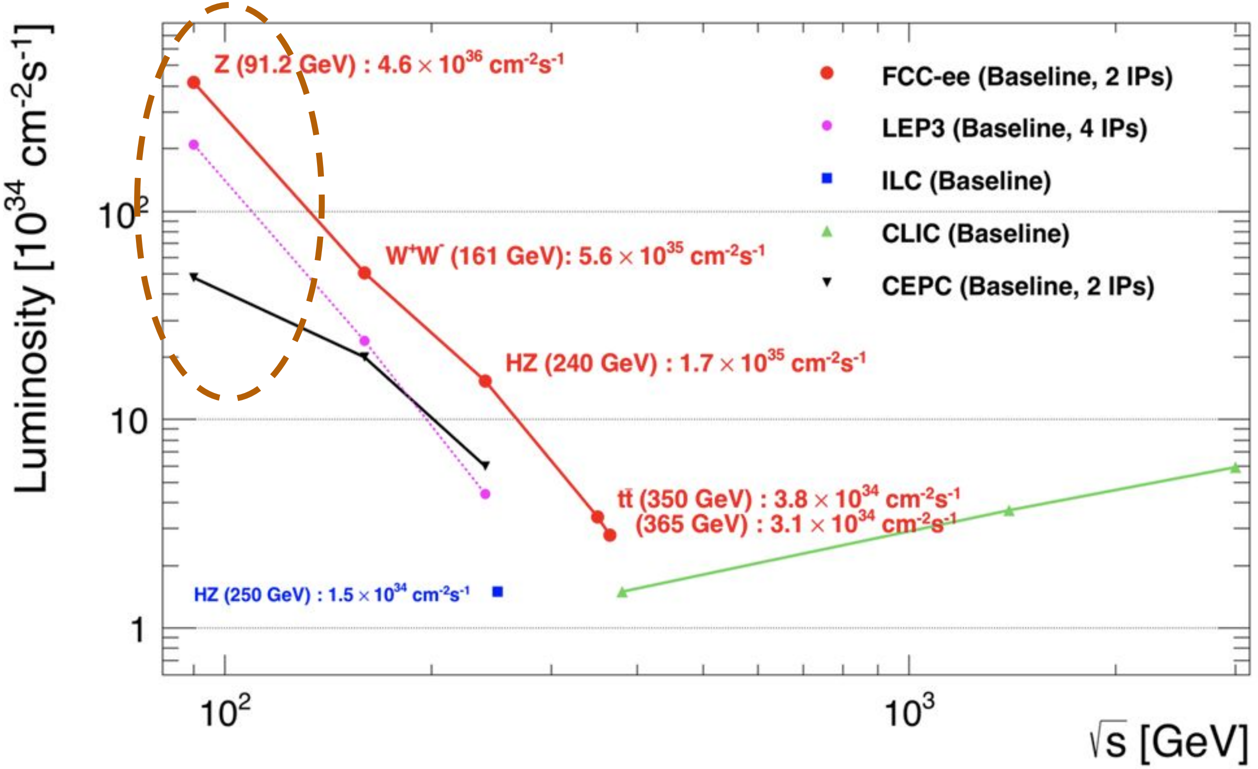
Precision Study @ Z-pole

- ❑ motivations and introduction
- ❑ a glimpse of 3-loop EWPOs
- ❑ from EWPOs to PHYSICS

Lisong Chen, KIT

~~Higgs~~ EW bosons Factory



- ~O(10^{12}) Z-bosons @ the circular ee collider.
- ~O(10^{11}) heavy-quark pairs and tau pairs (in boosted region!).
- constraints more SMEFT operators at once! Disentangle Higgs sector from EW sector.
- indirect/direct search for BSM(L-R neutrino mixing via Z-decay).
- etc..

❑ Z-pole Observables

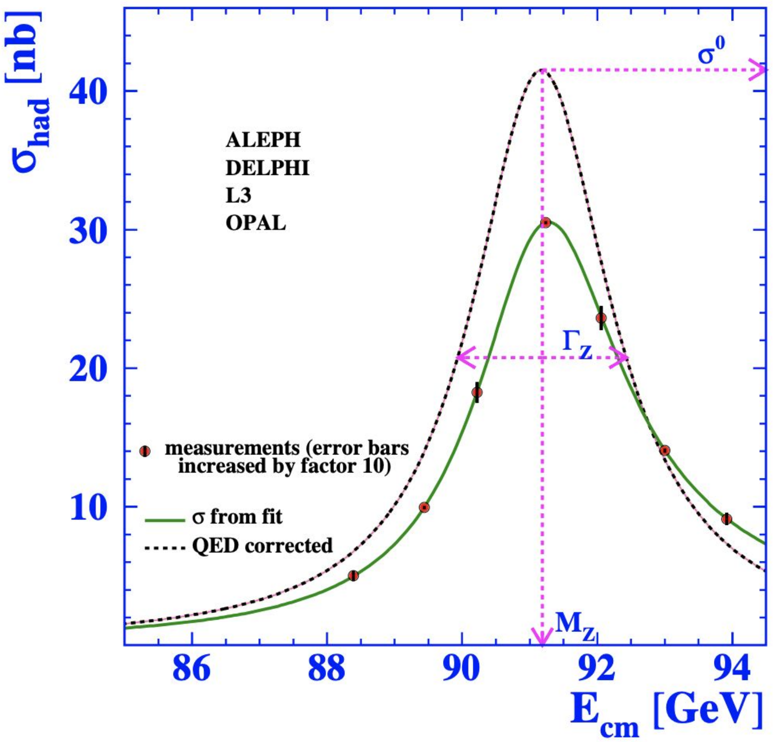
- ❑ cross section $\sigma(s = M_Z^2) \equiv \sigma_f^0$
- ❑ widths of Z boson.
- ❑ branching ratios.

$$\sigma^0_{had} = \sigma[e^+e^- \to hadrons]_{s=M_Z^2};$$

$$\Gamma_Z = \sum_f \Gamma[Z \to f\bar{f}],$$

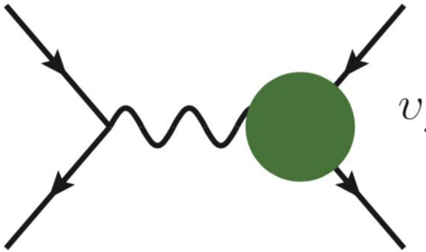
$$R_l = \Gamma[Z \to hadrons]/\Gamma[Z \to l^+l^-], \qquad (l = e, \mu, \tau);$$

$$R_q = \Gamma[Z \to q\bar{q}]/\Gamma[Z \to hadrons], \qquad (q = b, c, s, d, u);$$



❑ Asymmetries and effective weak-mixing angle

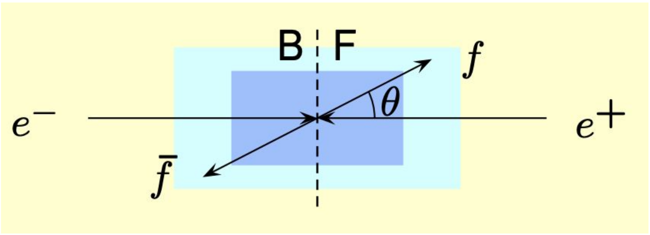
$$\bar{\psi}(v_f - a_f\gamma^5)\gamma^\mu Z_\mu \psi$$



$$A_{LR} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} = \frac{2\Re v_e/a_e}{1 + |v_e/a_e|^2} \equiv \mathcal{A}_e$$

$$A_{FB} \equiv \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B} = \frac{3\Re v_e/a_e}{1 + (v_e/a_e)^2}\frac{\Re v_f/a_f}{1 + (v_f/a_f)^2}$$

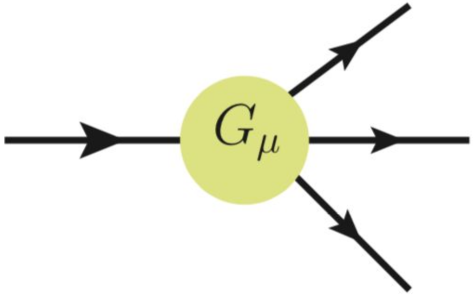
$$= \frac{3(1 - 4|Q_e|\sin^2\theta^e_{eff})}{1 + (1 - 4|Q_e|\sin^2\theta^e_{eff})^2}\frac{(1 - 4|Q_f|\sin^2\theta^f_{eff})}{1 + (1 - 4|Q_f|\sin^2\theta^f_{eff})^2}$$

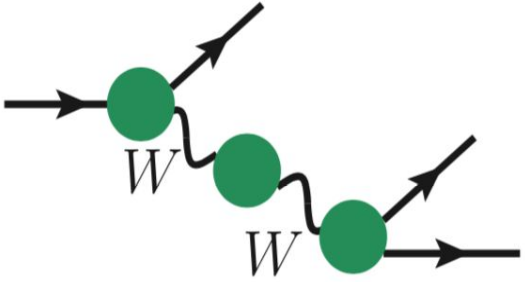


$$\sin^2\theta^f_{eff} = \frac{1}{4|Q_f|}\left(1 - \Re\frac{v_f}{a_f}\right) = \sin^2\theta_W(1 + \Delta\kappa).$$

radiative corrections.

Δr and M_W





$$\Gamma_\mu = \frac{{G_F}^2 m_\mu^5}{192\pi^2} F(\frac{m_e^2}{m_\mu})(1+\Delta q)$$

2-loop QED

$$G_\mu = \frac{\pi\alpha}{\sqrt{2} s_W^2 M_W^2}(1+\Delta r(M_H^2, M_t^2, \ldots))$$

$$M_W^2 = M_Z^2(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\alpha\pi}{\sqrt{2} G_\mu M_Z^2}(1+\Delta r)})$$

- We have seen parity-violating asymmetry can be determined by effective weak-mixing angle $\sin^2\theta^f_{eff}$. It relates to the ratio between dressed vector and axial-vector coupling.

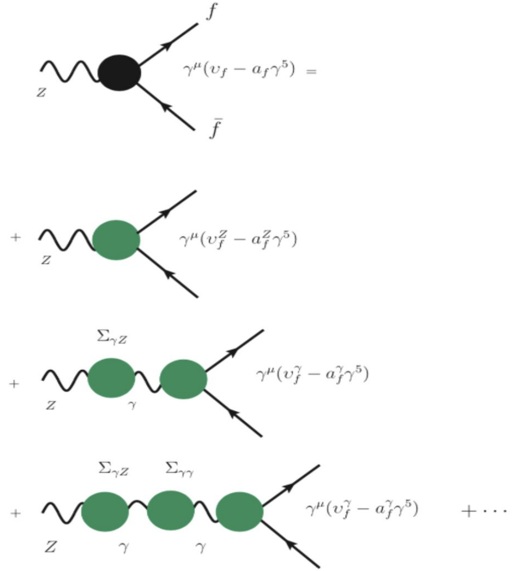
- Using the decay rate equation in terms of dressed vector and axial-vector couplings. We can derive the total and partial width of Z-boson.

$$\sin^2\theta^f_{eff} = \frac{1}{4|Q_f|}\Big(1 - \Re\frac{g_V^f}{g_A^f}\Big)_{s=\overline{M_Z}^2}$$

$$g_V^f = Z_e(v_f^Z - Q_f\sqrt{Z_{\gamma Z}}) - v_f^\gamma\frac{\hat{\Sigma}_{\gamma Z}}{s+\hat{\Sigma}_{\gamma\gamma}}$$

$$g_A^f = Z_e a_f^Z - a_f^\gamma\frac{\hat{\Sigma}_{\gamma Z}}{s+\hat{\Sigma}_{\gamma\gamma}}$$

Decompstion of the effective Zff vertex



Using optical theorem

$$\Im\Sigma_Z = \frac{1}{3\overline{M_Z}}\sum_f\sum_{spins}\int d\Phi(|g_V^f|^2 + |g_A^f|^2)$$

Plugging what we have from OS condition in pole scheme.

$$\overline{\Gamma}_Z = \frac{N_c^f}{12\pi\overline{M_Z}}C_Z(\mathcal{R}_V^f|g_V^f|^2 + \mathcal{R}_A^f|g_A^f|^2)$$

where C_Z features all self-energy contributions, and $\mathcal{R}^f_{V,A}$ feature final-state QCD and QED corrections.

SM Loop corrections

- 1-loop and leading 2-loop EW corrections
  Veltman, Passarino, Sirlin, Marciano, Bardin, Hollik, Riemann, Degrassi, Kniehl, …

- Full 2-loop corrections EW and mixed QCD-EW to Δr and Z-pole observables

  Djouai, Verzegnassi '87, Djouadi '88, Kniehl, Kühn, Stuart '99, Kniehl, Sirlin '93, Djouadi, Gambino '94, Halzen Kniehl '91, Chetyrkin, Kühn '96, Fleischer et al. '92 Freitas, Hollik, Walter, Weiglein '00, Awramik, Czakon '02, Onishchenko, Vertin '02,
  Awramik, Czakon, Freitas, Weiglein '04, Awramik, Czakon, Freitas '06, Hollik, Meier, Uccirati '05 '07, Awramik, Czakon, Freitas, Kniehl '08, Freitas, Huang '12, Freitas '13'14, Dubovyk, Freitas, Gluza, Riemann, Usovitsch '18

- Approximate 3- and 4-loop corrections to universal parameters (ρ parameter)

  Chetyrkin, Kühn, Steinhauser '95, Schröder, Steinhauser '05, Faisst, Kühn, Seidensticker, Veretin '03, Chetyrkin et al. '06, Boughezal, Tausk, v.d. Bij'05, Boughezal, Czakon '06

- Leading fermionic 3-loop EW&EW-QCD corrections to EWPOs. Chen, Freitas `20,

Experimental uncertainties given by future electron-positron colliders

	Exp	Current theo. error	CEPC	FCC-ee	ILC/GigaZ
M_W[MeV]	12	$4(\alpha^3, \alpha^2\alpha_s)$	1	$0.5 \sim 1$	2.5
Γ_Z[MeV]	2.3	$0.4(\alpha^3, \alpha^2\alpha_s, \alpha\alpha_s^2)$	0.5	0.1	1.0
$\sin^2\theta_{eff}^f$ [10^{-5}]	16	$4.5(\alpha^3, \alpha^2\alpha_s)$	< 1	0.6	1

The calculation of the next relevant order for the EWPOs will be indispensible!

Challenges of Theory Calculations of 3-loop EWPOs

- computer algebra (diagram generation, Lorentz/Dirac algebra, simplification, etc.)
- master integral reduction
- evaluation of master integrals (MIs)

} *entangled!*

$Z \to b\bar{b}$			
Number of topologies	1 loop	2 loops	3 loops
	1	5	50
Number of diagrams	15	1114	120187
Fermionic loops	0	150	**17580**
Bosonic loops	15	964	102607
QCD / EW	1 / 14	98 / 1016	10405 / 109782

Krzysztof Grzanka ‘22

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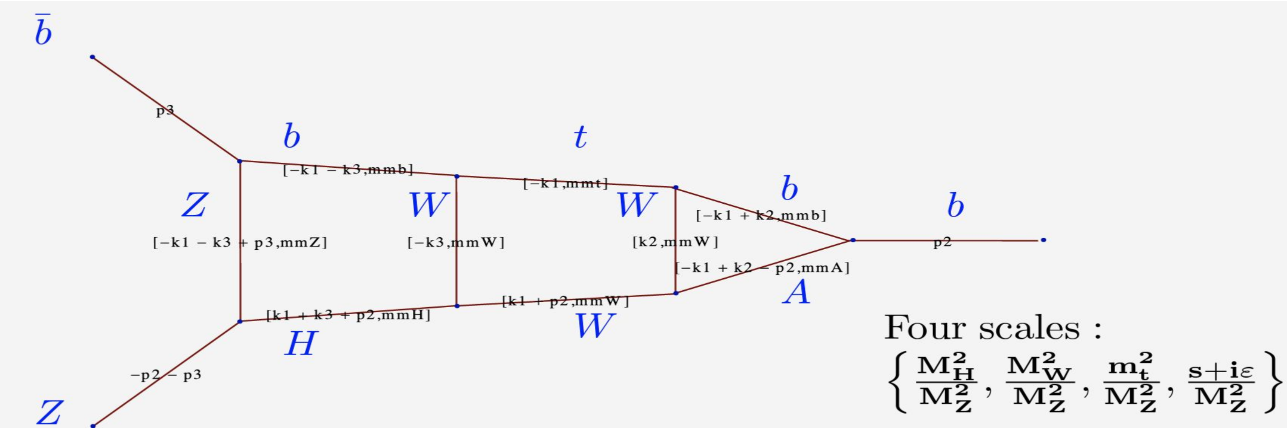

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IBP Reduction to MIs

- rapidly growing sys. of eq., consequently, large intermediate algebra, IBP reduction table, etc.
- optimal basis?

- Laporta-like approaches provide systematic way to automatize problems.
- New innovations:
  - finite fields (avoid large intermediate algebra. Finite Flow, CARAVEL,FireFly,etc.)
  - syzygy equations (reduce the complexity of the sys.)
  - Feynman trick
  - intersection theory
  - etc..

Evaluation of MIs

- Analytical (subject to num. of loops/legs/masses, knowledge of the space of special functions, etc) more insights from other talks!
- Approximation by expansion(asymptotic exp.)
  (subject to ratios between masses and kinematic variables)
- Numerical (different methods subject to different bottle necks)

numerical techniques are indispensable @N3LO EW loops.

- Sector decomposition (SD) method:
  - FIESTA5 [2012], [A.V.Smirnov]
  - pySecDec [2022], [Expansion by regions with pySecDec],
- The Mellin-Barnes (MB) method:
  - MB [M.Czakon, 2006]
  - MBnumerics [J.Usovitsch, I.Dubovyk, T.Riemann, 2015] - Minkowskian kinematics
- Differential equations (DEqs) method:
  - `DiffExp` [F. Moriello, 2019; M. Hidding, 2021],
  - `AMFlow` [X. Liu, Y.-Q. Ma, 2022],
  - `SeaSyde` [T. Armadillo, R. Bonciani, S. Devoto, N. Rana, A. Vi, 2022]

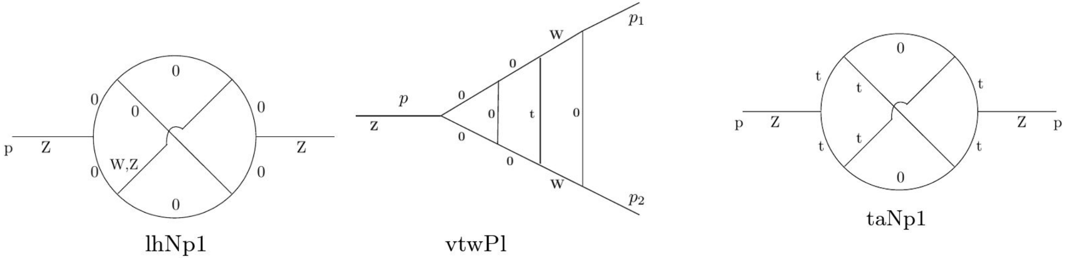
- Reductions at the integrand level;
- Expansions by regions; Taylor expansion in Feynman parameters;
- Loop-tree duality (G. Rodrigo et al, Weinzierl et al);
- Multi-loop amplitudes with numerical unitarity (Abreu et al.);
- Four-dimensional unsubtraction; Direct numerical evaluation of multi-loop integrals without contour deformation (R. Pittau et al.);
- Feynman parameters and dispersion relations (Song, Freitas); ***see Thursday's talk by Ayres***
- ...

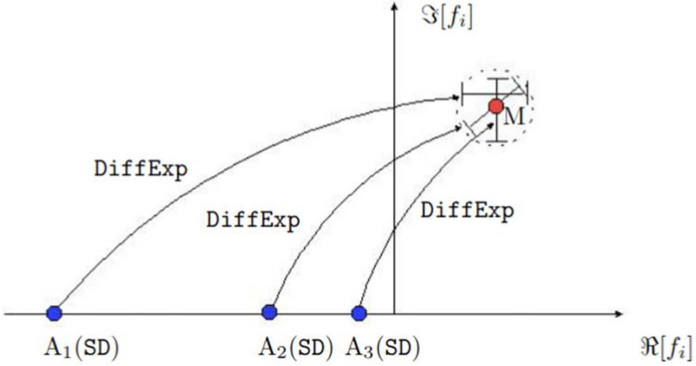
take-away:
Traditional SD/MB performed poorly at 3-loop EW (only few digits! Need improvements.

DE methods have systematical ways having UV/IR poles under well-controlled. Very good convergence and precision in general. Bottle neck: **IBP reductions!!**

An outpost of EWPOs at N3LO

I.Dubovyk,A.Freitas,J. Gluza, K.Grzanka, M.Hidding, J.Usovitsch, ‘Evaluation ofmulti-loop multi-scale Feynmanintegrals for precisionphysics’,2201.02576(PRD’2022)





can reach at least 10-digit accuracy.

—**enough for 3-loop EWPOs!**

Renormalization

- Conceptually well understood.

- On-Shell(OS) with complex pole mass
- OS+ $\overline{MS}$ for mixed QCD-EW
- other options: Bkgd field renormalization, complex mass renormalization, etc.

- Complex pole mass for *gauge-invariance* beyond one-loop.

- OS mass closely connects to experiments, while suffers from renormalon issue and non-perturbative QCD.
- $\overline{MS}$ top/bottom mass is preferable from theory point of view.

- masses calculated from two schemes related by a finite transformation.

complex-pole

$$s_0 \equiv \overline{M^2} - i\overline{M\Gamma}$$

The inverse dressed propagator (W/Z/H)

$$D(p^2) = p^2 - \overline{M^2} - \delta Z(p^2 - \overline{M^2}) + \Sigma(p^2) - \delta\overline{M^2}$$

yield mass counter term and widths

$$\delta\overline{M^2} = \frac{Re\Sigma(\overline{M^2} - i\overline{M\Gamma})}{Z} \qquad \overline{\Gamma} = \frac{Im\Sigma(\overline{M^2} - i\overline{M\Gamma})}{Z\overline{M}}$$

mass ratio between two schemes

$$\frac{M^{OS}}{M^{\overline{MS}}} = 1 + \alpha_s C_F \frac{3\log {M^{OS}}^2/\mu^2 - 4}{4\pi} + \mathcal{O}(\alpha_s^2)$$

Ward-Identity yields $Z_e = (\sqrt{Z_{\gamma\gamma}} + \frac{\sin\theta_W}{\cos\theta_W}\sqrt{Z_{Z\gamma}})^{-1}$

Weak-Mixing Angle $s_W + \delta s_W = \sqrt{1 - \frac{\overline{M_W^2} + \delta\overline{M_W^2}}{\overline{M_Z^2} + \delta\overline{M_Z^2}}}$

- Charge renormalization needs a special care. We need α around $q^2 \sim M_Z^2$, while it's defined at Thomson limit ($q^2 \sim 0$).
  (see new insights given by 2101.05154, S.Dittmaier)

- light-quark masses are inherently ill-defined in EW Lagrangian due to non-perturbative feature at the given mass scale $q^2 \sim m^2_{u,d,s}$.

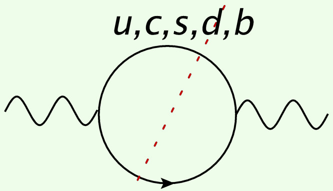
- Alternative methods needs to apply to carry out the contribution given by light quarks. Dispersion relation is the one frequently use. Other possible ways: Lattice QCD or Bhabha scattering.

$\delta Z_e = -\frac{1}{2}\delta Z_{\gamma\gamma} = \frac{1}{2}\Sigma'_{\gamma\gamma}(0)$ at one-loop level

$$\Sigma'_{\gamma\gamma}(0) \equiv \Pi(0) = \sum_f \frac{\alpha N_c Q_f^2}{3\pi}\left(\frac{2}{4-D} - \gamma_E - \log\frac{m_f^2}{4\pi\mu^2}\right)$$

$$\hat{\Pi}(s = M_Z^2) = \Pi(0) - \Re\Pi(M_Z^2) = \underbrace{\Pi^{lf}(0) - \Pi^{lf}(M_Z^2)}_{\Delta\alpha = \Delta\alpha_{lep} + \Delta_{had}} + \hat{\Pi}^{top}(M_Z^2)$$

Dispersion relation reads:



$$\Delta_{had} = -\frac{\alpha}{3\pi} s \int_{4m_\pi^2}^{\infty} ds' \frac{R_{\gamma\gamma}(s')}{s'(s'-s-i\epsilon)}\Big|_{s=M_Z^2}$$

$$R_{\gamma\gamma}(s') = \frac{\sigma(e^+e^- \to \gamma^* \to hadrons)}{\sigma(e^+e^- \to \gamma^* \to \mu^+\mu^-)}$$

- non-perturbative quantity, apply to ALL order.
- Good precision ~ 0.0001

- Mass counterterms: By assuming $\Gamma_{W,Z}/M_{W,Z} \sim \mathcal{O}(\alpha)$, the imaginary part contributes to counterterms.(*20,21 L.Chen A.Freitas*)

$$\delta\overline{M}^2_{Z(3)} = \mathrm{Re}\,\Sigma_{ZZ(3)}(\overline{M}^2_Z) + [\mathrm{Im}\,\Sigma_{ZZ(2)}(\overline{M}^2_Z)]\,[\mathrm{Im}\,\Sigma'_{ZZ(1)}(\overline{M}^2_Z)]$$
$$+ [\mathrm{Im}\,\Sigma_{ZZ(1)}(\overline{M}^2_Z)]\Big\{\mathrm{Im}\,\Sigma'_{ZZ(2)}(\overline{M}^2_Z) - [\mathrm{Im}\,\Sigma'_{ZZ(1)}(\overline{M}^2_Z)]\,[\mathrm{Re}\,\Sigma'_{ZZ(1)}(\overline{M}^2_Z)]$$
$$- \tfrac{1}{2}[\mathrm{Im}\,\Sigma_{ZZ(1)}(\overline{M}^2_Z)]\,[\mathrm{Re}\,\Sigma''_{ZZ(1)}(\overline{M}^2_Z)]$$
$$- \frac{\mathrm{Im}\,\Sigma_{\gamma Z(1)}(\overline{M}^2_Z)}{\overline{M}^2_Z}[2\,\mathrm{Re}\,\Sigma'_{\gamma Z(1)}(\overline{M}^2_Z) + \delta Z^{\gamma Z}_{(1)} + \delta Z^{Z\gamma}_{(1)}]\Big\}$$
$$+ \frac{\mathrm{Im}\,\Sigma_{\gamma Z(1)}(\overline{M}^2_Z)}{\overline{M}^2_Z}\Big\{2\,\mathrm{Im}\,\Sigma_{\gamma Z(2)}(\overline{M}^2_Z) - \frac{\mathrm{Im}\,\Sigma_{\gamma Z(1)}(\overline{M}^2_Z)}{\overline{M}^2_Z}[\mathrm{Im}\,\Sigma_{\gamma\gamma(1)}(\overline{M}^2_Z)]\Big\}$$
$$+ \tfrac{1}{2}\overline{M}^2_Z\,\delta Z^{\gamma Z}_{(1)}\,\delta Z^{\gamma Z}_{(2)}\,.$$

$$\delta\overline{M}^2_{Z(\alpha_s\alpha^2)} = \mathrm{Re}\,\Sigma_{ZZ(\alpha_s\alpha^2)}(\overline{M}^2_Z) + [\mathrm{Im}\,\Sigma_{ZZ(\alpha_s\alpha)}(\overline{M}^2_Z)]\,[\mathrm{Im}\,\Sigma'_{ZZ(\alpha)}(\overline{M}^2_Z)]$$
$$+ [\mathrm{Im}\,\Sigma_{ZZ(\alpha)}(\overline{M}^2_Z)]\,[\mathrm{Im}\,\Sigma'_{ZZ(\alpha_s\alpha)}(\overline{M}^2_Z)]$$
$$+ \frac{2}{\overline{M}^2_Z}[\mathrm{Im}\,\Sigma_{\gamma Z(\alpha_s\alpha)}(\overline{M}^2_Z)]\,[\mathrm{Im}\,\Sigma_{\gamma Z(\alpha)}(\overline{M}^2_Z)] + \frac{1}{2}\overline{M}^2_Z\,\delta Z^{\gamma Z}_{(\alpha)}\,\delta Z^{\gamma Z}_{(\alpha_s\alpha)}\,.$$

- Total width of Z-boson at 3-loop order (Pure EW)

$$\overline{\Gamma}_Z = \frac{1}{\overline{M}_Z}\Big\{\mathrm{Im}\,\Sigma_{Z(1)} + \mathrm{Im}\,\Sigma_{Z(2)} - (\mathrm{Im}\,\Sigma_{Z(1)})(\mathrm{Re}\,\Sigma'_{Z(1)})$$
$$+ \mathrm{Im}\,\Sigma_{Z(3)} - (\mathrm{Im}\,\Sigma_{Z(2)})(\mathrm{Re}\,\Sigma'_{Z(1)})$$
$$+ (\mathrm{Im}\,\Sigma_{Z(1)})\big[(\mathrm{Re}\,\Sigma'_{Z(1)})^2 - \mathrm{Re}\,\Sigma'_{Z(2)} - \tfrac{1}{4}(\delta Z^{\gamma Z}_{(1)})^2 - \tfrac{1}{2}(\mathrm{Im}\,\Sigma_{Z(1)})(\mathrm{Im}\,\Sigma''_{Z(1)})\big]$$
$$+ \mathrm{Im}\,\Sigma_{Z(4)} - (\mathrm{Im}\,\Sigma_{Z(3)})(\mathrm{Re}\,\Sigma'_{Z(1)})$$
$$+ (\mathrm{Im}\,\Sigma_{Z(2)})\big[(\mathrm{Re}\,\Sigma'_{Z(1)})^2 - \mathrm{Re}\,\Sigma'_{Z(2)} - \tfrac{1}{4}(\delta Z^{\gamma Z}_{(1)})^2 - (\mathrm{Im}\,\Sigma_{Z(1)})(\mathrm{Im}\,\Sigma''_{Z(1)})\big]$$
$$+ (\mathrm{Im}\,\Sigma_{Z(1)})\big[-(\mathrm{Re}\,\Sigma'_{Z(1)})^3 + 2(\mathrm{Re}\,\Sigma'_{Z(2)})(\mathrm{Re}\,\Sigma'_{Z(1)}) - \mathrm{Re}\,\Sigma'_{Z(3)}$$
$$- \tfrac{1}{2}\delta Z^{\gamma Z}_{(1)}\,\delta Z^{\gamma Z}_{(2)} + \tfrac{1}{2}(\mathrm{Re}\,\Sigma'_{Z(1)})(\delta Z^{\gamma Z}_{(1)})^2 - \tfrac{1}{2}(\mathrm{Im}\,\Sigma_{Z(1)})(\mathrm{Im}\,\Sigma''_{Z(2)})$$
$$+ \tfrac{3}{2}(\mathrm{Im}\,\Sigma_{Z(1)})(\mathrm{Re}\,\Sigma'_{Z(1)})(\mathrm{Im}\,\Sigma''_{Z(1)}) + \tfrac{1}{6}(\mathrm{Im}\,\Sigma_{Z(1)})^2(\mathrm{Re}\,\Sigma'''_{Z(1)})\big]\Big\}_{s=\overline{M}^2_Z}\,.$$

- Also one will obtain unstable particles' total widths by imposing on shell condition. (as a consequence of optical theorem)

Results?

1956, 1-loop, Behrends, Finkelstein Sirlin

43 y ↓

1999, 2-loops, van Ritbergen, Stuart

22 y ↓

2021, 3-loops

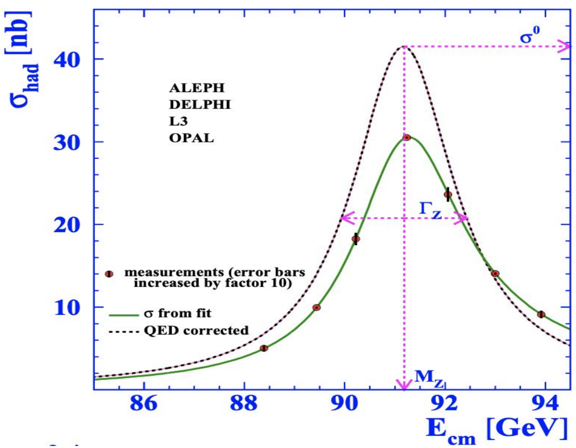
? ↓

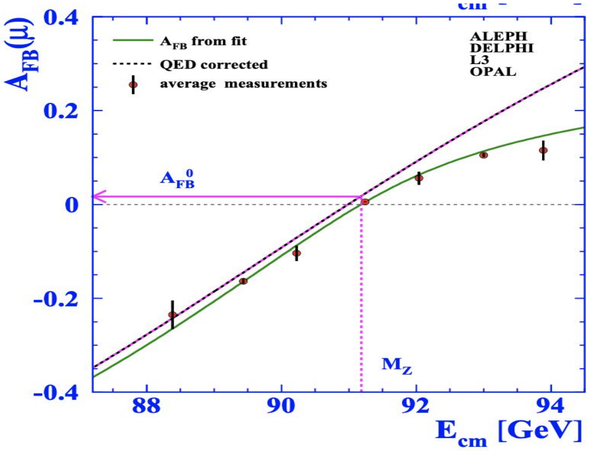
Janusz 2023 FCC week

Connect precision observables with measurements.

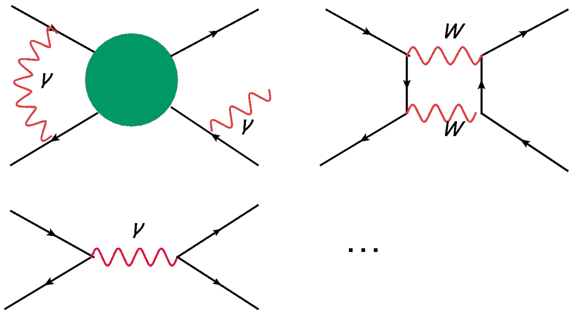
- EWPOs are "pseudo-observables".
- Most of them connect to the Z boson lineshape and asymmetries. ---need theory input to extract. (Fixed-order+resummations)

Implementation of QED effect:
1. Analytical
2. Monte Carlo tools.





LEP EWWG '05



$$\sigma[e^+e^- \to f\bar{f}] = \mathcal{R}_{ini}(s, s') \otimes \sigma_{hard}(s')$$

$$\sigma_{\rm hard} = \sigma_Z + \sigma_\gamma + \sigma_{\gamma Z} + \sigma_{\rm box}$$

shall be removed in determining EWPOs

…..

CERN-2019-003 (C4. by T.Riemman et al.)

In any case, we need to build a suitable theory framework. ZFITTER/DIZET will not be a useful basis for the FCC-ee, since it is structured to achieve consistent (1+1/2)-loop precision, but not beyond. No Laurent-series approach is foreseen in the kernel ZFITTER; but see Subsection C.4.5 on the SMATASY project and its applications to data. Further, later versions of the code lost modularity, owing to too-lazy additions concerning this item. We will have to begin developing a new program framework – probably object-oriented, e.g.,`C++` – that is general enough to be extended to any loop order and to different assumptions about QED and inputs. All the future calculations, covering up to weak three loops and QCD four loops should be performed to fit into this new framework.

- In LEP/SLD era

  ZFITTER/DIZET(D. Bardin et al) ,TOPAZ0(G.Passarino et al), and BHM/WOH(W.Hollik et al, not public)...

- In future electron-positron colliders' era

- Formally gauge invariant setup .
- Extendability that accommodates higher precision and new physics.

→ Motivates this project! (GRIFFIN: **G***auge*-**R***esonance*-**I***n*-**F***our*-**F***ermion*-**IN***teraction)*

Combining on- and off-resonance $ff \to f'\bar{f}'$ $(i,j) = (V,A)$

- Laurent expansion is suitable for describing the physics in the vicinity of the resonance. (R.Stuart 91')
  see the last part of Stefan's slides for more!
- Away from the resonance, non-expanded matrix elements (and non-Dyson-resummed ,real mass only) gives a better description.
- Full description of the Z-lineshape ?
  (simply using pole scheme is not applicable… delicated matching is required..)
- Alternative schemes? (complex-mass scheme, improved expansion of $\mathcal{A}_{ij}$, etc.)
- Ultimate goal: N3LO+leading N4LO@Z-pole, full NNLO of $ff \to f'\bar{f}'$.

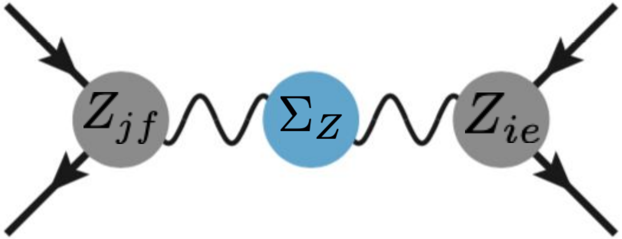
$$\mathcal{A}_{ij}\Big|_{s_0} = \frac{R_{ij}}{s - s_0} + S_{ij} + S'_{ij}(s - s_0)$$

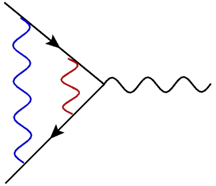
@NNLO @NLO @NLO

$$\mathcal{A}_{ij} = \mathcal{A}_{ij}\Big|_{s_0} + \mathcal{A}_{ij}^{noexp} - \mathcal{A}_{ij}\Big|_{M_Z^2}$$

Leading Pole Term R

$$\mathcal{A}_{ij} = \frac{R_{ij}}{s - s_0} + S_{ij} + (s - s_0) S'_{ij}$$



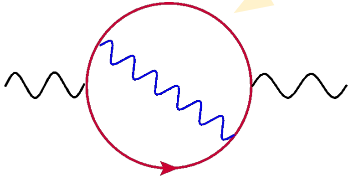


Freitas, Hollik, Walter, Weiglein'00;Amramik, Czakon '02;Onishchenko, Vertin '02; Dubovyk, Freitas, Gluza, Riemann Usovitsch '18; Freitas '14; 13Awramik, Czakon, Freitas, Weiglein '04;Hollik, Meier, Uccirati'05; Awramik,Czakon, Feritas '06...

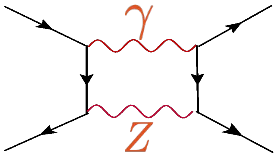
For NNLO

$$R_{ij} = \frac{Z_{ie} Z_{jf}}{1 + \Sigma'_Z} (1 + b^R_{\gamma Z})$$

already exists!

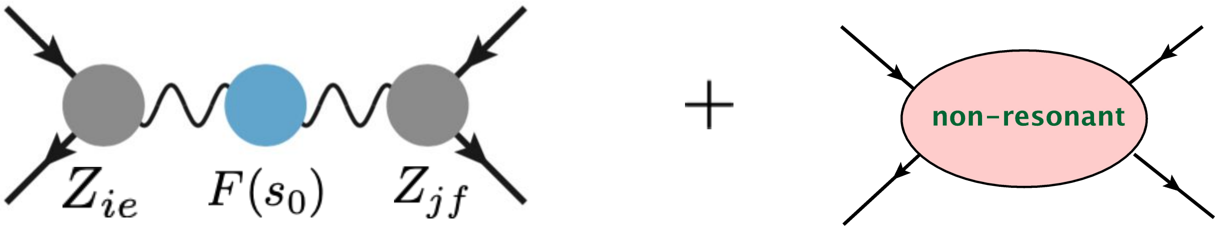


...



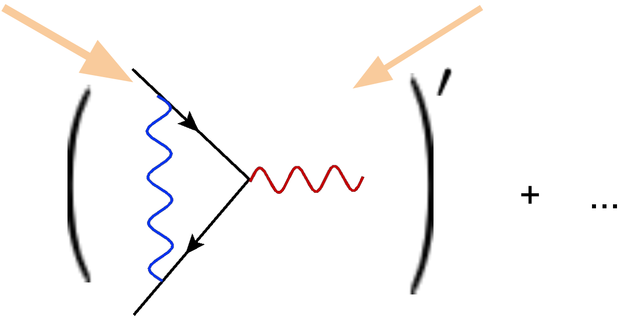
Non-resonant terms S , S'

$$\mathcal{A}_{ij} = \frac{R_{ij}}{s - s_0} + S_{ij} + (s - s_0) S'_{ij}$$



$$S_{ij} = Z'_{ie} Z_{jf} F(s_0) + Z_{ie} Z'_{jf} F(s_0) + Z_{ie} Z_{jf} F'(s_0) + B_{ij} + B_{ij}^{\gamma Z, S} \ldots$$

For NNLO :

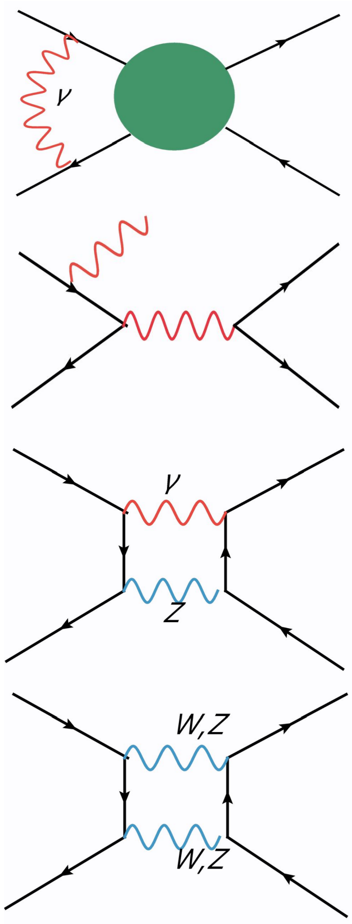


- Current state-of-art : *R*@NNLO+leading N3(4)LO,
  S@NLO,
  S'@LO.
  Off-resonance matrix elements @NLO

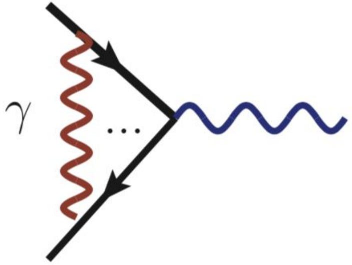
- Future projection, FCC, e.g., requires *at least one order higher* for each!)

- QED vertex contributions can be fully taken care by MC tools (e.g. KKMC S. Jadach, B.F.L.Ward, Z.Wąs).

- photon-Z boxes needs special care since they also contribute to resonant part.



- **IR subtraction scheme**: CEEX scheme (S. Jadach, B.F.L.Ward,Z.Was).

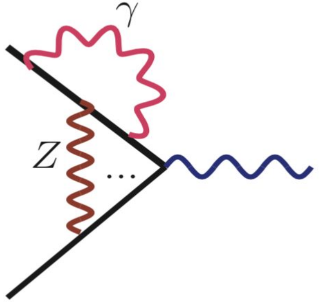


fully taken care by MC tools.

$$Z_{if}^{tot} = R_f^i \times Z_{if}$$

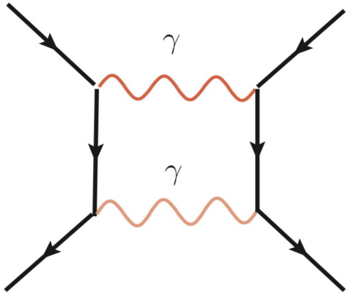
QED/QCD *factorizable* contributions excluded

$$R_f^{\mathrm{V}}(s) \equiv \frac{\mathcal{M}^{\mathrm{QED/QCD}}_{V^* \to f\bar{f}}}{\mathcal{M}^{\mathrm{Born}}_{V^* \to f\bar{f}}}, \qquad R_f^{\mathrm{A}}(s) \equiv \frac{\mathcal{M}^{\mathrm{QED/QCD}}_{A^* \to f\bar{f}}}{\mathcal{M}^{\mathrm{Born}}_{A^* \to f\bar{f}}}.$$



for non-factorizable vertex, IR finite, can be incorporated into Z_{if} order by order.

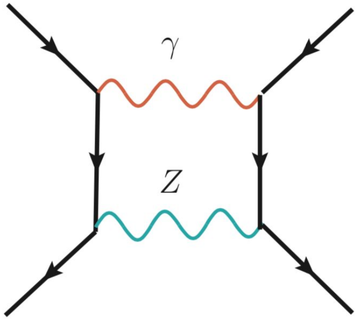
“do not obtain IR-safe obs. at given order according to Bloch-Nordsieck, instead, subtract IR parts from the amplitude if one watts to use MC generators with IR resummation.” —S. Jadach



$\gamma\gamma$ box:

$$B_{\rm VV(1)} = B^{\rm tot}_{\rm VV(1)} - S^{(0)}_{\rm VV}\,\frac{\alpha}{\pi} Q_f Q_{f'}\, f_{\rm IR}(m_\gamma, t, u)$$

$$f_{\rm IR}(m_\gamma, t, u) = \frac{2\pi}{\alpha}\left[R_{e(1)}(t) - R_{e(1)}(u)\right] = \ln\left(\frac{1-c_\theta}{1+c_\theta}\right)\left[\ln\left(\frac{2m_\gamma^2}{s\sqrt{1-c_\theta^2}}\right) + \frac{1}{2}\right]$$



$$B_{\gamma{\rm Z},ij(1)} = B^{\rm tot}_{\gamma{\rm Z},ij(1)} - \frac{R^{(0)}_{ij}}{s-s_0}\,\frac{\alpha}{\pi} Q_f Q_{f'}\left[f_{\rm IR}(m_\gamma, t, u) + \delta_G(s, t, u)\right].$$

$$\delta_G(s, t, u) = -2\ln\left(\frac{1-c_\theta}{1+c_\theta}\right)\,\ln\left(\frac{s_0 - s}{s_0}\right)$$

The Structure of the Library

`class inval`		`class psobs`	
input parameters (in the SM)		output observables	
Boson masses and widths	$M_{\mathrm{W,Z,H}}$ $\Gamma_{\mathrm{W,Z}}$	pesudo-observables defined at Z-peak	$F_{V,A}$, $\sin^2\theta^f_{eff}$ $\Gamma_{Z\to f\bar{f}}$, Δr, etc.
Fermion masses	$m^{\mathrm{OS}}_{\mathrm{e},\mu,\tau}$ $m^{\overline{\mathrm{MS}}}_{\mathrm{d,u,s,c}}(M_{\mathrm{Z}})$ $m^{\mathrm{OS}}_{\mathrm{t}}$	amplitude coefficients under pole scheme	R , S , and S'
Couplings	$\alpha(0)$ $\Delta\alpha \equiv 1-\alpha(0)/\alpha(M^2_{\mathrm{Z}})$ $\alpha^{\overline{\mathrm{MS}}}_s(M^2_{\mathrm{Z}})$, G_μ	(polarized) matrix element near or away Z-peak	M_{ij}

$$\sin^2\theta^f_{\mathrm{eff}} = \frac{1}{4|Q_f|}\left[1 - \mathrm{Re}\,\frac{Z_{Vf}}{Z_{Af}}\right]_{s=M^2_{\mathrm{Z}}},$$

$$F^f_A = \left[\frac{|Z_{Af}|^2}{1+\mathrm{Re}\,\Sigma'_Z} - \frac{1}{2}M_{\mathrm{Z}}\Gamma_{\mathrm{Z}}|a^{\mathrm{Z}}_{f(0)}|^2\,\mathrm{Im}\,\Sigma''_Z\right]_{s=M^2_{\mathrm{Z}}} + \mathcal{O}(\alpha^3),$$

$$F^f_V = \left[\frac{|Z_{Vf}|^2}{1+\mathrm{Re}\,\Sigma'_Z} - \frac{1}{2}M_{\mathrm{Z}}\Gamma_{\mathrm{Z}}|v^{\mathrm{Z}}_{f(0)}|^2\,\mathrm{Im}\,\Sigma''_Z\right]_{s=M^2_{\mathrm{Z}}} + \mathcal{O}(\alpha^3)$$

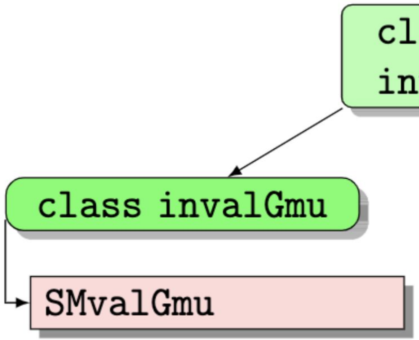
$$= F^f_A\left[(1-4|Q_f|\sin^2\theta^f_{\mathrm{eff}})^2 + \left(\mathrm{Im}\,\frac{Z_{Vf}}{Z_{Af}}\right)^2\right]$$

$$\Gamma_f[Z\to f\bar{f}] \equiv \frac{N^f_c M_Z}{12\pi}(\mathcal{R}_V F^f_V + \mathcal{R}_A F^f_A)$$

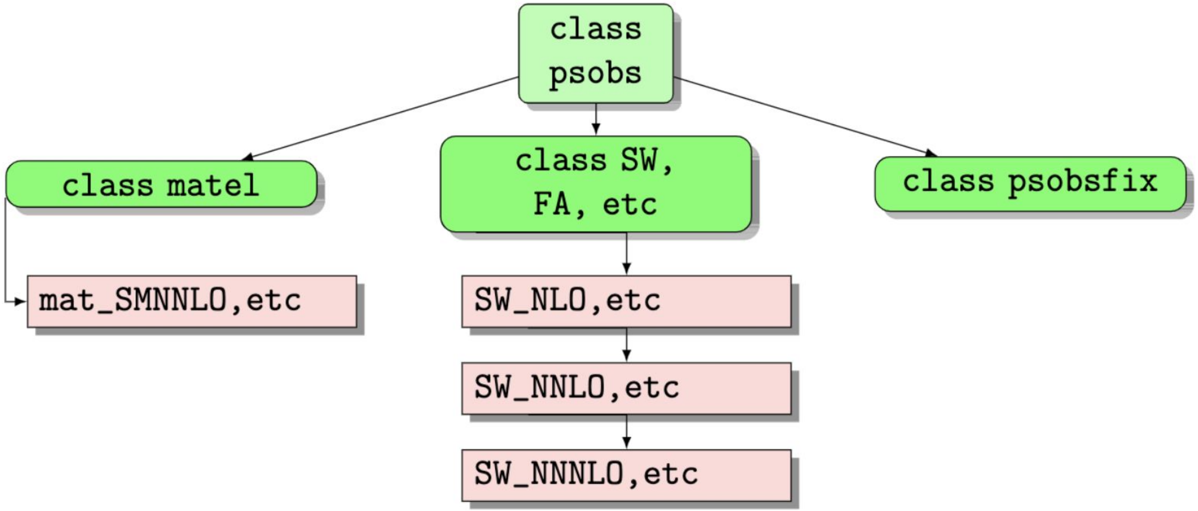
$$R = \mathcal{R}(F_A, \sin^2\theta_{eff}, b^R_{\gamma Z}, \ldots)$$

$$S = \mathcal{S}(Z_{ie}, Z_{jf}, Z'_{ie}, Z'_{jf}, \Sigma_Z, B_{ij}, \ldots)$$

- Class structure of the input:



- Class structure of the output:



Example of the Code

- setting the input.

$$\overline{M} = M^{\text{exp}}/\sqrt{1+(\Gamma^{\text{exp}}/M^{\text{exp}})^2}\ ,\overline{\Gamma} = M^{\text{exp}}/\sqrt{1+(\Gamma^{\text{exp}}/M^{\text{exp}})^2}.$$

```
#include "SMval.h"
int main()
{
  SMval myinput; //defining the input set as an object of class SMval
  myinput.set(MZ, 91.1876);
  myinput.set(GamZ, 2.4966);

  cout << myinput.get(MZc) << endl; //output the Z-boson mass in complex-
   pole mass scheme
}
```

Example 2.2.1 Setting input values, with conversion of the gauge-boson masses and widths from PDG value to complex-pole masses and widths

- defining the virtual function that evaluates the form factors or observables,...

Example 2.6.1 An example of defining function `result()` in the scope of the derived class `SW_SMLO`

```
Cplx SW_SMLO::result(void) const
{
  return(...); // the expression of effective weak-mixing angle defined at
    the LO SM
}
```

- the main file.

```
#include "EWOPZ2.h"
#include "SMval.h"
int main()
{
  SMval myinput; //defining the input set as an object of class SMval
  myinput.set(MZ, 91.1876);
  myinput.set(GamZ, 2.4966);
  ...            // more input parameters to be set up
  //defining objects from classes FA_SMLO and SW_SMLO
  FA_SMNNLO FA21(LEP, myinput);
  SW_SMNNLO SW21(LEP, myinput);
  //radiative correction for FA and SW due to delta_rho at O(alpha_t*alpha_s
    ^2)
  cout <<FA21.drho3aas2()<<endl;
  cout <<SW21.drho3aas2()<<endl;

  cout <<"the_leading-order_FA^lep"<< FA1.result() << endl; //output the F_A
    ^l at NNLO
  cout <<"the_leading-order_SW^lep"<< SW1.result() << endl; //output the SW^
    l at NNLO
}
```

Figure 3: The example of outputting numerical results of $F^f_{V,A}$ and $\sin^2\theta^f_{\rm eff}$ at NNLO in the SM.

- An example of output from `testmatel.cc`

the form factors/EWPOs used in building matrix elements.

evaluations of helicity matrix elements at demanded order.

differential x-section of Z-lineshape (hard) at demanded order

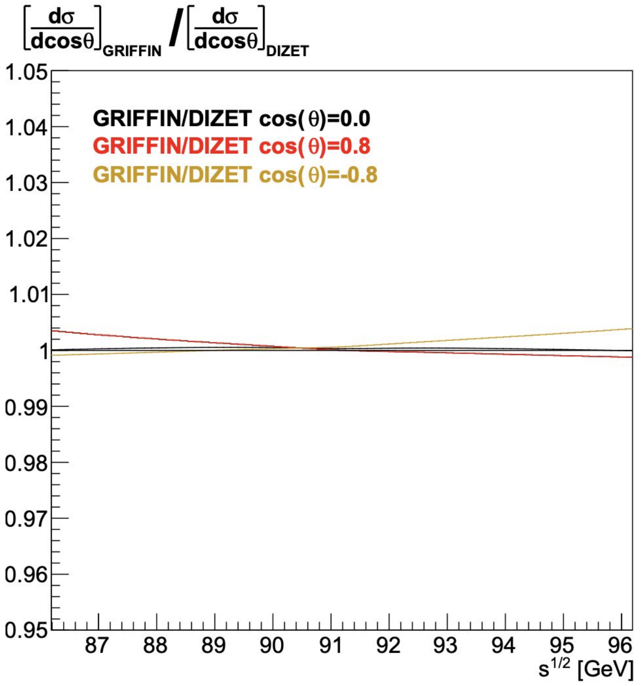
```
======================================================

======================================================
    ______ ____   ____ ______ ______ ____ _   __
   / ____// __ \ /  _// ____// ____//  _// | / /
  / / __ / /_/ / / / / /_   / /_    / / /  |/ /
 / /_/ // _, _/_/ / / __/  / __/  _/ / / /|  /
 \____//_/ |_|/___//_/    /_/    /___//_/ |_/
======================================================
                 version 1.0
======================================================

Complex-pole masses: MW=80.35, MZ=91.1535

=== Matrix element for ee->dd (i=e, f=d) ===

F_A^i (NNLO+) = (0.034499,0)
F_A^f (NNLO+) = (0.0345443,0)
sineff^i (NNLO+) = (0.231172,0)
sineff^f (NNLO+) = (0.230985,0)

SM matrix element M_VV for cos(theta)=0.5:
sqrt(s)         tot. result                 off-resonance contrib.
10       (0.000316739,-5.58082e-06)      (0.000309429,-5.53734e-06)
30       (3.53793e-05,-5.99317e-07)      (2.84458e-05,-5.59139e-07)
50       (1.25851e-05,-1.90789e-07)      (6.4247e-06,-1.59184e-07)
70       (6.07798e-06,-5.97311e-08)      (1.19433e-06,-4.81728e-08)
90       (-7.31188e-07,-3.55673e-06)     (8.7104e-09,-1.80673e-09)
110      (3.14635e-06,-1.62001e-07)      (4.59289e-07,1.10821e-08)
130      (2.12596e-06,-7.90095e-08)      (1.82894e-06,1.92144e-08)
150      (1.5668e-06,-5.34561e-08)       (3.83515e-06,2.49419e-08)
170      (1.20884e-06,-3.97403e-08)      (6.35319e-06,2.97998e-08)
190      (9.60973e-07,-3.33532e-08)      (9.31833e-06,3.12732e-08)

diff. cross-section for cos(theta)=0.5:
sqrt(s) dsig/dcos [nb]
10       0.14256
30       0.0133563
50       0.00341281
70       0.00700565
90       2.36867
110      0.0325627
130      0.0106802
150      0.00583374
170      0.00383656
190      0.00276578
```

Implementation of the higher-order contributions.

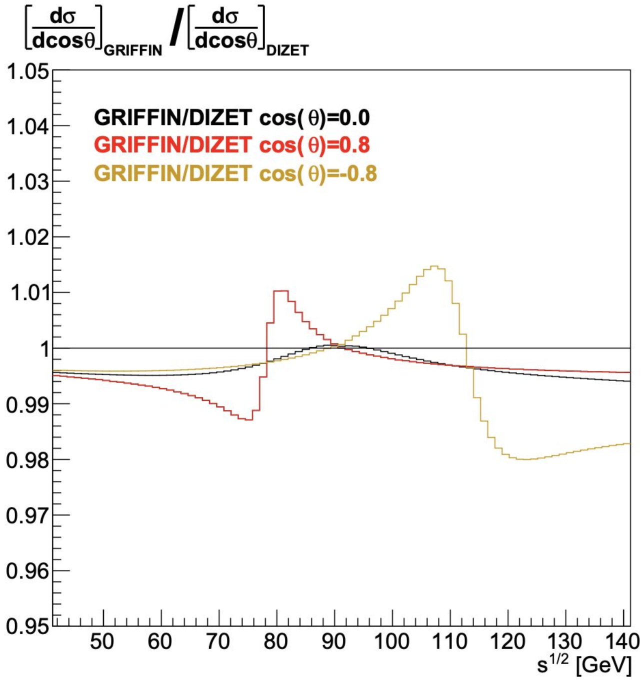
Corrections entering through $\delta\rho$:			
	drho2aas	$\mathcal{O}(\alpha_t \alpha_s)$	
	drho2a2	$\mathcal{O}(\alpha_t^2)$	
*	drho3aas2	$\mathcal{O}(\alpha_t \alpha_s^2)$	
*	drho3a2as	$\mathcal{O}(\alpha_t^2 \alpha_s)$	
*	drho3a3	$\mathcal{O}(\alpha_t^3)$	
*	drho3aas3	$\mathcal{O}(\alpha_t \alpha_s^3)$	
Full corrections to F_A^f, $\sin^2\theta_{\text{eff}}^f$:			
*	res2ff	$\mathcal{O}(\alpha_f^2)$	
*	res2fb	$\mathcal{O}(\alpha_f \alpha_b)$	
*	res2bb	$\mathcal{O}(\alpha_b^2)$	
*	res2aas	$\mathcal{O}(\alpha \alpha_s)$	(correction to internal gauge-boson self-energies)
*	res2aasnf	$\mathcal{O}(\alpha \alpha_s)$	(non-factorizable final-state corrections for $f = q$)
*	res3fff	$\mathcal{O}(\alpha_f^3)$	
*	res3ffa2as	$\mathcal{O}(\alpha_f^2 \alpha_s)$	

* asterisk indicates the contribution that can be summed up as a meaningful result.

On the Z-pole. δ=griffin/dizet < 0.001 *~N3LO*



Off-resonance region. δ ~ 0.001- 0.02 *~NNLO*



*the authors are indebt to **S.Jadach** and his group for providing the test program on KKMCee.

On-going MC-interfacing with YFS-Sherpa.

NNLO Corrections with GRIFFIN

Test Process $e^+e^- \to \mu^+\mu^-$ at 91.2GeV

GRIFFIN: A C++ library for EW radiative corrections 2211.16272

Developed by A. Freitas and L.Chen

Born	YFS	YFS+Recola	YFS+GRIFFIN
2114.5 pb	1463.09 pb	1494.7(8) pb	1497.5(7) pb

Order 0.1% difference between NLO and NNLO

Expected?

Ongoing validation effort with Ayres

GRIFFIN=**G**auge-**R**esonance-**I**n-**F**our-**F**ermion-**IN**teraction

Griffin Integration time: **30s** with 8 cores

Recola Integration time: **4mins** with 8 cores

Alan. Price Sherpa
2023. annual meeting

How to cook BSM models with GRIFFIN?

- ❑ **Step 1:** Define new parameters in `classes.h` (`class inval`)
- ❑ **Step 2:** Add new building blocks (form factors, self-energies, etc) in `ff.*` files.
- ❑ **Done! And have fun!**

$$R_{ij} = \left.\frac{Z_{ie}Z_{jf}}{1+\Sigma'_Z}\right|_{s=s_0} + B^R_{\gamma Z,ij} + B^{RL}_{\gamma Z,ij}\ln(1-\tfrac{s}{s_0})\,, \tag{9}$$

$$S_{ij} = \left[\frac{Z_{ie}Z'_{jf}+Z'_{ie}Z_{jf}}{1+\Sigma'_Z} - \frac{Z_{ie}Z_{jf}\Sigma''_Z}{2(1+\Sigma'_Z)^2} + \frac{G_{ie}G_{jf}}{s+\Sigma_{\gamma\gamma}} + B_{ij}\right]_{s=s_0} + B^S_{\gamma Z,ij} + B^{SL}_{\gamma Z,ij}\ln(1-\tfrac{s}{s_0})\,, \tag{10}$$

$$S'_{ij} = \left[\frac{Z_{ie}Z''_{jf}+Z''_{ie}Z_{jf}+2Z'_{ie}Z'_{jf}}{2(1+\Sigma'_Z)} - \frac{(Z_{ie}Z'_{jf}+Z'_{ie}Z_{jf})\Sigma''_Z + \frac{1}{3}Z_{ie}Z_{jf}\Sigma'''_Z}{2(1+\Sigma'_Z)^2} + \frac{Z_{ie}Z_{jf}(\Sigma''_Z)^2}{4(1+\Sigma'_Z)^3} + \frac{G_{ie}G'_{jf}+G'_{ie}G_{jf}}{s+\Sigma_{\gamma\gamma}} - \frac{G_{ie}G_{jf}(1+\Sigma'_{\gamma\gamma})}{(s+\Sigma_{\gamma\gamma})^2} + B'_{ij}\right]_{s=s_0} + B^{S'}_{\gamma Z,ij} + B^{S'L}_{\gamma Z,ij}\ln(1-\tfrac{s}{s_0})\,, \tag{11}$$

where

$$Z_{Vf}(s) = v^Z_f(s) + v^\gamma_f(s)\frac{\Sigma_{\gamma Z}(s)}{s+\Sigma_{\gamma\gamma}(s)}, \qquad G_{Vf}(s) \equiv v^\gamma_f(s), \tag{12}$$

$$Z_{Af}(s) = a^Z_f(s) + a^\gamma_f(s)\frac{\Sigma_{\gamma Z}(s)}{s+\Sigma_{\gamma\gamma}(s)}, \qquad G_{Af}(s) \equiv a^\gamma_f(s), \tag{13}$$

$$\Sigma_Z(s) = \Sigma_{ZZ}(s) - \frac{[\Sigma_{\gamma Z}(s)]^2}{s+\Sigma_{\gamma\gamma}(s)}. \tag{14}$$

Alternative scheme to depict the full SM Z-lineshape prediction?

A possible scheme works beyond NNLO?

$$\mathcal{A}_{ij} = \mathcal{A}_{ij}\Big|_{s_0} + \mathcal{A}_{ij}^{noexp} - \mathcal{A}_{ij}\Big|_{M_Z^2}$$

$$= \frac{\overline{R}'_{ij}}{(s - M_\mathrm{Z}^2)^2} + \frac{\overline{R}_{ij}}{s - M_\mathrm{Z}^2} + \overline{S}_{ij} + (s - M_\mathrm{Z}^2)\overline{S}'_{ij} + \ldots$$

Instead, we do…

$$M_{ij}^{\mathrm{exp},M_\mathrm{Z}^2} = \mathcal{T}_\alpha\left\{\left[\frac{R_{ij}}{s - s_0} + S_{ij} + (s - s_0)S'_{ij} + \ldots\right]_{s_0 = M_\mathrm{Z}^2 - iM_\mathrm{Z}\alpha\Gamma_\mathrm{Z}^{(1)}}\right\}$$

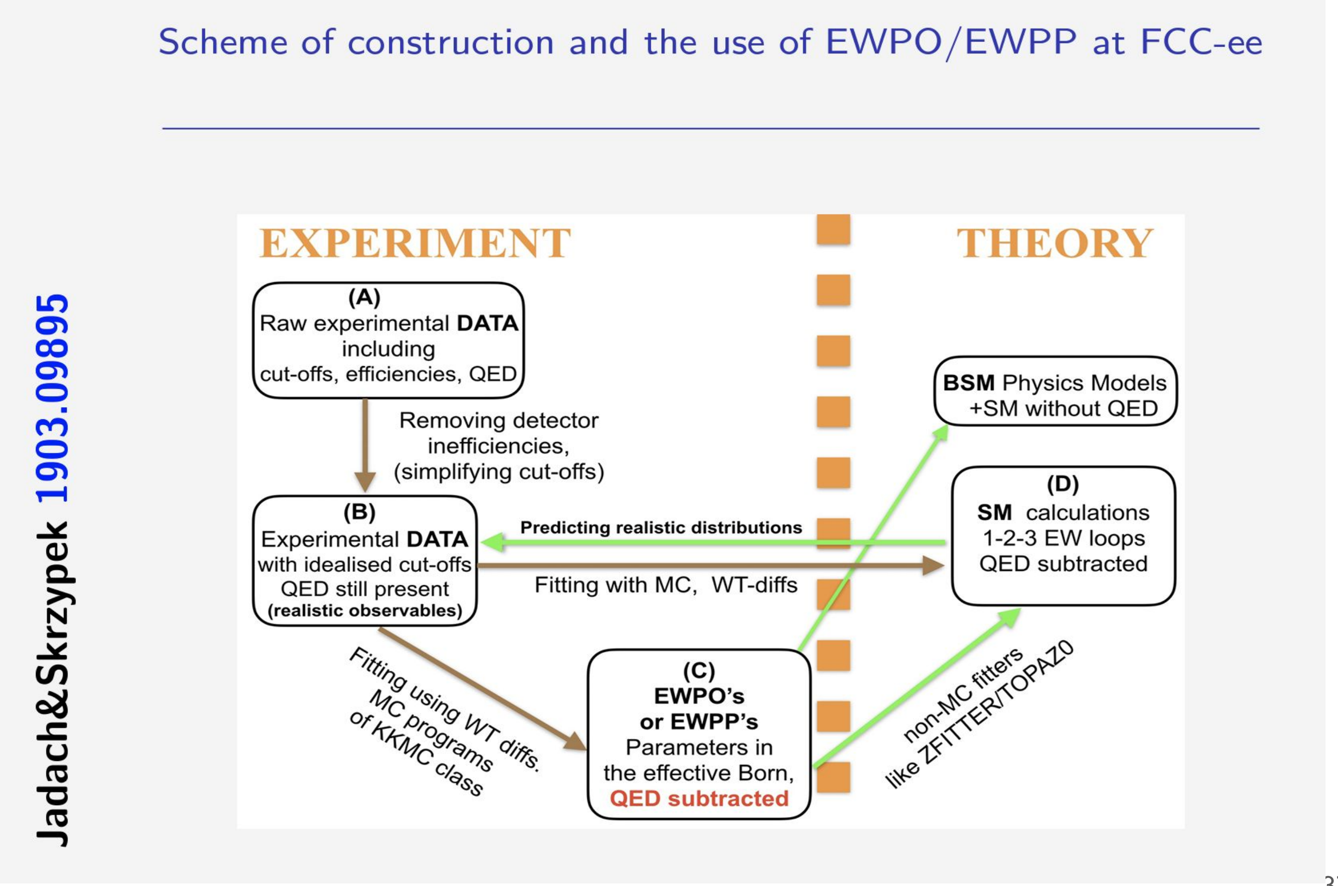
complex mass scheme? etc….

Retrospection on EWPO and/or EWPP

- ❑ Encapsulate UV.
- ❑ Insensitive to IR.
- ❑ Experimentally retrievable.
- ❑ Model-independent.
- ❑ sensitive to new physics.
- ❑ etc.

…

- ❑ *paradigm inherited from LEP may not suitable for FCCee!*



Future/On-going projections:

- ❑ Interfacing with MC tools (KKMC, YFS-Sherpa, POWHEG-EW, etc) ***on-going!***
- ❑ Alternative schemes regarding to the resonance, full-range, IR-subtractions/factorizations, renormalizations… ***on-going!***
- ❑ Including orders *beyond* NNLO @ Z-pole, NNLO *away from* Z-pole, *Bhabhar* ME, etc. ***on-going!***
- ❑ study of BSM, SMEFT. ***on-going!***
- ❑ Other 4-fermion interaction processes. (e.g. Drell-Yan at the HL-LHC)

https://github.com/lisongc/GRIFFIN/releases/tag/v1.0.0

we welcome feedbacks, suggestions, contributions/collaborations from the community!

Summary

- ❏ EWPOs need to be carried out at 3-loop EW/mixed QCD-EW, and leading 4-loop level to match the targeting precision given by future ee colliders.
- ❏ numerical techniques are **indispensable** in those calculations but challenges remain such as MIs reduction…
- ❏ To make the calculation useful, many other theoretical aspects should be stressed *is the profile of x-sec (QED-deconv) still valid?, better EWPOs/EWPPs? what is an optimal scheme to sketch the resonance(unstable particles)?*

- ❏ GRIFFIN, as a ***EW library***, provides a gauge-invariant, theoretically consistent description of 4-fermion scattering with a wider range of cme. It can systematically include higher-order contributions. The results has been validated and checked with **DIZET v6.45**(A. Arbuzov, J.Gluza, et al. '19&'23)

- ❏ GRIFFIN can also play as template for EWPOs fitting, a powerful tool to inspect the EWPO/EWPP candidates via B->D or B->C->D->B.

Backup Slides

Preliminary results and comparison with ZFITTER/DIZET

- ❑ **Benchmark inputs:**

GRIFFIN input parameters	
DIZET input parameters	DIZET output
$\alpha_s(M_Z^2) = 0.118, \quad \alpha = 1/137.035999084$	$\Gamma_Z = 2.495890$ GeV
$\Delta\alpha = 0.059, \quad M_Z = 91.1876$ GeV, $\quad G_\mu = 1.166137 \times 10^{-5}$	$M_W = 80.3599$ GeV
$m_t = 173.0$ GeV, $\quad M_H = 125.0$ GeV, $\quad m_{e,\mu,\tau,u,d,s,c,b} = 0$ GeV	$\Gamma_W = 2.090095$ GeV

- ❑ **using the W-mass and W-width output from dizet to minimize the parametrical shift between two schemes.**

❑ Numerical Results:

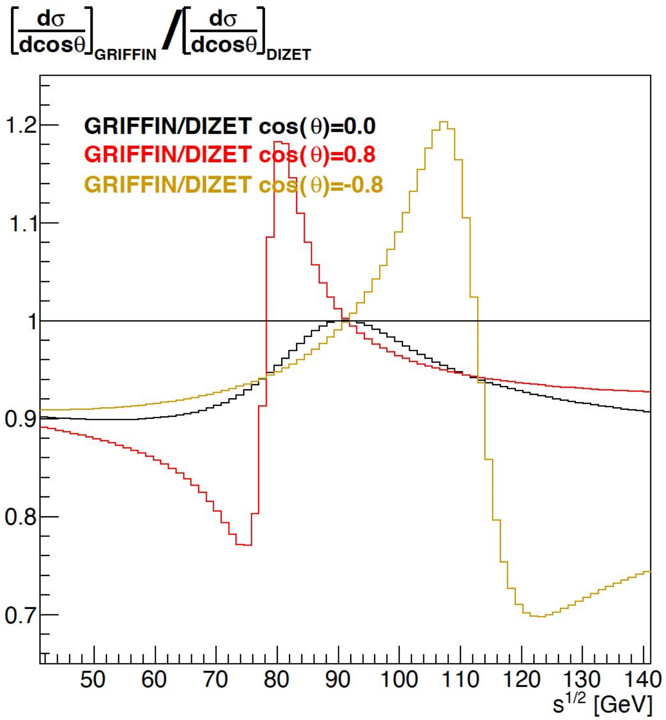
$$|\rho_Z^f| = \frac{2\sqrt{2}F_A^f}{G_\mu M_Z^2}$$

	$\lvert\rho_Z^f\rvert$		$\sin^2\theta_{\rm eff}^f$		$\Gamma_{Z\to f\bar{f}}$	
	DIZET 6.45	GRIFFIN	DIZET 6.45	GRIFFIN	DIZET 6.45	GRIFFIN
$\nu\bar{\nu}$	1.00800	1.00814	0.231119	NAN	0.167206	0.167197
$\ell\bar{\ell}$	1.00510	1.00519	0.231500	0.231534	0.083986	0.083975
$u\bar{u}$	1.00578	1.00573	0.231393	0.231420	0.299938	0.299958
$d\bar{d}$	1.00675	1.00651	0.231266	0.231309	0.382877	0.382846
$b\bar{b}$	0.99692	0.99420	0.232737	0.23292	0.376853	0.377432

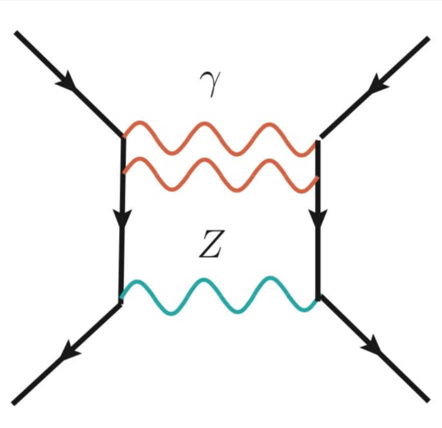
	DIZET 6.45	GRIFFIN all orders	GRIFFIN $\mathcal{O}(\alpha, \alpha^2, \alpha_t\alpha_s, \alpha_t\alpha_s^2)$
Δr	3.63947×10^{-2}	3.68836×10^{-2}	3.63987×10^{-2}

- ❑ Not a **one-one-one match.** (no leading N3LO implemented in dizet v.6.45)
- ❑ most numbers are in agreement up to at least **4-digit**. The actual discrepancy is in the realm of missing N3(4)LO.
- ❑ fictitious discrepancies stem from the input scheme/definition of the form factors/EWPOs.

Discrepancies between NLO/LO ~20-30%



Do we need?



- power counting $\alpha \sim \frac{\Gamma_Z}{M_Z}$
- For σ at N^{n}LO, we need n-loop at ***R,*** n-1-loop at ***S,*** n-2 at ***S'***
- Since

$$\delta_{ifi} = \frac{\alpha}{\pi}\frac{k_{0,min}}{E_{\text{beam}}} \ll \frac{\alpha}{\pi}\frac{\Gamma_Z}{M_Z} \sim \mathcal{O}(\alpha^2)$$

(S. Jadach et al. '00)

near the resonance. ***Is this valid at FCCee??***

In ZFITTER/DIZET:

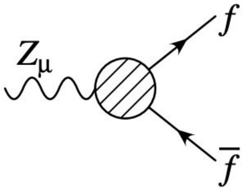
$$\Gamma_{Zf\bar{f}} = \Gamma_0 c_f \left|\rho_Z^f\right| \left(\left|g_Z^f\right|^2 R_V^f + R_A^f\right) + \delta_{\alpha\alpha_s}$$

$$\sin^2\theta_{eff}^f = \left(1 - \frac{M_W^2}{M_Z^2}\right)\left(1 + \Delta\kappa_f\right)$$

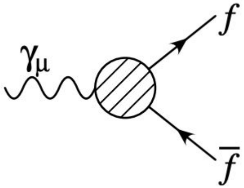
Conversion:

$$\left|\rho_Z^f\right| = \frac{2\sqrt{2}F_A^f}{G_\mu M_Z^2}$$

An example of the numerical impact given by non-consistenly using pole scheme (M. Awramik, M. Czakon, A. Freitas ‘06)



$$\equiv \Gamma[Z_\mu f \bar{f}] \equiv z_{\mathrm{f},\mu} = i\gamma_\mu(v_\mathrm{f} + a_\mathrm{f}\gamma_5)$$



$$\equiv \Gamma[\gamma_\mu f \bar{f}] \equiv g_{\mathrm{f},\mu} = i\gamma_\mu(q_\mathrm{f} + p_\mathrm{f}\gamma_5),$$

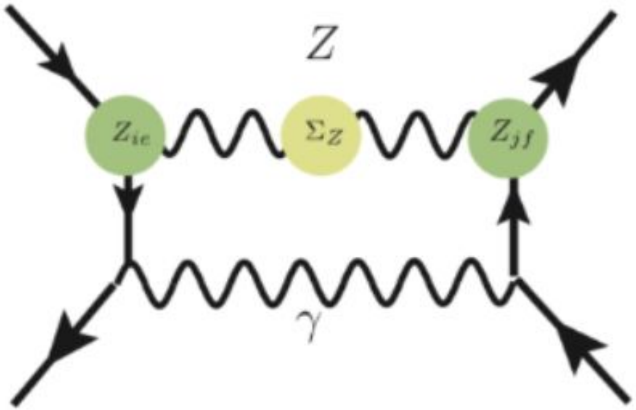
$$\delta s_W^2 = \sin^2\theta^f_{eff,ZFITTER} - \sin^2\theta^f_{eff,pole\,scheme}$$

$$= -\frac{\Gamma_Z}{M_Z}\frac{q_f^{(0)}}{a_e^{(0)}(a_f^{(0)} - v_f^{(0)})}(\Im p_e^{(1)} + \Im B_{ij}^{(1)}) \sim \mathcal{O}(10^{-6})$$

$$\delta^{exp}\sin^2\theta^f_{eff,\,FCC,\,CEPC} \sim \mathcal{O}(10^{-6})$$

❑ Pole scheme for gamma-Z box diagram.

$$B_{\gamma Z} \sim \int \frac{d^4q}{(2\pi)^4} \frac{\cdots}{q^2(\not{q}-\not{p}_2)(\not{q}-\not{k}_2)} \underbrace{\frac{Z_i(s',s_i)Z_f(s',s_f)}{s'-m_{Z0}^2+\Sigma_Z(s')}}_{W(s',s_i,s_f)}$$



$$s' = (q+p_2+p_1)^2\,, \quad s_i = (q+p_2)^2\,, \quad s_f = (q+k_2)^2$$

$$\begin{aligned} W(s',s_i,s_f) &= \frac{Z_i(s',s_i)Z_f(s',s_f)}{s'-m_Z^2+\Sigma_Z(s')} \\ &= \frac{Z_i(s_0,0)Z_f(s_0,0)+Z_i(s',s_i)Z_f(s',s_f)-Z_i(s_0,0)Z_f(s_0,0)}{s'-s_0+\Sigma_Z(s')-\Sigma_Z(s_0)} \\ &= \frac{Z_i(s_0,0)Z_f(s_0,0)}{(s'-s_0)(1+\Sigma_Z'(s_0))} + \frac{Z_i(s',s_i)Z_f(s',s_f)-Z_i(s_0,0)Z_f(s_0,0)}{s'-s_0+\Sigma_Z(s')-\Sigma_Z(s_0)} \\ &\equiv \frac{P(s_0)}{s'-s_0} + N(s',s_i,s_f) \end{aligned}$$