FOUR-LEPTON SCATTERING IN MASSIVE QED

BHABHA AND MØLLER SCATTERING UP TO TWO LOOPS

QCD meets EW CERN - 07/02/2024

Lorenzo Tancredi - Technical University Munich

[Collaboration with Delto, Duhr, Zhu — arXiv:2311.06385, arXiv:24xx.xxxxx] [and ongoing work with Duhr, Maggio, Nega, Wagner — arXiv:2305.14090, arXiv:24xx.xxxxx]

Technische Universität München

INTRODUCTION: BHABHA AND MØLLER SCATTERING

electroweak theory prediction at tree level in terms of the weak mixing angle is *Q^e*

Bhabha $e^+e^- \rightarrow e^+e^$ the 1-loop level [4–6] and becomes dependent on the energy scale at which the measurement is carried out, *i.e.* sin² ✓*^W*

Møller $e^-e^- \to e^-e^$ are well under control, and the planned future work will reinforce that conclusion. The planned future work will reinforce that conclusion α

W = 14 sin2 decompose at 2 sin
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Basic processes in QED, received a lot of attention since the birth of QFT *(see Landau's fourth book)*

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High-energy lepton colliders

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small angle scattering efficient tool for luminosity determination
a lepton colliders (radiative corrections OED dominated) *W* lepton colliders (radiative corrections QED dominated) The dominant e↵ect comes from the " *Z* mixing" diagrams depicted in Fig. 2 [5]. The prediction for *AP V* for the \sim ϵ 5 mall angle scattering efficient tool for luminosity determination
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electroweak theory prediction at tree level in terms of the weak mixing angle is *Q^e*

 $\sqrt{s} \sim \mathcal{O}(\text{GeV})$ colliders (flavour factories BELLE, BABAR, ...) + e^+e^- and e^+e^- and e^+e^- and e^+ in principle ILC. $\begin{array}{lll}\n\bullet^{\dagger} & \text{in principle ILC!} \\
\end{array}$ in principle ILC!

Møller $e^-e^- \rightarrow e^-e^-$

 $\overline{\mathbf{Z}}$

 $\overline{\mathbb{Z}}$

are well under control, and the planned future work will reinforce that controllers will reinforce that control in the planned future work will reinforce that conclusion control in the planned future work will reinforce th

 $\gamma \lesssim \chi$ Particularly relevant @ PRad-II (attempt to resolve proton radius - Dominant physical process in low-energy electron scattering $\frac{1}{2}$ arXiv:1903.09265) $\frac{1}{2}$ mass effects should not be mega e^- arXiv:1903.09265 en
E entities of 2.5 MeV (Sec. experiments, also used for luminosity monitoring. puzzle), and recently measured down to energies of 2.5 MeV (see

 \mathbf{v}

 $\overline{\mathbf{Z}}$

 $\overline{\mathbb{Z}}$

Also relevant to measure weak mixing angle …

 $v_{\rm e}$ Basic processes in QED, received a lot of attention since the birth of QFT *(see Landau's fourth book)*

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 $\overline{\mathbb{Z}}$

INTRODUCTION: BHABHA AND MØLLER SCATTERING

State-of-the-art in QED (ignoring other EW effects here)

NLO QED effects known exactly in Bhabha and Møller with full mass dependence

NNLO QED effects with *full mass dependence remain elusive* due to *missing twoloop amplitudes*

Leading order mass effects **[Becher, Melnikov '07]**

Leading power-suppressed mass effects also included **[Penin, Zerf '16]**

Next-to-soft stabilisation for real-virtual matrix elements **[Banerjee et al '21]**

NNLO Møller including leading order mass effects & next-to-soft stabilisation **[Banerjee et al '22]**

Fermionic loop corrections with full mass dependence in Bhabha **[Bonciani et al '15]**

To have full control on low energy / small angle regions, full mass dependence desirable ⟶**two-loop amplitudes remains last missing ingredient**

INTRODUCTION: HISTORY CALCULATION OF TWO-LOOP AMPLITUDE ULAIIUN UF TWU-LUUP AMPLITUDE

Full massless two loop amplitudes in terms of HPLs [Bern, Dixon, Ghinculov '00]

 $\frac{1}{2}$

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Ten years later, ladder planar integrals in terms of MPLs **[Henn, Smirnov, Smirnov '13]** p. Smirnov, Smirnov, ¹³

INTRODUCTION: HISTORY CALCULATION OF TWO-LOOP AMPLITUDE ULAIIUN UF TWU-LUUP AMPLITUDE $\frac{1}{2}$

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Eight more years for second planar family [Duhr, Smirnov, Tancredi '21]

d-logs but four square roots not rationalisable simultaneously

exploiting the fact that they don't mix, one can write results in terms of MPLs, but extremely cumbersome two-loop virtual correction to the 2 → 2 differential cross section. The rules for implementing CDR

MASSES AND GEOMETRY

What about the **non-planar integrals?**

.

*^T*³ ⁼ *^t*⁴ *, T*⁴ ⁼ *^m*² ⇥ *^t*⁵ *, T*⁵ = *m* ⇥ [*t*⁶ + *t*7] + *t*⁸ *, T*⁶ = *m* ⇥ [*t*⁶ + *t*7] *t*⁸ WHJJLJ AND ULUMLINI ¹ = *{*1*,* 1*} ,* ² = *{p/*3*,* 1*} ,* ³ = *{*1*, p/*2*} ,* ⁴ = *{p/*3*, p/*2*} ,* **MASSES AND GEOMETRY**

⁵ ⁼ *{^µ*¹

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Mathematically, things start becoming rather interesting in NPL sector In particular, we are interested in the non-planar family lilatically, things start becoming rather int $\frac{1}{\pi}$ in NDI sector lines correspond to massive propagators of mass *m*, Mathematically, things start becoming rather interesting in NPL sector

*, ^µ*¹ *} ,*

$$
I_{a_1a_2a_3a_4a_5a_6a_7a_8a_9} \left(D, \frac{s}{m^2}, \frac{t}{m^2}\right) =
$$

$$
e^{2\gamma_E \epsilon} (\mu^2)^{\sum\limits_{j=1}^9 a_j - D} \int \frac{d^D k_1}{i\pi^{\frac{D}{2}}} \frac{d^D k_2}{i\pi^{\frac{D}{2}}} \prod\limits_{j=1}^9 \frac{1}{P_j^{a_j}},
$$

$$
\frac{\partial^3}{\partial z^2}, \frac{t}{m^2}\bigg) = \n\begin{cases}\nP_1 = k_1^2 - m^2, & P_2 = (k_1 - k_2 - p_2)^2 - m^2, \\
P_3 = k_2^2 - m^2, & P_4 = (k_2 + p_1 + p_2)^2 - m^2, \\
P_5 = (k_1 + p_1)^2, & P_6 = (k_1 - k_2)^2, \\
\frac{D}{i\pi} \frac{D}{2} \frac{1}{i\pi \frac{D}{2}} \prod_{j=1}^9 \frac{1}{P_j^{a_j}}, & P_8 = (k_2 + p_1)^2, \\
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P_3 &= k_2^2 - m^2, & P_4 &= (k_2 + p_1 + p_2)^2 - m^2, \\
P_5 &= (k_1 + p_1)^2, & P_6 &= (k_1 - k_2)^2, & P_7 &= (k_2 - p_3)^2, \\
\frac{D}{i\pi^{\frac{D}{2}}} \frac{1}{i\pi^{\frac{D}{2}}} \prod_{j=1}^9 \frac{1}{P_j^{a_j}}, & P_8 &= (k_2 + p_1)^2, & P_9 &= (k_1 - p_3)^2.\n\end{aligned}
$$

 Δ loebraically " are mapped onto each other, i.e. *t*² \$ *t*³ and *t*⁶ \$ *t*7, where *^E* denotes the Euler-Mascheroni constant, *D* = Algebraically "simple" for today's standards **Superintegrals in the render Few III rendered integrals in the render Few III** Algebraically "simple" for today's standards: 2 din \overline{a} and two-loop integrals into \overline{a} erality, we may not *ionly* and *nasters* integrals patch [*y* : *z* : 1] with *y* = *s/m*² and *z* = *t/m*². For **Algebraically "simple"** for today's standards: 2 dimensionless ratios, "only" 52 masters integrals

MASSES AND GEOMETRY MASSES AND UEUME =

2 + ^q*y*(*y*+*z*4)

s > 4*m*² *,t <* 0 *,* though all results can also be easily

More in detail integrals can be expressed in the left graph of \mathcal{M}

pearance of new mathematical functions of elliptic type.

 $\begin{array}{cc} \hline \begin{array}{ccc} 6 & 6 & 7 \end{array} & \text{6 propagator graph:} & \text{I}_{110111100} \end{array}$ 6 master integrals in top sector (+ sub-topologies) α , α , β , i.e., α is a six independent master integrals in top sector (plus) $\begin{array}{r} 6, & \dots, & 6 \end{array}$ 6 master integrals in top sector (+ sub-topologies) d~I = *A* (✏*, y, z*)~I*.* (9) τ $\frac{1}{110111100}$

arithmic di↵erential forms, this matrix is said to be in

ance of additional transcendental integrals. In this way,

I eading singularities (maximally iterated integr dashed lines correspond to massless propagators. Leading singularities (maximally iterated integrand residues) fulfil homogeneous **d** equation a \mathbf{I} $\tilde{\mathbf{C}}$ μ , be done to but α is control, starting from the six-point of α differential equation and can be used to build space of solutions [Primo, Tancredi '16,'17]

> *t* ◆ cutting all propagators (max cut). Conveni Start cutting all propagators (max cut). Convenient in Baikov **[Frellesvig, Papadopoulos '17]**

$$
\text{MaxCut}_{\mathcal{C}}\left[I_{110111100}\right] \sim \int_{\mathcal{C}} \frac{dz_2 \wedge dz_1}{z_2 \sqrt{(z_1 - s - z_2)(z_1 - s + 4m^2 - z_2)} \sqrt{(tz_1 - st + sz_2)^2 - 4m^2(tz_1^2 + s(t - z_2)^2)}}.
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$$

One extra residue! Max cut is not the end of the story, we can "**cut again**" taking residue at $z_2 = 0$

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 $\begin{array}{ccccc}\n\hline\n\text{F1} & \text{F2} & \text{F3} \\
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$$

left with a **one-fold integral on a square root of a quartic polynomial**: no extra residue but two independent branch cuts which provide the solutions to the homogeneous differential equation [Primo, Tancredi '16,'17] t_{S} is independent puncture $[5, 1]$. Can $[5, 1]$. Can $[5, 1]$. Can can be considered as $[5, 1]$.

MASSES AND GEOMETRY planud due to the co E **MACCEC AND CEOMETDV** To solve the system, it is useful to search for a basis \mathcal{L} transformation to a so-called ✏*-factorized form*:

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J*,* ~

2 + ^q*y*(*y*+*z*4)

~I*.* (10)

J = U(*y, z,* ✏)

More in detail Ω such a system can be found by a path-order by a path-order order by a path-order order by a path-order order by a path-order of Ω

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> *D, ^s* G om Geometry is an *elliptic curve* Geometry is an elliptic curve ϵ content μ is an empire curve

$$
\mathcal{E}_4:Y^2=(X-e_1)(X-e_2)(X-e_3)(X-e_4)
$$

$$
e_1 = y - 4, \quad e_2 = -\frac{yz + 2\sqrt{yz(y + z - 4)}}{4 - z},
$$

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e_3 = -\frac{yz - 2\sqrt{yz(y + z - 4)}}{4 - z}, \quad e_4 = y.
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~

J(*y, z,* ✏) = ^Pexp

d~

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• Teneredi '16 '171 By further taking the residue at *z*² = 0 in (12), one is

> *D, ^s* G om r Geometry is an elliptic curve Geometry is an elliptic curve ϵ content μ is an empire curve Geometry is an **elliptic curve**

Periods obtained integrating on two branch cuts pariode obtained integrating on two branch cute reflux *butance megrating* on two branch cuts *P*eriods obtained integrating on two branch cuts 4 *z* g on two branch cuts We choose as first *period* for *E*⁴ the integral and
 $\overline{}$ *Y* $\frac{1}{\sqrt{2}}$ *^E*⁴ : *^Y* ² = (*^X ^e*1)(*^X ^e*2)(*^X ^e*3)(*^X ^e*4)*,* (13) with the four roots given by an α eriods obtained integrating on two branch cuts $\mathbf S$

leveraging many of these developments. In particular,

A $\ddot{}$ ~

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for the planar topologies, and for all polylogarithmic

J0(✏*, y*0*, z*0)*,* (11)

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(*y*0*, z*0) to a generic point (*y, z*). In the polylogarithmic

$$
\mathcal{E}_4:Y^2=(X-e_1)(X-e_2)(X-e_3)(X-e_4)
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$$

$$
\Psi_0(y, z) \equiv 2 \int_{e_2}^{e_3} \frac{dX}{Y} = \frac{4K(\lambda)}{\sqrt{(e_1 - e_3)(e_2 - e_4)}}
$$
\n
$$
\lambda = \frac{4}{2 + \sqrt{\frac{-y(y + z - 4)}{-z}}}
$$
\n
$$
\Psi_1(y, z) \equiv 2 \int_{e_2}^{e_1} \frac{dX}{Y} = \frac{4K(1 - \lambda)}{\sqrt{(e_1 - e_3)(e_2 - e_4)}}
$$

ance of additional transcendental integrals. In this way,

integrals generalizations of these methods to genus-one

for the planar topologies, and for all poly

for the planar topologies, and for all polylogarithmic

HOW DO WE COMPUTE THESE INTEGRALS?

(Intermezzo on differential equations and canonical forms)

DIFFERENTIAL EQUATIONS

Most powerful technique to compute Feynman integrals: **differential equations method**

[Kotikov '93][Remiddi '97] [Gehrmann, Remiddi '00]

We compute Feynman integrals as series in $\epsilon = (4 - d)/2$

DIFFERENTIAL EQUATIONS

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We compute Feynman integrals as series in $\epsilon = (4 - d)/2$

Iterative structure in ϵ made manifest by differentiation

$$
\mathcal{F} = \prod_{l=1}^{L} \frac{d^D k_l}{(2\pi)^D} \frac{S_1^{b_1} \dots S_m^{b_m}}{D_1^{a_1} \dots D_n^{a_n}}
$$

with $S_i \in \{k_i \cdot k_j, \ldots, k_i \cdot p_j\}$

Integration by Parts etc

$$
\int \prod_{l=1}^{L} \frac{d^D k_l}{(2\pi)^D} \frac{\partial}{\partial k_l^{\mu}} \left[v_{\mu} \frac{S_1^{b_1} \dots S_m^{b_m}}{D_1^{a_1} \dots D_n^{a_n}} \right] = 0
$$

(Scalar) Feynman Integrals Basis of Master Integrals (MIs)

 $I = \{I_1(z, \epsilon), \ldots, I_N(z, \epsilon)\}\$

DIFFERENTIAL EQUATIONS

Most powerful technique to compute Feynman integrals: **differential equations method [Kotikov '93][Remiddi '97] [Gehrmann, Remiddi '00]**

By differentiating and reducing to masters we obtain a **linear system of differential equations**

 $dI = GM(z, \epsilon)I$ In this form, iterative structure *hidden* in arbitrary dependence on ϵ

DIFFERENTIAL EQUATIONS specified by the topology of the Feynman graph under consideration. The *{N*1*, N*2*,...,Nm}* are a minimal set of irreducible scalar products in the products in the products in the products in the products in

Most powerful technique to compute Feynman integrals: **differential equations method [Kotikov '93][Remiddi '97]** the structure of a finite-dimensional vector space, whose basis we refer to as *master integrals*. [Gehrmann, Remiddi '00] \mathbf{F} as outlined in the integrals that belong the integrals that belong the integrals of \mathbf{F}

loop as a loop momenta that cannot be written as a linear compilation of the propagators. We set the propagator
.

By differentiating and reducing to masters we obtain a **linear system of differential equations** By differentiating and reducing to masters we obtain a *linear system of differential equations*

 $dI = GM(z, \epsilon)I$ loini, iterative structure *niquem* in arbitrary dependence on ϵ $\overline{\mathbf{u}}$ and it takes the general formulation $\overline{\mathbf{u}}$ In this form, iterative structure *hidden* in arbitrary dependence on ϵ

Imagine to be able to perform a series of **rotations** R_i **on the original basis** magine to be able to perform a series of **rotations** R_i **on the original basis**

$$
\underline{J} = \mathbf{R}(\underline{z}, \epsilon) \underline{I} \quad \text{with} \quad \mathbf{R}(\underline{z}, \epsilon) = \mathbf{R}_r(\underline{z}, \epsilon) \cdots \mathbf{R}_2(\underline{z}, \epsilon) \mathbf{R}_1(\underline{z}, \epsilon)
$$

 Out[1] that Out[2] Such that

 $dJ = G\mathbf{M}$ (α) *J* \rightarrow $G\mathbf{M}$ (α) $(\mathbf{R}(\alpha, \alpha)G\mathbf{M}(\alpha, \alpha) + d\mathbf{R}(\alpha, \alpha) \mathbf{D}(\alpha, \alpha) = 1$ $\mathrm{d}\underline{J}=\epsilon\,\mathbf{GM}\,\left(\underline{z}\right)\underline{J}\,,\quad\text{where}\quad\epsilon\,\mathbf{GM}\,\left(\underline{z}\right)=\left[\mathbf{R}(\underline{z},\epsilon)\mathbf{GM}(\underline{z},\epsilon)+\mathrm{d}\mathbf{R}(\underline{z},\epsilon)\right]\mathbf{R}(\underline{z},\epsilon)^{-1}\,.$

DIFFERENTIAL EQUATIONS differential system and it takes the general form

Most powerful technique to compute Feynman integrals: **differential equations method** d*I* = GM(*z,* ✏)*I .* (2.2)

 $[Kotikov '93] [Remiddi '97]$ **[Gehrmann, Remiddi '00]**

$\mathrm{d}\underline{J}=\epsilon\,\mathbf{GM}\,\left(\underline{z}\right)\underline{J}\,,\quad\text{where}\quad\epsilon\,\mathbf{GM}\,\left(\underline{z}\right)=[\mathbf{R}(\underline{z},\epsilon)\mathbf{GM}(\underline{z},\epsilon)+\mathrm{d}\mathbf{R}(\underline{z},\epsilon)]\,\mathbf{R}(\underline{z},\epsilon)^{-1}\,.$ The crucial property of eq. (2.4) is that the new matrix \mathcal{L} is that the new matrix \mathcal{L} the Gauss-Manin system is in this particular form we call it to be ✏*-factorised*. In addition,

Since GM(z) does not depend on ϵ , the iterative structure in ϵ becomes manifest

We refer to such a basis as in *epsilon-factorised form* [Kotikov '10; J. Henn '13; Lee '13, ...]

CANONICAL AND EPS-FACTORISED BASES

What can we say about GM(z) ?

 $dJ = \epsilon$ *GM*(*z*) J

- Is **GM(z) unique** ?

- Are there *ϵ*-factorised bases that are **better than others***?*

Can we define an **optimal basis** of master integrals for a given problem? We understand the problem well in the **polylogarithmic case**

CANONICAL AND EPS-FACTORISED BASES that allow us to say that the basis we are starting from is not too far from a *good basis*. To this aim, it is useful to start from the much better understood polylogarithmic case.

What can we say about $GM(z)$?

For a rather large class of problems that can be solved in terms of α

- $d\underline{J} = \epsilon \, GM(\underline{z}) \underline{J}$ Is GM(z) unique ? $\mathbf{r} = \mathbf{r} \cdot \mathbf{r}$
	- Is **GM(z) unique** ?

 $-$ Are there ϵ -factorised bases that are **better than others**?

can we define an **optimal basis** of master integrals for a given problem? to determine a suitable basis for general values of $\mathbf W$ We understand the problem well in the **polylogarithmic case**

[Arkani Hamed et al '10; Kotikov '10; J. Henn '13] σ and all naturally decompositions of logarithms. This analysis is the complete and one minimum and one might word be generalised by \mathbb{R}^n *Conjecturally,* these integrals fulfil *canonical differential equations*

CANONICAL BASES: THE POLYLOGARITHMIC CASE The analysis is usually performed in *d* = *d*02✏ dimensions, with typically *d*⁰ = 2*,* 4*,* 6. As a **CANUNICAL BASES:** THE POLYLOGARITHMIC CASE f intuitive later the understanding the understanding the unit of the understanding $\frac{1}{2}$

Leading Singularities \sim *iterative residues*
of the integrand in all integration variables if the integrand is in definition is in definition order corrections in \mathcal{C} **Leading Singularities** ~ *iterative residues*

[Arkani Hamed et al '10; Kotikov '10; J. Henn '13] σ and all naturally decompositions of logarithms. This analysis is the complete and one minimum and one might word be generalised by \mathbb{R}^n *Conjecturally,* these integrals fulfil *canonical differential equations*

leading singularities has a general multi-parameter $\text{Recipe (in a nutshell)}$:

- case and for increasing numbers of loops, it becomes computationally extremely difficult to and the subset integrate whose meghanisms have only simple poles and a 1. choose integrals whose *integrands* have only **simple poles and are in d-log form**
- 2. choose integrals whose *iterated residues* at all simple poles can be **normalized to numbers** didates for a canonical integral. More important integral integrals. The parameter of α **[Arkani-Hamed et al'10; Henn, Mistlberger, Smirnov, Wasser '20]**

CANONICAL BASES: THE POLYLOGARITHMIC CASE

What do these conditions imply?

$$
G(c_1, c_2, ..., c_n, x) = \int_0^x \frac{dt_1}{t_1 - c_1} G(c_2, ..., c_n, t_1)
$$

=
$$
\int_0^x \frac{dt_1}{t_1 - c_1} \int_0^{t_1} \frac{dt_2}{t_2 - c_2} ... \int_0^{t_{n-1}} \frac{dt_n}{t_n - c_n}
$$

n = number of integrations or **transcendental weight**

MPLs are iterated integrals over d-log forms (with rational entries)

The requirements before, guarantees that Feynman integrals are written as **pure, uniform weight combinations of MPLs**

Note: this makes sense, since forms with single poles span the full first de Rham cohomology, or in other words **MPLs are generated by dlogs!**

BEYOND POLYLOGARITHMS: CONCEPTUAL DIFFERENCES

Even with MPLs, insisting on *simple poles* in the integrand *(neglecting integration contour)* is too strong of a requirement, as it forces us to **exclude any squared propagator**!

Physics:

Double poles often imply power-like singularities in the IR which should be excluded in gauge theories

Typically true when dealing with massless propagators

Massive propagators can be squared at will, *without changing IR behaviour* and (actually) *improving UV behaviour*

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Mathematics:

Differential forms with *simple poles* are intrinsically *not enough* to span full space for more general problems *(elliptic curves or Tori, K3, Calabi-Yaus etc)*

$$
\rightarrow
$$

Think about *independent integrands* in the elliptic case:

 $K(x) = \int_0^{\frac{u}{\sqrt{(1 - t^2)(1 - xt^2)}}}$ has **no poles** while $E(x) = \int_0^{\frac{v}{\sqrt{1 - t^2}}} dt \frac{\sqrt{1 - x^2}}{\sqrt{1 - t^2}}$ has **double pole at infinity** 1 θ *dt* $(1 - t^2)(1 - xt^2)$ $E(x) = \int$ 1 $\overline{0}$ *dt* $1 - xt^2$ $1 - t^2$

A DIFFERENT PERSPECTIVE ON MPLS?

UNIPOTENT FUNCTIONS AND DIFFERENTIAL EQUATIONS

Canonical integrals in polylogarithmic case give rise to **pure combinations of MPLs**

$$
G(c_1, c_2, ..., c_n, x) = \int_0^x \frac{dt_1}{t_1 - c_1} G(c_2, ..., c_n, t_1)
$$

=
$$
\int_0^x \frac{dt_1}{t_1 - c_1} \int_0^{t_1} \frac{dt_2}{t_2 - c_2} ... \int_0^{t_{n-1}} \frac{dt_n}{t_n - c_n}
$$

[…,Remiddi, Vermaseren '99, Goncharov '00,…]

MPLs are unipotent: they fulfil particularly simple differential equations

$$
\frac{d}{dx}G(c_1,\ldots,c_n;x) = \frac{1}{x-c_1}G(c_2,\ldots,c_n;x)
$$

by diff. we lower the weight & length

UNIPOTENT FUNCTIONS AND DIFFERENTIAL EQUATIONS discarding any contributions from integrals whose own differential equations do not couple

the maximal cuts of the integrals provide a solution to it [9–11, 14, 15], and to W also Canomical micgrais in polylogarithmic case give rise to pu Canonical integrals in polylogarithmic case give rise to **pure combinations of MPLs**

$$
G(c_1, c_2, ..., c_n, x) = \int_0^x \frac{dt_1}{t_1 - c_1} G(c_2, ..., c_n, t_1)
$$

=
$$
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$$

to the block. In the following, we refer to this system also as the *maximal cut system*, as

[…,Remiddi, Vermaseren '99, Goncharov '00,…]

MPLs are unipotent: they fulfil particularly simple differential equations and therefore was limited to at most two orders in $\mathcal{L}_\mathcal{A}$ at most two orders in $\mathcal{L}_\mathcal{A}$

$$
\frac{d}{dx}G(c_1,\ldots,c_n;x) = \frac{1}{x-c_1}G(c_2,\ldots,c_n;x)
$$
 by diff. we lower the weight & length

General definition is: *W^u* unipotent if it fulfils system of diff equations with Nilpotent matrices

$$
d\mathbf{W}^{u} = \left(\sum_{i} \mathbf{U}_{i}(\underline{z}) d z_{i}\right) \mathbf{W}^{u}, \qquad \text{Where } U_{i}(\underline{z}) \text{ are } \mathbf{Nilpotent} \text{ matrices: } U_{i} \cdot U_{i} \cdot \dots \cdot U_{i} = 0
$$

BEYOND POLYLOGARITHMS: CONCEPTUAL DIFFERENCES namely: *A function is called pure if it is unipotent and its total di*↵*erential involves only*

Same condition is fulfilled by **Elliptic polylogarithms (eMPLs)**

[Brown Levin '11; Brödel, Mafra, Matthes, Schlotterer '14]
[Brödel, Dulat, Duhr, Penante, Tancredi '17, '18] [Brödel, Dulat, Duhr, Penante, Tancredi '17, '18]

discarding any contributions from integrals whose own differential equations do not couple

We can insist on single poles \leftrightarrow logarithmic singularities (Gauge Theory) the solutions such that the powers of logarithms appearing in the powers of logarithms appearing in the power series expansions of α

$$
\mathcal{E}_4(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x, \vec{a}) = \int_0^x dt \Psi_{n_1}(c_1, t, \vec{a}) \mathcal{E}_4(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; t, \vec{a})
$$

with the requirement that all integrals must have at most logarithmic singularities leads in the requirement of

into a *semi-simple* part Wss and *unipotent* part Wu, i.e.

to an infinite tower of **transcendental kernels** *Jean't* be obtained from "residue of integrand"¹ t , where to the torus description, where t is the contribution are generated by eq. (2.33). Price to pay: infinite tower of **transcendental kernels** [can't be obtained from "residue of integrand"]

Still fulfil unipotent diff equation: at the basis of definition of symbol!

unipotent diff equation: at the basis of definition of symbol!
$$
dW^u = \left(\sum_i U_i(\underline{z}) dz_i\right) W^u
$$
,

procedure to all orders in $\mathcal A$

CAN WE USE THE UNIPOTENCE CONDITION?

EXAMPLE: POLYLOGARITHMIC CASE Instead of following this standard approach, let us pretend that we were unable to **EXAMPLE:** POLYLOGARITHMIC CASE

It works in the (simple) polylogarithmic case: Sunrise with **2 massive and 1 massless propagator** *I*¹ *WOTKS* and Euler's theorem on homogeneous functions functions functions functions functions \mathcal{A} 1.000 million and the contract of the second product of the second p

 $I_1 = I_{0,1,1,0,0}$, $I_2 = I_{1,1,1,0,0}$ and $I_3 = I_{1,1,2,0,0}$

use this example this example this example to illustrate the steps in our procedure. As we will see, they will see, t

Differential equations read: $dI = [A_0 + \epsilon A_1]I$

 $d-2$ Homogeneous equation in $d=2$

$$
\frac{\partial}{\partial m^2} \begin{pmatrix} \mathfrak{I}_2 \\ \mathfrak{I}_3 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ \frac{1}{m^2(s-4m^2)} & \frac{-s+10m^2}{m^2(s-4m^2)} \end{pmatrix} \begin{pmatrix} \mathfrak{I}_2 \\ \mathfrak{I}_3 \end{pmatrix}
$$

one with respect to *s* follows at each step from a scaling relation implied by dimensional

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$$

one with respect to *s* follows at each step from a scaling relation implied by dimensional

 $\binom{1}{2}$ Matrix of **homogeneous solutions** contains **algebraic functions and logs**

$$
dW = AW \rightarrow W = \begin{pmatrix} \frac{1}{r(s,m^2)} & \frac{1}{r(s,m^2)} \log \left(\frac{s - r(s,m^2)}{s + r(s,m^2)} \right) \\ \frac{s}{r(s,m^2)^3} & \frac{s}{2m^2r(s,m^2)^2} + \frac{s \log \left(\frac{s - r(s,m^2)}{s + r(s,m^2)} \right)}{r(s,m^2)^3} \end{pmatrix} \text{ with } r(s,m^2) = \sqrt{s(s-4m^2)}
$$

EXAMPLE: POLYLOGARITHMIC CASE Instead of following this standard approach, let us pretend that we were unable to **EXAMPLE:** POLYLOGARITHMIC CASE

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Homogeneous equation in d=2
$$
\frac{\partial}{\partial m^2} \begin{pmatrix} \mathfrak{I}_2 \\ \mathfrak{I}_3 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ \frac{1}{m^2(s-4m^2)} & \frac{-s+10m^2}{m^2(s-4m^2)} \end{pmatrix} \begin{pmatrix} \mathfrak{I}_2 \\ \mathfrak{I}_3 \end{pmatrix}
$$

 \overline{C} Matrix of **homogeneous solutions** contains **algebraic functions and logs**

ations are linear in ϵ , we cot Since differential equations are linear in ϵ , we could be "tempted" to just "rotate W away"

$$
d\underline{I} = [A_0(\underline{z}) + \epsilon A_1(\underline{z})] \underline{I} \longrightarrow \underline{I} = W \cdot \underline{J} \qquad d\underline{J} = \epsilon [W^{-1} \cdot A_1(\underline{z}) \cdot W] \underline{J}
$$

 \overline{n} $\frac{1}{1}$ *s* $|c|$ *m*² *s* and basi *s* ◆2 S^{\dagger} $\frac{1}{2}$ 3 ✓*m*² *s* bina ✓*m*² *s* $\frac{1}{4}$ *,* New matrix not in dlog form (logs not dlogs !) and basis *J* is not pure combination of UT MPLs…

EXAMPLE: POLYLOGARITHMIC CASE step does not lead to a canonical basis, not lead to a factorisation of \mathcal{M} . Moreover, the coefficient of \mathcal{M}

Incteed we will rotate away only a "part" of the homogeneous solution: sessions, we wan retail away only a part of the homogeneous scrittion. Instead, we will rotate away only a "part" of the homogeneous solution: a tadpole and two in the top sector. In this case, the underlying geometry is a Riemannian geome

cient functions in the differential equations will be of mixed the different will be of mixed transcendental w
Instead, we instead, we ight a set of mixed transcendental weight. Instead, we ight a set of mixed to be a set

Split it in **semi-simple** and **unipotent** $W = W^{ss} \cdot W^u$ unit leading singularity in *d* = 2 space-time dimensions. It is immediate to do this analysis \mathbf{v}

$$
\mathbf{W}^{\text{ss}} = \begin{pmatrix} \frac{1}{r(s,m^2)} & 0\\ \frac{s}{r(s,m^2)^3} & \frac{1}{2m^2(s-4m^2)} \end{pmatrix} \quad \text{and} \quad \mathbf{W}^{\text{u}} = \begin{pmatrix} 1 & \log\left(\frac{s-r(s,m^2)}{s+r(s,m^2)}\right) \\ 0 & 1 \end{pmatrix} \quad r(s,m^2) = \sqrt{s(s-4m^2)}
$$
\nunipotent part contains transcendental solution

For *m*²

² = *m*²

³ =: *m*² and *m*²

 ω ¹ oraic part in semi-sim $\frac{1}{2}$ motrix only algebraic part in semi-simple matrix $\hspace{1cm}$

EXAMPLE: POLYLOGARITHMIC CASE step does not lead to a canonical basis, not lead to a factorisation of \mathcal{M} . Moreover, the coefficient of \mathcal{M} \blacksquare Semi-simple part Western Wi. This works and an upper-triangular university of the USA of the Wu. This works because our works becaus

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$$
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$$
\nunipotent part contains transcendental solution

\nonly algebraic part in semi-simple matrix

For *m*²

choice for the first integral in the top sector *I*² already has uniform transcendental weight

² = *m*²

³ =: *m*² and *m*²

 ω ¹ only algebraic part in semi-simple matrix

Rotate away **only semi-simple part** $I' = \begin{bmatrix} 0 \\ \frac{1}{1} \end{bmatrix}$ $I' = \begin{bmatrix} 0 \\ \frac{1}{1} \end{bmatrix}$

Rotate away only semi-simple part

\n
$$
\underline{I}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (\mathbf{W}^{\text{ss}})^{-1} \end{pmatrix} \underline{I}
$$

and the series of the seri End basis corresponds to matrix W^u : one master has \bf{weight} 0, the other has \bf{weight} 1, \bf{weight} $10 - 212$ rstood for ϵ rstood for elliptic curves and Calabi-Yau generalizations! 1 Monodromy), which is well understood for elliptic curves and Calabi-Yau generalizations ! *mixing disentangled* —> this behaviour is typical at a so-called **MUM point** (Maximal Unipotent

EXAMPLE 1: POLYLOGARITHMIC CASE $\mathsf{A}\mathsf{S}\mathsf{E}$ is, however, and algebraic function algebraic func

 \sim \sim

0
0
0
0
0

Clean up remaining non-factorised dependence with a rotation

$$
\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2(s+2m^2)}{r(s,m^2)} & 1 \end{pmatrix} \begin{pmatrix} \epsilon^2 & 0 & 0 \\ 0 & \epsilon^2 & 0 \\ 0 & 0 & \epsilon \end{pmatrix}
$$

and it is therefore easily integrated out. Therefore easily integrated out. These manipulations can be summari

$$
d\underline{J} = \epsilon \text{ GM}^{\epsilon} \underline{J}
$$
 with $\underline{J} = (J_1, J_2, J_3)^T = \mathbf{T} \underline{I}'$,

$$
GM^{\epsilon} = \begin{pmatrix}\n-2\alpha_1 & 0 & 0 \\
0 & 2\alpha_1 - \alpha_2 - 3\alpha_3 & \alpha_4 \\
2\alpha_1 - 2\alpha_2 & -6\alpha_4 & -3\alpha_1 + \alpha_2\n\end{pmatrix}
$$

$$
\alpha_1 = d \log(m^2) \, , \ \ \alpha_2 = d \log(s) \, , \ \ \alpha_3 = d \log\left(s - 4m^2\right) \, , \ \ \alpha_4 = d \log\left(\frac{s - r(s, m^2)}{s + r(s, m^2)}\right)
$$

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$$
GM^{\epsilon}=\left(\begin{array}{ccc}-2\alpha_1&0&0\\0&2\alpha_1-\alpha_2-3\alpha_3&\alpha_4\\2\alpha_1-2\alpha_2&-6\alpha_4&-3\alpha_1+\alpha_2\end{array}\right)
$$

$$
\alpha_1 = d \log(m^2), \ \alpha_2 = d \log(s), \ \alpha_3 = d \log(s - 4m^2), \ \alpha_4 = d \log\left(\frac{s - r(s, m^2)}{s + r(s, m^2)}\right)
$$

and it is therefore easily integrated out. These manipulations can be summarised in the

alysing leading singularities with **DLogBasis** find the same basis up to cons *J*¹ = *M*¹ *, J*² = *M*² *, J*³ = *M*¹ + 3*M*³ *.* (3.20) simple constant rotation that is given by We would like to stress the following points concerning the application of our procedure in **[P. Wasser '19,'20]***J*₂ *M*₂ *J*₃ *J*₄ *J*₄ *J*₄ *M*₂ *M*₂ *M*₂ *M*₄ *314 .* (3.20) NB: by analysing leading singularities with **DLogBasis** find the same basis up to constant rotation!

$$
J_1 = M_1 \,, \quad J_2 = M_2 \,, \quad J_3 = -M_1 + 3M_3
$$

STRATEGY SUCCESSFUL IN MANY NON-TRIVIAL CASES <u>JUL IIN MANI NUNTINIVIAL UA</u> which can be understood as follows: First, the additional scale that the additional scale th on does not increase the number of master integrals in the top sector. This can be interpreted as the fact that, contrary to the sunrise, the integral residue in the integral residue in the integral sector considered, the introduction of new functions might be required. These are likewise (iterated) integrals of functions already present in the differential equations.

Strategy is general and **does not have to do with d** ϵ and ϵ that the basis of the period in ϵ is the top in ϵ in ϵ is the period ϵ basis from the literature \mathbf{C} . Notice that in this problem, no additional new functions \mathbf{C} differential equations. Second, there is instead a different singularity structure in the sub-Strategy is general and **does not have to do with details of the geometry***

Applied it successfully to elliptic sunrise (equal or uccessfully to elliptic sunrise (equal or different masses) \overline{p} \overline{p} Applied it successfully to elliptic sunrise (equal or different masses)

were needed to achieve the $\mathcal I$ indication that all differential forms are independent under integration by parts identities. Nevertheless, we have not proven that we have not proven this last statement for \sim equations for the top sector. Integrating over this pole gives a contribution formally similar or different masses) $p \left(k_2 \right)$ p $\frac{1}{\sqrt{1-\frac{1$ $k₁$ *k*2 $k_1 + k_2$ *p* / k_2 / *p*

of the maximal cut, which is linearly independent under integration by parts. This implies the part of this implicit in part of the maximal cut.

Many other multi-scale elliptic problems

two-loop sunrise graph, showing how our procedure allows us to obtain ✏-factorised systems

 $Deyona$ is emplied can be Even cases beyond 1 elliptic curve

BACK TO MASTER INTEGRALS FOR BHABHA AND MØLLER

Following the strategy above, we obtain a fully *ϵ*-factorised system of differential equations

Boundary conditions can be fixed by using regularity conditions (absence of pseudo thresholds) or, equivalently, large mass expansion

The result can then be written in terms of iterated integrals over many differential forms which involve the period and quasi period of the elliptic curve, and integrals over it *z* any differential forms which

$$
T_1(y, z) = \int dy \left[\frac{-z}{y} (4y^2 + 4y(z - 4) + z(z - 4))\Psi_0 \right]
$$

\n
$$
-8z \frac{(y + z - 4)(y + z)}{(t + 2y - 4)} \partial_y \Psi_0 \right]
$$

\n
$$
+dz \left[\frac{-z}{4 - z} \frac{-48 + 4y + 2y^2 + 12z + yz}{z + y - 4} \Psi_0 \right]
$$

\n
$$
T_2(y, z) = \sqrt{4 - z} \sqrt{-z} \int dy \left[\frac{z}{y} \frac{4 + 2y - y^2 - z - yt}{2(y + z - 4)} \Psi_0 \right]
$$

\n
$$
- \frac{1}{2}z(1 + y)\partial_y \Psi_0 \right] + \sqrt{-z} \sqrt{4 - z} dz \left[\Psi_0 \right]
$$

\n
$$
\times \frac{y - 4}{2(y + z - 4)} + \frac{(y - 4)y(1 + y)}{2(-4 + 2y + z)} \partial_y \Psi_0 \right],
$$

BACK TO MASTER INTEGRALS FOR BHABHA AND MØLLER

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BACK TO MASTER INTEGRALS FOR BHABHA AND MØLLER 1 2 *z*(1 + *y*)@*^y* ⁰ ⁺ p*^z* ^p⁴ *^z* ^d*^z* ⁰ \boldsymbol{X} to magnetic the integration of 2(*^y* ⁺ *^z* 4) ⁺ (*^y* 4)*y*(1 + *^y*) 2(4+2*y* + *z*) Currently, there are no public numerical routines to can parameterize the kinematical variables by can parameterize the kinematical variables by the construction are immaterial for this paper and are WADUA AND MAILED integrals in terms of Chen iterated integrals. Λ and matrix are immaterial for the construction are immaterial for the set of the set discussed entry control in the same of \mathbf{S} The construction are in the construction and are in the construction and are in the construction and are in the UN DIIADIIA AND MULLLIN **RACK TO MACTED INTECDAI C EOD** *y >* 4 and *z <* 0 for definiteness. While the details of planar sector. We therefore obtain generalized series ex-R BHABHA AND MØLLER from the di \sim from the di \sim factorized for \sim $-$ a $-$ factorized for \sim $t_{\rm eff}$ the construction are immaterial for this paper are immaterial for this paper and are immaterial for this paper and are immaterial for the construction of the construction of the construction of the construction o can parameterize the kinematical variables by which are a second to express the objects required to express the objects required to express the objects required to express the contribution of the contribution of the contribution of the contribution of the contribution matrix *A* in (10). Again, formulas are given assuming *y >* 4 and *z <* 0 for definiteness. While the details of

These differential forms look pretty complicated (and there are worse ones) but they can be simplified! These differential forms lealy repatty complicated (and a fillipse anner citital forms fook pretty compileated (and These differential forms look pretty complicated (and there are worse ones) but they can be simplified! *,* (19) the individual master integrals. In particular, we obe are worse ones) but they can be simplified! rse ones) but they can be *y* and $\frac{1}{2}$ and $\frac{1}{2}$ and $\frac{1}{2}$ and $\frac{1}{2}$ and $\frac{1}{2}$

Co to cononical coordinates of elliptic curve do to canonical coordinates of emptic carve Go to **canonical coordinates** of elliptic curve

$$
y = 2\frac{(1-x)(1+t_4)}{t_4-x}, \quad z = 4\frac{t_4(1-x^2)}{x^2-t_4^2}
$$
\n
$$
\Psi_0(x,t_4) = \frac{2(x^2-t_4)}{-Y} K(t_4)
$$

in (15) becomes \mathbf{C} series expansions allow for fast numerical evaluation, \mathbf{C} \mathbf{A} we find In these variables, **integrals become "simple"** and we find $\sum_{k=1}^{\infty}$ there is subjected integral become $\sum_{k=1}^{\infty}$ In these variables, **integrals become "simple"** and w \int

$$
T_1(x, t_4) = 8t_4 \frac{K(t_4)}{\pi} \left[(1 - t_4) \mathcal{F}(x, t_4) - \frac{x^2 - 1}{(1 + t_4)Y} \right],
$$

\n
$$
T_2(x, t_4) = \frac{1}{\pi} \sqrt{\frac{t_4}{1 + t_4}} \frac{t_4(3 - 2x) - 3x + 2}{t_4 - x} K(t_4) - \frac{f(t_4)}{2\pi},
$$

in the description of the ancient of the ancient state along with the ancient state along with the ancient state α T1(*x, t*4) =8*t*⁴ where where

^y = 2(1 *^x*)(1 + *^t*4)

⇡

$$
\partial_{t_4} f = 2 \frac{1 - t_4}{\sqrt{t_4}(1 + t_4)^{3/2}} \mathbf{K}(t_4)
$$

$$
\mathcal{F}(x, t_4) = \mathbf{K}(t_4) \partial_{t_4} \left[\frac{1}{\mathbf{K}(t_4)} \int_{-1}^x \frac{dX}{\sqrt{(X^2 - 1)(X^2 - t_4)}} \right]
$$

 \mathcal{F}^2 in particular, we obtain

der to algorithmically obtain a small mass expansion for *, z* = 4*t*4(1 *^x*²)

the construction are immaterial for this paper and are

(Derivative of) Abel's Map *F*(*Derivative of*) Abel's (Derivative of) Abel's Map

BACK TO MASTER INTEGRALS FOR BHABHA AND MØLLER 1 2 *z*(1 + *y*)@*^y* ⁰ ⁺ p*^z* ^p⁴ *^z* ^d*^z* ⁰ ⇥ *^y* ⁴ 2(*^y* ⁺ *^z* 4) ⁺ (*^y* 4)*y*(1 + *^y*) ◆ *,* (18) WADUA AND MAILED integrals in terms of Chen iterated integrals. Λ and matrix are immaterial for the construction are immaterial for the set of the set discussed entry control in the same of \mathbf{S} The construction are in the construction and are in the construction and are in the construction and are in the UN DIIADIIA AND MULLLIN *y >* 4 and *z <* 0 for definiteness. While the details of **TRACK TH MASTER INTEGRALS FILK** \mathcal{B} , it such that \mathcal{B} is such that \mathcal{B} is such that one say that one say that one say that one say that \mathcal{B} \blacksquare and \blacksquare IADNA AND MØLLEN **obtain a** the individual master integrals. In particular, we ob-CAN PARAMETERIZE THE KINEMATICAL VA *<u><i>x* \overline{A} **,** \overline{A} \over $t_{\rm{max}}$ individual master integrals. In particular, we obtain tanin a generalized power series (including logarithms of the contract of the contract of the contract of the the mass), whose coecients can be expressed in terms of harmonic polylogarithms [86]. We obtain results that

*x*² *t*² 4

These differential forms look pretty complicated (and there are worse ones) but they can be simplified! These differential forms lealy repatty complicated (and a fillipse anner citital forms fook pretty compileated (and e differential forms look pretty complicated (and there are worse ones) but they can be simplified! *,* (19) rential forms look pretty complicated (and there are worse ones) but they can be simplified! of harmonic polynomials $\mathcal{S}(\mathcal{S})$. We obtain $\mathcal{S}(\mathcal{S})$ tendal forms look pretty complicated (and then Møller scattering. As a cross check, we compared individre worse ones) but they can be simplified:

in (15) becomes

Co to cononical coordinates of elliptic curve do to canonical coordinates of emptic carve Go to *canonical coordinates* of elliptic curve **1 amates** of emptic

can parameterize the kinematical variables by

 $y = 2\frac{(1-x)(1+t_4)}{t}$ $t_4 - x$ $z = 4\frac{t_4(1-x^2)}{2-t^2}$ $x^2 - t_4^2$ *,* (19) $\sqrt{1-\eta}$ t_4-x ⁷ *^Y* K(*t*4)*.* (20) $\sqrt{(1+1)^2}$ the congruence of the congruence $\sqrt{(1+2)^2}$ $\frac{z-x(1+ t_4)}{z}$, $z = 4\frac{t_4(1-x^2)}{z}$ $t_4=x$, $z=\pm \sqrt{x^2-t^2}$ α as the two transcendents in α the two transcendents in α

$$
Y^2 = (x^2 - 1)(x^2 - t_4)
$$

can parameterize the kinematical variables by

Currently, there are no public numerical routines to

can parameterize the kinematical variables by

of harmonic polynomials \mathbb{R}^n . We obtain \mathbb{R}^n results that \mathbb{R}^n

with $\mathcal{B}(\mathcal{B})$ and $\mathcal{B}(\mathcal{B})$ and $\mathcal{B}(\mathcal{B})$ and $\mathcal{B}(\mathcal{B})$ and $\mathcal{B}(\mathcal{B})$ and $\mathcal{B}(\mathcal{B})$ and $\mathcal{B}(\mathcal{B})$

Using the master integrals calculated above, as well as

ence subgroup 1(4) \sim SL2(Z) \sim

^e, mb, s, t, ✏)*,* (24)

in the description of the ancillary files along with the

the final, simplified form is essential, simplified form is essential to ecoes the control of the control of t
In planet of the control of the con

(1 *^t*4)*F*(*x, t*4) *^x*² ¹

2*A*(↵⁰

⇡

$$
\Psi_0(x, t_4) = \frac{2(x^2 - t_4)}{-Y} \,\mathrm{K}(t_4)
$$

in (15) becomes Similarly, all other differential forms become products of Similarly, all other differential forms become products ^T2(*x, t*4) = ¹ *ifferential forms become products* rly, all other

2(4+2*y* + *z*)

*t*⁴ *x*

$$
\text{This become products of } \begin{cases} \{\sqrt{x^2 - 1}, \sqrt{x^2 - t_4}, \sqrt{1 + t_4}, \sqrt{t_4}, \sqrt{1 - t_4}\} \\ \{\text{K}(t_4), f(t_4), \mathcal{F}(x, t_4)\} \end{cases}
$$

(↵*e, m, s, t,* ✏) = *Z*²

We can identifying *t*⁴ with a *Hauptmodul* for the congru-

where
$$
x^2 = \frac{m^2}{s}
$$

\n
$$
\mathcal{F}(x, t_4) = \mathcal{K}(t_4) \partial_{t_4} \left[\frac{1}{\mathcal{K}(t_4)} \int_{-1}^x \frac{dX}{\sqrt{(X^2 - 1)(X^2 - t_4)}} \right]
$$
\nTotal of 87 differ
\n $\phi_{t_4} f = 2 \frac{1 - t_4}{\sqrt{t_4 (1 + t_4)^{3/2}}} \mathcal{K}(t_4)$

⁰

 \mathbf{v} simplified and all double integrals over the per-

 U_1 U_2 U_3 U_4 U_5 U_6 U_7 U_8 U_9 U_9 U_9 U_9 U_9 U_9 U_9 U_9 U_9 of the alphabet: al 01 **o7 unierential** iol Total of 87 differential forms

~ "letters of the alphabet" ? Total of 87 differential forms $\frac{u_1}{v_1}$ order on $\frac{u_2}{v_2}$ divergences can then belonged between $\frac{u_1}{v_1}$ rences of the archi-Total of 87 di α ["]letter \mathcal{F} *µ*2 ◆✏ *Ze*↵*e*(*µ*)*, m^b* = *Zmm .* (25) Γ in Γ or Γ in Γ in Γ in Γ in Γ in Γ in Γ is the solution of the s dida of **o** *c* differential forms \sim "letters of the alphabet" ? Total of **87 differential** forms

*A*r

BACK TO MASTER INTEGRALS FOR BHABHA AND MØLLER 1 2 *z*(1 + *y*)@*^y* ⁰ ⁺ p*^z* ^p⁴ *^z* ^d*^z* ⁰ ⇥ *^y* ⁴ 2(*^y* ⁺ *^z* 4) ⁺ (*^y* 4)*y*(1 + *^y*) ◆ *,* (18) WADUA AND MAILED integrals in terms of Chen iterated integrals. Λ and matrix are immaterial for the construction are immaterial for the set of the set discussed entry control in the same of \mathbf{S} The construction are in the construction and are in the construction and are in the construction and are in the UN DIIADIIA AND MULLLIN *y >* 4 and *z <* 0 for definiteness. While the details of **TRACK TH MASTER INTEGRALS FILK** \mathcal{B} , it such that \mathcal{B} is such that \mathcal{B} is such that one say that one say that one say that one say that \mathcal{B} \blacksquare and \blacksquare IADNA AND MØLLEN **obtain a** the individual master integrals. In particular, we ob-CAN PARAMETERIZE THE KINEMATICAL VA *<u><i>x* \overline{A} **,** \overline{A} \over $t_{\rm{max}}$ individual master integrals. In particular, we obtain tanin a generalized power series (including logarithms of the contract of the contract of the contract of the the mass), whose coecients can be expressed in terms of harmonic polylogarithms [86]. We obtain results that

*x*² *t*²

4

These differential forms look pretty complicated (and there are worse ones) but they can be simplified! These differential forms lealy repatty complicated (and a fillipse anner citital forms fook pretty compileated (and e differential forms look pretty complicated (and there are worse ones) but they can be simplified! *,* (19) rential forms look pretty complicated (and there are worse ones) but they can be simplified! of harmonic polynomials $\mathcal{S}(\mathcal{S})$. We obtain $\mathcal{S}(\mathcal{S})$ tendal forms look pretty complicated (and then Møller scattering. As a cross check, we compared individre worse ones) but they can be simplified:

Currently, there are no public numerical routines to

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 \mathcal{F}^2 in particular, we obtain

 $\frac{V^2}{a^2}$ $\frac{1}{a^2}$ $\frac{1}{a^2}$ $\frac{1}{b^2}$

Co to cononical coordinates of elliptic curve do to canonical coordinates of emptic carve Go to *canonical coordinates* of elliptic curve **1 amates** of emptic

can parameterize the kinematical variables by

2(4+2*y* + *z*)

*t*⁴ *x*

 $y = 2\frac{(1-x)(1+t_4)}{t}$ $t_4 - x$ $z = 4\frac{t_4(1-x^2)}{2-t^2}$ $x^2 - t_4^2$ *,* (19) in (15) becomes We checked agains **AMFlow** [Liu, Ma '22] in differer ence subgroup 1, \mathcal{L} (\mathcal{L}) \mathcal{L} is turns to turn subgroup \mathcal{L} out that by changing variables to the canonical coordinates, one can easily see that the two transcendents of the two transcendental integrals in (18) are just combinations of simpler functions $Y^2 = (x^2 - 1)(x^2 - t_4).$ of harmonic polynomials \mathbb{R}^n . We obtain \mathbb{R}^n results that \mathbb{R}^n $2(x^2-t_4)$ $\Psi_0(x,t_4) = \frac{dy}{dx}$ K(t_4) \mathbf{u} with $\mathcal{B}(\mathcal{B})$ and $\mathcal{B}(\mathcal{B})$ and $\mathcal{B}(\mathcal{B})$ and $\mathcal{B}(\mathcal{B})$ and $\mathcal{B}(\mathcal{B})$ and $\mathcal{B}(\mathcal{B})$ and $\mathcal{B}(\mathcal{B})$ ing kinematics, and found agreement to high precision and found agreement to high precision. $\begin{bmatrix} t_4 \end{bmatrix}$ appropriate for phenomenological studies. A precise description of the numerical implementations can be found in the description of the anciellary files along with the anciellary files along with the second with the second \mathcal{L} α in and the third of this manuscript. UV RENORMALIZATION AND IRE USE OF THE USE OF of the alphabet: $(x^2 - t_4)$ We can identifying *t*⁴ with a *Hauptmodul* for the congruence subgroup 1(4) \sim SL2(Z) \sim out that by changing variables to the canonical coordi- \overline{a} as in the canonical in-definition of $\overline{t_4}$ tegrals in (18) are just combinations of simple \mathcal{N} and simple \mathcal{N} \overline{a} (1 *^t*4)*F*(*x, t*4) *^x*² ¹ r *t*⁴ T1(*x, t*4) =8*t*⁴ *t*4(3 2*x*) 3*x* + 2 *^t*⁴ *^x* K(*t*4) *^f*(*t*4) where *f*(*t*4) is given by $Y^2 = (x^2 - 1)(x^2 - t_4)$ in (15) becomes $\Psi_0(x, t_4) = \frac{2(x^2 - t_4)}{V}$ $\frac{Y^{(2)} - Y}{Y}$ K(*t*₄) We can identifying *t*⁴ with a *Hauptmodul* for the congru e substituting the substitution of $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$. out that by changing variables to the canonical coordinates to the canonical coordinates to the canonical coordinates of t_4 nates, one can easily see that the two transcendents of the two transcendental integrals in (18) are just combinations of simpler functions regions (Bhabha and Møller) (1 + *t*4)*Y* ⇡ 1 + *t*⁴ *^t*⁴ *^x* K(*t*4) *^f*(*t*4) Similarly, all other differential forms become products of $\sqrt{1-\eta}$ t_4-x ⁷ *^Y* K(*t*4)*.* (20) Sim¹¹ and the two transcendents of cannot the two transcendents in the two transcendents $t_{\rm eff}$ are just combinations of simpler functions α T1(*x, t*4) =8*t*⁴ **49210 / 1888** AMFlow II.in Ma ^T2(*x, t*4) = ¹ r *t*⁴ where $\overline{}$ $2(\alpha^2 + 1)$ $\Psi_0(x,t_4) = \frac{\Xi(x-\mu_4)}{2\pi}$ K(t_4) \overline{y} the description of the anciellary files along with the anciellary files along with the anciellary field with UV RENORMALIZATION AND IRRETARY AND IRRE Red agains **Amfrow** [End, Ma 22] in different kinematic regions (Bilabila and Møner) the planar integrals from [34, 35], we can obtain an analytic result for the bare amplitude for both polarized and under scattering scattering. The UV divergences can then between U rences of the archi- $\sqrt{(1+1)^2}$ the congruence of the congruence $\sqrt{(1+2)^2}$ $\frac{z-x(1+ t_4)}{z}$, $z = 4\frac{t_4(1-x^2)}{z}$ $t_4=x$, $z=\pm \sqrt{x^2-t^2}$ α as the two transcendents in α the two transcendents in α ^T2(*x, t*4) = ¹ ⇡ 1 + *t*⁴ *^t*⁴ *^x* K(*t*4) *^f*(*t*4) 2⇡ (23) in the description of the anciellary files along with the anciellary files along with the anciellary files along with the set of the E_{t} t_{A} = $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $K(t_{A})$ $-Y$ Using the master integrals calculated above, as well as the planar integrals from [34, 35], we can obtain an ana- $\vert t_4 \rangle$ unpolarized scattering. The UV divergences can then benefits can then benefits \mathbf{V} divergences can then benefits of \mathbf{V} *A*r (↵*e, m, s, t,* ✏) = *Z*² 2*A*(↵⁰ *^e, mb, s, t,* ✏)*,* (24) where 8*^z* (*^y* ⁺ *^z* 4)(*^y* ⁺ *^z*) with the relation bare and physical quantities and physical quantities and physical quantities and physical qua ⁴⇡ ⁼ ↵⁰ 4⇡ grals [85] and we fix all boundary conditions imposing di↵erential equation is expressed by Chen iterated inte-~ "letters of the alphabet" ? *^e* = *Ze*↵*e*(*µ*)*, m^b* = *Zmm .* (25) two functions of the contract
The contract of the contract o **AMFlow** [Liu, Ma '22] in different kinematic regions (Bhabha and Møller) ^T2(*y, z*) =p⁴ *^z* p*^z* d*y y* 2(*^y* ⁺ *^z* 4) ⁰ *{* $\overline{}$ *x*₂ 1 **p**_{$\frac{1}{2}$ *t*} ^p1 + *^t*⁴ *,* ^p*t*⁴ *,* $\left\{\overline{t_4}\right\}$ three transcendental functions *{*K(*t*4)*, f*(*t*4)*, F*(*x, t*4)*}*. We want to stress that the choice of canonical coordinates that the choice of canonical coordinates of canonical coordinates in (19) is not merely an academic curiosity, and $\frac{1}{2}$. The numerical evaluation of the integrals integrals in the integral of $\frac{1}{2}$ described below. To explicitly solve the integrals, we first expanding (11) in \mathcal{A} is \mathcal{A} at each order, the solution of th di↵erential equation is expressed by Chen iterated inte- \overline{f} *^y* (4*^y* 8*^z* (*^y* ⁺ *^z* 4)(*^y* ⁺ *^z*) (*^t* + 2*^y* 4) @*^y* ⁰ red agains **AMFlow** [Liu, Ma '22] in different kinematic regions (Bhabha 48 + 4*^y* + 2*y*² + 12*^z* ⁺ *yz* Ξ *z* 4+2*^y ^y*² *^z yt* the final, simplified form is essential to ecoes \mathcal{L}_c to economic imple*s* checked agains **AMFlow [Liu, Ma '22]** in different kinematic regions (Bhabha and Møller)

$$
\partial_{t_4} f = 2 \frac{1 - t_4}{\sqrt{t_4} (1 + t_4)^{3/2}} \, \mathcal{K}(t_4)
$$

WHAT ABOUT THE AMPLITUDE?

AMPLITUDES AND TENSOR DECOMPOSITION

We use the fact that **equal lepton scattering** (Bhabha & Møller) can be obtained from **scattering of different flavour** by crossing, schematically:

$$
(e^+e^- \to e^+e^-) = (e_1^+e_1^- \to e_2^+e_2^-) + (s \leftrightarrow t)
$$

We perform a tensor decomposition with external states in $D=4$ dimensions to retain full dependence on the electron polarizations [Peraro, Tancredi '19, '21]

$$
\mathcal{A}(1_{e^+},2_{e^-},3_{e^-},4_{e^+})=\sum_{i=1}^8\mathcal{F}_i\,T_i
$$

By working in $D = 4$, we are guaranteed to have as many tensors as many different polarizations: 16/2 = 8, only a **physically relevant number of combinations is computed** *The art guaranteed to have as many tensors merges in a property a physically relevant number of c ^T*³ ⁼ *^t*⁴ *, T*⁴ ⁼ *^m*² ⇥ *^t*⁵ *,* CANONICAL BASES FOR THE NON-PLANAR

AMPLITUDES AND TENSOR DECOMPOSITION ² *i*=1 where, due to momentum conservation, *s* + *t* + *u* = 4*m*². se veral topologies. On the technical topologies. On the technical level, on the technical level, our control WHI FILADEA W master integrals. This is achieved for the tensors as a chief of tensors as a chief of tensors as a chief of t $\frac{1}{2}$, improved by field techniques $\frac{1}{2}$. \blacksquare *^T*¹ ⁼ *^m*² ⇥ *^t*¹ *, T*² ⁼ *^m* ⇥ [*t*² ⁺ *^t*3] **TENCOD DECOMBOCITION** SIONAL REGULARIZATION SCHEME ET DECOMPOSITION SCHEME IN TERRETAILLE

Tensor structures can be chosen conveniently as follows: $\frac{1}{5}$ tender directated can be enoden conveniently ad R \overline{S} computer algebra system Form \mathcal{S} *^T*³ ⁼ *^t*⁴ *, T*⁴ ⁼ *^m*² ⇥ *^t*⁵ *,* factors *Fi*, *T*⁵ = *m* ⇥ [*t*⁶ + *t*7] + *t*⁸ *, T*⁶ = *m* ⇥ [*t*⁶ + *t*7] *t*⁸

nal states and loop momenta are treated in *D* dimensions.

$$
\mathcal{A}(1_{e^+},2_{e^-},3_{e^-},4_{e^+})=\sum_{i=1}^8\mathcal{F}_i\,T_i
$$

the scattering amplitude into eight independent Lorentz-

$$
t_i = \overline{U}_e(p_2) \Gamma_i^{(1)} V_e(p_1) \times \overline{U}_e(p_3) \Gamma_i^{(2)} V_e(p_4)
$$

$$
\Gamma_i = {\Gamma_i^{(1)}, \Gamma_i^{(2)}}
$$

$$
T_1 = m^2 \times t_1, \t T_2 = m \times [t_2 + t_3] \t \Gamma_1 = \{1, 1\}, \t T_2 = \{\psi_3, 1\},
$$

\n
$$
T_3 = t_4, \t T_4 = m^2 \times t_5, \t T_5 = m \times [t_6 + t_7] + t_8, \t T_6 = m \times [t_6 + t_7] - t_8 \t \Gamma_7 = \{\gamma^{\mu_1}, \gamma_{\mu_1}\}, \t T_8 = m \times [t_6 - t_7],
$$

\n
$$
T_9 = m^2 \times t_1, \t T_1 = \{1, 1\}, \t T_1 = \{\gamma^{\mu_1}, \gamma^{\mu_2}\}, \t T_2 = \{\gamma^{\mu_1}, \gamma^{\mu_2}\}, \t T_3 = \{\gamma^{\mu_1}, \gamma^{\mu_2}\}, \t T_4 = \{\gamma^{\mu_1}, \gamma^{\mu_2}\}, \t T_5 = \{\gamma^{\mu_1}, \gamma^{\mu_2}\}, \t T_6 = \{\gamma^{\mu_1}, \gamma^{\mu_2}\}, \t T_7 = \{\gamma^{\mu_1}, \gamma^{\mu_2}\}, \t T_8 = \{\gamma^{\mu_3}, \gamma^{\mu_1}, \gamma^{\mu_2}\}, \t T_9 = \{\gamma^{\mu_1}, \gamma^{\mu_3}\}, \t T_1 = \{1, 1\}, \t T_2 = \{\gamma^{\mu_1}, \gamma^{\mu_2}\}, \t T_3 = \{\gamma^{\mu_1}, \gamma^{\mu_2}\}, \t T_4 = \{\gamma^{\mu_1}, \gamma^{\mu_2}\}, \t T_5 = \{\gamma^{\mu_1}, \gamma^{\mu_2}\}, \t T_6 = \{\gamma^{\mu_1}, \gamma^{\mu_2}\}, \t T_7 = \{\gamma^{\mu_1}, \gamma^{\mu_2}\}, \t T_8 = \{\gamma^{\mu_1}, \gamma^{\mu_2}\}, \t T_9 = \{\gamma^{\mu_1}, \gamma^{\mu_3}\}, \t T_1 = \{1, 1\}, \t T_2 = \{\gamma^{\mu_1}, \gamma^{\mu_2}\}, \t T_4 = \{1, 1\}, \t T_5 = \{\gamma^{\mu_1}, \gamma^{\mu_2}\}, \t T_6 = \{\gamma^{\mu_1}, \gamma^{\mu_2}\}, \t T_7
$$

tensors are odd under $p_2 \leftrightarrow p_2$, $p_1 \leftrightarrow p_4$, under which the amplitude must be invariant. which implies that $\mathcal{F}_7 = \mathcal{F}_8 = 0$ to all orders in perturbation theory! Two tensors are odd under $p_2 \leftrightarrow p_3 ~~ p_1 \leftrightarrow p_4$, under which the amplitude must be invariant, FIG. 1: The non-planar topology (left) and its Two tensors are odd under $p_2 \leftrightarrow p_3 ~~ p_1 \leftrightarrow p_4$, under which the amplitude must be invariant, all our cases in perturbation theory is $n_1 = \frac{1}{\sqrt{3}}$. which implies that $\mathscr{F}_7 = \mathscr{F}_8 = 0$ to **all orders in perturbation theory**! ider $p_2 \leftrightarrow p_2$ $p_1 \leftrightarrow p_4$, under which the amplitude must be invariant, $rac{P}{C}$ 1

*, p/*2*^µ*¹ *} ,* ⁸ ⁼ *{p/*3*^µ*¹

7

*, p/*2*^µ*¹ *} .* (6)

 $\frac{1}{2}$ $\frac{1}{2}$

 $p_2 = \frac{1}{2}$

der the simultaneous exchange *p*² \$ *p*³ and *p*¹ \$ *p*4. We

1

 $\overline{\Omega}$

 \mathcal{D} \setminus

5

 $p_1 \longrightarrow \frac{1}{\sqrt{2\pi}}$

to the number of \mathfrak{g} independent configurations to \mathfrak{g} in \mathfrak{g} in \mathfrak{g}

5

6

 $\frac{1}{\sqrt{2}}$ are 24 $\frac{1}{\sqrt{2}}$ and 24 $\frac{1}{\sqrt{2}}$ are 24

independent in a particular in a particular theory such as Ω , Ω , Ω

 $p_1 \longrightarrow p_2$

*p*1

*^T*³ ⁼ *^t*⁴ *, T*⁴ ⁼ *^m*² ⇥ *^t*⁵ *,*

computer algebra system FORM [57–60], we insert Feyn-

*^T*¹ ⁼ *^m*² ⇥ *^t*¹ *, T*² ⁼ *^m* ⇥ [*t*² ⁺ *^t*3]

D, ^s

*p*2

 \sim

 \overline{a}

 \sim

 α ote: with this choice, we obtain ampl 1 3 5 p_1
 p₂
 p₂
 p₂
 p₂
 p₃
 p₃
 p₃
 p₁
 p₂
 p₁
 p₁
 p₁ Note: with this choice, we obtain amplitude directly in $\sum_{i=1}^{n}$ in define [Peraro, Tancredi '19, '21]^{p_1} $\sum_{i=1}^{n} p_1 = \frac{1}{\sqrt{1-\frac{1}{n}}} \frac{3}{\sqrt{1-\frac{1}{n}}} p_2$ **HV SCHEME** Letaro, lancreal 19, 21 $65x^2 + 6$ ³ = *{*1*, p/*2*} ,* ⁴ = *{p/*3*, p/*2*} ,* ⁵ ⁼ *{^µ*¹ *, ^µ*¹ *} ,* ⁶ ⁼ *{p/*3*^µ*¹ *, ^µ*¹ *} ,* Note: with this choice, we obtain amplitude **directly in tHV scheme [Peraro, Tancredi '19, '21]**

AMPLITUDES AND TENSOR DECOMPOSITION

[Nogueira]

From the form factors one can easily obtained both polarized and unpolarized amplitudes

We use standard programs **QGRAF, FORM, Mathematica, Reduze2, Kira (with FireFly)**

[Manteuffel, Studerus]

[Meierhöfer, Usovitsch; Klappert et al]

[Vermaseren]

All in all, including planar integrals and crossings, there are **252 masters**

PL integrals can be expressed as MPLs. NPL integrals as iterated integrals over **elliptic differential forms**. Before discussing evaluation strategy, what checks have we done?

CHECKS: UV & IR FACTORIZATION *^t*⁴ *^x* K(*t*4) *^f*(*t*4) lytic result for the bare amplitude for both polarized and *V* & IR FACTORIZATION 12 ✏ + ✓↵*e*(*µ*) 4⇡ $\mathbb N$

All master in *A*r (↵*e, m, s, t,* ✏) = *Z*² 2*A*(↵⁰ *^Z^m* =1 + ✓↵*e*(*µ*) ้รน *D*✏ 3 ✏ ⁴ ⁸✏ ¹⁶✏ All master integrals checked versus **AMFlow [Liu, Ma '22]**

renormalized according to

◆

 $\ddot{}$

UV renormalization requires renormalizing coupling, electron mass and wave function. We perform renormalization on-shell and *F*(*x, t*4) is the derivative of the *Abel map*: \overline{a} malization on-shell discussed bare and physical quantities r \overline{r} *equires* report 1 O ulizing coupling, electron SS. wave f anction. 1011 011 - 311

$$
\frac{e^2}{4\pi} = \alpha_e^0 = \left(\frac{e^{\gamma_E}}{4\pi}\mu^2\right)^{\epsilon} Z_e \alpha_e(\mu), \quad m_b = Z_m m
$$

◆

 W_{α} are then \ldots simplified and all double integrals over the personal double integrals over Writh **IR** noles that are one-loon exact reference police chat are one loop chate. \overline{W}_{α} are then left with \overline{IP} poles that are one-loop-exact, We are then left with **IR poles** that **are one-loop exact**

$$
\mathcal{A}^{OS}(\alpha, m, s, t, \epsilon) = e^{\frac{\frac{\alpha}{4\pi}Z_1^{\text{IR}}}{\epsilon}} \mathcal{C}(\alpha, m, s, t, \epsilon)
$$
\n
$$
Z_1^{\text{IR}} = \frac{4(-2m^2 + s)}{\sqrt{-s}\sqrt{4m^2 - s}} \ln\left(1 - \frac{s}{2m^2} - \frac{1}{2}\sqrt{\frac{-s}{m^2}}\sqrt{4 - \frac{s}{m^2}}\right) + \frac{4(-2m^2 + t)}{\sqrt{-t}\sqrt{4m^2 - t}} \ln\left(1 - \frac{t}{2m^2} - \frac{1}{2}\sqrt{\frac{-t}{m^2}}\sqrt{4 - \frac{t}{m^2}}\right) - \frac{4(-2m^2 + u)}{\sqrt{-u}\sqrt{4m^2 - u}} \ln\left(1 - \frac{u}{2m^2} - \frac{1}{2}\sqrt{\frac{-u}{m^2}}\sqrt{4 - \frac{u}{m^2}}\right) - 4,
$$

From ϵ -factorised differential equations, it is "easy" to obtain **series expansions** in any kinematical region

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From ϵ -factorised differential equations, it is "easy" to obtain **series expansions** in any kinematical region

For most applications the **electron mass can be considered small** \rightarrow we perform a small mass expansion of the individual master integrals and of the whole amplitude

$$
\mathscr{A}(s,t,m^2) = \sum_{ijk} (m^2)^{i\epsilon} \log^j(m^2) \epsilon^k A_{ij}^{(k)}(s,t)
$$

Coefficients of the series $A_{ij}^{(k)}(s,t)$ can be written in terms of **harmonic polylogarithms [Remiddi, Vermaseren '19]**

Boundary conditions can be all fixed by **regularity and eigenvalue conditions** (which should then be transported to the region $m^2 \ll s$, $|t|$)

Series converges very well in the bulk of the phase-space, but one must **take special care in considering forward or backward limit** $t \to 0$ or $u \to 0$ (scattering angle going to 0 or π)

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$$

$$
A_{ij}^{(k)}(s, t) = \sum_{n,m} A_{ij}^{(k,n,m)}(s) (t^n \log^m(-t/s))
$$

logarithms log (−*t/s*) can *spoil the convergences of the mass expansion* As expected, "Regge" limit does not commute with small mass limit…

SMALL MASS EXPANSION & NUMERICAL EVALUATION SMALL MASS EXPANSION & NIIMERICAL EVALUATION intermediate energy of *E*CM = 32*m*. We highlight the

This effect can be seen clearly plotting the 2Re ($\mathcal{C}^{(2)}\mathcal{C}^{(0)*}$) in *extreme regions* (here for **Møller scattering**) ι s ι

Effects are stronger at very low energies ($E_{CM} = 2.5 MeV$)

Here compare 2 $Re\left(\mathscr{C}^{(1)}\mathscr{C}^{(0)^*}\right)$ expanded versus exact, and separately 2 $Re\left(\mathscr{C}^{(2)}\mathscr{C}^{(0)^*}\right)$ function of θ

energies. A function of the size of the size of the size of the SNLO QED size of the size of the SNLO QED size of the SNLO QED size of the NNLO QUARK III (α) and the NNLO QED size of the SNLO QED size of the NNLO QED

CONCLUSIONS AND OUTLOOK

- ➤ Bhabha and Møller scattering are fundamental "standard candles" in QED, both for **phenomenological applications** and as **experimental ground for new techniques**
- ➤ Pushing the calculation to two loops required new techniques to handle integrals of elliptic type

- \blacktriangleright We derived an ϵ -factorised basis leveraging new algorithms that can be extended beyond polylogs
- ➤ Having differential equations in this form, it becomes in principle straightforward to obtain series expansions
- ➤ For Bhabha and Møller, we constructed a small mass expansion
- ➤ We proved that it converges extremely well in the bulk of the phase space, but non-trivial effects can be observed at the boundaries (forward / backward region)
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Stay tuned :-)

and thank you for your attention!