FOUR-LEPTON SCATTERING IN MASSIVE QED

BHABHA AND MØLLER SCATTERING UP TO TWO LOOPS

QCD meets EW
CERN – 07/02/2024

Lorenzo Tancredi – Technical University Munich

[Collaboration with Delto, Duhr, Zhu — arXiv:2311.06385, arXiv:24xx.xxxxx]

[and ongoing work with Duhr, Maggio, Nega, Wagner — arXiv:2305.14090, arXiv:24xx.xxxxx]

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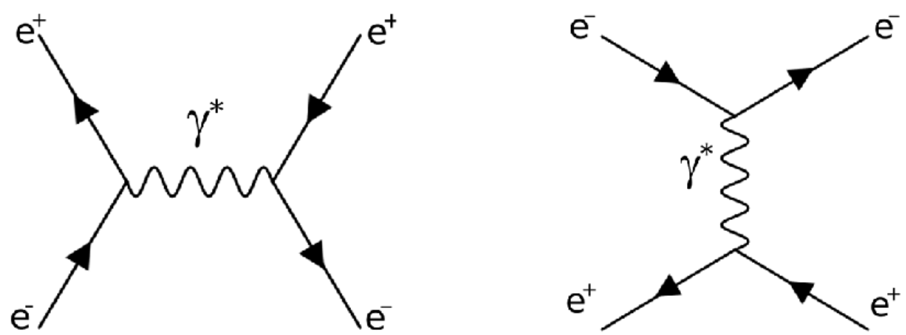
LMU TUM ORIGINS Excellence Cluster

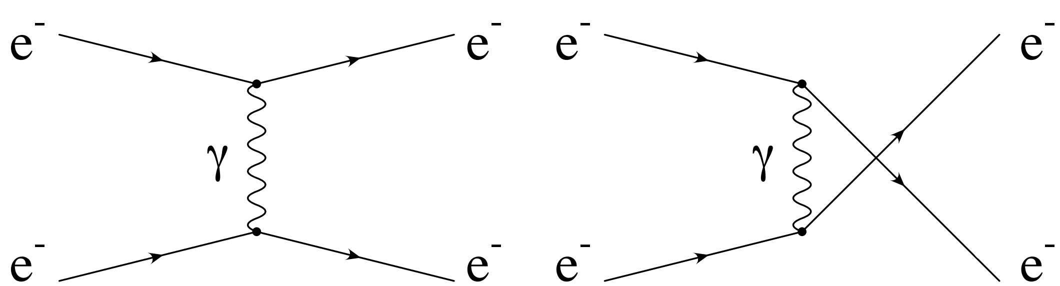
Technische Universität München

INTRODUCTION: BHABHA AND MØLLER SCATTERING

Bhabha $e^+e^- \to e^+e^-$



Møller $e^-e^- \to e^-e^-$



Basic processes in QED, received a lot of attention since the birth of QFT *(see Landau's fourth book)*

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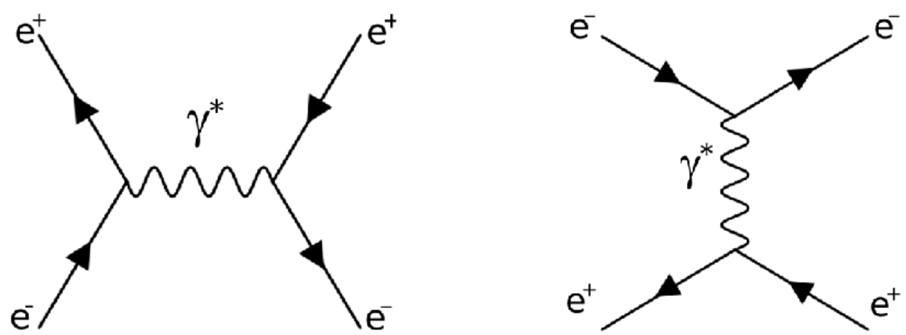
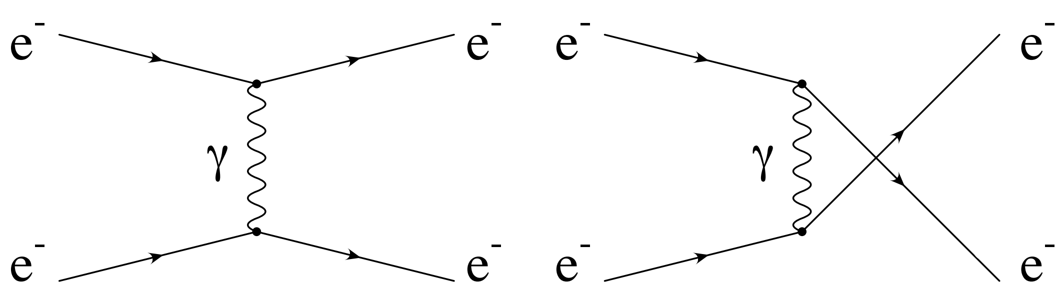


High-energy lepton colliders

Small angle scattering efficient tool for luminosity determination @ lepton colliders (radiative corrections QED dominated)

Large angle used to measure integrated luminosity at $\sqrt{s} \sim \mathcal{O}(\mathrm{GeV})$ colliders (flavour factories BELLE, BABAR, …) + in principle ILC!

Møller $e^-e^- \to e^-e^-$



Low-energy lepton colliders

Dominant physical process in low-energy electron scattering experiments, also used for luminosity monitoring.

Particularly relevant @ PRad-II (attempt to resolve proton radius puzzle), and recently measured down to energies of 2.5 MeV (see arXiv:1903.09265) — mass effects should not be neglected

Also relevant to measure weak mixing angle …

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INTRODUCTION: BHABHA AND MØLLER SCATTERING

State-of-the-art in QED (ignoring other EW effects here)

NLO QED effects known exactly in Bhabha and Møller with full mass dependence

NNLO QED effects with ***full mass dependence remain elusive*** due to ***missing two-loop amplitudes***

Leading order mass effects **[Becher, Melnikov '07]**

Leading power-suppressed mass effects also included **[Penin, Zerf '16]**

Next-to-soft stabilisation for real-virtual matrix elements **[Banerjee et al '21]**

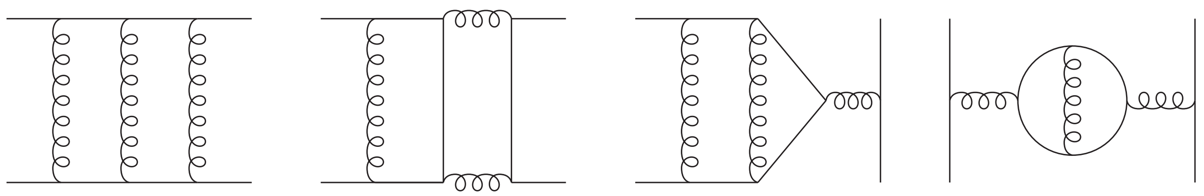
NNLO Møller including leading order mass effects & next-to-soft stabilisation **[Banerjee et al '22]**

Fermionic loop corrections with full mass dependence in Bhabha **[Bonciani et al '15]**

To have full control on low energy / small angle regions, full mass dependence desirable
$\longrightarrow$ **two-loop amplitudes remains last missing ingredient**

INTRODUCTION: HISTORY CALCULATION OF TWO-LOOP AMPLITUDE

Full massless two loop amplitudes in terms of HPLs [Bern, Dixon, Ghinculov '00]



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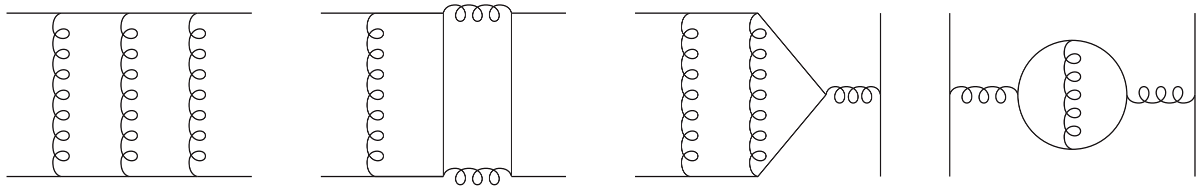
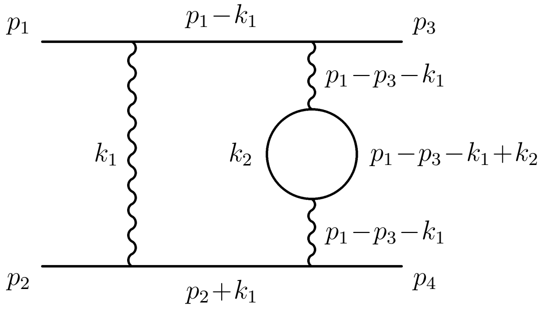


Form factor integrals and purely fermionic contributions [Bonciani et al '03, '04]



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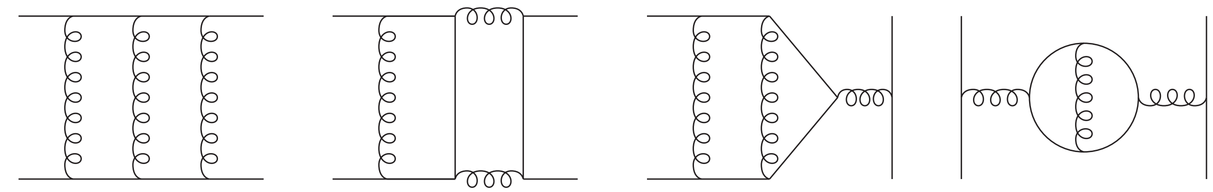

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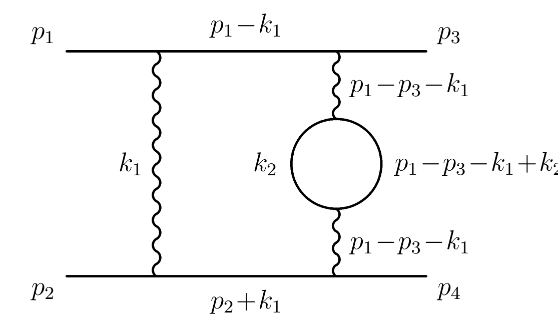
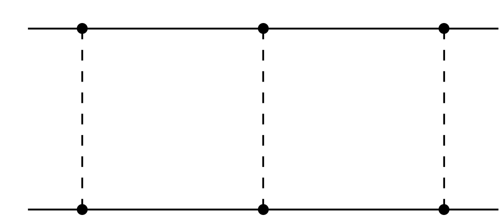


Ten years later, ladder planar integrals in terms of MPLs **[Henn, Smirnov, Smirnov '13]**



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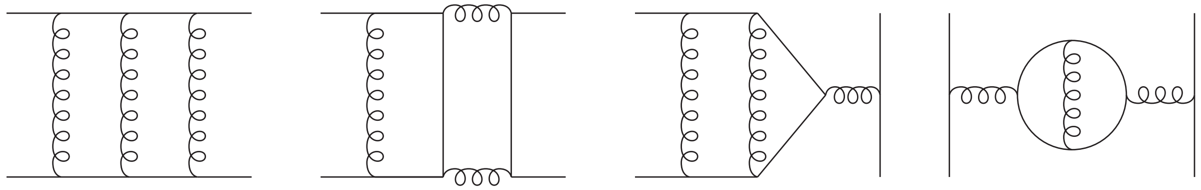

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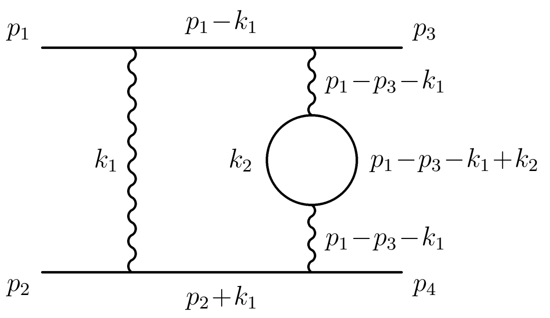

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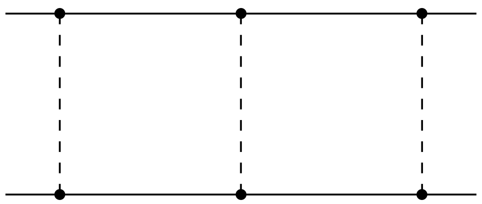
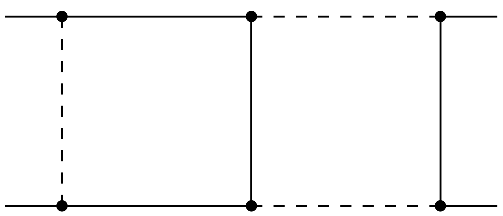


Eight more years for second planar family **[Duhr, Smirnov, Tancredi '21]**

d-logs but four square roots not rationalisable simultaneously

exploiting the fact that they don't mix, one can write results in terms of MPLs, but extremely cumbersome



MASSES AND GEOMETRY

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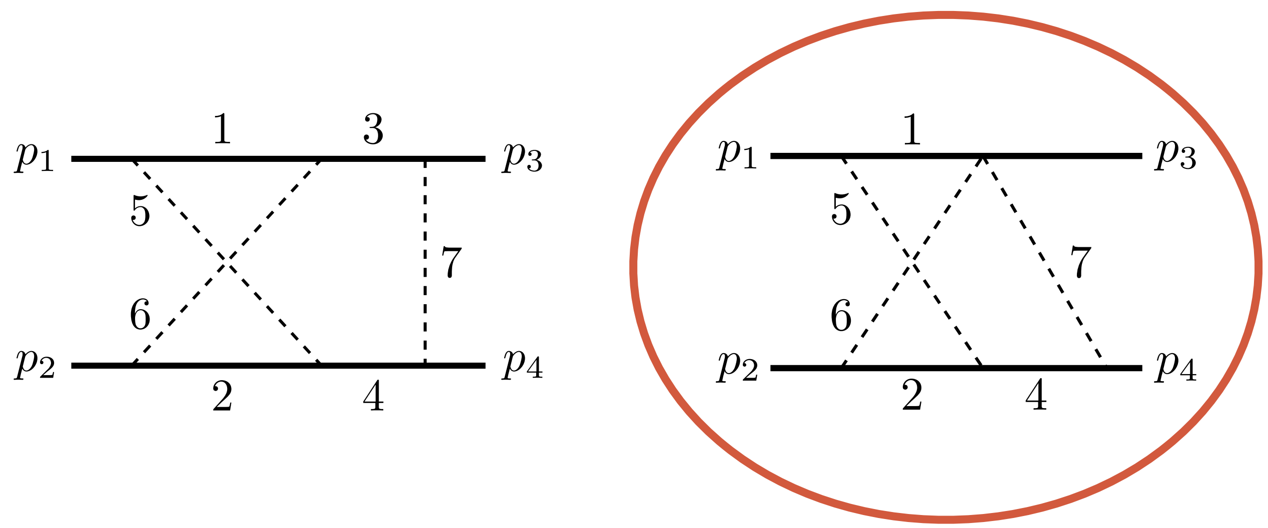
Mathematically, things start becoming rather interesting in NPL sector



$$\mathrm{I}_{a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9}\left(D, \frac{s}{m^2}, \frac{t}{m^2}\right) = e^{2\gamma_E \epsilon} \left(\mu^2\right)^{\sum_{j=1}^{9} a_j - D} \int \frac{\mathrm{d}^D k_1}{i\pi^{\frac{D}{2}}} \frac{\mathrm{d}^D k_2}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{9} \frac{1}{P_j^{a_j}},$$

$$\begin{aligned}
&P_1 = k_1^2 - m^2, \quad P_2 = (k_1 - k_2 - p_2)^2 - m^2, \\
&P_3 = k_2^2 - m^2, \quad P_4 = (k_2 + p_1 + p_2)^2 - m^2, \\
&P_5 = (k_1 + p_1)^2, \quad P_6 = (k_1 - k_2)^2, \quad P_7 = (k_2 - p_3)^2 \\
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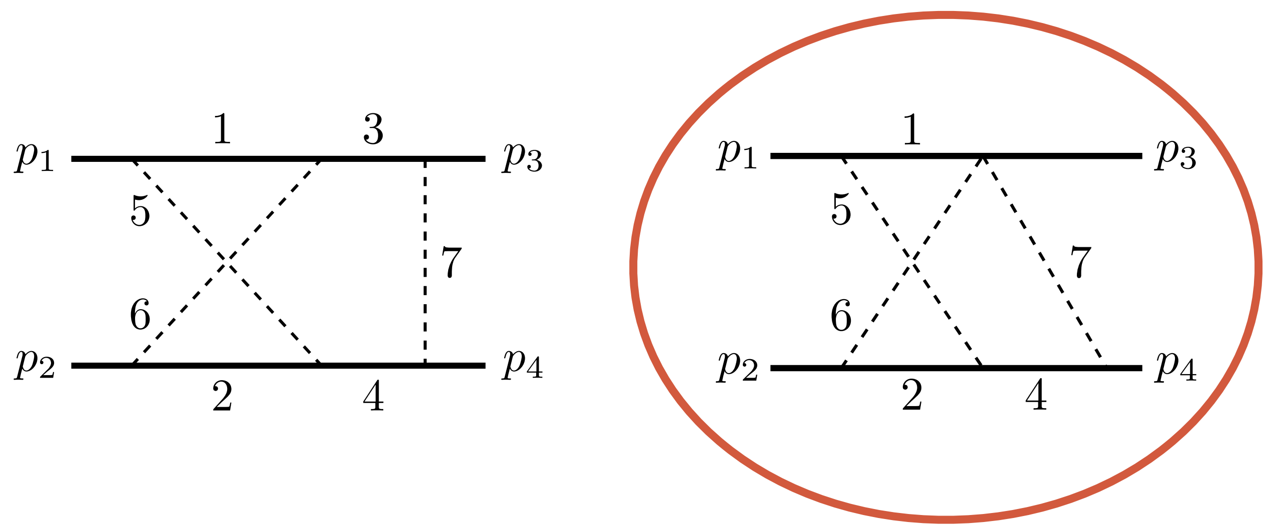

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$$P_1 = k_1^2 - m^2, \quad P_2 = (k_1 - k_2 - p_2)^2 - m^2,$$
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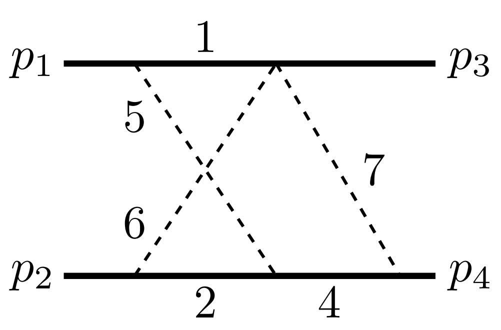
Algebraically "simple" for today's standards: 2 dimensionless ratios, "only" 52 masters integrals

MASSES AND GEOMETRY

More in detail



6 propagator graph: $\mathrm{I}_{110111100}$

6 master integrals in top sector (+ sub-topologies)

Leading singularities (maximally iterated integrand residues) fulfil **homogeneous differential equation** and can be used to build space of solutions **[Primo, Tancredi '16,'17]**

Start cutting all propagators (max cut). Convenient in Baikov **[Frellesvig, Papadopoulos '17]**

$$\mathrm{MaxCut}_{\mathcal{C}}\left[\mathrm{I}_{110111100}\right] \sim \int_{\mathcal{C}} \frac{\mathrm{d}z_2 \wedge \mathrm{d}z_1}{z_2\sqrt{(z_1-s-z_2)(z_1-s+4m^2-z_2)}\sqrt{(tz_1-st+sz_2)^2-4m^2(tz_1^2+s(t-z_2)^2)}} \,.$$

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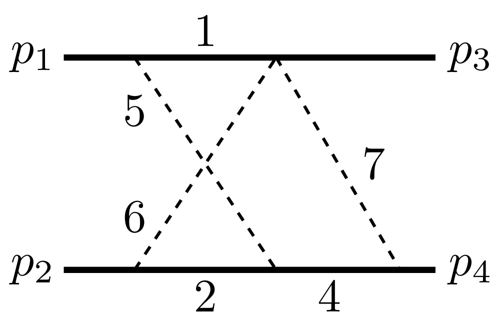

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One extra residue! Max cut is not the end of the story, we can **"cut again"** taking residue at $z_2 = 0$

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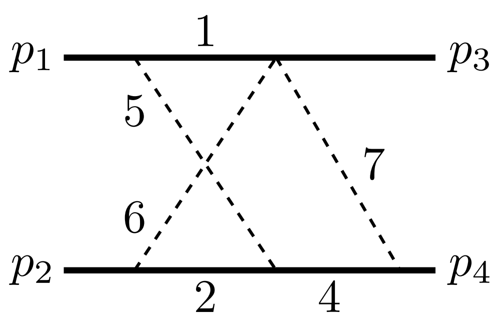

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left with a **one-fold integral on a square root of a quartic polynomial**: no extra residue but two independent branch cuts which provide the solutions to the homogeneous differential equation **[Primo, Tancredi '16,'17]**

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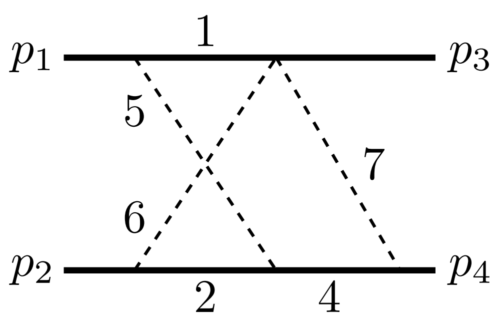

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Geometry is an **elliptic curve**

$$\mathcal{E}_4 : Y^2 = (X - e_1)(X - e_2)(X - e_3)(X - e_4)$$

$$e_1 = y - 4\,, \quad e_2 = -\frac{yz + 2\sqrt{y\,z(y+z-4)}}{4-z}\,,$$

$$e_3 = -\frac{yz - 2\sqrt{y\,z(y+z-4)}}{4-z}\,, \quad e_4 = y\,.$$

Periods obtained integrating on two branch cuts

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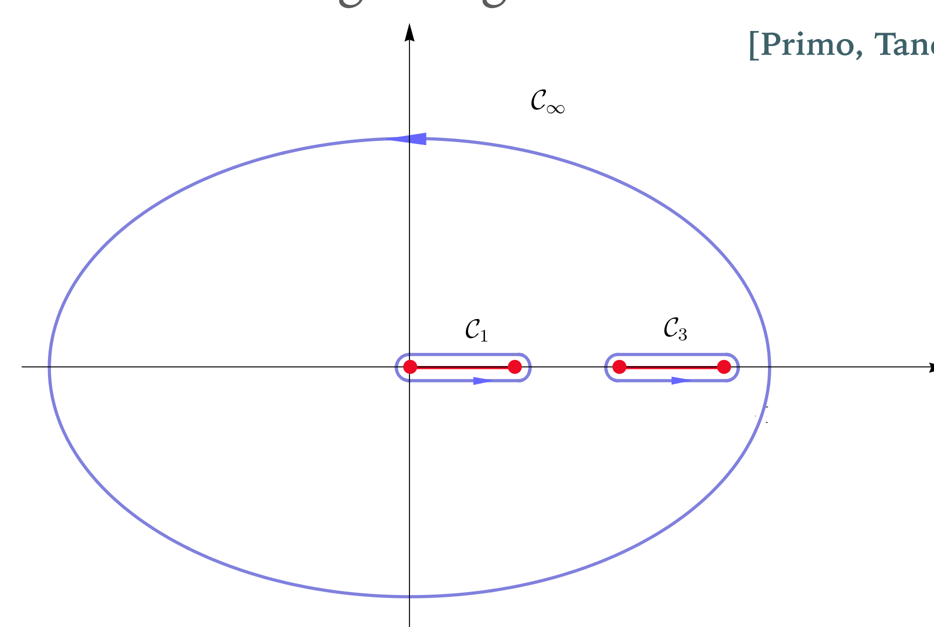

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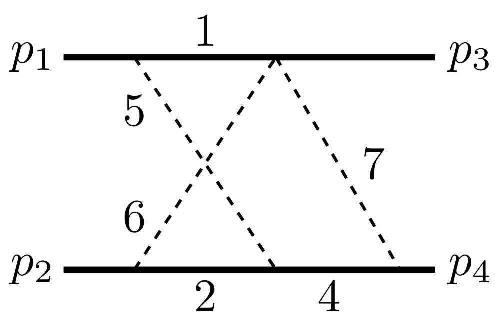

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Periods obtained integrating on two branch cuts

$$\Psi_0\,(y\,,z) \equiv 2\int_{e_2}^{e_3} \frac{\mathrm{d}X}{Y} = \frac{4\,\mathrm{K}(\lambda)}{\sqrt{(e_1-e_3)(e_2-e_4)}}$$

$$\lambda = \frac{4}{2 + \sqrt{\frac{-y(y+z-4)}{-z}}}$$

$$\Psi_1\,(y\,,z) \equiv 2\int_{e_2}^{e_1} \frac{\mathrm{d}X}{Y} = \frac{4\,\mathrm{K}(1-\lambda)}{\sqrt{(e_1-e_3)(e_2-e_4)}}$$

HOW DO WE COMPUTE THESE INTEGRALS?

(Intermezzo on differential equations and canonical forms)

DIFFERENTIAL EQUATIONS

Most powerful technique to compute Feynman integrals: **differential equations method**

[Kotikov '93] [Remiddi '97]
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We compute Feynman integrals as series in $\epsilon = (4-d)/2$

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Iterative structure in ϵ made manifest by differentiation

(Scalar) Feynman Integrals

$$\mathcal{I} = \int \prod_{l=1}^{L} \frac{d^D k_l}{(2\pi)^D} \frac{S_1^{b_1} \ldots S_m^{b_m}}{D_1^{a_1} \ldots D_n^{a_n}}$$

with $S_i \in \{k_i \cdot k_j, \ldots, k_i \cdot p_j\}$

→

Integration by Parts etc

$$\int \prod_{l=1}^{L} \frac{d^D k_l}{(2\pi)^D} \frac{\partial}{\partial k_l^{\mu}} \left[v_\mu \frac{S_1^{b_1} \ldots S_m^{b_m}}{D_1^{a_1} \ldots D_n^{a_n}} \right] = 0$$

Basis of Master Integrals (MIs)

$$\underline{I} = \{I_1(\underline{z}, \epsilon), \ldots, I_N(\underline{z}, \epsilon)\}$$

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By differentiating and reducing to masters we obtain a **linear system of differential equations**

$$d\underline{I} = GM(\underline{z}, \epsilon)\, \underline{I}$$

In this form, iterative structure *hidden* in arbitrary dependence on ϵ

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Imagine to be able to perform a series of **rotations R_i on the original basis**

$$\underline{J} = \mathbf{R}(\underline{z}, \epsilon)\underline{I} \quad \text{with} \quad \mathbf{R}(\underline{z}, \epsilon) = \mathbf{R}_r(\underline{z}, \epsilon) \cdots \mathbf{R}_2(\underline{z}, \epsilon)\mathbf{R}_1(\underline{z}, \epsilon)$$

Such that

$$\mathrm{d}\underline{J} = \epsilon\,\mathbf{GM}\ (\underline{z})\underline{J}\,, \quad \text{where} \quad \epsilon\,\mathbf{GM}\ (\underline{z}) = [\mathbf{R}(\underline{z}, \epsilon)\mathbf{GM}(\underline{z}, \epsilon) + \mathrm{d}\mathbf{R}(\underline{z}, \epsilon)]\,\mathbf{R}(\underline{z}, \epsilon)^{-1}$$

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Since GM(z) does not depend on ϵ, the iterative structure in ϵ becomes manifest

We refer to such a basis as in ***epsilon-factorised form*** [Kotikov '10; J. Henn '13; Lee '13, …]

CANONICAL AND EPS-FACTORISED BASES

What can we say about GM(z) ?

$$d\underline{J} = \epsilon\, GM(\underline{z})\, \underline{J}$$

- Is **GM(z)** **unique** ?
- Are there ϵ-factorised bases that are **better than others**?

→ Can we define an **optimal basis** of master integrals for a given problem?

We understand the problem well in the **polylogarithmic case**

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$$I \sim \int \prod_{i=1}^{n} \mathrm{d}x_i\, \mathcal{F}(x_i, \underline{z}) \quad \xrightarrow[\text{can be written as}]{\text{if}} \quad \sim \sum_i c_i \int d\log f_1^{\,i} \int d\log f_2^{\,i} \ldots \int d\log f_n^{\,i}\,; \quad c_i \in \mathbb{Q}$$

↑

Leading Singularities ~ *iterative residues*
of the integrand in all integration variables

Conjecturally, these integrals fulfil ***canonical differential equations*** [Arkani Hamed et al '10; Kotikov '10; J. Henn '13]

CANONICAL BASES: THE POLYLOGARITHMIC CASE

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Recipe (in a nutshell):

1. choose integrals whose ***integrands*** have only **simple poles and are in d-log form**
2. choose integrals whose ***iterated residues*** at all simple poles can be **normalized to numbers**

[Arkani-Hamed et al'10; Henn, Mistlberger, Smirnov, Wasser '20]

CANONICAL BASES: THE POLYLOGARITHMIC CASE

What do these conditions imply?

$$G(c_1, c_2, ..., c_n, x) = \int_0^x \frac{dt_1}{t_1 - c_1} G(c_2, ..., c_n, t_1)$$
$$= \int_0^x \frac{dt_1}{t_1 - c_1} \int_0^{t_1} \frac{dt_2}{t_2 - c_2} ... \int_0^{t_{n-1}} \frac{dt_n}{t_n - c_n}$$

n = number of integrations or **transcendental weight**

MPLs are iterated integrals over d-log forms (with rational entries)

The requirements before, guarantees that Feynman integrals are written as **pure, uniform weight combinations of MPLs**

Note: this makes sense, since forms with single poles span the full first de Rham cohomology, or in other words **MPLs are generated by dlogs!**

BEYOND POLYLOGARITHMS: CONCEPTUAL DIFFERENCES

Even with MPLs, insisting on ***simple poles*** in the integrand *(neglecting integration contour)* is too strong of a requirement, as it forces us to **exclude any squared propagator**!

Physics:

Double poles often imply power-like singularities in the IR which should be excluded in gauge theories

→ Typically true when dealing with massless propagators

Massive propagators can be squared at will, *without changing IR behaviour* and (actually) *improving UV behaviour*

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Mathematics:

Differential forms with ***simple poles*** are intrinsically ***not enough*** to span full space for more general problems ***(elliptic curves or Tori, K3, Calabi-Yaus etc)***

→ Think about *independent integrands* in the elliptic case:

$$K(x) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-xt^2)}}$$ has **no poles** while $$E(x) = \int_0^1 dt \frac{\sqrt{1-xt^2}}{\sqrt{1-t^2}}$$ has **double pole at infinity**

A DIFFERENT PERSPECTIVE ON MPLS?

UNIPOTENT FUNCTIONS AND DIFFERENTIAL EQUATIONS

Canonical integrals in polylogarithmic case give rise to **pure combinations of MPLs**

$$G(c_1, c_2, ..., c_n, x) = \int_0^x \frac{dt_1}{t_1 - c_1} G(c_2, ..., c_n, t_1)$$
$$= \int_0^x \frac{dt_1}{t_1 - c_1} \int_0^{t_1} \frac{dt_2}{t_2 - c_2} ... \int_0^{t_{n-1}} \frac{dt_n}{t_n - c_n}$$

[…,Remiddi, Vermaseren ’99, Goncharov ’00,…]

MPLs are unipotent: they fulfil particularly simple differential equations

$$\frac{d}{dx} G(c_1, \ldots, c_n; x) = \frac{1}{x - c_1} G(c_2, \ldots, c_n; x)$$

by diff. we lower the weight & length

UNIPOTENT FUNCTIONS AND DIFFERENTIAL EQUATIONS

Canonical integrals in polylogarithmic case give rise to **pure combinations of MPLs**

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by diff. we lower the weight & length

General definition is: W^u unipotent if it fulfils system of diff equations with Nilpotent matrices

$$\mathrm{d}\mathbf{W}^{\mathrm{u}} = \left(\sum_i \mathbf{U}_i(\underline{z})\, \mathrm{d}z_i \right) \mathbf{W}^{\mathrm{u}},$$

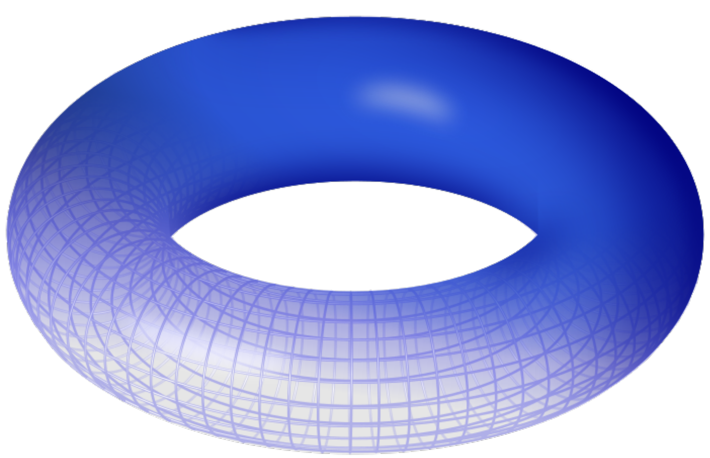
Where $U_i(\underline{z})$ are **Nilpotent matrices**: $\underbrace{U_i \cdot U_i \cdot \cdots \cdot U_i}_{n} = 0$

BEYOND POLYLOGARITHMS: CONCEPTUAL DIFFERENCES

Same condition is fulfilled by **Elliptic polylogarithms (eMPLs)**

[Brown Levin '11; Brödel, Mafra, Matthes, Schlotterer '14]
[Brödel, Dulat, Duhr, Penante, Tancredi '17, '18]

We can insist on **single poles** ↔ logarithmic singularities (Gauge Theory)



$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \Psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

Price to pay: infinite tower of **transcendental kernels** [can't be obtained from "residue of integrand"]

Still fulfil unipotent diff equation: at the basis of definition of symbol!

$$\mathrm{d}\mathbf{W}^{\mathrm{u}} = \left(\sum_i \mathbf{U}_i(\underline{z})\, \mathrm{d}z_i\right) \mathbf{W}^{\mathrm{u}},$$

CAN WE USE THE UNIPOTENCE CONDITION?

EXAMPLE: POLYLOGARITHMIC CASE

It works in the (simple) polylogarithmic case: Sunrise with **2 massive and 1 massless propagator**

$$I_1 = I_{0,1,1,0,0}\,, \quad I_2 = I_{1,1,1,0,0} \quad \text{and} \quad I_3 = I_{1,1,2,0,0}$$

Differential equations read: $d\underline{I} = \left[A_0 + \epsilon A_1\right] \underline{I}$

Homogeneous equation in d=2

$$\frac{\partial}{\partial m^2}\begin{pmatrix} \mathfrak{I}_2 \\ \mathfrak{I}_3 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ \frac{1}{m^2(s-4m^2)} & \frac{-s+10m^2}{m^2(s-4m^2)} \end{pmatrix}\begin{pmatrix} \mathfrak{I}_2 \\ \mathfrak{I}_3 \end{pmatrix}$$

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Matrix of **homogeneous solutions** contains **algebraic functions and logs**

$$dW = AW \qquad \rightarrow \qquad W = \begin{pmatrix} \frac{1}{r(s,m^2)} & \frac{1}{r(s,m^2)} \log\left(\frac{s-r(s,m^2)}{s+r(s,m^2)}\right) \\ \frac{s}{r(s,m^2)^3} & \frac{s}{2m^2 r(s,m^2)^2} + \frac{s\log\left(\frac{s-r(s,m^2)}{s+r(s,m^2)}\right)}{r(s,m^2)^3} \end{pmatrix} \qquad \text{with} \quad r(s,m^2) = \sqrt{s(s-4m^2)}$$

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Matrix of **homogeneous solutions** contains **algebraic functions and logs**

Since differential equations are linear in ϵ, we could be "tempted" to just "rotate W away"

$$d\underline{I} = \left[A_0(\underline{z}) + \epsilon\, A_1(\underline{z})\right]\underline{I} \longrightarrow \quad \underline{I} = W \cdot \underline{J} \quad\quad d\underline{J} = \epsilon\, \left[W^{-1}\cdot A_1(\underline{z})\cdot W\right]\underline{J}$$

New matrix not in dlog form (logs not dlogs !) and basis $\underline{J}$ is not pure combination of UT MPLs…

EXAMPLE: POLYLOGARITHMIC CASE

Instead, we will rotate away only a "part" of the homogeneous solution:

Split it in **semi-simple** and **unipotent** $W = W^{ss} \cdot W^{u}$

$$\mathbf{W}^{\mathrm{ss}} = \begin{pmatrix} \frac{1}{r(s,m^2)} & 0 \\ \frac{s}{r(s,m^2)^3} & \frac{1}{2m^2(s-4m^2)} \end{pmatrix} \quad \text{and} \quad \mathbf{W}^{\mathrm{u}} = \begin{pmatrix} 1 & \log\left(\frac{s-r(s,m^2)}{s+r(s,m^2)}\right) \\ 0 & 1 \end{pmatrix} \qquad r(s,m^2) = \sqrt{s(s-4m^2)}$$

unipotent part contains transcendental solution

only algebraic part in semi-simple matrix

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Rotate away **only semi-simple part** $$\underline{I}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{(\mathbf{W}^{\mathrm{ss}})^{-1}} \end{pmatrix} \underline{I}$$

End basis corresponds to matrix W^u: one master has **weight 0**, the other has **weight 1**, ***weight mixing disentangled*** —> this behaviour is typical at a so-called **MUM point** (Maximal Unipotent Monodromy), which is well understood for elliptic curves and Calabi-Yau generalizations !

EXAMPLE 1: POLYLOGARITHMIC CASE

Clean up remaining non-factorised dependence with a rotation $\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2(s+2m^2)}{r(s,m^2)} & 1 \end{pmatrix} \begin{pmatrix} \epsilon^2 & 0 & 0 \\ 0 & \epsilon^2 & 0 \\ 0 & 0 & \epsilon \end{pmatrix}$

$\mathrm{d}\underline{J} = \epsilon\, \mathbf{GM}^\epsilon \underline{J}$ with $\underline{J} = (J_1, J_2, J_3)^T = \mathbf{T}\, \underline{I}'$,

$$\mathbf{GM}^\epsilon = \begin{pmatrix} -2\alpha_1 & 0 & 0 \\ 0 & 2\alpha_1 - \alpha_2 - 3\alpha_3 & \alpha_4 \\ 2\alpha_1 - 2\alpha_2 & -6\alpha_4 & -3\alpha_1 + \alpha_2 \end{pmatrix}$$

$$\alpha_1 = \mathrm{d}\log(m^2)\,, \quad \alpha_2 = \mathrm{d}\log(s)\,, \quad \alpha_3 = \mathrm{d}\log\left(s - 4m^2\right)\,, \quad \alpha_4 = \mathrm{d}\log\left(\frac{s - r(s,m^2)}{s + r(s,m^2)}\right)$$

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NB: by analysing leading singularities with **DLogBasis** find the same basis up to constant rotation!

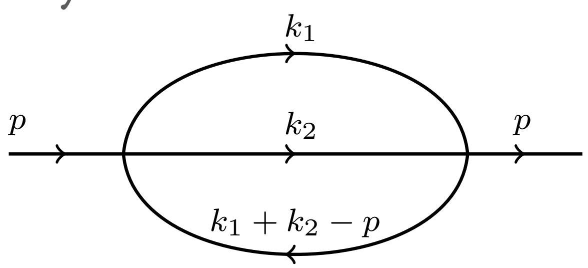
[P. Wasser ’19,’20]

$$J_1 = M_1\,, \quad J_2 = M_2\,, \quad J_3 = -M_1 + 3M_3$$

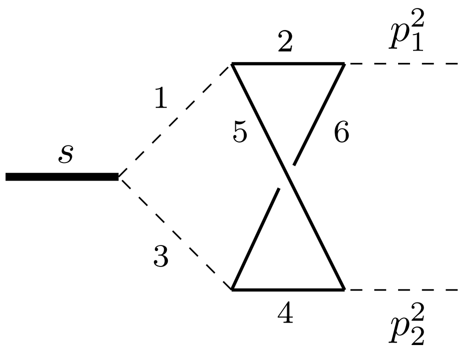
STRATEGY SUCCESSFUL IN MANY NON-TRIVIAL CASES

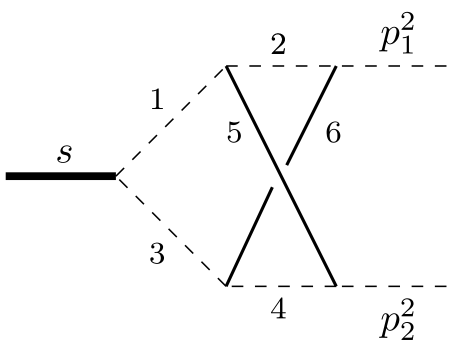
Strategy is general and **does not have to do with details of the geometry***

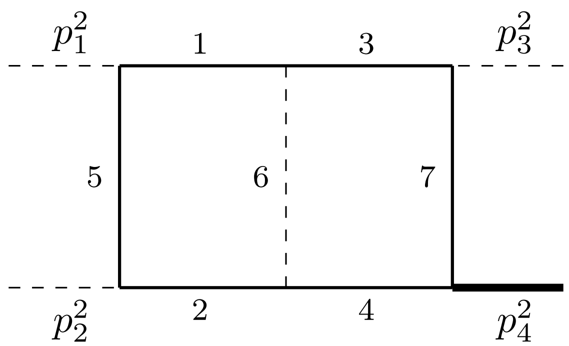
Applied it successfully to elliptic sunrise (equal or different masses)



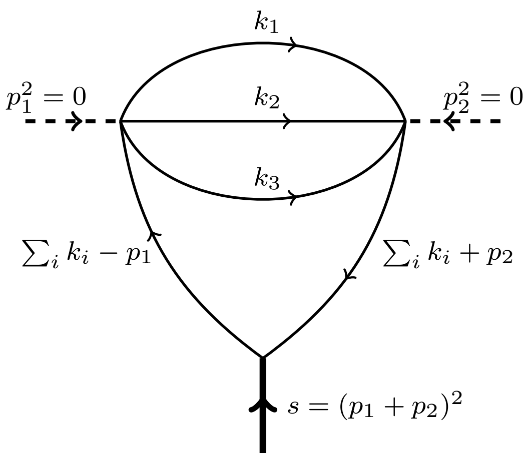
Many other multi-scale elliptic problems

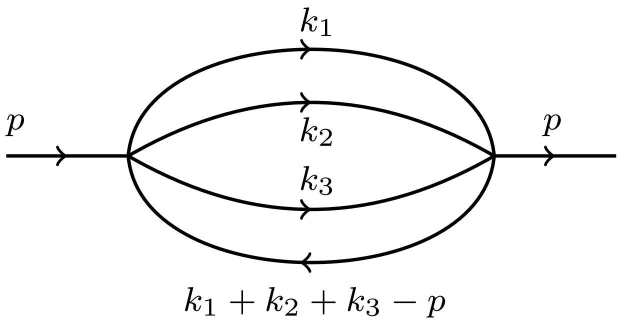






Even cases beyond 1 elliptic curve





BACK TO MASTER INTEGRALS FOR BHABHA AND MØLLER

Following the strategy above, we obtain a fully ϵ-factorised system of differential equations

Boundary conditions can be fixed by using regularity conditions (absence of pseudo thresholds) or, equivalently, large mass expansion

The result can then be written in terms of iterated integrals over many differential forms which involve the period and quasi period of the elliptic curve, and integrals over it

$$\mathrm{T}_1(y,z) = \int \mathrm{d}y \left[\frac{-z}{y}(4y^2+4y(z-4)+z(z-4))\Psi_0 -8z\frac{(y+z-4)(y+z)}{(t+2y-4)}\partial_y\Psi_0 \right] +\mathrm{d}z\left[\frac{-z}{4-z}\frac{-48+4y+2y^2+12z+yz}{z+y-4}\Psi_0\right]$$

$$\mathrm{T}_2(y,z) = \sqrt{4-z}\sqrt{-z}\int \mathrm{d}y \left[\frac{z}{y}\frac{4+2y-y^2-z-yt}{2(y+z-4)}\Psi_0 -\frac{1}{2}z(1+y)\partial_y\Psi_0\right] + \sqrt{-z}\sqrt{4-z}\,\mathrm{d}z\left[\Psi_0 \times \frac{y-4}{2(y+z-4)} + \frac{(y-4)y(1+y)}{2(-4+2y+z)}\partial_y\Psi_0\Big)\right],$$

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$$
\begin{aligned}
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$$

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& -\frac{1}{2}z(1+y)\partial_y\Psi_0\Bigg] + \sqrt{-z}\sqrt{4-z}\,\mathrm{d}z\Bigg[\Psi_0 \\
& \times \frac{y-4}{2(y+z-4)} + \frac{(y-4)y(1+y)}{2(-4+2y+z)}\partial_y\Psi_0\Bigg)\Bigg],
\end{aligned}
$$

quasi-period

period

BACK TO MASTER INTEGRALS FOR BHABHA AND MØLLER

These differential forms look pretty complicated (and there are worse ones) but they can be simplified!

Go to **canonical coordinates** of elliptic curve

$$y = 2\frac{(1-x)(1+t_4)}{t_4 - x}\,, \qquad z = 4\frac{t_4(1-x^2)}{x^2 - t_4^2}$$

→

$$Y^2 = (x^2-1)(x^2-t_4)$$

$$\Psi_0(x,t_4) = \frac{2(x^2-t_4)}{-Y}\,\mathrm{K}(t_4)$$

In these variables, **integrals become "simple"** and we find

$$\mathrm{T}_1(x,t_4) = 8t_4\frac{\mathrm{K}(t_4)}{\pi}\left[(1-t_4)\mathcal{F}(x,t_4) - \frac{x^2-1}{(1+t_4)Y}\right],$$

$$\mathrm{T}_2(x,t_4) = \frac{1}{\pi}\sqrt{\frac{t_4}{1+t_4}}\,\frac{t_4(3-2x)-3x+2}{t_4-x}\,\mathrm{K}(t_4) - \frac{f(t_4)}{2\pi}\,,$$

where

$$\partial_{t_4} f = 2\frac{1-t_4}{\sqrt{t_4}(1+t_4)^{3/2}}\,\mathrm{K}(t_4)$$

$$\mathcal{F}(x,t_4) = \mathrm{K}(t_4)\partial_{t_4}\left[\frac{1}{\mathrm{K}(t_4)}\int_{-1}^{x}\frac{\mathrm{d}X}{\sqrt{(X^2-1)(X^2-t_4)}}\right]$$

(Derivative of) Abel's Map

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$\longrightarrow$

$$Y^2 = (x^2-1)(x^2-t_4)$$

$$\Psi_0(x, t_4) = \frac{2(x^2-t_4)}{-Y}\,\mathrm{K}(t_4)$$

Similarly, all other differential forms become products of

$$\{\sqrt{x^2-1}\,,\ \sqrt{x^2-t_4}\,,\ \sqrt{1+t_4}\,,\ \sqrt{t_4}\,,\ \sqrt{1-t_4}\}$$

$$\{\mathrm{K}(t_4)\,,\ f(t_4)\,,\ \mathcal{F}(x, t_4)\}$$

where $x^2 = \frac{m^2}{s}$

$$\mathcal{F}(x,t_4) = \mathrm{K}(t_4)\partial_{t_4}\left[\frac{1}{\mathrm{K}(t_4)}\int_{-1}^{x}\frac{\mathrm{d}X}{\sqrt{(X^2-1)(X^2-t_4)}}\right]$$

$$\partial_{t_4} f = 2\frac{1-t_4}{\sqrt{t_4}(1+t_4)^{3/2}}\,\mathrm{K}(t_4)$$

Total of **87 differential** forms
$\sim$ “letters of the alphabet” ?

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$$Y^2 = (x^2-1)(x^2-t_4)$$

$$\Psi_0(x,t_4) = \frac{2(x^2-t_4)}{-Y}\,\mathrm{K}(t_4)$$

checked agains **AMFlow** [Liu, Ma '22] in different kinematic regions (Bhabha and Møller)

$\sim$ "letters of the alphabet" ?

$$\partial_{t_4} f = 2\frac{1-t_4}{\sqrt{t_4}(1+t_4)^{3/2}}\,\mathrm{K}(t_4)$$

WHAT ABOUT THE AMPLITUDE?

AMPLITUDES AND TENSOR DECOMPOSITION

We use the fact that **equal lepton scattering** (Bhabha & Møller) can be obtained from **scattering of different flavour** by crossing, schematically:

$$\left(e^+e^- \to e^+e^-\right) = \left(e_1^+e_1^- \to e_2^+e_2^-\right) + (s \leftrightarrow t)$$

We perform a tensor decomposition with external states in $D = 4$ dimensions to retain full dependence on the electron polarizations [Peraro, Tancredi '19, '21]

$$\mathcal{A}(1_{e^+}, 2_{e^-}, 3_{e^-}, 4_{e^+}) = \sum_{i=1}^{8} \mathcal{F}_i \, T_i$$

By working in $D = 4$, we are guaranteed to have as many tensors as many different polarizations: $16/2 = 8$, only a **physically relevant number of combinations is computed**

AMPLITUDES AND TENSOR DECOMPOSITION

Tensor structures can be chosen conveniently as follows:

$$\mathcal{A}(1_{e^+}, 2_{e^-}, 3_{e^-}, 4_{e^+}) = \sum_{i=1}^{8} \mathcal{F}_i \, T_i$$

$$t_i = \overline{U}_e(p_2)\,\Gamma_i^{(1)}\,V_e(p_1) \times \overline{U}_e(p_3)\,\Gamma_i^{(2)}\,V_e(p_4)$$

$$\mathbf{\Gamma}_i = \{\Gamma_i^{(1)}, \Gamma_i^{(2)}\}$$

$$\begin{aligned} T_1 &= m^2 \times t_1\,, & T_2 &= m \times [t_2 + t_3] \\ T_3 &= t_4\,, & T_4 &= m^2 \times t_5\,, \\ T_5 &= m \times [t_6 + t_7] + t_8\,, & T_6 &= m \times [t_6 + t_7] - t_8 \\ T_7 &= m \times [t_2 - t_3]\,, & T_8 &= m \times [t_6 - t_7]\,, \end{aligned}$$

$$\begin{aligned} \mathbf{\Gamma}_1 &= \{1, 1\}\,, & \mathbf{\Gamma}_2 &= \{\not{p}_3, 1\}\,, \\ \mathbf{\Gamma}_3 &= \{1, \not{p}_2\}\,, & \mathbf{\Gamma}_4 &= \{\not{p}_3, \not{p}_2\}\,, \\ \mathbf{\Gamma}_5 &= \{\gamma^{\mu_1}, \gamma_{\mu_1}\}\,, & \mathbf{\Gamma}_6 &= \{\not{p}_3\gamma^{\mu_1}, \gamma_{\mu_1}\}\,, \\ \mathbf{\Gamma}_7 &= \{\gamma^{\mu_1}, \not{p}_2\gamma_{\mu_1}\}\,, & \mathbf{\Gamma}_8 &= \{\not{p}_3\gamma^{\mu_1}, \not{p}_2\gamma_{\mu_1}\} \end{aligned}$$

Two tensors are odd under $p_2 \leftrightarrow p_3 \;\; p_1 \leftrightarrow p_4$, under which the amplitude must be invariant, which implies that $\mathscr{F}_7 = \mathscr{F}_8 = 0$ to **all orders in perturbation theory**!

Note: with this choice, we obtain amplitude **directly in tHV scheme** [Peraro, Tancredi '19, '21]

AMPLITUDES AND TENSOR DECOMPOSITION

From the form factors one can easily obtained both polarized and unpolarized amplitudes

We use standard programs **QGRAF, FORM, Mathematica, Reduze2, Kira (with FireFly)**

[Nogueira] **[Vermaseren]** **[Manteuffel, Studerus]** **[Meierhöfer, Usovitsch; Klappert et al]**

This allows us to easily express our amplitudes in terms of master integrals

All in all, including planar integrals and crossings, there are **252 masters**

PL integrals can be expressed as MPLs. NPL integrals as iterated integrals over **elliptic differential forms**. Before discussing evaluation strategy, what checks have we done?

CHECKS: UV & IR FACTORIZATION

All master integrals checked versus **AMFlow** [Liu, Ma '22]

UV renormalization requires renormalizing coupling, electron mass and wave function.
We perform renormalization on-shell

$$\frac{e^2}{4\pi} = \alpha_e^0 = \left(\frac{e^{\gamma_E}}{4\pi}\mu^2\right)^{\epsilon} Z_e \alpha_e(\mu)\,, \quad m_b = Z_m m$$

We are then left with **IR poles** that **are one-loop exact**

$$\mathcal{A}^{\mathrm{OS}}(\alpha, m, s, t, \epsilon) = e^{\frac{\frac{\alpha}{4\pi} Z_1^{\mathrm{IR}}}{\epsilon}} \mathcal{C}(\alpha, m, s, t, \epsilon)$$

$$\begin{aligned} Z_1^{\mathrm{IR}} =& \frac{4(-2m^2+s)}{\sqrt{-s}\sqrt{4m^2-s}} \ln\left(1 - \frac{s}{2m^2} - \frac{1}{2}\sqrt{\frac{-s}{m^2}}\sqrt{4 - \frac{s}{m^2}}\right) \\ &+ \frac{4(-2m^2+t)}{\sqrt{-t}\sqrt{4m^2-t}} \ln\left(1 - \frac{t}{2m^2} - \frac{1}{2}\sqrt{\frac{-t}{m^2}}\sqrt{4 - \frac{t}{m^2}}\right) \\ &- \frac{4(-2m^2+u)}{\sqrt{-u}\sqrt{4m^2-u}} \ln\left(1 - \frac{u}{2m^2} - \frac{1}{2}\sqrt{\frac{-u}{m^2}}\sqrt{4 - \frac{u}{m^2}}\right) \\ &-4\,, \end{aligned}$$

SMALL MASS EXPANSION & NUMERICAL EVALUATION

From ϵ-factorised differential equations, it is "easy" to obtain **series expansions** in any kinematical region

SMALL MASS EXPANSION & NUMERICAL EVALUATION

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For most applications the **electron mass can be considered small** → we perform a small mass expansion of the individual master integrals and of the whole amplitude

$$\mathscr{A}(s,t,m^2) = \sum_{ijk} (m^2)^{i\epsilon} \log^j(m^2) \epsilon^k \, A_{ij}^{(k)}(s,t)$$

Coefficients of the series $A_{ij}^{(k)}(s,t)$ can be written in terms of **harmonic polylogarithms**

[Remiddi, Vermaseren '19]

Boundary conditions can be all fixed by **regularity and eigenvalue conditions** (which should then be transported to the region $m^2 \ll s, |t|$)

SMALL MASS EXPANSION & NUMERICAL EVALUATION

Series converges very well in the bulk of the phase-space, but one must **take special care in considering forward or backward limit $t \to 0$ or $u \to 0$** (scattering angle going to 0 or π)

SMALL MASS EXPANSION & NUMERICAL EVALUATION

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$$\mathscr{A}(s,t,m^2) = \sum_{ijk} (m^2)^{i\epsilon} \log^j(m^2) \epsilon^k A_{ij}^{(k)}(s,t)$$

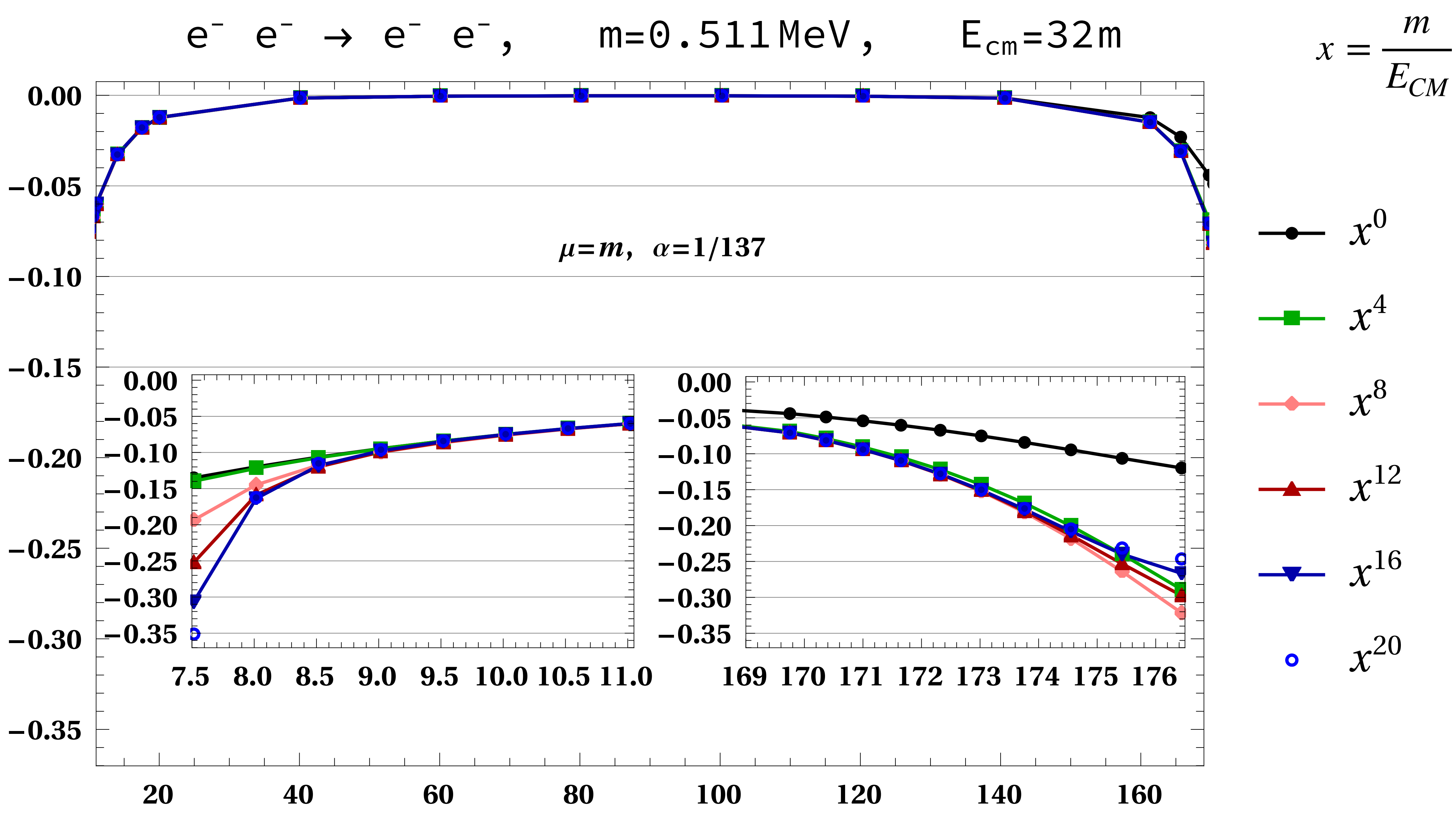
$$A_{ij}^{(k)}(s,t) = \sum_{n,m} A_{ij}^{(k,n,m)}(s) \left(t^n \log^m(-t/s)\right)$$

logarithms $\log(-t/s)$ can ***spoil the convergences of the mass expansion***

As expected, “Regge” limit does not commute with small mass limit…

SMALL MASS EXPANSION & NUMERICAL EVALUATION

This effect can be seen clearly plotting the $2Re\left(\mathscr{C}^{(2)}\mathscr{C}^{(0)*}\right)$ in ***extreme regions*** (here for **Møller scattering**)



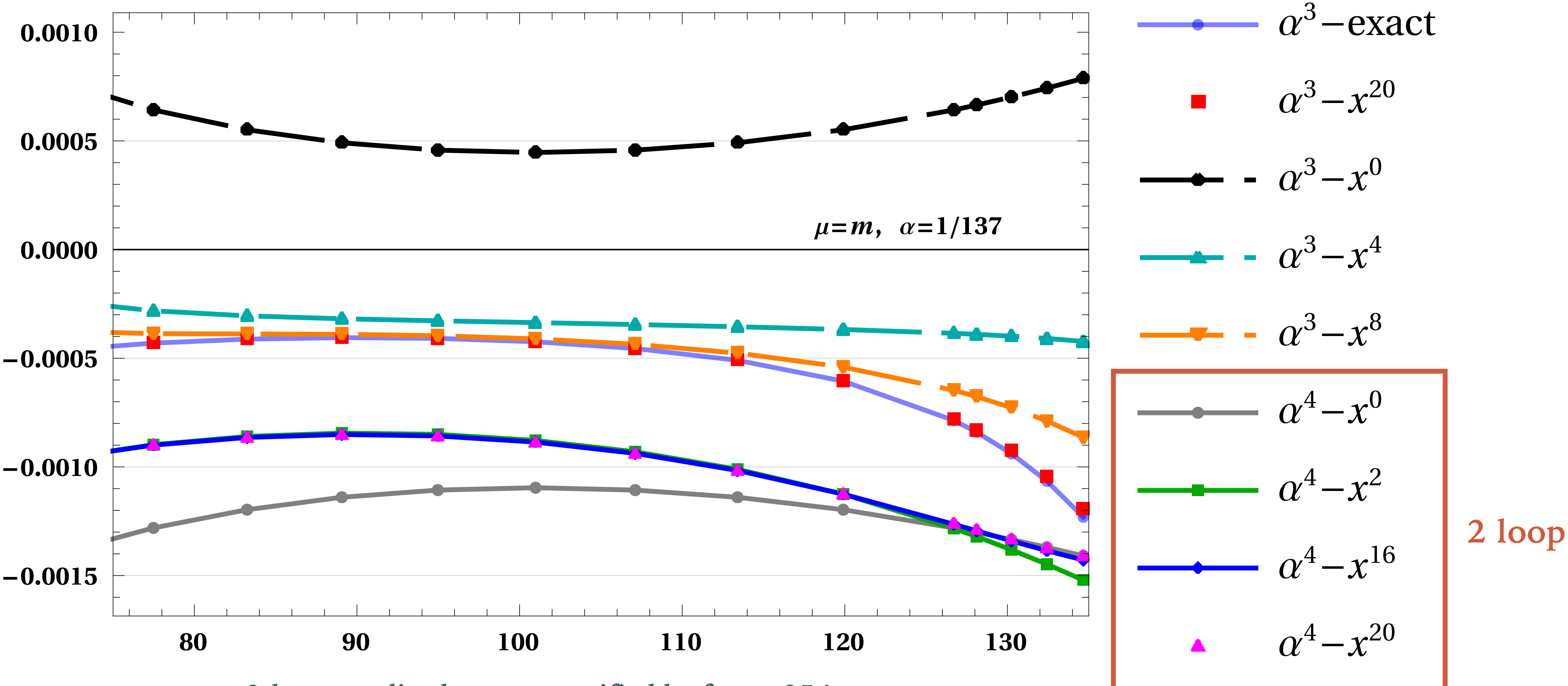
SMALL MASS EXPANSION & NUMERICAL EVALUATION

Effects are stronger at very low energies ($E_{CM} = 2.5 MeV$)

Here compare $2Re\left(\mathscr{C}^{(1)}\mathscr{C}^{(0)*}\right)$ expanded versus exact, and separately $2Re\left(\mathscr{C}^{(2)}\mathscr{C}^{(0)*}\right)$ function of θ

$e^- e^- \to e^- e^-$, m=0.511 MeV, E_{cm}=5m

$$x = \frac{m}{E_{CM}}$$



2-loop amplitudes are magnified by factor 25 !

CONCLUSIONS AND OUTLOOK

- Bhabha and Møller scattering are fundamental "standard candles" in QED, both for **phenomenological applications** and as **experimental ground for new techniques**
- Pushing the calculation to two loops required new techniques to handle integrals of elliptic type
- We derived an ϵ-factorised basis leveraging new algorithms that can be extended beyond polylogs
- Having differential equations in this form, it becomes in principle straightforward to obtain series expansions
- For Bhabha and Møller, we constructed a small mass expansion
- We proved that it converges extremely well in the bulk of the phase space, but non-trivial effects can be observed at the boundaries (forward / backward region)
- Next step will be implementation of our results in McMule framework for NNLO studies

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Stay tuned :-)

and thank you for your attention!