Two-Loop Feynman Integrals for Multi-Scale QCD Corrections

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QCD Meets Electroweak

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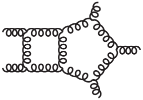
Scattering Amplitudes and Feynman Integrals

Aim: Compute the amplitude: a sum of Feynman integrals.

$$A^{(2)}(p_1, \ldots, p_5) =$$  $$+ \mathcal{O}(10000) \text{ diagrams.}$$

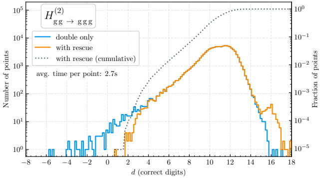
- Standard approach: reduce $A^{(2)}$ to master integrals:

$$A^{(2)}(p_i \cdot p_j, p_i \cdot \varepsilon_j) = \sum_k \underbrace{\mathcal{C}_k(p_i \cdot p_j, p_i \cdot \varepsilon_j)}_{\text{rational functions}} \underbrace{\mathcal{I}_k(p_i \cdot p_j)}_{\text{master integrals}} .$$

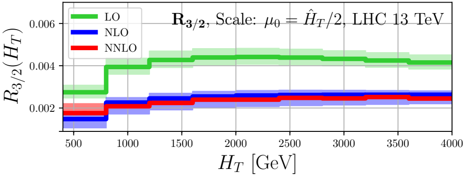
- Key challenges in computing master integrals:
  - Integrals with many IR/UV divergences $\Rightarrow$ deep ϵ expansion.
  - Instabilities in $\mathcal{C}_k$ combination $\Rightarrow$ need strong precision control.
  - Want $\mathcal{I}_k$, such that $\mathcal{C}_k$ are as easy as possible to compute.

Amplitudes Suitable for Phenomenology

Amplitudes are stable and fast. E.g. NNLO 3-jet production.



[Abreu et al '21]



[Czakon et al].

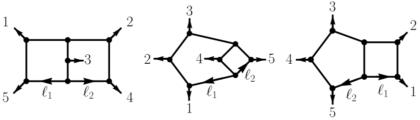
Many NNLO five-point cross-sections available:

- 3γ [Kallweit et al; Czakon et al].
- $3j$ [Czakon et al].
- $\gamma\gamma j$ [Chawdhry et al; Badger et al].
- γjj [Badger et al].
- $Wb\bar{b}$ [Hartanto et al; Buonocore et al]
- $t\bar{t}H$ (approx) [Catani et al]
- $t\bar{t}W$ (approx) [Buonocore et al]

Amplitudes even used in α_S determination. [ATLAS].

State of the Art Two-Loop Five-Point Integrals

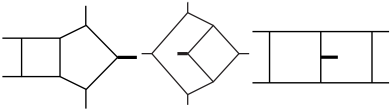
- Massless (5-scale)



[Papadopoulos, Tommasini, Wever '15]
[Gehrmann, Henn, lo Presti '18]
[Abreu, Page, Zeng '18]
[Chicherin, Gehrmann, Henn, lo Presti, Mitev, Wasser '18]

[Abreu, Dixon, Herrmann, Page, Zeng '18]
[Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia '18]
[Gehrmann, Henn, lo Presti '18]
[Chicherin, Sotnikov '20]

- One-Mass (6-scale)

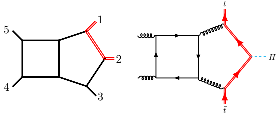


[Papadopoulos, Tommasini, Wever '15]
[Abreu, Ita, Moriello, Page Tschernow, Zeng '20]
[Canko Papadopoulos, Syrrakos '20]
[Chicherin, Sotnikov, Zoia '21]

[Abreu, Ita, Page, Tschernow '21]
[Kardos, Papadopoulos, Smirnov, Syrrakos, Wever '22]
[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia '23]

- $t\bar{t}+j$ family [Badger, Becchetti, Chaubey, Marzucca '22] (6-scale)
- $t\bar{t}+H$ families [Febres, Figueiredo, Kraus, BP, Reina '23] (7-scale)

[Agarwal, Heinrich, Jones, Kerner, Klein, Lang, Magerya, Olsson '24] (numeric)



Integral Computation Wish List

Analytic wishes:

- Must ϵ expand to some set of functions up to $\mathcal{O}(\epsilon)$.

$$\mathcal{I}_j(p_a \cdot p_b, \epsilon) = \sum_{k=-4}^{0} d_{jkl}\epsilon^k h_l(p_a \cdot p_b) + \mathcal{O}(\epsilon).$$

- Must understand algebra of relations between h_l.

$$\sum_i \alpha_i h_i = 0 \;\Leftrightarrow\; \alpha_i = 0, \quad \text{and} \quad h_i h_j = c_{ijk} h_k.$$

Numerical wishes:

- Must cover all of physical phase space.
- Large phase space $\Rightarrow$ need high speed evaluation.
- Ability to increase precision in dangerous phase-space regions.

Differential Equations

[Gehrmann Remiddi '01; Henn '13]

In this talk, we will attack this problem using differential equations.

- Part 1: Constructing Differential Equations.
- Part 2: Solving Differential Equations (pentagon functions).

Differential Equations and their Construction

Differential Equations Summary

- Feynman integrals satisfy coupled system of first-order DEs:

$$\mathrm{d}\mathcal{I}_k = \underbrace{\tilde{\mathbf{M}}_{kl}(\epsilon, \vec{s})}_{\text{differential forms}} \; \mathcal{I}_l.$$

[Gehrmann, Remiddi '01]

- Conjecture: Exists $\mathcal{J}_i = U_{ij}\mathcal{I}_j$, satisfying ϵ-factorized DE:

[Henn '13]

$$\mathrm{d}\mathcal{J}_k = \epsilon \mathbf{M}_{kl}(\vec{s})\mathcal{J}_l.$$

- This trivializes ϵ expansion!

$$\mathcal{J}_k = \sum_{n=0}^{\infty} \epsilon^n \mathcal{J}_k^{(n)} \quad \Rightarrow \quad \mathcal{J}_k^{(n)} = \underbrace{\mathcal{J}_k^{(n)}(\gamma_0)}_{\text{(numeric) boundary}} + \underbrace{\int_\gamma \mathbf{M}_{kl}(\gamma)\mathcal{J}_l^{(n-1)}(\gamma)}_{\text{iterated integration}}.$$

Crucial Components

$\mathcal{J}_k$, $\mathbf{M}_{kl}$, $\mathcal{J}_k^{(n)}(\gamma_0)$ and iterated integral understanding.

Constructing Differential Equations

1. Construct a basis of "pure" integrals. $\Leftarrow$ Hardest step.

2. Construct a basis of differential forms $\Leftarrow$ quasi-systematic.

$$\mathbf{M}_{kl} = \underbrace{M_{klm}}_{\text{rational numbers}} \underbrace{\omega_m}_{\text{differential forms}}$$

3. Use ω_m as Ansatz. Fit with evaluations of DE. $\Leftarrow$ Easy.

Important Caveat

Only strong theory understanding for polylogarithmic integrals.

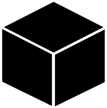
The Computational Backbone: Analytic Reconstruction

- We will constantly study the DE using Ansaetze.

$$\mathcal{F}_k(\vec{s}) = \sum_{j=1}^{M} f_{jk}\,\mathfrak{a}_{jk}(\vec{s}), \qquad f_{jk} \in \mathbb{Q}.$$

- Finite-field evaluations constrain unknown f_{jk}.

$$\vec{s}^{(0)} \longrightarrow$$



$$\longrightarrow \{\mathcal{F}_1(\vec{s}^{(0)}), \ldots, \mathcal{F}_M(\vec{s}^{(0)})\}.$$

[Schabinger, von Manteuffel '14; Peraro '16]

- Often extract properties on "univariate slice". Fix numeric $\vec{a}, \vec{b}$

$$\vec{s} = \vec{a} + \vec{b}t \qquad \mathcal{F}_k(\vec{s}) \to \mathcal{F}_k(t).$$

Numerical Differential Equations

Differential equations constructed with algebraic operations:

$$d\mathcal{I}_i = \mathbf{b}_{ia}(\epsilon, \vec{s})\tilde{\mathcal{I}}_a, \qquad \text{and} \qquad \tilde{\mathcal{I}}_a = c_{aj}(\epsilon, \vec{s})\mathcal{I}_j$$

With finite field IBPs, can easily compute c_{aj} for numeric ϵ, $\vec{s}$.

$$\mathbf{M}_{kl}(\epsilon_0, \vec{s}_0) = \mathbf{b}_{\mathbf{ia}}(\epsilon_0, \vec{s}_0)c_{aj}(\epsilon_0, \vec{s}_0).$$

Numerical evaluations of $\mathbf{M}_{kl}(\epsilon_0, \vec{s}_0)$ have multiple uses:

- ϵ-**factorization check**: $\mathbf{M}_{kl}(\epsilon_1, \vec{s}_0)/\epsilon_1 = \mathbf{M}_{kl}(\epsilon_2, \vec{s}_0)/\epsilon_2$.
- Compute **alphabet dimension**: linearly independent $\mathbf{M}_{kl}$.
- **Data for Ansatz**-based computation of (pieces of) $\mathbf{M}_{kl}$.

Pure Basis Construction: General Summary

Automatic Techniques (Mostly well suited for low scale count):

- Many codes: Libra, Fuchsia, Epsilon, INITIAL, DLogBasis.

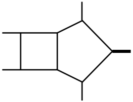
  [Lee; Gituliar, Magerya; Prausa; Dlapa et al; Henn et al]

The Ad-hoc Approach

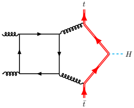
1. ϵ-factorize on "maximal cut".
   - "Unit leading singularities".

     [Cachazo '08; Arkani-Hamed, Bourjaily, Cachazo, Trnka '10; Henn '13]
   - Magnus expansion.

     [Argeri, Di Vita, Mastrolia, Mirabella, Schlenk, Schubert, Tancredi '14]
   - Educated guess from literature experience. E.g.



$\left[(\ell_1^\epsilon \cdot \ell_2^\epsilon)(p_1 \cdot p_2)\sqrt{\Delta_5}\right] \rightarrow$

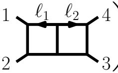


$\left[(\ell_1^\epsilon \cdot \ell_2^\epsilon)(p_1 \cdot p_2)\sqrt{\Delta_5}\right]$

2. Release cut conditions and fix mistakes.

Leading Singularities*: The Massless Double Box

- Start with "good" integral representation of max cut:

max-cut (

)

$$= \int \Omega(\epsilon), \quad \Omega(0) = \frac{d\rho_8 d\rho_9 \mathcal{N}(s,t)}{\underbrace{2s\rho_8\rho_9[(s+\rho_8)\rho_9 + s(\rho_8 - t)]}_{\text{Baikov polynomial}}}$$

[Frellesvig, Papadopoulos '17]

- Leading singularities are repeated residues of $\Omega(0)$.

$$\mathrm{Res}_{\rho_9=\rho_9^i}\left[\mathrm{Res}_{\rho_8=\rho_8^i}\Omega(0)\right], \quad \begin{array}{l}(\rho_8,\rho_9)^0 = (0,0),\\ (\rho_8,\rho_9)^1 = (0,t),\\ \quad\dots\end{array}$$

- Fix $\mathcal{N}$ by requiring residues to be rational numbers.

$$\mathcal{N} = s^2 t.$$

*Pedagogical resource: [Wasser '18] (Master's thesis).

Magnus Expansion (i)

[Argeri, Di Vita, Mastrolia, Mirabella, Schlenk, Schubert, Tancredi '14]

- Let us assume that the precanonical DE is linear in ϵ

$$\tilde{\mathbf{M}}(\epsilon, \vec{s}) = \tilde{\mathbf{M}}^{(0)}(\vec{s}) + \epsilon \tilde{\mathbf{M}}^{(1)}(\vec{s}).$$

- A basis $\mathcal{J}_i = U_{ij}\mathcal{I}_j$ satisfies an ϵ-factorized DE if

$$\mathrm{d}U_{ij} = \tilde{\mathbf{M}}^{(\mathbf{0})}_{\mathbf{ix}} U_{xj}.$$

- A particular solution is given by the Magnus expansion.

$$U = \exp\left(\sum_{i=0}^{\infty} \Omega_i\right), \qquad \Omega_1 = \int_\gamma \tilde{\mathbf{M}}^{(\mathbf{0})}(\gamma), \quad \Omega_2 = \frac{1}{2}\int_{\gamma_1}\int_{\gamma_2} [\tilde{\mathbf{M}}^{(\mathbf{0})}(\gamma_1), \tilde{\mathbf{M}}^{(\mathbf{0})}(\gamma_2)].$$

- Practical strategy: reconstruct just $\tilde{\mathbf{M}}^{(\mathbf{0})}$ from samples.

Magnus (ii): Beyond the Maximal Cut

- If max cuts are canonicalized, next-to-max cut DE takes form

$$\tilde{\mathbf{M}}^{(\mathbf{0})} = \begin{pmatrix} 0 & 0 \\ \tilde{\mathbf{B}} & 0 \end{pmatrix}$$

- For simplicity, let $\tilde{\mathbf{b}} = \tilde{\mathbf{B}}$ be 1×1. Magnus expansion simple!

$$dU = \begin{pmatrix} 0 & 0 \\ \tilde{\mathbf{b}} & 0 \end{pmatrix} U \qquad \Rightarrow U = \begin{pmatrix} 1 & 0 \\ \int \tilde{\mathbf{b}} & 1 \end{pmatrix}$$

- Same strategy: reconstruct only $\tilde{\mathbf{b}}$ from samples of $\tilde{\mathbf{M}}^{(\mathbf{0})}$.

- Proceed iteratively. Challenges: reconstruction/integrations.

Structure of Canonical Differential Equations

- Many entries of DE matrix are linearly dependent.

$$\mathbf{M}_{kl}(\vec{s}) = \underbrace{M_{klm}}_{\in\mathbb{Q}} \underbrace{\boldsymbol{\omega_m}(\vec{s})}_{\text{alphabet}}$$

- In considered cases, the differential forms are "$\mathrm{d}\log$"

$$\boldsymbol{\omega_m} = \mathrm{d}\log(W_m).$$

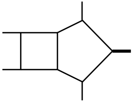
- W_m are algebraic functions of the kinematics, e.g.

$$W_{29} = m_t^2(v_{15} - v_{34})^2 + q^2 v_{15} v_{34} \qquad W_{35} = \frac{v_{12} - \sqrt{v_{12}^2 - 4m_t^2 q^2}}{v_{12} + \sqrt{v_{12}^2 - 4m_t^2 q^2}}.$$

examples from [Febres Cordero, Figuereido, Kraus, BP, Reina '23]

Algebraic Structures and Galois Groups

- Pure basis often involves introduction of square roots, e.g.



$$\left[(\ell_1^\epsilon \cdot \ell_2^\epsilon)(p_1 \cdot p_2)\sqrt{\Delta_5}\right]$$

- Nested square roots can also arise.

$$\sqrt{N_\pm} = \sqrt{N_b + \sqrt{N_b^2 - N_c}}.$$

[Febres Cordero, Figueiredo, Kraus, BP, Reina '23; Becchetti et al (to appear)]

- Useful basis property: representation of "Galois group".

$$\underbrace{\alpha}_{\text{Galois transformation}} : \sqrt{\Delta_5} \to -\sqrt{\Delta_5} \qquad \Rightarrow \alpha(I_{\text{pb}}) = -I_{\text{pb}}.$$

- $\mathbf{M}_{kl}$ then also organize into representations of Galois group.

Multi-scale Alphabet Construction

- Even* W_i are denominators of $\mathbf{M}_{kl}$. Analytic reconstruction!

$$\mathrm{d}\log(W_i) = \frac{\mathrm{d}W_i}{W_i}.$$

- Observation: denominators of odd W_i product of even W_i:

$$\mathrm{d}\log\left(\frac{a+\sqrt{b}}{a-\sqrt{b}}\right) = \frac{1}{\sqrt{b}}\frac{2b(\mathrm{d}a) - a(\mathrm{d}b)}{a^2-b}, \qquad a^2 - b = \prod_i W_i^{\text{even}}.$$

- Find relevant W_i^{even} on univariate slice. Fix a by Ansatz.

$$\mathrm{Num}(a)^2 - b\,\mathrm{Den}(a)^2 \mod \prod_i W_i^{\text{even}} = 0.$$

[Abreu, Ita, BP, Tschernow '21; Febres Cordero, Figueiredo, Kraus, BP, Reina '23]

* Assume Galois group is $\mathbb{Z}_2$. Otherwise, open problem.

Solving DEs and “Pentagon Functions”

DE Solutions and Iterated Integrals

- As solution to DE, integrals satisfy recursion relation

$$\mathcal{J}_k^{(n)} = \mathcal{J}_k^{(n)}(\gamma_0) + \int_\gamma M_{klm}\boldsymbol{\omega}_m(\gamma)\mathcal{J}_l^{(n-1)}(\gamma).$$

- This evolves solutions from boundary point to target point.

$$\gamma(t) \quad : \quad \gamma(0) = \vec{s}_{\text{boundary}}, \quad \gamma(1) = \vec{s},$$

- To describe the solutions, one defines iterated integrals.

$$[\boldsymbol{\omega}_{a_1}, \ldots, \boldsymbol{\omega}_{a_n}](\vec{s}) = \int_0^1 \boldsymbol{\omega}_{a_n}(\gamma[t_n])[\boldsymbol{\omega}_{a_1}, \ldots, \boldsymbol{\omega}_{a_{n-1}}](\vec{s}(\gamma[t_n])).$$

[Chen '72]

- Solutions are naturally expressed in terms of iterated integrals.

$$\mathcal{J}_k^{(n)} = c^k_{i_1,\ldots,i_n}[\omega_{i_1}, \ldots, \omega_{i_n}] + \mathcal{O}(\text{boundary}), \qquad c_{i_1,\ldots,i_n} = M_{kji_n} c^j_{i_1,\ldots,i_{n-1}}.$$

Algebraic Properties of Iterated Integrals

- Iterated integrals products controlled by "shuffle algebra".

$$[\omega_{a_1}, \ldots, \omega_{a_m}][\omega_{b_1}, \ldots, \omega_{b_n}] = \sum_{c \in a \sqcup\!\sqcup b} [\omega_{c_1}, \ldots, \omega_{c_{m+n}}].$$

- Linear independence of functions from ω_i independence

$$\text{"}\alpha_i \omega_i = d\eta \;\Rightarrow\; \alpha_i = 0\text{"} \quad \Rightarrow \quad [\omega_{i_1}, \ldots, \omega_{i_n}] \text{ are linearly independent.}$$

[Chen '72]

- For $\mathrm{d}\log$-forms, all relations are "logarithm-like".

- Iterated integrals inherit Galois properties of alphabet.

$$\alpha(\omega_i) = (-1)^{\alpha_i}\omega_j \quad \Rightarrow \quad \alpha([\omega_{i_1}, \ldots, \omega_{i_n}]) = (-1)^{\sum_k \alpha_{i_k}}[\omega_{j_1}, \ldots, \omega_{j_n}].$$

This understanding was a large chunk of our wishlist!

Numerical Iterated Integrals

We follow the approach of [Chicherin, Sotnikov '20].

- Weights 1, 2: Solve explicitly in terms of logs and Li_2s.

  [Duhr, Gangl, Rhodes '11]

- Weight 3: Numerical integration with tanh-sinh quadrature

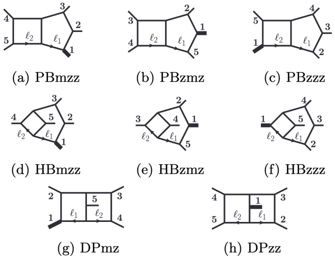
$$\mathcal{J}_i^{(3)}(\vec{s}) = \mathcal{J}_i^{(3)}(\vec{s}_0) + \int_0^1 \mathrm{d}t M_{ijk} \frac{\partial \log(W_k[\gamma(t)])}{\partial t} \mathcal{J}_j^{(2)}[\gamma(t)].$$

- Weight 4: Perform 1 integral analytically $\Rightarrow$ one-fold integral.

$$\int_0^1 \mathrm{d}t_n \frac{\partial \log[W_i(t_n)]}{\partial t_n} \int_0^{t_n} \mathrm{d}t_{n-1} g(t_{n-1}) = \int_0^1 \mathrm{d}t_{n-1} g(t_{n-1}) \underbrace{\left[\int_{t_{n-1}}^1 \mathrm{d}t_n \frac{\partial \log[W_i(t_n)]}{\partial t_n}\right]}_{\log(W_i(1)) - \log(W_i(t_{n-1}))}.$$

[Caron-Huot, Henn '14]

Evaluation Performance



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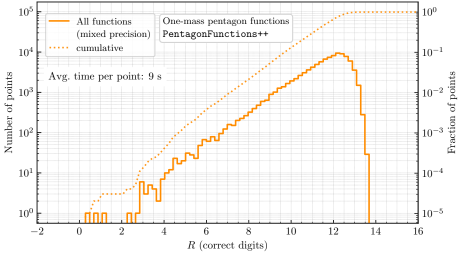


[Abreu, Chicherin, Ita, BP, Sotnikov, Tschernow, Zoia '23]

- All integrals for $pp \to H + 2j$.

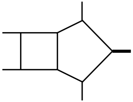
  See also [Kardos, Papadopoulos, Smirnov, Syrrakos, Wever '22]

- 9s/point: Double precision (+ quad precision rescue system) .

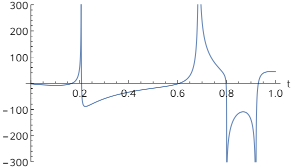
Series Solutions of DEs

recently popularized by [Moriello '19]

- Solve DE with collection of power series:



$$\sim \sum_{j_1,j_2} (t-t_0)^{j_1/2} \log(t-t_0)^{j_2} \sim$$



[Abreu, Ita, Moriello, BP, Tschernow, Zeng '20]

- Many public implementations: DiffExp/SeaSyde/AMFlow.

  [Hidding '20] [Armadillo et al '22] [Liu, Ma '22] [See Simone's talk]

- Only game in town for elliptic five-point.
- Low algebra control. How to optimize for $2 \to 3$ phase space?
- NB: Can use series tools to evaluate pentagon functions

  [Badger, Hartanto, Zoia '21]

Boundary Values (i): Regularity Constraints

- Lots of approaches! Will discuss numerical implementation of [Abreu, Ita, Moriello, BP, Tschernow, Zeng '20]
- Consider DE on path $\gamma(t)$ going through singularity at t_0.

$$\mathbf{M}[\gamma(t)] = \frac{M_{-1}}{t - t_0}\mathrm{d}t + \mathcal{O}(t - t_0)^0.$$

- Singularities in the DE might cause branch points of solution:

$$\mathcal{J}^{(n+1)} = \int \mathrm{d}\log(t - t_0)(M_{-1}\mathcal{J}^{(n)}(\gamma[t_0])) + \mathcal{O}(t - t_0)^0.$$

- Absent singularity [Sebastian's talk] $\Rightarrow$ constraint!

$$M_{-1}\mathcal{J}^{(n)}[\gamma(t_0)] = 0.$$

- Combine constraints from multiple surfaces $\Rightarrow \mathcal{J}^{(n)}(\vec{s}_0)$.

Boundary Values (ii)

The Pragmatic Solution

High precision AMFlow evaluations.

[Liu, Ma '22], See also [Hidding, Usovitsch '22]

Discussion/Summary

Bottlenecks in (Polylogarithmic) Integral Calculation

- Growing integral count: efficiently automate canonicalization?
  - In multi-scale case residue techniques very desirable.

$$\oint_{\{|f_i(\ell)|=\epsilon\}} \frac{d^4\ell_1 d^4\ell_2}{D_1 \cdots D_n}.$$

[Cachazo '08]

  - Can we improve understanding and automate?

See dlogBasis [Henn, Mistlberger, Smirnov, Wasser '20]

- Symbol alphabets without the DE?
  - Even letters $\sim$ Landau equations. [see Sebastian's talk].
  - Odd letters "from even". Algebraic factorizations?

$$W_{\mathrm{odd}}\overline{W}_{\mathrm{odd}} = \prod_i W_{i,\mathrm{even}}^{\alpha_i}.$$

see also [Heller, von Manteuffel, Schabinger '19], [Jiang, Liu, Xu, Yang '24]

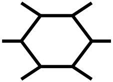
Take-Home Messages

- Large progress in multi-scale Feynman integral calculation.
- Integrals understanding required for amplitude computation.
- Canonical differential equations give strong analytic control.
- Numerical constructions of DE very useful!
- "Pentagon functions approach" has good stability/efficiency.

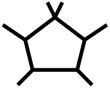
How Many Integral Topologies?

- One-loop integrals with 4D externals stop at pentagons.
- No solution for 6-prop cut $\Rightarrow$ identity:

$$1 = \sum_{i=1}^{6} c_i D_i \qquad \Rightarrow$$



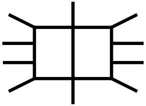
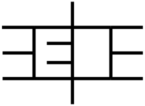
$$= \sum_{i=1}^{6} c_i$$



$${}_i \; .$$

[Hilbert's Weak Nullstellensatz]

- At two loops, there is no solution to 12-propagator cut:

$\Rightarrow$  ,  * , $\ldots$

*Masters potentially $O(\epsilon)$, but analysis more intricate.

Many more integral topologies to compute!