Tensor Reduction of Loop Integrals

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QCD Meets EW
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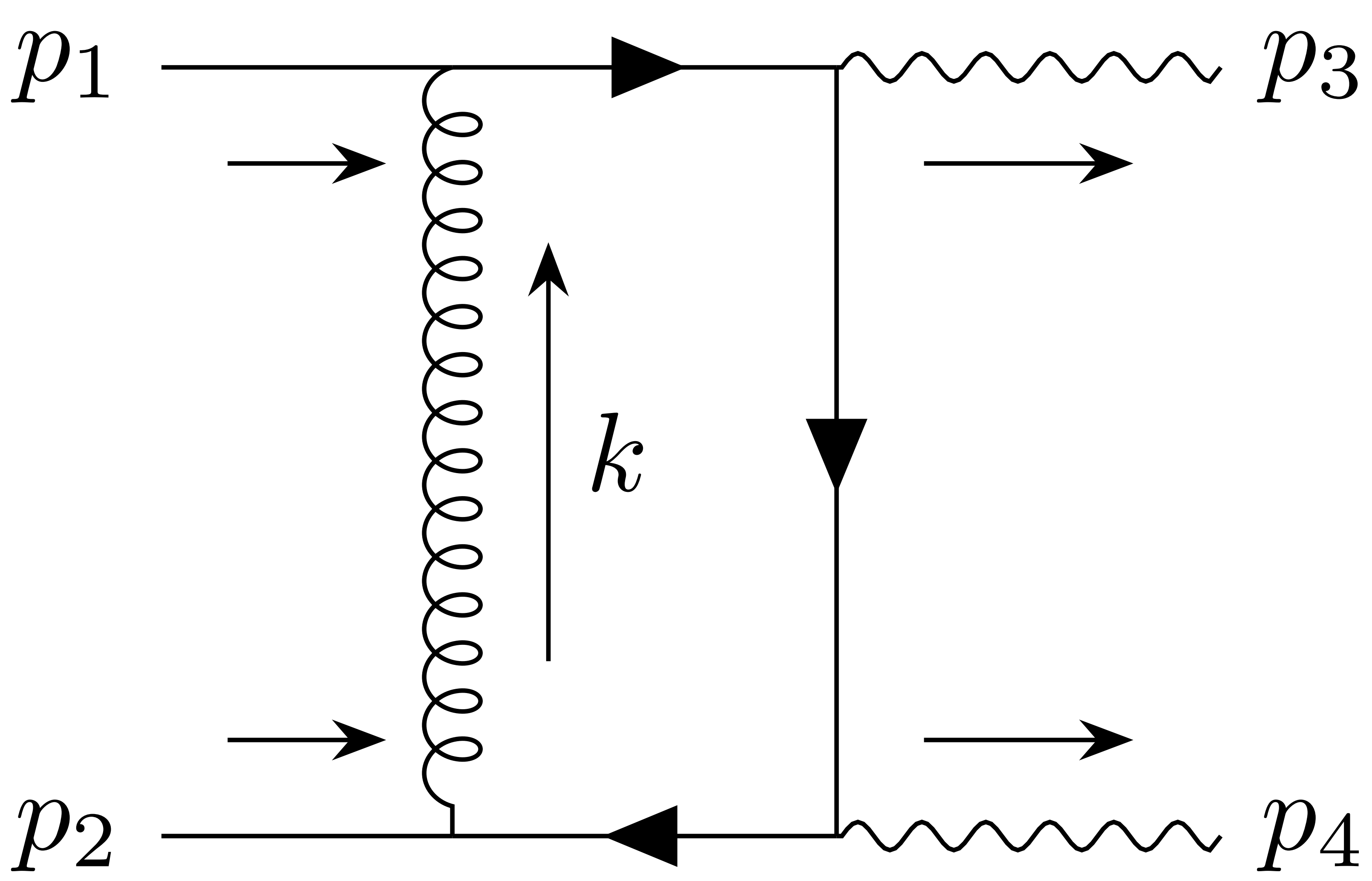
1. Introduction

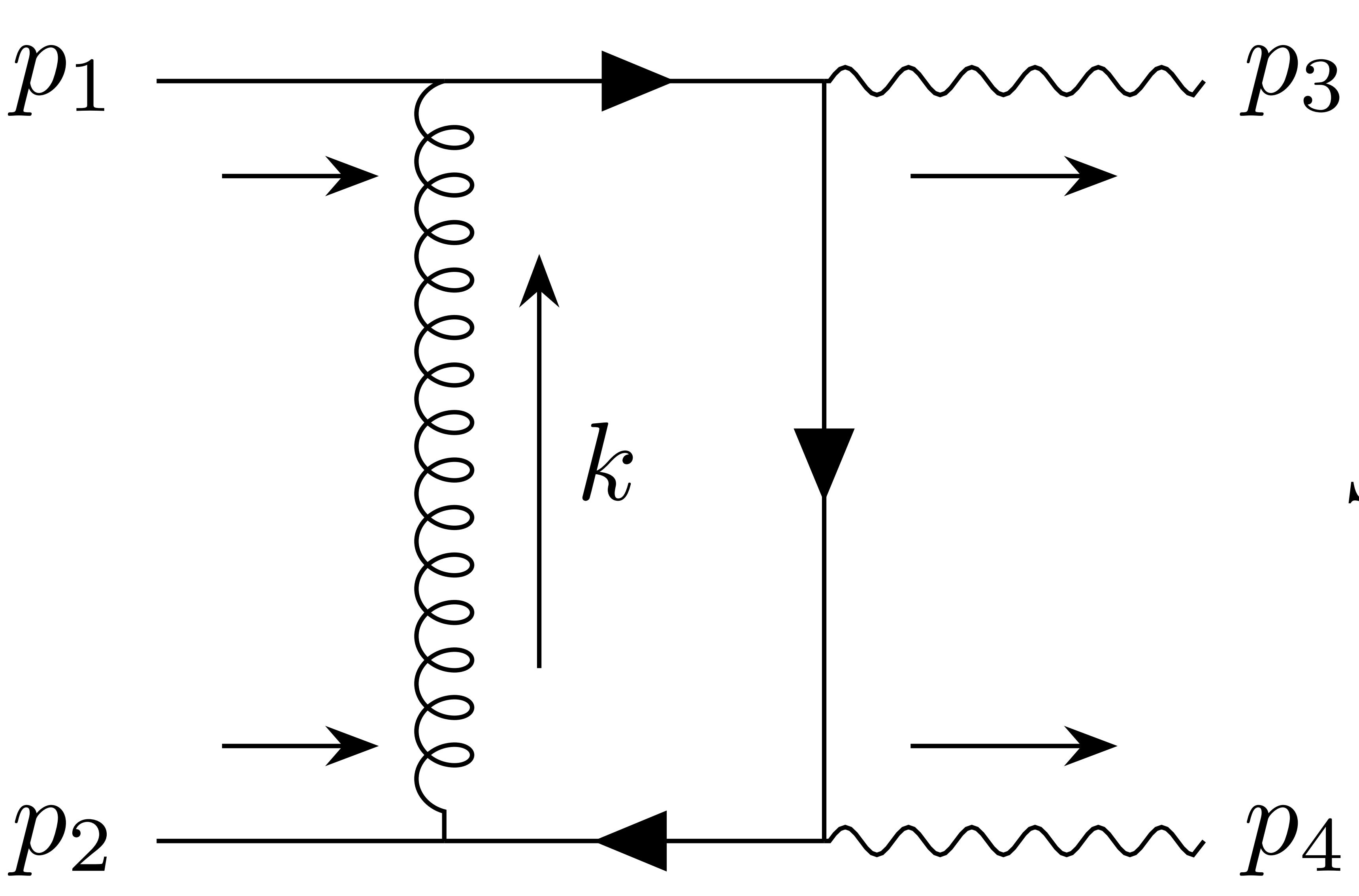
Let's start from an example



TENSOR INTEGRAL : $I^{\nu\rho\sigma}$

$$\sim I := \epsilon_3^{*,\delta_2}\epsilon_4^{*,\delta_1}\bar{v}(p_2)\gamma^\mu\gamma_\nu\gamma_{\delta_1}\gamma_\rho\gamma_{\delta_2}\gamma_\sigma\gamma_\mu u(p_1)\int\frac{dk^D}{(2\pi)^D}\frac{(k-p_2)^\nu(k+p_1-p_3)^\rho(k+p_1)^\sigma}{k^2(k-p_2)^2(k+p_1)^2(k+p_1-p_3)^2}$$

Expansion around a threshold



$$p_4^2 = m_{\gamma^*}^2\,, \quad p_{i\neq 4}^2 = 0\,,$$

10.1007/JHEP12(2013)088, Anastasiou, Duhr, Dulat, Herzog, Mistlberger

$$s = \frac{m_{\gamma^*}^2}{z}\,,\ t = s\delta\lambda\,,\ \ u = s\delta(1-\lambda)$$

k hard

$p_{1,2} \sim \sqrt{s}$

$p_3 \sim \delta\sqrt{s}$

$\delta \to 0$

$$I^{\nu\rho\sigma} = \int \frac{dk^D}{(2\pi)^D} \frac{(k-p_2)^\nu (k+p_1-p_3)^\rho (k+p_1)^\sigma}{k^2 (k-p_2)^2 (k+p_1)^2 (k+p_1-p_3)^2}$$

$\longrightarrow$ $I^{\mu\nu\rho}$ around production threshold of γ^*

Expansion around a threshold

$$p_4^2 = m_{\gamma*}^2\,, \qquad p_{i\neq 4}^2 = 0\,, \qquad k \text{ hard}$$

$$p_{1,2} \sim \sqrt{s} \qquad p_3 \sim \delta\sqrt{s} \qquad \delta \to 0$$

$$I^{\nu\rho\sigma} = \int \frac{dk^D}{(2\pi)^D} \frac{(k-p_2)^\nu (k+p_1-p_3)^\rho (k+p_1)^\sigma}{k^2 (k-p_2)^2 (k+p_1)^2 (k+p_1-p_3)^2}$$

$$\frac{(k+p_1-p_3)^\rho}{(k+p_1-p_3)^2} = \frac{(k+p_1)^\rho}{(k+p_1)^2} + \underbrace{\frac{2(k+p_1)^\rho p_3\cdot(k+p_1)}{((k+p_1)^2)^2} - \frac{p_3^\rho}{(k+p_1)^2}}_{\mathcal{O}(\delta)}$$

$$\underbrace{-\frac{2(p_3)^\rho p_3\cdot(k+p_1)}{((k+p_1)^2)^2} + 4\frac{(k+p_1)^\rho (p_3\cdot(k+p_1))^2}{((k+p_1)^2)^3}}_{\mathcal{O}(\delta^2)} + \mathcal{O}(\delta^3)$$

After the expansion, the topology is the triangle.
Independent external momenta: p_1, p_2.

$\Longrightarrow$ We would like to have efficient tensor reduction!

Overview

One-Loop corrections for e^+e^- annihilation into $\mu^+\mu^-$ in the Weinberg model,
Passarino, Veltman (1979)

One-loop

10.1016/0010-4655(92)90125-I, Ezawa et al. (1992)

0509141, Denner, Dittmaier (2005)

0810.0992, Binoth et al. (2008)

0905.1005, van Hameren (2009)

0903.4665, van Hameren, Papadopoulos, Pittau (2009)

1111.5206, Cascioli, Maierhofer, Pozzorini (2011)

…

Helicity amplitudes

1904.00705, Chen (2019)

1906.03298, Peraro, Tancredi (2019)

2012.0082, Peraro, Tancredi (2020)

…

Parametric integrals

9606018, Tarasov (1996)

9912251, Anastasiou, Glover, Oleari (1999)

…

2. The Passarino-Veltman Tensor Reduction Problem

Classical Example

$$I^{\mu\nu} = \int dk^D \frac{k^\mu k^\nu}{k^2(k+p)^2} \overset{\text{ANSATZ}}{=} A g^{\mu\nu} + B p^\mu p^\nu$$

$$I^{\mu\nu} g_{\mu\nu} = AD + Bp^2$$

$$I^{\mu\nu} p_\mu p_\nu = Ap^2 + B(p^2)^2$$

$$\overset{\text{INVERSION}}{\Longrightarrow} \quad A = \cdots \quad B = \cdots$$

One-Loop corrections for e^+e^- annihilation into $\mu^+\mu^-$ in the Weinberg model, Passarino, Veltman (1979)

PV gets messy

$$
\begin{aligned}
I^{\mu_1\mu_2\mu_3\mu_4} = & \sum_{i_1,i_2,i_3,i_4=1}^{N_\mathrm{p}} c^{(0)}_{i_1i_2i_3i_4}\, p_{i_1}^{\mu_1} p_{i_2}^{\mu_2} p_{i_3}^{\mu_3} p_{i_4}^{\mu_4} + g^{\mu_1\mu_2} \sum_{i_3,i_4=1}^{N_\mathrm{p}} c^{(12)}_{i_3i_4}\, p_{i_3}^{\mu_3} p_{i_4}^{\mu_4} \\
& + g^{\mu_1\mu_3} \sum_{i_2,i_4=1}^{N_\mathrm{p}} c^{(13)}_{i_2i_4}\, p_{i_2}^{\mu_2} p_{i_4}^{\mu_4} + g^{\mu_1\mu_4} \sum_{i_2,i_3=1}^{N_\mathrm{p}} c^{(14)}_{i_2i_3}\, p_{i_2}^{\mu_2} p_{i_3}^{\mu_3} \\
& + g^{\mu_2\mu_3} \sum_{i_1,i_4=1}^{N_\mathrm{p}} c^{(23)}_{i_1i_4}\, p_{i_1}^{\mu_1} p_{i_4}^{\mu_4} + g^{\mu_2\mu_4} \sum_{i_1,i_3=1}^{N_\mathrm{p}} c^{(24)}_{i_1i_3}\, p_{i_1}^{\mu_1} p_{i_3}^{\mu_3} \\
& + g^{\mu_3\mu_4} \sum_{i_1,i_2=1}^{N_\mathrm{p}} c^{(34)}_{i_1i_2}\, p_{i_1}^{\mu_1} p_{i_2}^{\mu_2} + c^{(12,34)} g^{\mu_1\mu_2} g^{\mu_3\mu_4} \\
& + c^{(13,24)} g^{\mu_1\mu_3} g^{\mu_2\mu_4} + c^{(14,23)} g^{\mu_1\mu_4} g^{\mu_2\mu_3} .
\end{aligned}
$$

INVERSION

$$\Longrightarrow \quad c^{(\dots)}_{\dots} = \cdots$$

The Passarino-Veltman Tensor Reduction Problem

Different ansatz

Easier Inversion

$$I^{\mu\nu} = \int dk^D \frac{k^\mu k^\nu}{k^2(k+p)^2} \overset{\text{ANSATZ}}{=} A\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) + B\frac{p^\mu p^\nu}{(p^2)^2}$$

$$I^{\mu\nu}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) = A(D-1)$$

$$I^{\mu\nu}p_\mu p_\nu = B$$

$$\overset{\text{TRIVIAL}}{\Longrightarrow}$$

$$A = \frac{I^{\mu\nu}}{D-1}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)$$

$$B = I^{\mu\nu}p_\mu p_\nu$$

TR = Projection onto basis

$$I^{\mu\nu} = \int dk^D \frac{k^\mu k^\nu}{k^2(k+p)^2} = A\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) + B\frac{p^\mu p^\nu}{(p^2)^2}\,, \quad A = \frac{I^{\mu\nu}}{D-1}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\,, \quad B = I^{\mu\nu} p_\mu p_\nu$$

$\rightarrow$ Define

$$g_\perp^{\mu\nu} := \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)$$

$$\rightarrow \quad I^{\mu\nu} = \left(I_{\alpha\beta}\frac{g_\perp^{\alpha\beta}}{D-1}\right) g_\perp^{\mu\nu} + I_{\alpha\beta}\left(\frac{p^\alpha p^\mu}{p^2}\right)\left(\frac{p^\beta p^\nu}{p^2}\right)$$

TR = Projection onto basis

$$I^{\mu\nu} = \left(I_{\alpha\beta} \frac{g_\perp^{\alpha\beta}}{D-1} \right) g_\perp^{\mu\nu} + I_{\alpha\beta} \left(\frac{p^\alpha p^\mu}{p^2} \right) \left(\frac{p^\beta p^\nu}{p^2} \right)$$

Rewrite $I^{\mu\nu}$ as:

$$I^{\mu\nu} = (I_{\alpha\beta} \langle g_\perp^{\alpha\beta} \rangle) g_\perp^{\mu\nu} + \left(I_{\alpha\beta} \langle p^\alpha \rangle \langle p^\beta \rangle \right) p^\mu p^\nu \,, \quad \langle g_\perp^{\mu\nu} \rangle := \frac{g_\perp^{\mu\nu}}{D-1} \,, \quad \langle p^\mu \rangle := \frac{p^\mu}{p^2}$$

Verify projection conditions:

$$\langle g_\perp^{\mu\nu} \rangle g_{\perp,\mu\nu} = 1 \qquad \langle g_\perp^{\mu\nu} \rangle p_\mu = 0 \qquad \langle p_\mu \rangle g_\perp^{\mu\nu} = 0 \qquad \langle p_\mu \rangle p^\mu = 1$$

TR = Projection onto basis

$$I^{\mu\nu} = \left(I_{\alpha\beta} \frac{g_\perp^{\alpha\beta}}{D-1} \right) g_\perp^{\mu\nu} + I_{\alpha\beta} \left(\frac{p^\alpha p^\mu}{p^2} \right) \left(\frac{p^\beta p^\nu}{p^2} \right)$$

Rewrite $I^{\mu\nu}$ as:

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Verify projection conditions:

$$\langle g_\perp^{\mu\nu} \rangle g_{\perp,\mu\nu} = 1 \qquad \langle g_\perp^{\mu\nu} \rangle p_\mu = 0 \qquad \langle p_\mu \rangle g_\perp^{\mu\nu} = 0 \qquad \langle p_\mu \rangle p^\mu = 1$$

What we would like to have:

For any rank R, N_p ext. momenta

PROJECTION

$$I^{\mu_1 \dots \mu_R} = I_{\alpha_1 \dots \alpha_R} \sum_a \langle T_a \rangle^{\alpha_1 \dots \alpha_R} T_a^{\mu_1 \dots \mu_R} \,, \qquad \langle T_a \rangle \cdot T_b := \langle T_a \rangle^{\mu_1 \dots \mu_R} T_{b,\mu_1 \dots \mu_R} = \delta_{ab}$$

How to make it systematic

The Dual Basis

$$I^{\mu_1\dots\mu_R} = I_{\alpha_1\dots\alpha_R}\sum_a \langle T_a\rangle^{\alpha_1\dots\alpha_R} T_a^{\mu_1\dots\mu_R},$$

$$\langle T_a\rangle \cdot T_b := \langle T_a\rangle^{\mu_1\dots\mu_R} T_{b,\mu_1\dots\mu_R} = \delta_{ab}$$

Dual momenta

Large loop integrals,
van Neerven, Vermaseren (1984)
1105.4319, Ellis, Kunszt, Melnikov, Zanderighi
...

Dual metric tensors

1801.06084, Ruijl , Herzog, Ueda, Vermaseren, Vogt

How to make it systematic

The Dual Basis

$$I^{\mu_1\ldots\mu_R} = I_{\alpha_1\ldots\alpha_R}\sum_a \langle T_a\rangle^{\alpha_1\ldots\alpha_R} T_a^{\mu_1\ldots\mu_R},$$

$$\langle T_a\rangle \cdot T_b := \langle T_a\rangle^{\mu_1\ldots\mu_R} T_{b,\mu_1\ldots\mu_R} = \delta_{ab}$$

ANALOGY

$$|\psi\rangle = \sum_a |a\rangle\langle a|\psi\rangle$$

$$\langle a|b\rangle = \delta_{a,b}$$

Dual momenta

Large loop integrals,
van Neerven, Vermaseren (1984)
1105.4319, Ellis, Kunszt,
Melnikov, Zanderighi
...

Dual metric tensors

1801.06084, Ruijl , Herzog,
Ueda, Vermaseren, Vogt

Dual momenta

Large loop integrals, van Neerven, Vermaseren (1984)

$N_p \times N_p$ matrix, for N_p linearly independent external momenta in D dimensions

$$\Pi_{ij} := p_i \cdot p_j \,, \qquad \Delta := \Pi^{-1}$$

$$\langle p_i \rangle^\mu := \sum_{j=1}^{N_p} \Delta_{ij}\, p_j^\mu \,, \qquad \langle p_{i_1}{}^{\mu_1} \ldots p_{i_R}^{\mu_R} \rangle = \langle p_{i_1}{}^{\mu_1} \rangle \ldots \langle p_{i_R}{}^{\mu_R} \rangle$$

Verify orthogonality:

$$p_i \cdot \langle p_j \rangle = \sum_{k=1}^{N_p} \Delta_{kj}\, p_i \cdot p_k = \sum_{k=1}^{N_p} \Pi_{ik} \Delta_{kj} = \delta_{ij}$$

Dual momenta

Unity in the space of momenta:

$$u^{\mu\nu} := \sum_{i=1}^{N_p} p_i^{\mu} \langle p_i \rangle^{\nu} \qquad\qquad p_{k,\mu} u^{\mu\nu} = p_k^{\nu}$$

With this we can easily define the transverse metric in general:

$$g_{\perp}^{\mu\nu} := g^{\mu\nu} - u^{\mu\nu}$$

Dual metric tensor

1801.06084, Ruijl , Herzog, Ueda, Vermaseren, Vogt

Generic rank R even

$$T_a^{\mu_1 \cdots \mu_R} = g_\perp^{\mu_1 \mu_2} g_\perp^{\mu_3 \mu_4} \cdots g_\perp^{\mu_{R-1} \mu_R}$$

There are $N_{metric} = \dfrac{R!}{2^{R/2}(R/2)!}$ independent metric tensors

How do we find the corresponding dual $\langle T_a \rangle^{\mu_1 \cdots \mu_R}$?

Dual metric tensor

$\{T_a^{\mu_1\cdots\mu_R} = g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} \cdots g_\perp^{\mu_{R-1}\mu_R}, \langle T_a \rangle^{\mu_1\cdots\mu_R} = \langle g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} \cdots g_\perp^{\mu_{R-1}\mu_R} \rangle\}$

For $R = 2$ we use $T_a \cdot \langle T_a \rangle = 1$ and immediately find:

$$\left\{ T_a^{\mu_1\mu_2} = g_\perp^{\mu_1\mu_2}, \langle T_a \rangle^{\mu_1\mu_2} = \frac{g_\perp^{\mu_1\mu_2}}{D_\perp} \right\}$$

Dual metric tensor

$$\{T_a^{\mu_1\cdots\mu_R} = g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} \cdots g_\perp^{\mu_{R-1}\mu_R}, \langle T_a \rangle^{\mu_1\cdots\mu_R} = \langle g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} \cdots g_\perp^{\mu_{R-1}\mu_R} \rangle\}$$

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For $R = 4$ we have $T_a^{\mu_1\mu_2\mu_3\mu_4} \in \{g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4}, g_\perp^{\mu_1\mu_3} g_\perp^{\mu_2\mu_4}, g_\perp^{\mu_1\mu_4} g_\perp^{\mu_2\mu_3}\}$

ANSATZ

$$\langle g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} \rangle = A_1 g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} + A_2 g_\perp^{\mu_2\mu_3} g_\perp^{\mu_1\mu_4} + A_3 g_\perp^{\mu_2\mu_4} g_\perp^{\mu_1\mu_3}$$

Dual metric tensor

$\{T_a^{\mu_1\cdots\mu_R} = g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} \cdots g_\perp^{\mu_{R-1}\mu_R}, \langle T_a \rangle^{\mu_1\cdots\mu_R} = \langle g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} \cdots g_\perp^{\mu_{R-1}\mu_R} \rangle\}$

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ANSATZ

$$\langle g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} \rangle = A_1 g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} + A_2 g_\perp^{\mu_2\mu_3} g_\perp^{\mu_1\mu_4} + A_3 g_\perp^{\mu_2\mu_4} g_\perp^{\mu_1\mu_3}$$

Only 2 coefficients are necessary!

Dual metric tensor

$$\{T_a^{\mu_1\cdots\mu_R} = g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} \cdots g_\perp^{\mu_{R-1}\mu_R}, \langle T_a \rangle^{\mu_1\cdots\mu_R} = \langle g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} \cdots g_\perp^{\mu_{R-1}\mu_R} \rangle\}$$

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ANSATZ

$$\langle g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} \rangle = A_1 g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} + A_2 (g_\perp^{\mu_2\mu_3} g_\perp^{\mu_1\mu_4} + g_\perp^{\mu_2\mu_4} g_\perp^{\mu_1\mu_3})$$

ORTHOGONALITY $\Longrightarrow$

$$A_1 = \frac{D_\perp + 1}{D_\perp (D_\perp - 1)(D_\perp + 2)},$$

$$A_2 = -\frac{1}{D_\perp (D_\perp - 1)(D_\perp + 2)}$$

Dual metric tensor

$$\{T_a^{\mu_1 \cdots \mu_R} = g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} \cdots g_\perp^{\mu_{R-1}\mu_R}, \langle T_a \rangle^{\mu_1 \cdots \mu_R} = \langle g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} \cdots g_\perp^{\mu_{R-1}\mu_R} \rangle\}$$

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ANSATZ

$$\langle g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} \rangle = A_1 g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} + A_2 (g_\perp^{\mu_2\mu_3} g_\perp^{\mu_1\mu_4} + g_\perp^{\mu_2\mu_4} g_\perp^{\mu_1\mu_3})$$

ANSATZ $\leftrightarrow$ groups considering symmetry under transformation of indices

Dual metric tensor

$\{T_a^{\mu_1\cdots\mu_R} = g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} \cdots g_\perp^{\mu_{R-1}\mu_R}, \langle T_a \rangle^{\mu_1\cdots\mu_R} = \langle g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} \cdots g_\perp^{\mu_{R-1}\mu_R} \rangle\}$

For $R = 6$, shorthand notation

$$g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} g_\perp^{\mu_5\mu_6} =: (12)(34)(56)$$

Equivalence classes for transformation of indices, e.g.

$$[(1\underbrace{2)(3}4)(56)] := (1\underbrace{2)(3}4)(56) + (1\underbrace{2)(5}6)(34) + (3\underbrace{4)(5}6)(12)\,.$$

ANSATZ

$$\langle g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} g_\perp^{\mu_5\mu_6} g_\perp^{\mu_7\mu_8} \rangle =$$

$[(12)(34)(56)]$ $\quad A_1 g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} g_\perp^{\mu_5\mu_6}$

$[(1\underbrace{2)(3}4)(56)]$ $\quad +A_2 \left(g_\perp^{\mu_1\mu_3} g_\perp^{\mu_2\mu_4} g_\perp^{\mu_5\mu_6} + \ldots \right)$

$[(1\underbrace{2)(3}\underbrace{4)(5}6)]$ $\quad +A_3 \left(g_\perp^{\mu_1\mu_3} g_\perp^{\mu_2\mu_5} g_\perp^{\mu_4\mu_6} + \ldots \right)$

Dual metric tensor $\{T_a^{\mu_1\cdots\mu_R} = g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} \cdots g_\perp^{\mu_{R-1}\mu_R}, \langle T_a \rangle^{\mu_1\cdots\mu_R} = \langle g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} \cdots g_\perp^{\mu_{R-1}\mu_R} \rangle\}$

For $R = 8$ we have

$$g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} g_\perp^{\mu_5\mu_6} g_\perp^{\mu_7\mu_8} =: (12)(34)(56)(78)$$

All the transformations of indices:

$$[(12)(34)(56)(78)]$$

$$[(1\underbrace{2)(3}4)(56)(78)]$$

$$[(1\underbrace{2)(3}\underbrace{4)(5}6)(78)]$$

$$[(1\underbrace{2)(3}\underbrace{4)(5}\underbrace{6)(7}8)]$$

Dual metric tensor

$\{T_a^{\mu_1\cdots\mu_R} = g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} \cdots g_\perp^{\mu_{R-1}\mu_R}, \langle T_a \rangle^{\mu_1\cdots\mu_R} = \langle g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} \cdots g_\perp^{\mu_{R-1}\mu_R} \rangle\}$

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For $R = 8$ we have

ANSATZ

$$\langle g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} g_\perp^{\mu_5\mu_6} g_\perp^{\mu_7\mu_8} \rangle =$$

$[(12)(34)(56)(78)]$ $\quad A_1 g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} g_\perp^{\mu_5\mu_6} g_\perp^{\mu_7\mu_8}$

$[(12)(34)(56)(78)]$ $\quad +A_2 \left(g_\perp^{\mu_1\mu_3} g_\perp^{\mu_2\mu_4} g_\perp^{\mu_5\mu_6} g_\perp^{\mu_7\mu_8} + \ldots \right)$

$[(12)(34)(56)(78)]$ $\quad +A_3 \left(g_\perp^{\mu_1\mu_3} g_\perp^{\mu_2\mu_5} g_\perp^{\mu_4\mu_6} g_\perp^{\mu_7\mu_8} + \ldots \right)$

$[(12)(34)(56)(78)]$ $\quad +A_4 \left(g_\perp^{\mu_1\mu_3} g_\perp^{\mu_2\mu_4} g_\perp^{\mu_5\mu_7} g_\perp^{\mu_6\mu_8} + \ldots \right)$

$[(12)(34)(56)(78)]$ $\quad +A_5 \left(g_\perp^{\mu_1\mu_3} g_\perp^{\mu_2\mu_7} g_\perp^{\mu_4\mu_6} g_\perp^{\mu_5\mu_8} + \ldots \right)$

Dual metric tensor

$\{T_a^{\mu_1\cdots\mu_R} = g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4}\cdots g_\perp^{\mu_{R-1}\mu_R}, \langle T_a\rangle^{\mu_1\cdots\mu_R} = \langle g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4}\cdots g_\perp^{\mu_{R-1}\mu_R}\rangle\}$

ANSATZ

$$\langle g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} g_\perp^{\mu_5\mu_6} g_\perp^{\mu_7\mu_8}\rangle =$$

[(12)(34)(56)(78)]	$A_1 g_\perp^{\mu_1\mu_2} g_\perp^{\mu_3\mu_4} g_\perp^{\mu_5\mu_6} g_\perp^{\mu_7\mu_8}$
[(12)(34)(56)(78)]	$+A_2 \left(g_\perp^{\mu_1\mu_3} g_\perp^{\mu_2\mu_4} g_\perp^{\mu_5\mu_6} g_\perp^{\mu_7\mu_8} + \ldots \right)$
[(12)(34)(56)(78)]	$+A_3 \left(g_\perp^{\mu_1\mu_3} g_\perp^{\mu_2\mu_5} g_\perp^{\mu_4\mu_6} g_\perp^{\mu_7\mu_8} + \ldots \right)$
[(12)(34)(56)(78)]	$+A_4 \left(g_\perp^{\mu_1\mu_3} g_\perp^{\mu_2\mu_4} g_\perp^{\mu_5\mu_7} g_\perp^{\mu_6\mu_8} + \ldots \right)$
[(12)(34)(56)(78)]	$+A_5 \left(g_\perp^{\mu_1\mu_3} g_\perp^{\mu_2\mu_7} g_\perp^{\mu_4\mu_6} g_\perp^{\mu_5\mu_8} + \ldots \right)$

Number of terms in the ansatz is in 1-to-1 correspondence between the unique integer partitions of $R/2$:

$$[1+1+1+1] \quad [2+1+1] \quad [3+1] \quad [2+2] \quad [4]$$

Dual metric tensors

Useful properties

$$\langle g_{\perp,\mu_1\mu_2}\cdots g_{\perp,\mu_{2n-1}\mu_{2n}}\rangle \neq \langle g_{\perp,\mu_1\mu_2}\rangle\cdots\langle g_{\perp,\mu_{2n-1}\mu_{2n}}\rangle$$

Dual metric tensors

Useful properties

$$g_{\perp}^{\mu_1\mu_2}\cdots g_{\perp}^{\mu_{2m-1}\mu_{2m}}\langle g_{\perp,\mu_1\mu_2}\cdots g_{\perp,\mu_{2m-1}\mu_{2m}} g_{\perp,\rho_1\rho_2}\cdots g_{\perp,\rho_{2n-1}\rho_{2n}}\rangle = \langle g_{\perp,\rho_1\rho_2}\cdots g_{\perp,\rho_{2n-1}\rho_{2n}}\rangle$$

(Prove it using orthogonality relations)

Now we have the building blocks…

What about their combinatorics?

Rank R, N_p ext. momenta

As usual, we can list all possible tensor structures as in usual PV reduction.

But now, each tensor structure is accompanied by its dual

$$I^{\mu_1\ldots\mu_R} = I_{\alpha_1\ldots\alpha_R}\sum_a \langle T_a\rangle^{\alpha_1\ldots\alpha_R} T_a^{\mu_1\ldots\mu_R},$$

Look at all possible structures **without transverse metric tensor**:

$$\sum_a T_a^{\alpha_1\cdots\alpha_R}\langle T_a\rangle^{\mu_1\cdots\mu_R} \supset \sum_{i_1,i_2,\cdots,i_R=1}^{N_p} p_{i_1}{}^{\alpha_1}\cdots p_{i_R}^{\alpha_R}\langle p_{i_1}{}^{\mu_1}\cdots p_{i_R}^{\mu_R}\rangle = \sum_{i_1,i_2,\cdots,i_R=1}^{N_p} p_{i_1}{}^{\alpha_1}\cdots p_{i_R}^{\alpha_R}\langle p_{i_1}{}^{\mu_1}\rangle\cdots\langle p_{i_R}^{\mu_R}\rangle$$

$$= \prod_{k=1}^{R}\left(\sum_{i=1}^{N_p} p_i^{\alpha_k}\langle p_i\rangle^{\mu_k}\right) = \prod_{k=1}^{R} u^{\alpha_k\mu_k}$$

Rank R, N_p ext. momenta

$$I^{\mu_1\cdots\mu_R} = I_{\alpha_1\ldots\alpha_R}\sum_a \langle T_a\rangle^{\alpha_1\ldots\alpha_R} T_a^{\mu_1\cdots\mu_R},$$

Look at all possible structures **with at least one transverse metric tensor**:

$$\sum_a T_a^{\alpha_1\cdots\alpha_R}\langle T_a\rangle^{\mu_1\cdots\mu_R} \supset \sum_{n=2}^{2\lfloor R/2\rfloor}\sum_{i_{n+1},\ldots,i_R=1}^{N_p} g_\perp^{\alpha_1\alpha_2}\cdots g_\perp^{\alpha_{n-1}\alpha_n} p_{i_{n+1}}^{\alpha_{n+1}}\cdots p_{i_R}^{\alpha_R}\langle g_\perp^{\mu_1\mu_2}\cdots g_\perp^{\mu_{n-1}\mu_n} p_{i_{n+1}}^{\mu_{n+1}}\cdots p_{i_R}^{\mu_R}\rangle$$

$$= \sum_{n=2}^{2\lfloor R/2\rfloor} g_\perp^{\alpha_1\alpha_2}\cdots g_\perp^{\alpha_{n-1}\alpha_n}\langle g_\perp^{\mu_1\mu_2}\cdots g_\perp^{\mu_{n-1}\mu_n}\rangle \prod_{k=n+1}^{R} u^{\alpha_k\mu_k} = \sum_{n=2}^{2\lfloor R/2\rfloor}\prod_{i=1}^{n/2} \underbrace{u^{\alpha_{2i-1}\mu_{2i-1}} u^{\alpha_{2i}\mu_{2i}}} \prod_{k=n+1}^{R} u^{\alpha_k\mu_k}$$

4. Wick operation analogy

Define

$$\prod_i \underbrace{u^{\alpha_i\mu_i}u^{\beta_i\nu_i}} := \left(\prod_i g_\perp^{\alpha_i\beta_i}\right)\left\langle \prod_i g_\perp^{\mu_i\nu_i} \right\rangle$$

With the contraction operation, it is now easy to list all the terms present:

$$\sum_a T_a^{\alpha_1\cdots\alpha_R} \langle T_a \rangle^{\mu_1\cdots\mu_R} = u^{\alpha_1\mu_1}\ldots u^{\alpha_R\mu_R} + \text{all contractions}$$

WICK CONTRACTIONS

$$= \mathcal{T}\left\{u^{\alpha_1\mu_1}\ldots u^{\alpha_R\mu_R}\right\}$$

TIME ORDERING

Going back to an easy example

$$I^{\mu\nu} = I_{\alpha\beta}\left(g_{\perp}^{\alpha\beta}\frac{g_{\perp}^{\mu\nu}}{D-1} + \left(\frac{p^{\alpha}p^{\mu}}{p^2}\right)\left(\frac{p^{\beta}p^{\nu}}{p^2}\right)\right)$$

$$I^{\mu\nu} = I_{\alpha\beta}\left(g_{\perp}^{\alpha\beta}\langle g_{\perp}^{\mu\nu}\rangle + u^{\alpha\mu}u^{\beta\nu}\right)$$

Or, more suggestively

$$I^{\mu\nu} = I_{\alpha\beta}\left(\underbrace{u^{\alpha\mu}u^{\beta\nu}} + u^{\alpha\mu}u^{\beta\nu}\right)$$

Example R=4

$$I^{\mu_1\mu_2\mu_3\mu_4} = I_{\alpha_1\alpha_2\alpha_3\alpha_4}\mathscr{T}\left\{u^{\alpha_1\mu_1}u^{\alpha_2\mu_2}u^{\alpha_3\mu_3}u^{\alpha_4\mu_4}\right\}$$

$$\mathscr{T}\left\{u^{\alpha_1\mu_1}u^{\alpha_2\mu_2}u^{\alpha_3\mu_3}u^{\alpha_4\mu_4}\right\} = u^{\alpha_1\mu_1}u^{\alpha_2\mu_2}u^{\alpha_3\mu_3}u^{\alpha_4\mu_4}$$

$$+u^{\alpha_1\mu_1}u^{\alpha_2\mu_2}u^{\alpha_3\mu_3}u^{\alpha_4\mu_4} + u^{\alpha_1\mu_1}u^{\alpha_2\mu_2}u^{\alpha_3\mu_3}u^{\alpha_4\mu_4} + u^{\alpha_1\mu_1}u^{\alpha_2\mu_2}u^{\alpha_3\mu_3}u^{\alpha_4\mu_4}$$

$$+u^{\alpha_1\mu_1}u^{\alpha_2\mu_2}u^{\alpha_3\mu_3}u^{\alpha_4\mu_4} + u^{\alpha_1\mu_1}u^{\alpha_2\mu_2}u^{\alpha_3\mu_3}u^{\alpha_4\mu_4} + u^{\alpha_1\mu_1}u^{\alpha_2\mu_2}u^{\alpha_3\mu_3}u^{\alpha_4\mu_4}$$

$$+u^{\alpha_1\mu_1}u^{\alpha_2\mu_2}u^{\alpha_3\mu_3}u^{\alpha_4\mu_4} + u^{\alpha_1\mu_1}u^{\alpha_2\mu_2}u^{\alpha_3\mu_3}u^{\alpha_4\mu_4} + u^{\alpha_1\mu_1}u^{\alpha_2\mu_2}u^{\alpha_3\mu_3}u^{\alpha_4\mu_4}$$

Number of terms

For a rank-R tensor and N_p independent external momenta, tensor reduction generates

$$\sum_{n=0}^{\lfloor \frac{R}{2} \rfloor} \frac{R!}{2^n\, n!(R-2n)!} N_{\mathrm{p}}^{R-2n} \qquad \text{terms}$$

Rank	Ext. mom.	# terms
5	3	558
6	4	8641
8	4	240809

In a practical amplitude computation…

Reduce the proliferation of terms using

$$g_\perp^{\mu\nu} p_{i\nu} = 0$$

$$\langle g_\perp \rangle^{\mu\nu} p_{i\nu} = 0$$

$$\gamma_\perp^\mu := g_\perp^{\mu\nu} \gamma_\nu,$$

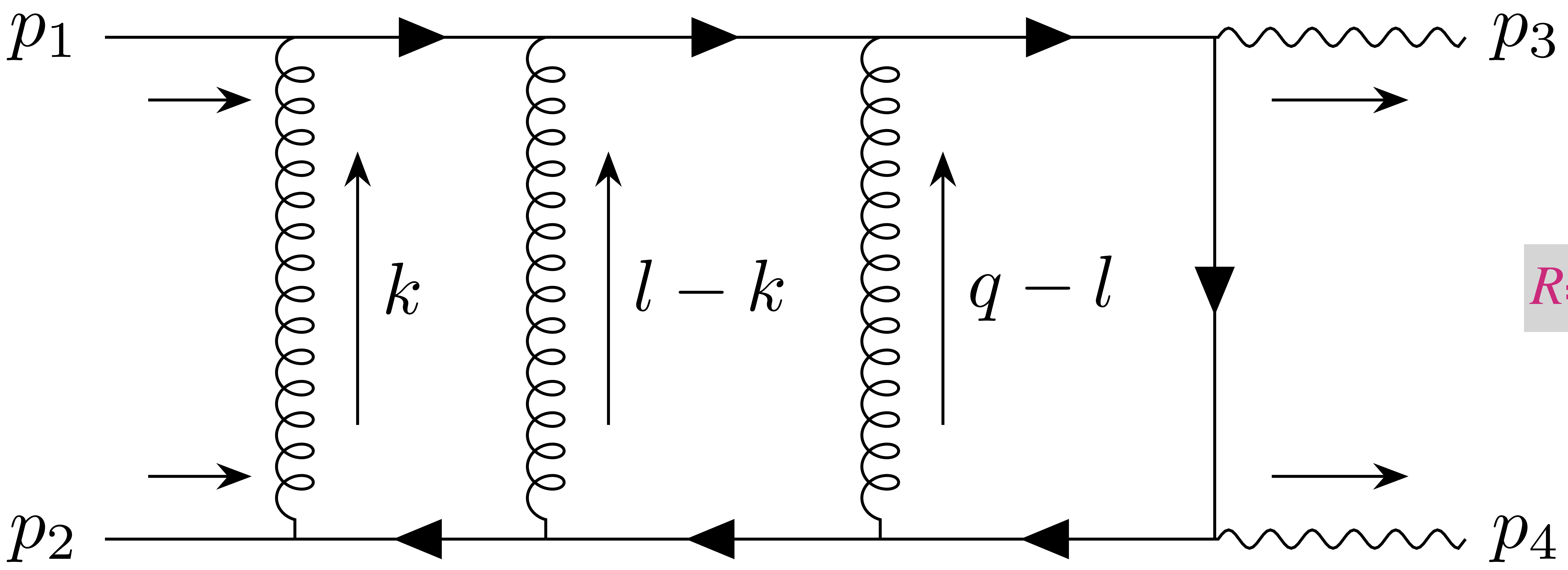
$$\left\{ \gamma_\perp^\mu, \gamma^\nu \right\} = \left\{ \gamma_\perp^\mu, \gamma_\perp^\nu \right\} = 2 g_\perp^{\mu\nu} \mathbf{1}_{4\times4}$$

$$u^{\mu\nu} p_{i\nu} = p_i^\mu$$

Substitute $\langle g_\perp \rangle$, $g_\perp$ only at the very end

Substitute u only at the very end

Tests



R=7 , N_p=3 massless externals

We verified TR gives the same result in interference with the tree summed over spins and polarisations

0010025, New features of FORM, Vermaseren (2000), ...

Summary

$$u^{\alpha_1 \mu_1} \ldots u^{\alpha_R \mu_R}$$

metrics in the space of momenta

Summary

$$\mathcal{T}\left\{u^{\alpha_1\mu_1}\ldots u^{\alpha_R\mu_R}\right\}$$

metrics in the space of momenta

+

metrics in the transverse space

Summary

$$I_{\alpha_1 \ldots \alpha_R} \mathcal{T}\{u^{\alpha_1 \mu_1} \ldots u^{\alpha_R \mu_R}\}$$

metrics in the space of momenta

+

metrics in the transverse space

+

scalar integrals

Summary

$$I^{\mu_1\ldots\mu_R} = I_{\alpha_1\ldots\alpha_R}\mathcal{T}\{u^{\alpha_1\mu_1}\ldots u^{\alpha_R\mu_R}\}$$

metrics in the space of momenta

+

metrics in the transverse space

+

scalar integrals

=

tensor integrals

Why we needed TR

We came across this as a part of a larger project, the **numerical integration of locally finite subtracted amplitudes.**

Locally finite | Divergent

$$\mathscr{M} = \int dk^D [m - r] + \int dk^D r$$

Numerically in $D = 4$

↓

LTD/cFF,

Threshold subtraction

Analytically in D dim

↓

TR,

IBPs, master integrals

1812.03753, Anastasiou, Sterman

2008.12293, Anastasiou, Haindl, Sterman, Yang, Zeng

2212.12162, Anastasiou, Sterman

soon, Anastasiou, Karlen, Sterman, Venkata

0804.3170, Catani, Gleisberg, Krauss, Rodrigo, Winter

…

1906.06138, Capatti, Hirshi, Kermanschah, Ruijl

2211.09653, Capatti

…

0912.3495, Kilian, Kleinschmidt

2110.06869, Kermanschah

10.1016/0370-2693(81)90288-4, Tkachov (1981)

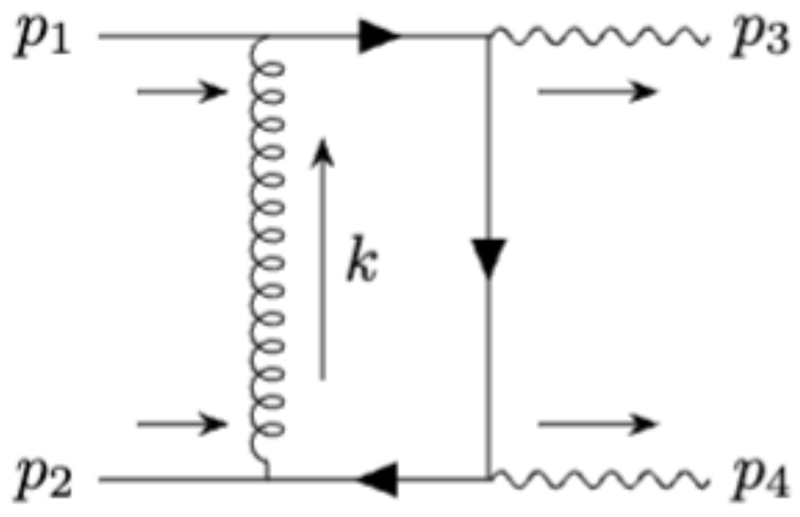
10.1016/0550-3213(81)90199-1, Chetyrkin, Tkachov (1981)

0102033, Laporta (2000)

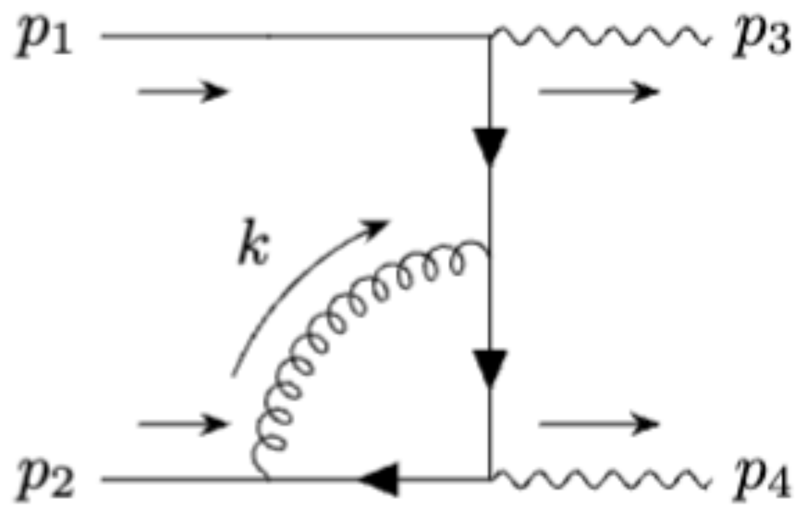
…

Example

$\mathscr{M}_{finite} =$

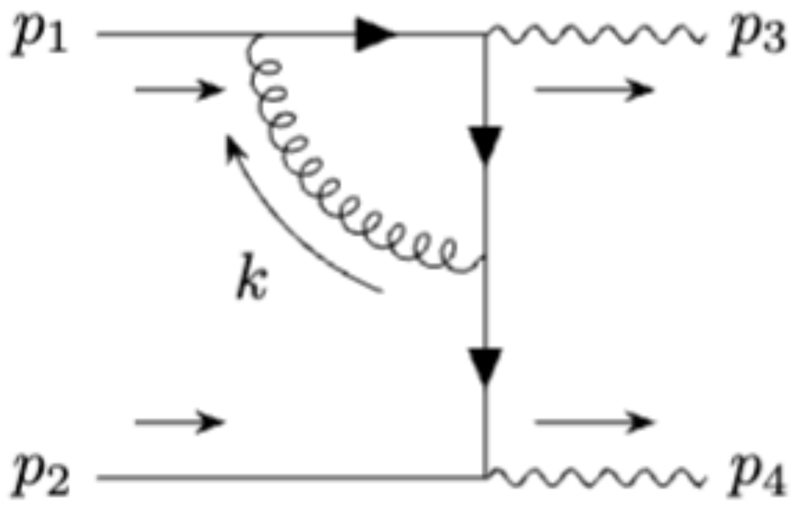


$+ \ (1 - \mathcal{R}_k)$

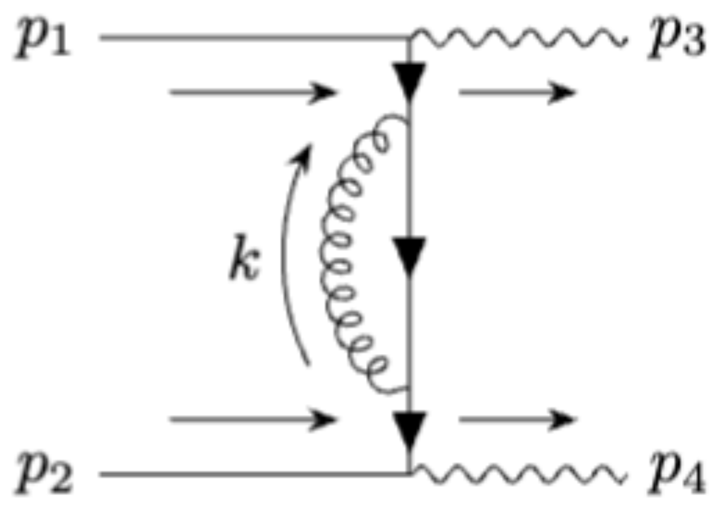


$+ \ (1 - \mathcal{R}_k)$

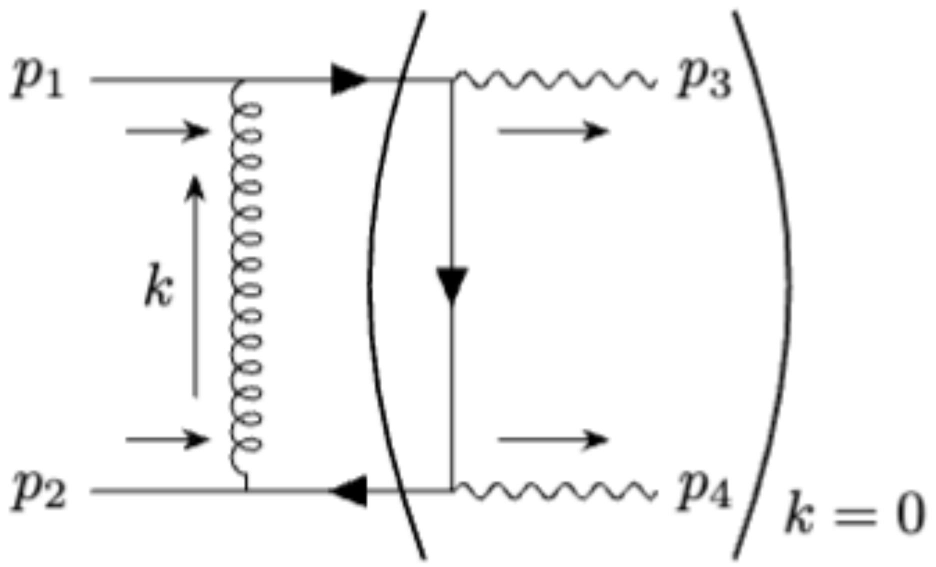
UV ctm



$+ \ (1 - \mathcal{R}_k)$



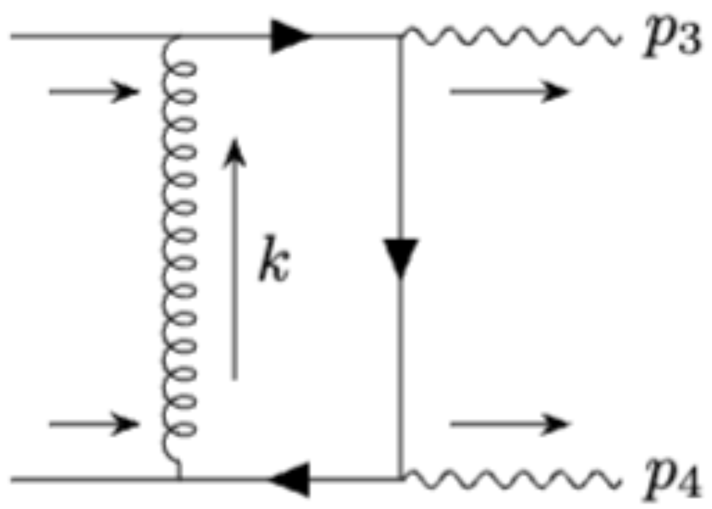
$- \ (1 - \mathcal{R}_k)$



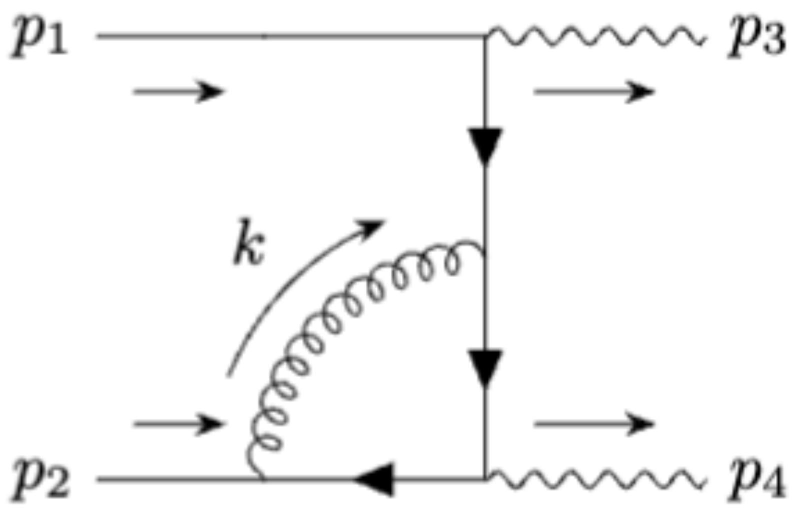
IR ctm

Example

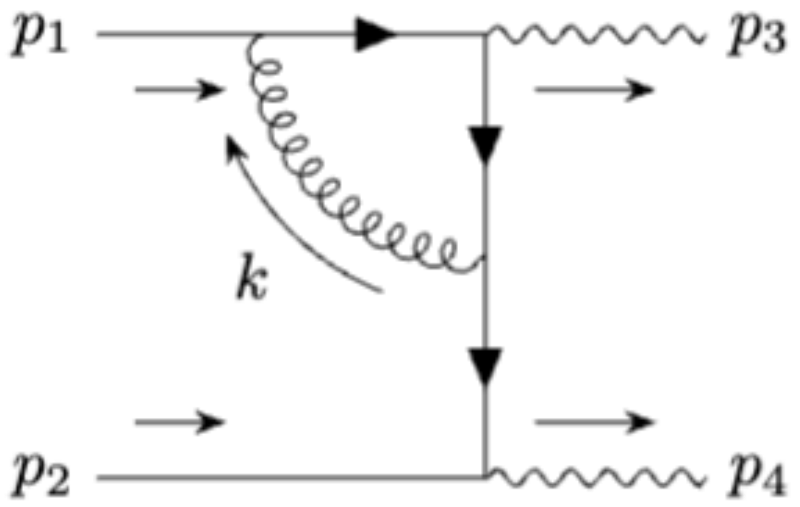
$\mathscr{M}_{finite} =$



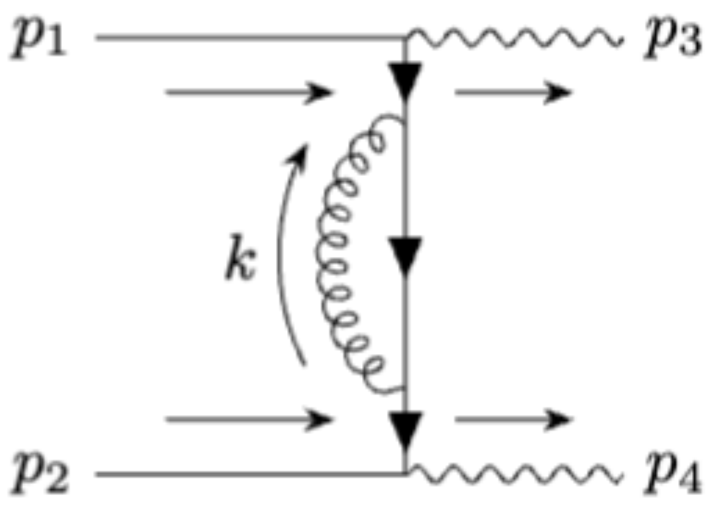
$+ (1 - \mathcal{R}_k)$



$+ (1 - \mathcal{R}_k)$

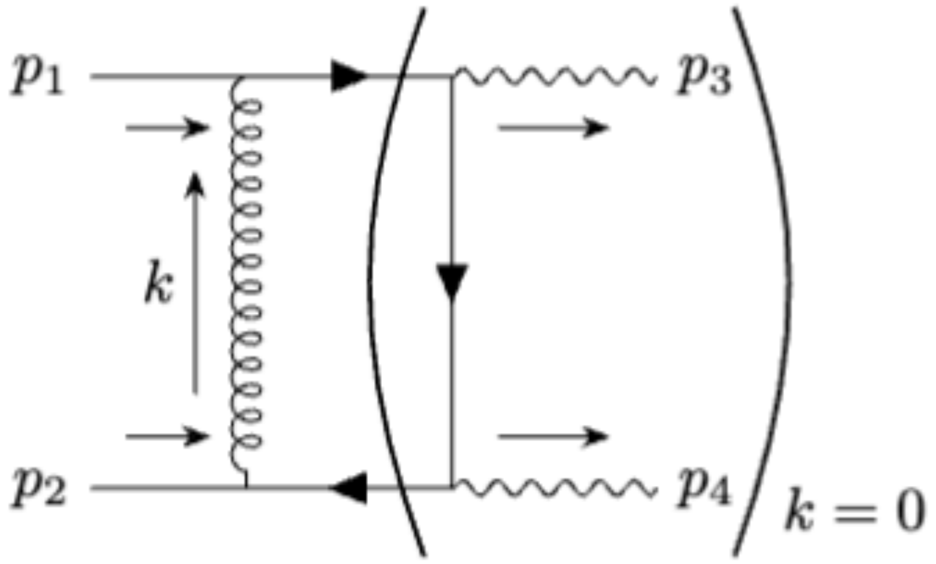


$+ (1 - \mathcal{R}_k)$



UV ctm

$- (1 - \mathcal{R}_k)$

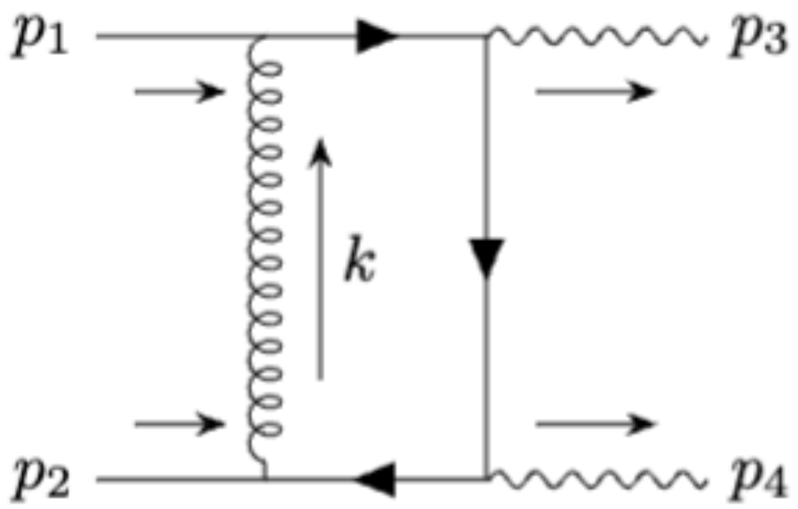


IR ctm

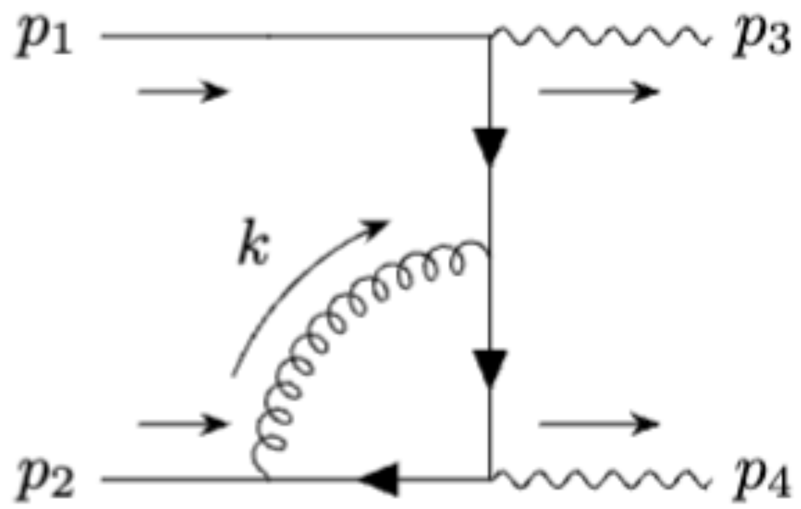
Same structure for N photons!

Example

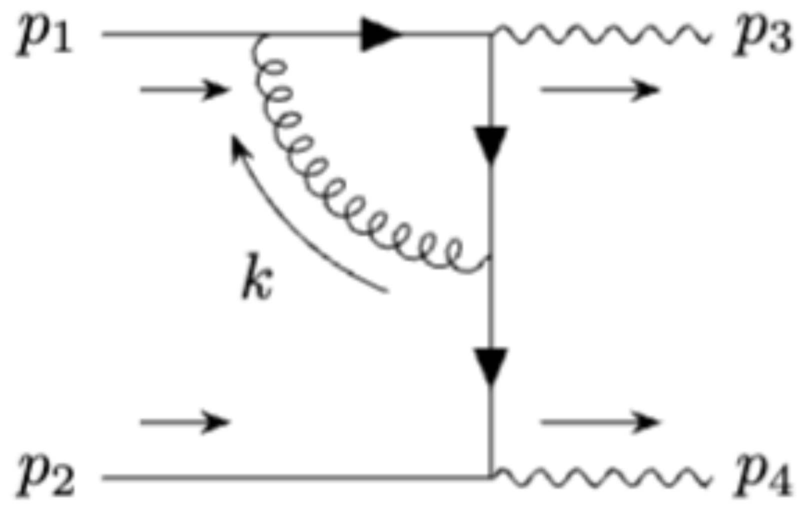
$$\mathscr{M}_{finite} =$$



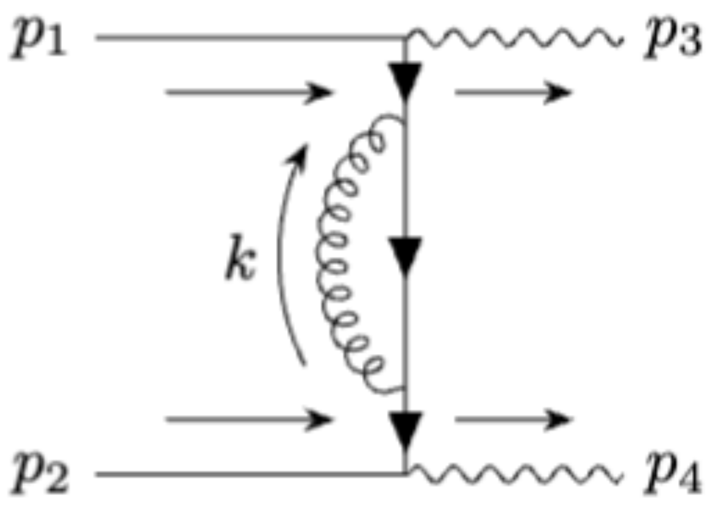
$$+ (1 - \mathcal{R}_k)$$



$$+ (1 - \mathcal{R}_k)$$

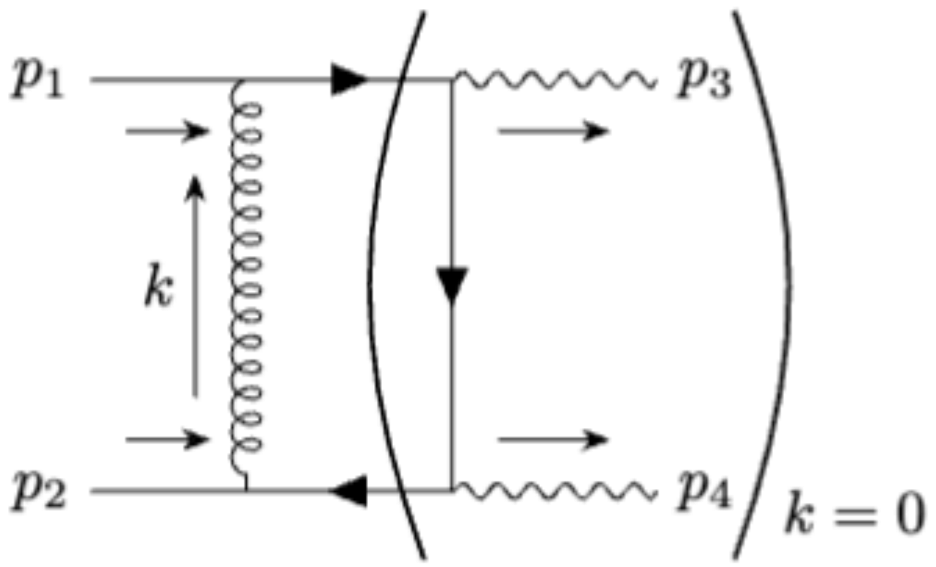


$$+ (1 - \mathcal{R}_k)$$



UV ctm

$$- (1 - \mathcal{R}_k)$$



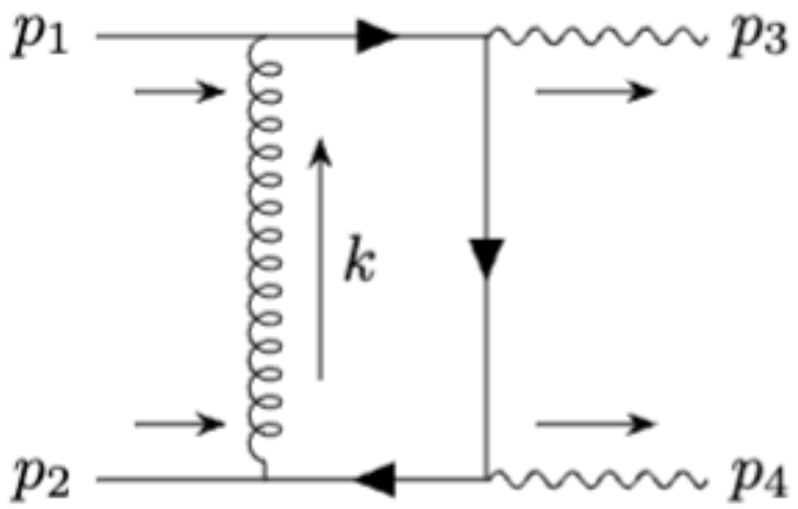
IR ctm

Same structure for N photons!

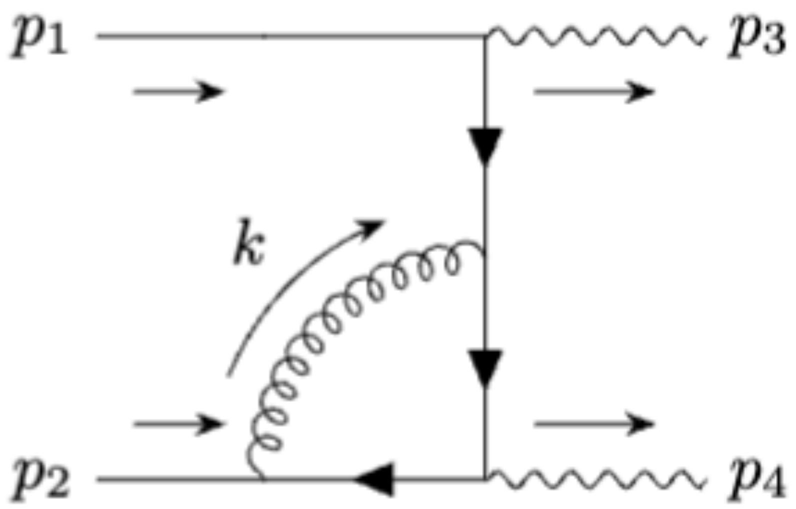
And γ^* !

Example

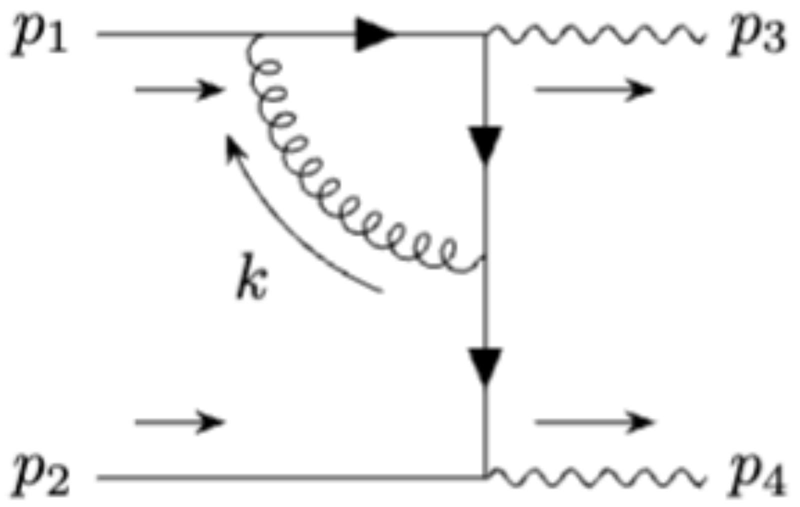
$\mathscr{M}_{finite} =$



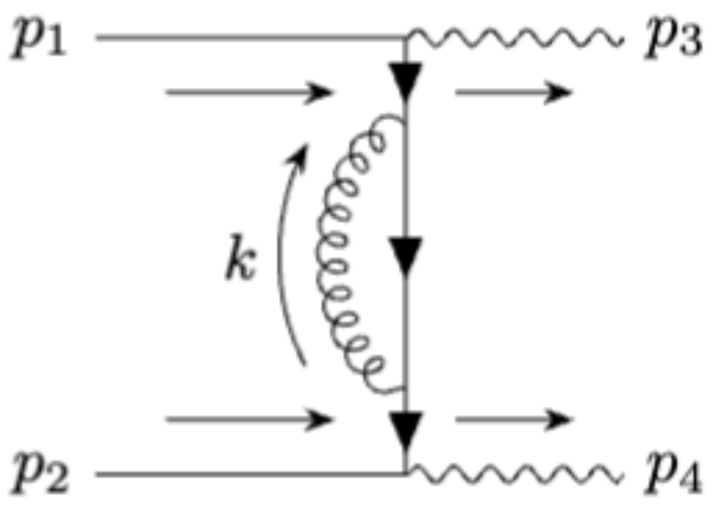
$+ (1 - \mathcal{R}_k)$



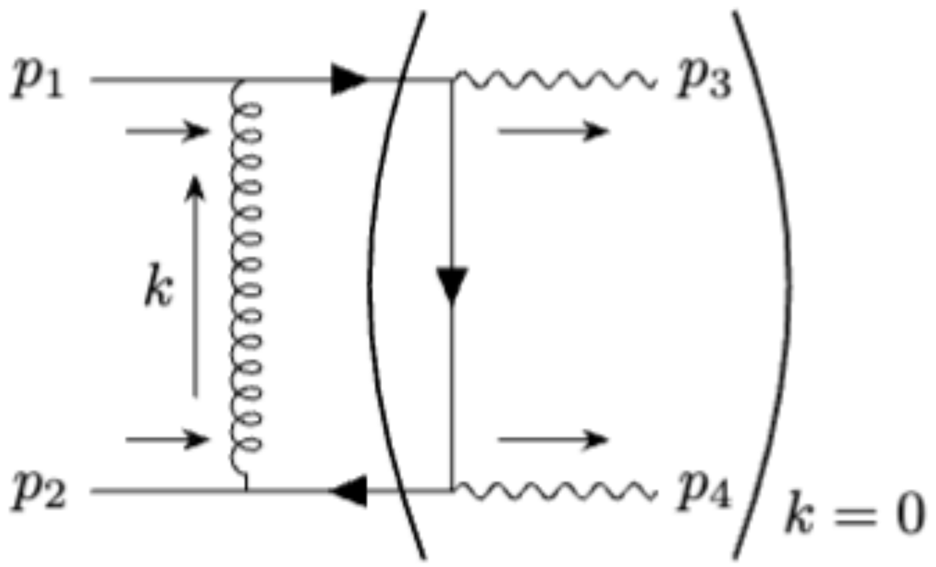
$+ (1 - \mathcal{R}_k)$



$+ (1 - \mathcal{R}_k)$



$- (1 - \mathcal{R}_k)$



UV ctm

IR ctm

Same structure for N photons!

And γ^* !

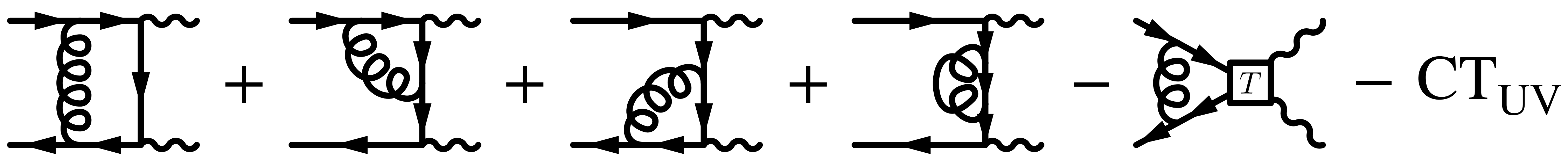
$\Downarrow$

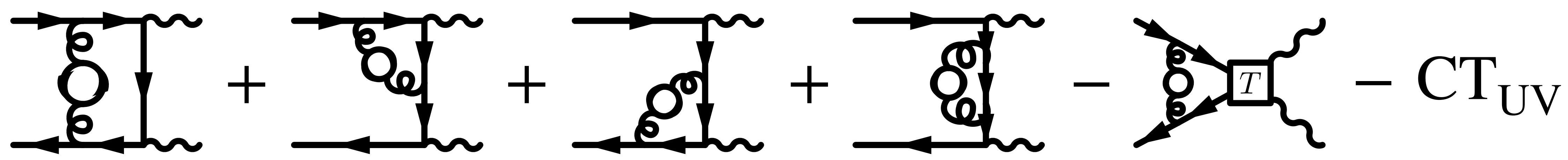
Needed compact expression integrated ctms for any N

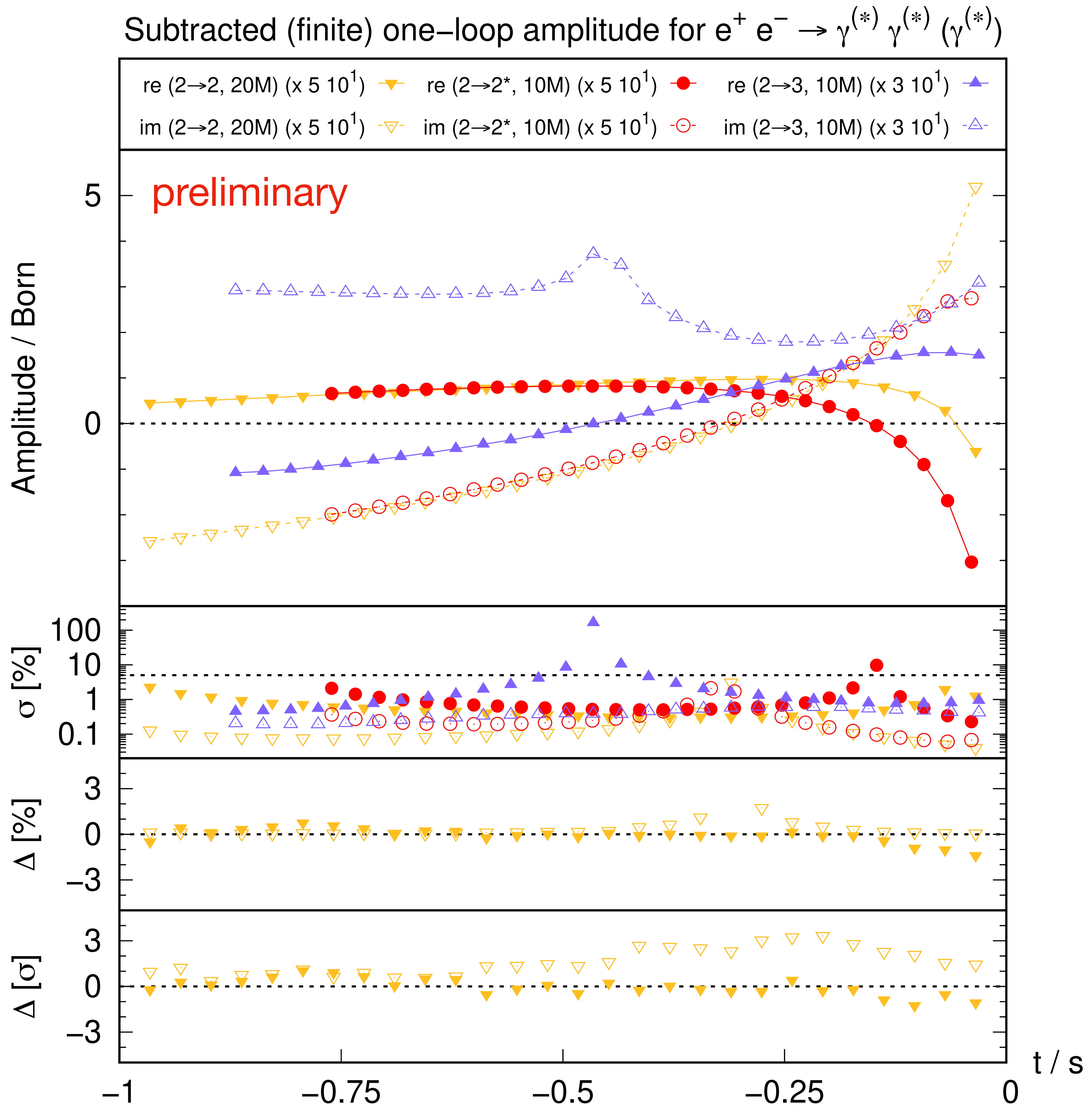
Some results…

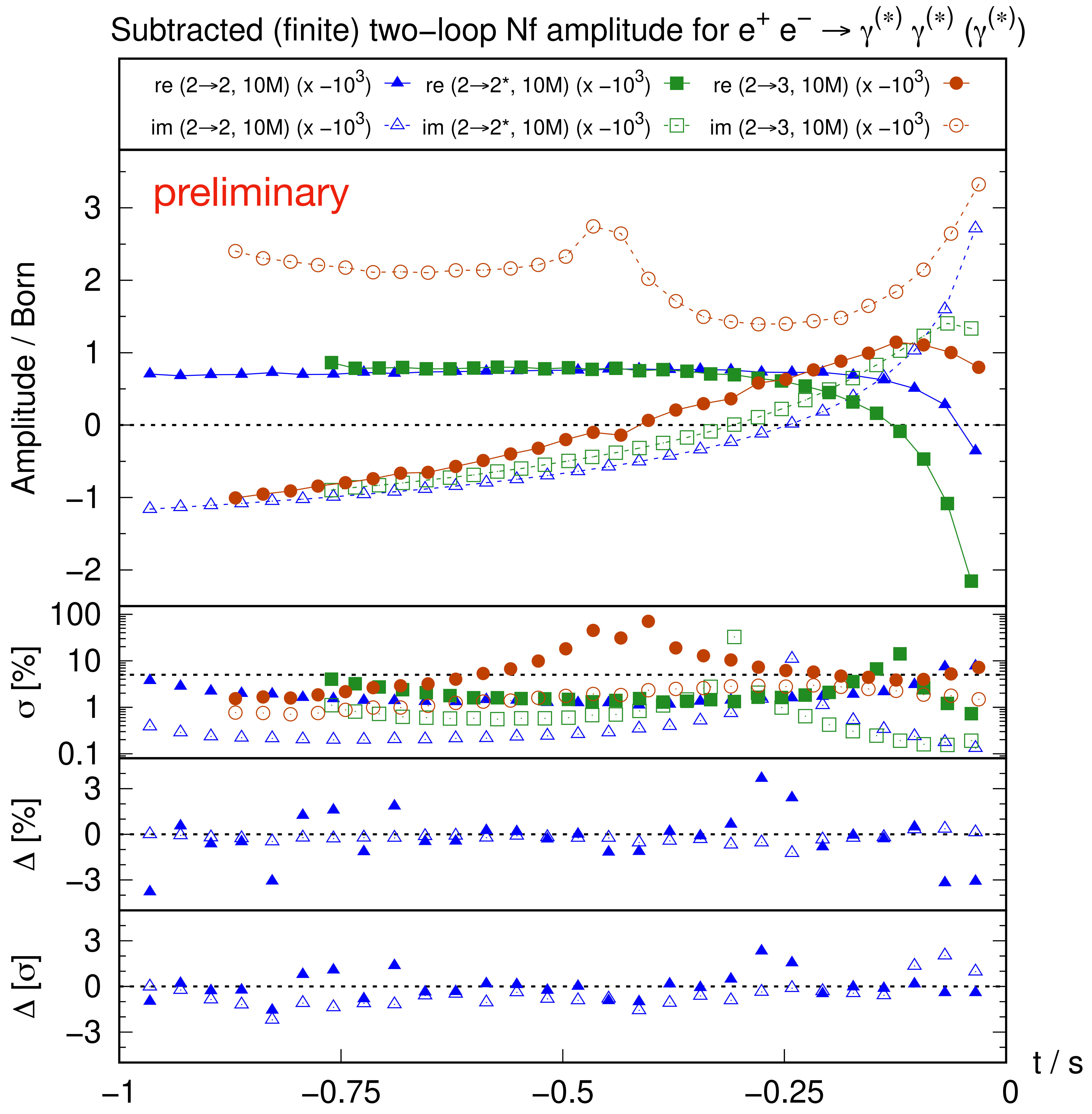
Slide from D. Kermanschah

Phase space scan of amplitude (single final state ordering)

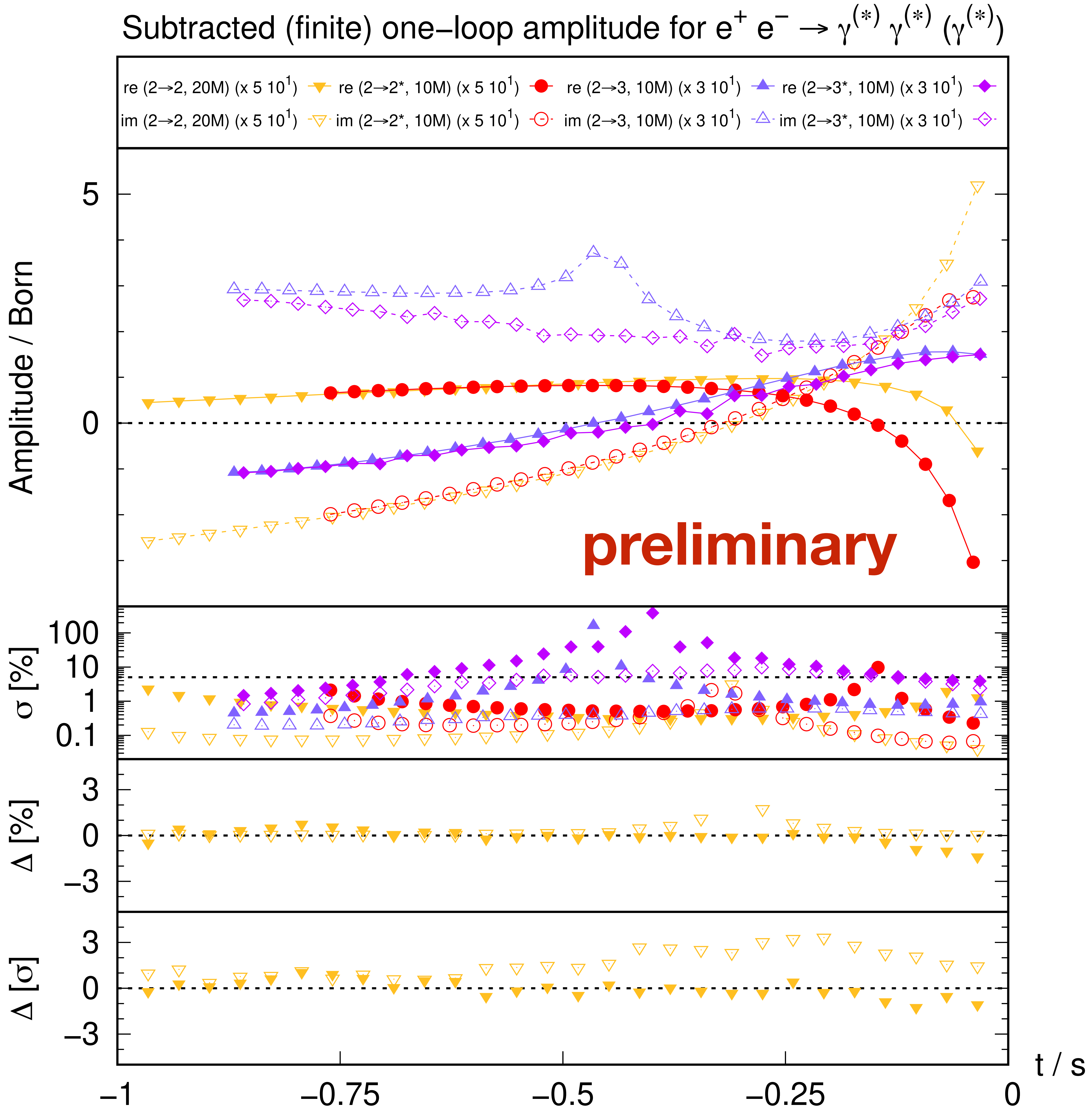


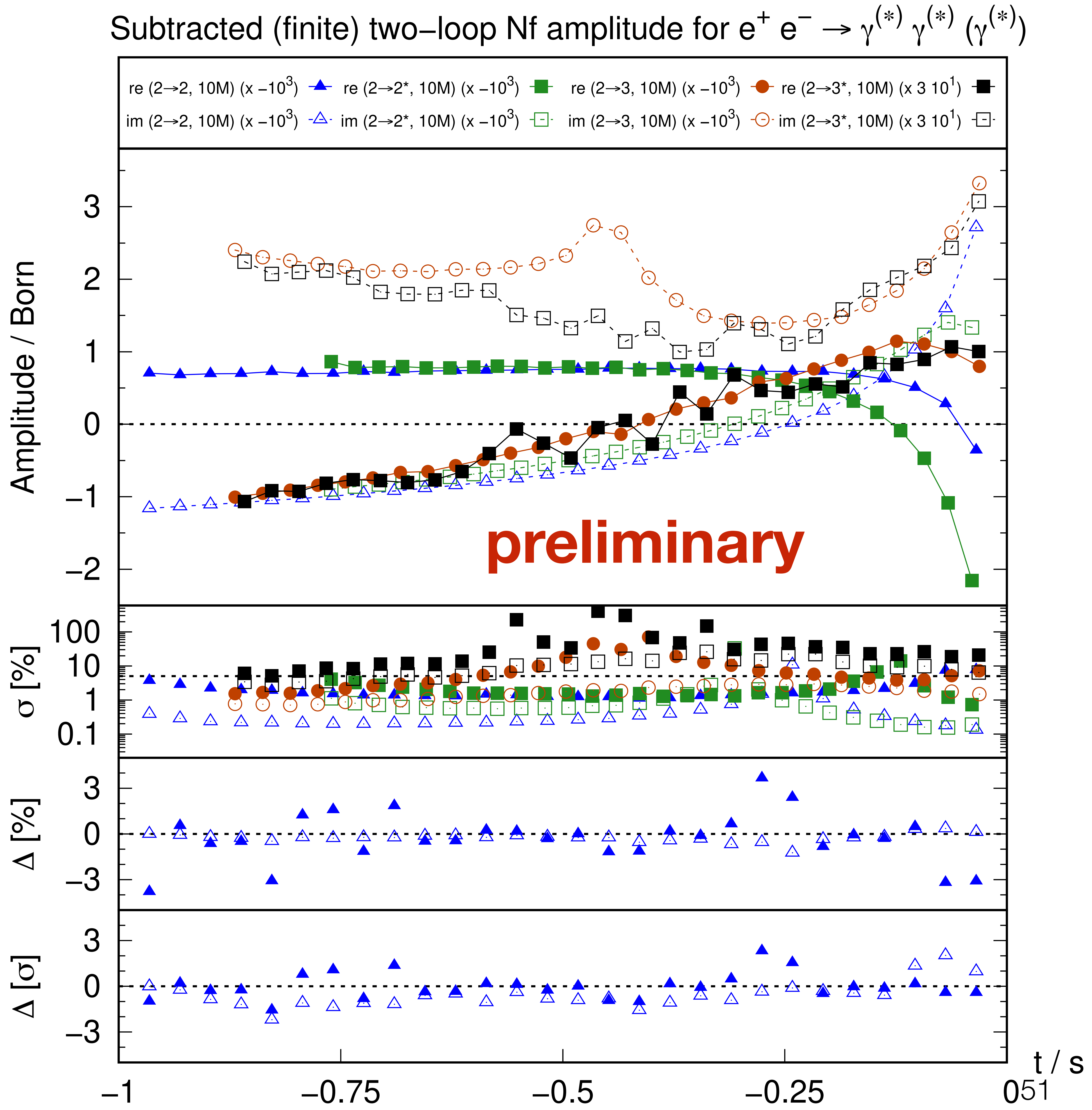






Phase space scan of amplitude (single final state ordering)





Slide from D. Kermanschah

Comparison with external results

One-loop interferences

with 10M MC samples on 48 cores

cross checked with OpenLoops [1907.13071]
thanks to Federico Buccioni

preliminary	Rambo PSP $\sqrt{s} = 1\,\mathrm{TeV}$	finite remainder (MC + integrated CTs)	$\Delta\,[\%]$	time/sample	total time
$q\,\bar{q} \to \gamma\,\gamma$	#1	-3.5882e-05 ± 0.0151e-05 GeV^-1	0.4	0.02 ms	3 min
	#2	-3.4318e-05 ± 0.0149e-05 GeV^-1	0.4		
$q\,\bar{q} \to \gamma^*\gamma^*$	#1	-3.7127e-05 ± 0.0149e-05 GeV^-1	0.4	0.04 ms	7 min
	#2	-3.5459e-05 ± 0.0141e-05 GeV^-1	0.4		
$q\,\bar{q} \to \gamma\,\gamma\,\gamma$	#1	-1.7300e-10 ± 0.0251e-10 GeV^-2	1.5	0.24 ms	40 min
	#2	-0.1784e-10 ± 0.0152e-10 GeV^-2	8.5		
$q\,\bar{q} \to \gamma^*\gamma^*\gamma^*$	#1	-4.7035e-11 ± 0.0737e-11 GeV^-2	1.6	2.19 ms	6 h
	#2	1.9957e-11 ± 0.0943e-11 GeV^-2	4.7		

Comparison with external results

Finite remainder

$$\mathcal{M}_R^{(0)} = \mathcal{F}^{(0)}$$

$$\mathcal{M}_R^{(1)} = \mathbf{Z}^{(1)}\mathcal{M}_R^{(0)} + \mathcal{F}^{(1)}$$

$$\mathcal{M}_R^{(2)} = \mathbf{Z}^{(2)}\mathcal{M}_R^{(0)} + \mathbf{Z}^{(1)}\mathcal{F}^{(1)} + \mathcal{F}^{(2)}$$

9605323, Catani, Seymour
0201274, Anastasiou, Glover, Tejeda-Yeomans

Compared $\mathcal{F}^{(2,N_f)}$ for $q\,\bar{q} \to \gamma\gamma\gamma$ with the known fully analytical result.

2010.15834, Abreu, Page, Pascual, Sotnikov
2012.13553, Chawdhry, Czakon, Mitov, Poncelet
2305.17056, Abreu, De Laurentis, Ita, Klinkert, Page, Sotnikov

Soon, $\mathcal{F}^{(2,N_f)}$ for $q\,\bar{q} \to \gamma^*\gamma^*\gamma^*$

soon, Anastasiou, Lazopoulos, Kermanschah, MV

5. Conclusions

- Large number of terms is unavoidable, but we have general answer in closed form

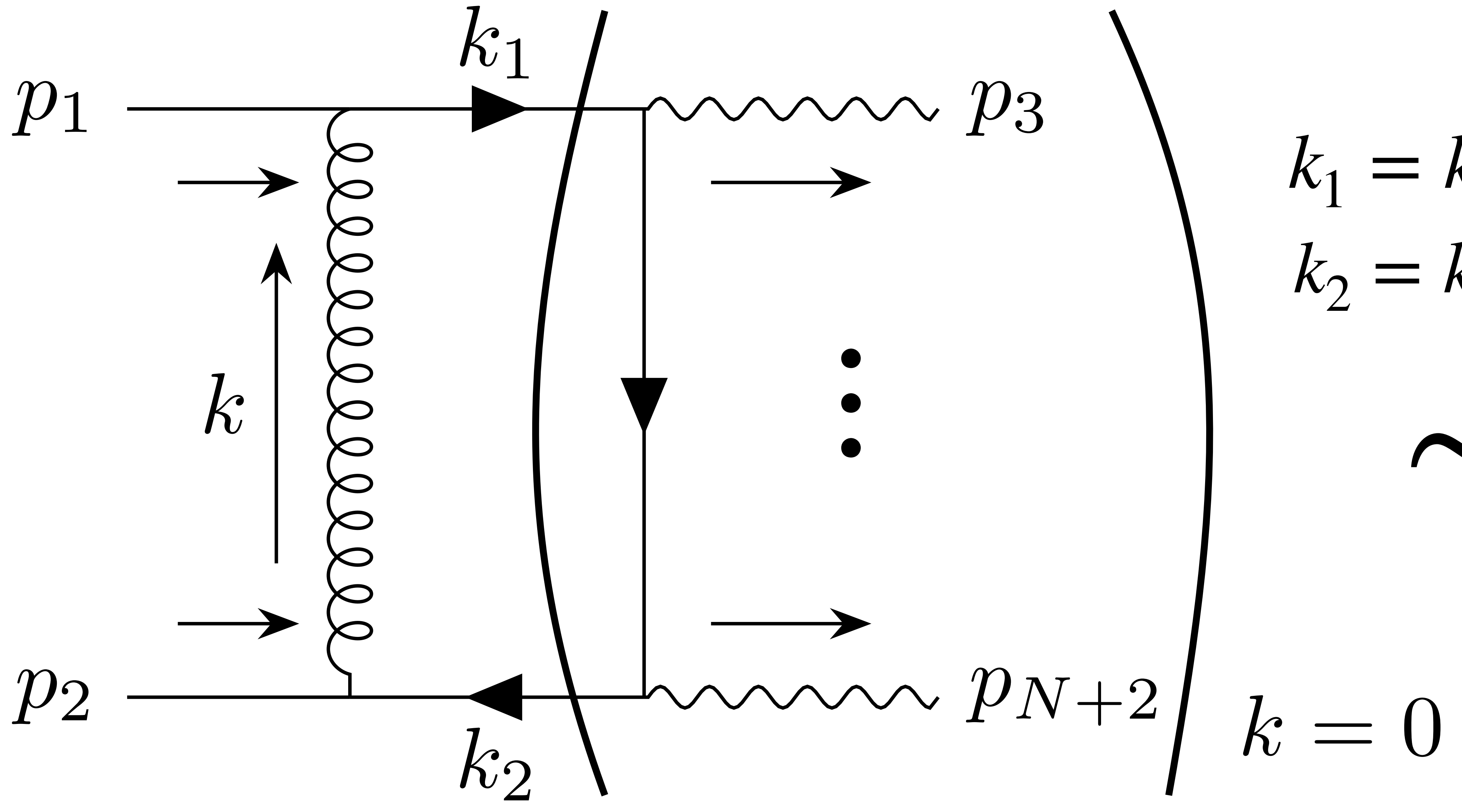
$$I^{\mu_1\ldots\mu_R} = I_{\alpha_1\ldots\alpha_R}\mathcal{T}\left\{u^{\alpha_1\mu_1}\ldots u^{\alpha_R\mu_R}\right\}$$

- Passarino-Veltman reduction employed in full generality, inversion of large matrix avoided, algorithmic approach

- Possibility of avoiding the full expansion in real amplitude calculations

Backup slides

Integration of the IR ctm for the one-loop amplitude for N photons



$$\sim \mathscr{R}[\tilde{M}^{(0)}] := S^{\alpha\beta} \frac{k_{2,\alpha} k_{1,\beta}}{k^2 k_1^2 k_2^2} \,,$$

$$S^{\alpha\beta} := \bar{v}_2 \gamma^\mu \gamma^\alpha \mathbb{P}[\ldots]\mathbb{P} \gamma^\beta \gamma_\mu u_1 \,, \qquad \mathbb{P} := \frac{\not{p}_1 \not{p}_2}{s_{12}}$$

Backup slides

Integration of the IR ctm for the one-loop amplitude for N photons

$$\mathscr{R}[\tilde{M}^{(0)}] := S^{\alpha\beta}\frac{k_{2,\alpha}k_{1,\beta}}{k^2k_1^2k_2^2}\,,$$

To start tensor reduction:

$$\Pi = \begin{bmatrix} 0 & p_1 \cdot p_2 \\ p_1 \cdot p_2 & 0 \end{bmatrix}, \qquad \begin{bmatrix} \langle p_1 \rangle^\mu \\ \langle p_2 \rangle^\mu \end{bmatrix} := \begin{bmatrix} 0 & \frac{1}{p_1 \cdot p_2} \\ \frac{1}{p_1 \cdot p_2} & 0 \end{bmatrix} \begin{bmatrix} p_1^\mu \\ p_2^\mu \end{bmatrix},$$

Backup slides

TR of the IR ctm for the one-loop amplitude for N photons

$$\mathscr{R}[\tilde{M}^{(0)}] = S_{\alpha\beta}\mathscr{T}[u^{\alpha\mu}u^{\beta\nu}]\frac{k_{2,\mu}k_{1,\nu}}{k^2k_1^2k_2^2},$$

tree-level

tree-level

$$\mathscr{R}[\tilde{M}^{(0)}] = \Bigg[4\bar{v}_2[\ldots]u_1\frac{p_1^\mu p_2^\nu}{p_1\cdot p_2} + \Big(-4\bar{v}_2[\ldots]u_1$$

$$+\bar{v}_2\gamma^\sigma\gamma^\rho\mathbb{P}[\ldots]\mathbb{P}\gamma_\rho\gamma_\sigma u_1\Big)\frac{g_\perp^{\mu\nu}}{D_\perp}\Bigg]\frac{k_{2,\mu}k_{1,\nu}}{k^2k_1^2k_2^2}$$