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Outline

1 Motivation

2 Lattice setup

3 Data analysis

4 Model average and final results

5 Conclusions and outlook
Motivation

- Precision matters if lattice QCD is to have an impact on the proton radius puzzle
- In lattice QCD as in the context of scattering experiments: radii extracted from the slope of the electromagnetic form factors at $Q^2 = 0$,
  \[
  \langle r^2 \rangle = -\frac{6}{G(0)} \left. \frac{\partial G(Q^2)}{\partial Q^2} \right|_{Q^2=0}
  \]  
  (1)
- Assume SU(2) isospin symmetry (isospin-breaking corrections are small) $\Rightarrow$ quark-disconnected diagrams cancel in isovector combination, but not in the isoscalar one
- Full calculation of the proton and neutron form factors separately necessitates explicit treatment of the numerically challenging quark-disconnected contributions
- Neglected in many previous lattice studies, in particular no simultaneous control of all relevant systematics (continuum and infinite-volume extrapolation)
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QCD on the lattice

- Coupling of QCD is large at large distances / low energies
- Low-energy regime of QCD (typical hadronic scales) is hence inaccessible to perturbative methods
- Powerful tool for the non-perturbative study: lattice QCD
- Replace space-time by a four-dimensional Euclidean lattice
- Gauge-invariant UV-regulator for the quantum field theory due to the momentum cut-off
- Path integral becomes finite-dimensional and can be computed numerically
- Allows a systematic extrapolation to the continuum and infinite-volume limit, \( a \to 0 \) and \( V \to \infty \)
Ensembles

**Coordinated Lattice Simulations (CLS)** \(^1\)

- Non-perturbatively $O(a)$-improved Wilson fermions
- $N_f = 2 + 1$: 2 degenerate light quarks ($m_u = m_d$), 1 heavier strange quark ($m_s > m_{u,d}$)
- $\text{tr} \, M_q = 2m_l + m_s = \text{const.}$
- Tree-level improved Lüscher-Weisz gauge action
- $O(a)$-improved conserved vector current

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\(^1\)Bruno et al. 2015 [JHEP 2015 (2), 43]; Bruno, Korzec, and Schaefer 2017 [PRD 95, 074504].
Nucleon two- and three-point correlation functions

- Measure the two- and three-point correlation functions of the nucleon
- For three-point functions, Wick contractions yield connected and disconnected contribution
- Compute the quark loops via a stochastic estimation using a frequency-splitting technique\(^2\)
- Extract the effective form factors \(C_{E,M}^{\text{eff}}\) using the ratio method\(^3\)

\(^2\)Giusti et al. 2019 [EPJC 79, 586]; Cè et al. 2022 [JHEP 2022 (8), 220]; \(^3\)Korzec et al. 2009 [PoS 066, 139].
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Excited-state analysis

- Cannot construct exact interpolating operator for the proton (any hadron) on the lattice
- All possible states with the same quantum numbers contribute
- Effect of heavier excited states suppressed exponentially with the distance between operators in Euclidean time
- For baryons, the relative statistical noise grows also exponentially with the source-sink separation $t_{\text{sep}} = y_0 - x_0$
- Explicit treatment of the excited-state systematics required
- Apply summation method with varying starting values $t_{\text{sep}}^{\text{min}}$ for the linear fit
- Perform weighted average over $t_{\text{sep}}^{\text{min}}$ with weights given by a smooth window function$^4$
- Reduced human bias (same window on all ensembles), conservative error estimate

$^4$Djukanovic et al. 2022 [PRD 106, 074503]; Agadjanov et al. 2023 [PRL 131, 261902].
Direct Baryon $\chi$PT fits

- $\langle r^2 \rangle = -\frac{6}{G(0)} \left. \frac{\partial G(Q^2)}{\partial Q^2} \right|_{Q^2=0}$, $\mu_M = G_M(0) \Rightarrow$ parametrize $Q^2$-dependence of FFs

- Combine this with the chiral, continuum, and infinite-volume extrapolation

- Use expressions from covariant chiral perturbation theory\(^5\) to perform a simultaneous fit to the pion-mass, $Q^2$-, lattice-spacing, and finite-volume dependence of the form factors

- Include contributions from the $\rho$ ($\omega$ and $\phi$) mesons in the isovector (isoscalar) channel

- Reconstruct proton and neutron observables from separate fits to the isovector and isoscalar form factors

- Perform fits with various cuts in $M_\pi$ and $Q^2$, as well as with different models for the lattice-spacing and finite-volume dependence, in order to estimate systematic uncertainties

- Large number of degrees of freedom $\Rightarrow$ improved stability against lowering the $Q^2$-cut

\(^5\)Bauer, Bernauer, and Scherer 2012 [PRC 86, 065206].
Zemach radius from the lattice

- Atomic physics: hydrogen hyperfine splitting (HFS) influenced by proton’s EM structure
- Relevant parameter deduced from the HFS: Zemach radius $r^p_Z$,

$$ r^p_Z = -\frac{4}{\pi} \int_0^\infty \frac{dQ}{Q^2} \left( \frac{G^p_E(Q^2)G^p_M(Q^2)}{\mu^p_M} - 1 \right) = -\frac{2}{\pi} \int_0^\infty \frac{dQ^2}{(Q^2)^{3/2}} \left( \frac{G^p_E(Q^2)G^p_M(Q^2)}{\mu^p_M} - 1 \right) $$

(2)

- Firm theoretical prediction of the Zemach radius required both to guide the atomic spectroscopy experiments and for the interpretation of their results
- $B\chi PT$ including vector mesons only trustworthy for $Q^2 \lesssim 0.6\text{ GeV}^2$
- Tail of the integrand suppressed: contribution of the form factors above $0.6\text{ GeV}^2$ to $r_Z$ less than $0.9\%$
- Extrapolate $B\chi PT$ fit results using a $z$-expansion ansatz

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Model average

- Perform a weighted average over the results of all fit variations, using weights derived from the Akaike Information Criterion\textsuperscript{9},

\[ w_i = \exp \left( -\frac{1}{2} \text{BAIC}_i \right) / \sum_j \exp \left( -\frac{1}{2} \text{BAIC}_j \right), \quad \text{BAIC}_i = \chi^2_{\text{noaug, min, } i} + 2n_{f,i} + 2n_{c,i}, \]

where \( n_f \) is the number of fit parameters and \( n_c \) the number of cut data points

- Strongly prefers fits with low \( n_c \), \textit{i.e.}, the least stringent cut in \( Q^2 \) \( \Rightarrow \) apply a flat weight over the different \( Q^2 \)-cuts to ensure strong influence of our low-momentum data

- Determine the final cumulative distribution function (CDF) from the weighted sum of the bootstrap distributions\textsuperscript{10}

- Quote median of this CDF together with the central 68\% percentiles

Slope of the electric form factor closer to that of PRad\textsuperscript{11} than to that of A1\textsuperscript{12}

Good agreement with A1 for the magnetic form factor

\textsuperscript{11}Xiong et al. 2019 [Nature 575, 147]; \textsuperscript{12}Bernauer et al. 2014 [PRC 90, 015206].
(Mostly) compatible with the collected experimental world data\textsuperscript{13} within our errors

\textsuperscript{13}Ye et al. 2018 [PLB 777, 8].
Magnetic moments reproduced, low value for $\sqrt{\langle r_E^2 \rangle_p}$ clearly favored, $\sqrt{\langle r_M^2 \rangle_p}$ agrees with A1
Zemach radius

- Model-averaged result:
  \[ r_Z^p = 1.013(15) \text{ fm} \]
  \[ \Rightarrow \text{low value favored} \]

- Agrees within \(2\sigma\) with most of the experimental determinations

- Our estimate is \(\sim 80\%\) correlated with the electromagnetic radii (based on the same form factor data)

- Low result for \(r_Z^p\) expected, no independent puzzle
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Conclusions

- Determination of the electromagnetic form factors of the proton and neutron from lattice QCD including connected and disconnected contributions, as well as a full error budget
- Chiral, continuum, and infinite volume extrapolation by fitting our form factors to the expressions from covariant baryon chiral perturbation theory
- Magnetic moments of the proton and neutron agree well with the experimental values
- Small electric \( \text{and} \) magnetic radii of the proton favored
- Competitive errors, in particular for the magnetic radii
- First lattice calculation of the Zemach radius of the proton \( \rightarrow \) small value favored (80\% correlation with electromagnetic radii)
- Further investigations required, in particular for the magnetic proton radius
Backup slides
From correlation functions to form factors

- Average over the forward- and backward-propagating nucleon and over x-, y-, and z-polarization for the disconnected part
- Calculate the ratios

\[
R_{V_{\mu}}(q; t_{\text{sep}}, t) = \frac{C_{3, V_{\mu}}(q; t_{\text{sep}}, t)}{C_{2}(0; t_{\text{sep}})} \sqrt{\frac{\bar{C}_{2}(q; t_{\text{sep}} - t)C_{2}(0; t)C_{2}(0; t_{\text{sep}})}{C_{2}(0; t_{\text{sep}} - t)\bar{C}_{2}(q; t)\bar{C}_{2}(q; t_{\text{sep}})}},
\]

(4)

where \( t_{\text{sep}} = y_{0} - x_{0}, \ t = z_{0} - x_{0}, \) and \( \bar{C}_{2}(q; t_{\text{sep}}) = \sum_{\tilde{q} \in q} C_{2}(\tilde{q}; t_{\text{sep}})/\sum_{\tilde{q} \in q} 1 \)

- At zero sink momentum, the effective form factors can be computed from the ratios as

\[
G_{E}^{\text{eff}}(Q^{2}; t_{\text{sep}}, t) = \sqrt{\frac{2E_{q}}{m + E_{q}}} R_{V_{0}}(q; t_{\text{sep}}, t),
\]

(5)

\[
G_{M}^{\text{eff}}(Q^{2}; t_{\text{sep}}, t) = \sqrt{2E_{q}(m + E_{q})} \sum_{j, k} \epsilon_{ijk} q_{k} \text{Re} R_{V_{j}}^{\Gamma_{i}}(q; t_{\text{sep}}, t) \frac{\sum_{j \neq i} q_{j}^{2}}{\sum_{j \neq i} q_{j}^{2}},
\]

(6)
Excited-state analysis

- Sum the effective form factors over the operator insertion time,

\[ S_{E,M}(Q^2; t_{sep}) = \sum_{t=t_{skip}}^{t_{sep}-t_{skip}} G_{E,M}^{\text{eff}}(Q^2; t, t_{sep}), \quad t_{skip} = 2a \] (7)

- For \( t_{sep} \to \infty \), the slope as a function of \( t_{sep} \) is given by the ground-state form factor,

\[ S_{E,M}(Q^2; t_{sep}) \xrightarrow{t_{sep} \to \infty} C_{E,M}(Q^2) + \frac{1}{a}(t_{sep} + a - 2t_{skip})G_{E,M}(Q^2) \] (8)

- Perform a weighted average over \( t_{sep}^{\text{min}} \), where the weights are given by a smooth window function,

\[ \hat{G} = \frac{\sum_i w_i G_i}{\sum_i w_i}, \quad w_i = \tanh \left( \frac{t_i - t_{w}^{\text{low}}}{\Delta t_w} \right) - \tanh \left( \frac{t_i - t_{w}^{\text{up}}}{\Delta t_w} \right), \] (9)

where \( t_i \) is the value of \( t_{sep}^{\text{min}} \) in the \( i \)-th fit, \( t_{w}^{\text{low}} = 0.9 \) fm, \( t_{w}^{\text{up}} = 1.1 \) fm and \( \Delta t_w = 0.08 \) fm
Excited-state analysis: window average on E300

E300 ($M_\pi = 176$ MeV, $a = 0.049$ fm)
Excited-state analysis: window average on D450

D450 ($M_\pi = 218$ MeV, $a = 0.076$ fm)

Graph showing $G_{E}^{d}(Q^{2} = 0.065$ GeV$^{2}$) and $G_{M}^{d}(Q^{2} = 0.065$ GeV$^{2}$) with window average.
$Q^2$-dependence of the isovector form factors on E300

- Direct $B\chi$PT fit describes data very well
- Reduced error due to the inclusion of several ensembles in one fit
$Q^2$-dependence of the isoscalar form factors on E300

Covariant BχPT fit on E300 ($M_r = 176$ MeV)

\[ G_E^{u+d}(Q^2) = \frac{G_E^{u+d}(Q^2)}{G_E^{u+d}(0)} \]

- BχPT fit
- original summation data
- corrected summation data

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$Q^2$-dependence of the isovector form factors on E250

Covariant BχPT fit on E250 ($M_r = 130$ MeV)

BχPT fit

original summation data

corrected summation data

Covariant BχPT fit on E250 ($M_r = 130$ MeV)

BχPT fit

original summation data

corrected summation data
$Q^2$-dependence of the isoscalar form factors on E250

Covariant BχPT fit on E250 ($M_r = 130$ MeV)

- BχPT fit
- original summation data
- corrected summation data

**Graphs:**

- $G_E^{u+d}(Q^2)/G_E^{u+d}(0)$
- $G_{M}^{u+d}(Q^2)/G_{M}^{u+d}(0)$

**Axes:**

- $t_0Q^2$
- $0.00$ to $0.30$

- $G_E^{u+d}(Q^2)$
- $0.3$ to $1.0$

- $G_{M}^{u+d}(Q^2)$
- $-0.2$ to $1.4$
Residuals of the B\(\chi\)PT fits

B\(\chi\)PT fit to the isovector form factors

B\(\chi\)PT fit to the isoscalar form factors

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Histograms

BχPT fit to the isovector form factors

BχPT fit to the isoscalar form factors

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Proton EM FFs and radii from lattice QCD

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Q-Q plots

BχPT fit to the isovector form factors

Empirical value vs Expected value

KS-test p-value: 73.17%

BχPT fit to the isoscalar form factors

Empirical value vs Expected value

KS-test p-value: 80.45%
Disambiguating the statistical and systematic uncertainties

- Scale the statistical variances of the individual fit results by a factor of $\lambda = 2$
- Repeat the model averaging procedure
- Assumptions:
  - Above rescaling only affects the statistical error of the averaged result
  - Statistical and systematic errors add in quadrature
- Contributions of the statistical and systematic errors to the total error,
  \[
  \sigma_{\text{stat}}^2 = \frac{\sigma_{\text{scaled}}^2 - \sigma_{\text{orig}}^2}{\lambda - 1}, \quad \sigma_{\text{syst}}^2 = \frac{\lambda \sigma_{\text{orig}}^2 - \sigma_{\text{scaled}}^2}{\lambda - 1}
  \] (10)
- Consistency check: results are almost independent of $\lambda$ (if it is chosen not too small)
CDFs of the electromagnetic radii and magnetic moment of the proton

\[ \langle r^2 \rangle_p \text{ [fm}^2 \text{]} \]

\[ \langle r^2 \rangle_M \text{ [fm}^2 \text{]} \]

\[ \mu_p \text{ [MeV} \text{fm}] \]

\[ P(y) \]

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\( \langle r_E^2 \rangle^{u-d} = (0.785 \pm 0.022 \pm 0.026) \text{ fm}^2 \)
\( \langle r_M^2 \rangle^{u-d} = (0.663 \pm 0.011 \pm 0.008) \text{ fm}^2 \)
\( \mu_M^{u-d} = 4.62 \pm 0.10 \pm 0.07 \)

\( \langle r_E^2 \rangle^{u+d-2s} = (0.554 \pm 0.018 \pm 0.013) \text{ fm}^2 \)
\( \langle r_M^2 \rangle^{u+d-2s} = (0.657 \pm 0.030 \pm 0.031) \text{ fm}^2 \)
\( \mu_M^{u+d-2s} = 2.47 \pm 0.11 \pm 0.10 \)

\( \langle r_E^2 \rangle^p = (0.672 \pm 0.014 \pm 0.018) \text{ fm}^2 \)
\( \langle r_M^2 \rangle^p = (0.658 \pm 0.012 \pm 0.008) \text{ fm}^2 \)
\( \mu_M^p = 2.739 \pm 0.063 \pm 0.018 \)

\( \langle r_E^2 \rangle^n = (-0.115 \pm 0.013 \pm 0.007) \text{ fm}^2 \)
\( \langle r_M^2 \rangle^n = (0.667 \pm 0.011 \pm 0.016) \text{ fm}^2 \)
\( \mu_M^n = -1.893 \pm 0.039 \pm 0.058 \)
$z$-expansion

- $z$-expansion: model-independent description of the $Q^2$-dependence of the form factors
- Map domain of analyticity of the form factors onto the unit circle,

$$
z(Q^2) = \frac{\sqrt{\tau_{\text{cut}} + Q^2} - \sqrt{\tau_{\text{cut}} - \tau_0}}{\sqrt{\tau_{\text{cut}} + Q^2} + \sqrt{\tau_{\text{cut}} - \tau_0}},
$$

where $\tau_{\text{cut}} = 4M^2_{\pi,\text{phys}}$, and we employ $\tau_0 = 0$

- Expand the form factors as

$$G_E(Q^2) = \sum_{k=0}^{n} a_k z(Q^2)^k, \quad G_M(Q^2) = \sum_{k=0}^{n} b_k z(Q^2)^k$$

- We fix $G_E(0) = a_0 = 1$, use $n = 7$, and incorporate the 4 sum rules for each form factor
- \( z \)-expansion agrees very well with \( B\chi PT \) parametrization in the region where it is fitted
- For the integration, smoothly replace the \( B\chi PT \) parametrization by the \( z \)-expansion
Crosscheck of direct fits with $z$-expansion: proton magnetic moment

- Use $n = 2$ and no sum rules (focus on low-momentum region)
- Magnetic moment significantly smaller than direct fits which are compatible with experiment
- Direct fits use more data in one fit $\Rightarrow$ increased stability against statistical fluctuations

![Graph showing comparison between direct fits and $z$-expansion results]

$z$-expansion, $Q_{\text{cut}}^2 = 0.6 \text{ GeV}^2$
Crosscheck of direct fits with $z$-expansion: proton electromagnetic radii

Radii in good agreement with direct fits, albeit with significantly larger errors