LOCALIZATION AT LARGE N AND ADS/CFT

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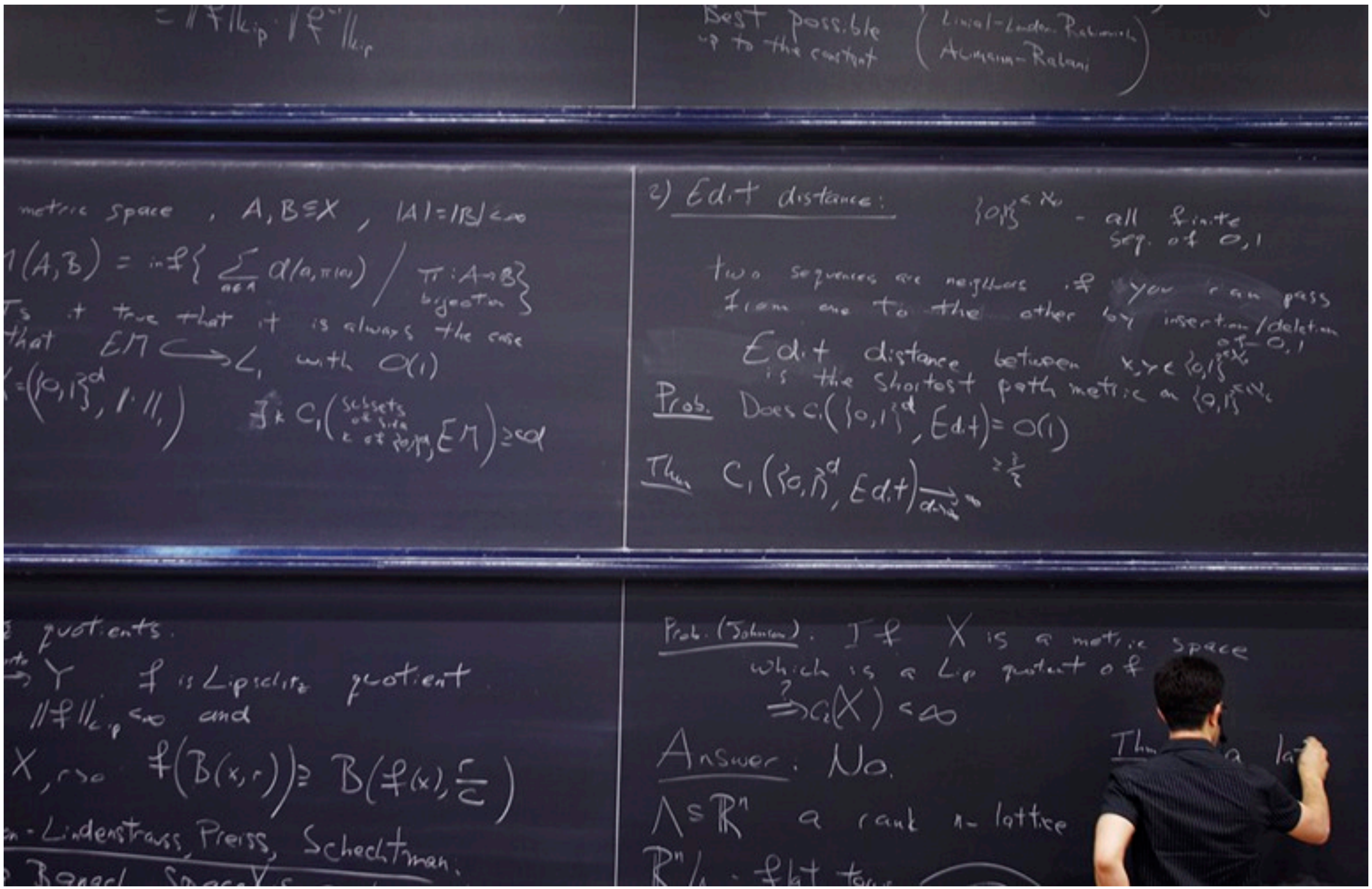
Lecture notes: [M.M. 1104.0783]

REST IN PEACE



Characteristic feature of non-abelian gauge theories:

the physics at *strong coupling* is qualitatively different from the physics at *weak coupling*

The weak coupling regime is usually well understood thanks to perturbation theory, but in general we have much less control over the strong coupling regime

In gauge theories with an AdS dual, the strong coupling physics is encoded in a weakly coupled string or gravity theory, and this leads to *predictions* for the strong coupling behavior

Conversely, if we are able to derive the AdS prediction for the strong coupling behavior *directly in the gauge theory*, we have a *test* of the corresponding AdS/CFT duality

More ambitiously, we would like to have *exact results* which *interpolate* between the weak coupling regime, where perturbation theory holds, and the strong coupling regime described by the dual string

In these lectures I will describe a new set of techniques to address the problem of strong-weak coupling interpolation in a large class of 3d CFTs with AdS duals

One problem that motivated the development of these techniques is the theory of *N* M2 branes. This theory is supposed to have a *U(N)* gauge symmetry, so at weak coupling the number of degrees of freedom grows like N^2

However, the AdS dual predicts that at strong coupling this number should grows like $N^{3/2}$

One concrete goal of these lectures will be to explain this puzzling scaling at strong coupling from first principles

Perturbation theory

In these lectures we will look at non-abelian gauge theories with gauge group *U(N)* whose action has the schematic structure

$$S = \frac{1}{g_s} \int \mathrm{tr} \left(\frac{1}{2} A \mathcal{D} A + A^3 + \cdots \right)$$

gauge coupling differential operator interactions

Example: Yang-Mills theory

$$S = \frac{1}{4 g_{\mathrm{YM}}^2} \int \mathrm{tr} \left(F_{\mu\nu} F^{\mu\nu} \right)$$

second order
differential operator

Our favourite example will be however a non-abelian gauge theory in three dimensions, *Chern-Simons theory*, and their supersymmetric extensions

$$S_{\mathrm{CS}} = -\underbrace{\frac{k}{4\pi}}_{\text{CS level}} \int_M \mathrm{tr}\Big(A \wedge \underbrace{\mathrm{d}}_{\text{first order differential operator}} A + \frac{2\mathrm{i}}{3} A \wedge A \wedge A\Big)$$

CS level

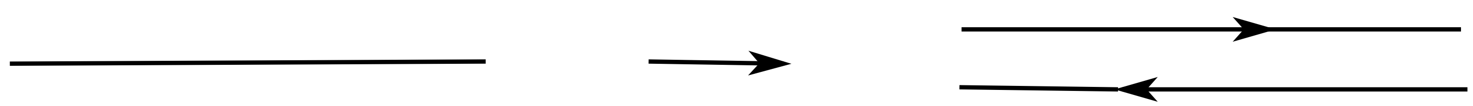
first order differential operator

We will denote $$g_s = \frac{2\pi\mathrm{i}}{k}$$

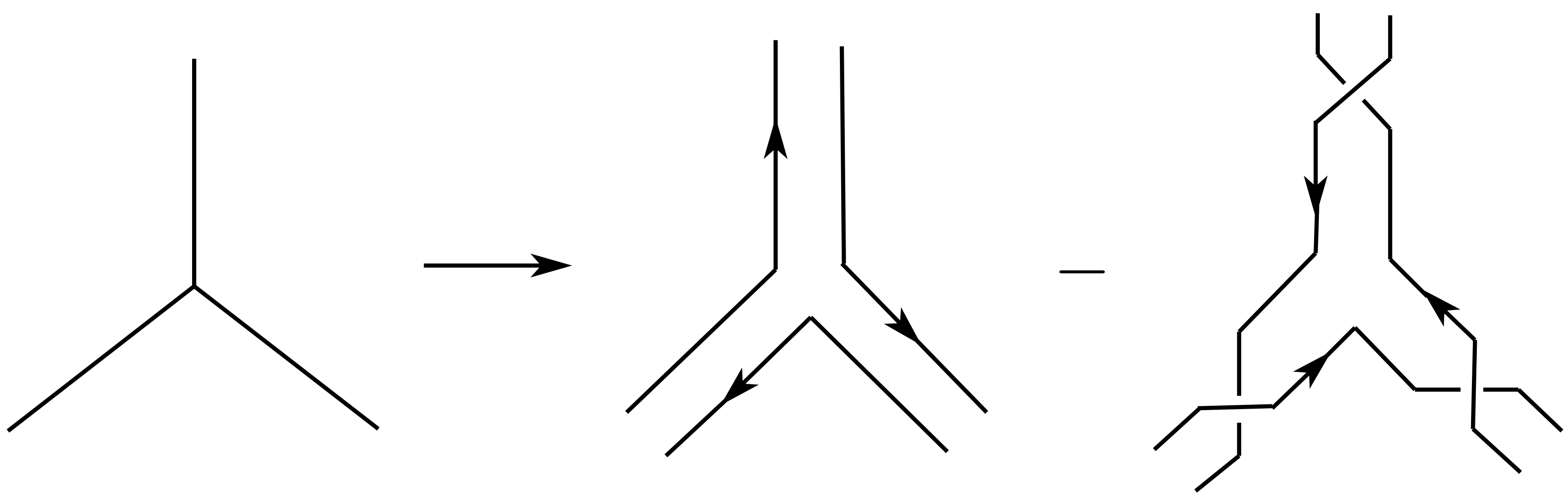
1) *k* has to be an integer so that $\mathrm{e}^{\mathrm{i}S_{\mathrm{CS}}}$ is invariant under large gauge transformations

2) the theory based on this action is exactly solvable and topological [Witten 1988],

Both Yang-Mills theory and Chern-Simons theory can be treated in perturbation theory. One way to go beyond standard perturbation theory is to use the $1/N$ expansion of 't Hooft



gluon propagator



gluon vertex

In these lectures we will look at free energies of *U(N)* theories with fields in the adjoint/bifundamental, on different manifolds, in particular on the three-sphere

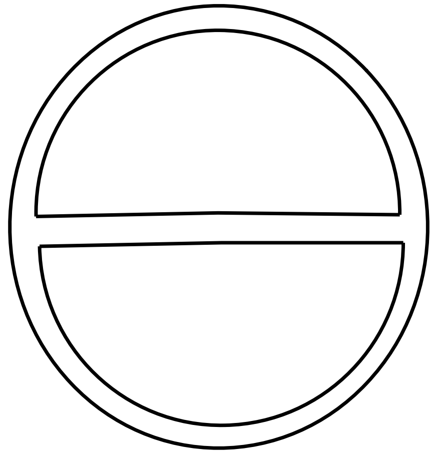
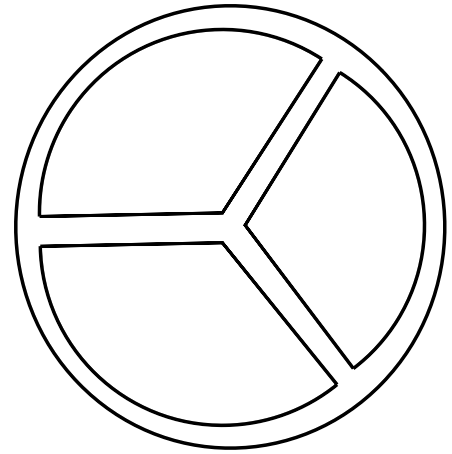
$$Z(\mathbb{S}^3) = \int \mathcal{D}A \cdots \mathrm{e}^{-S(A,\cdots)}$$

$$F(\mathbb{S}^3) = \log Z(\mathbb{S}^3) = \sum_{g=0}^{\infty} N^{2-2g} F_g(t)$$

$t = g_s N$ ‘t Hooft parameter

genus *g* free energy $F_g(t) =$ sum of double-line, connected vacuum diagrams of genus *g*

genus *g=0*: *planar* free energy

$$F_0(t) =$$  $+$  $+ \cdots$

t^3 $\qquad$ t^4

In general $\qquad F_g(t) = \sum_{n \geq 0} a_{g,n} t^n$

This is an expansion at *weak 't Hooft coupling*, and it is essentially equivalent to perturbation theory (but restricted to genus g diagrams)

Warning: this expansion does not include one-loop terms and terms coming from the path-integral measure, which lead sometimes to singular terms at $t = 0$

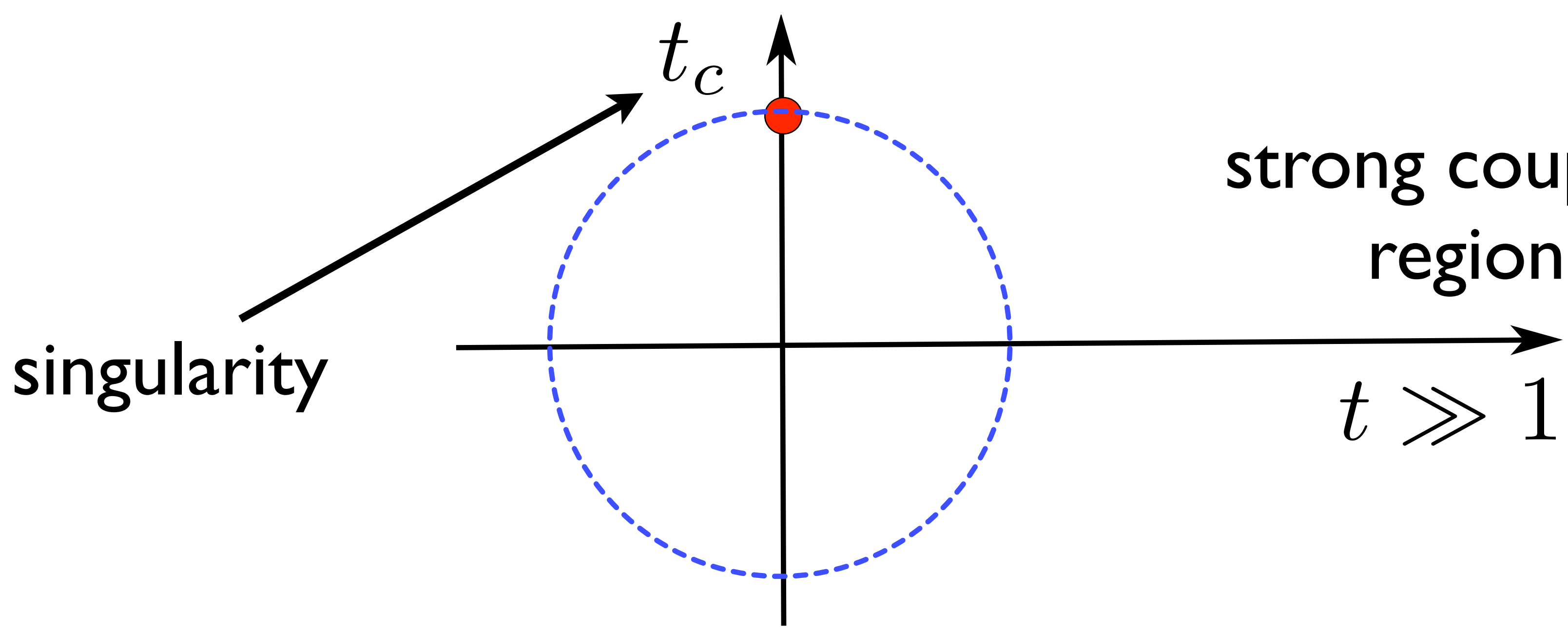
In many interesting theories (CS, ABJM, ...) the weak-coupling expansion is *analytic* at *t=0* with a *finite radius of convergence*

$$a_{g,n} \sim |t_c|^{-n}, \qquad n \gg 1 \qquad \text{common to all g}$$

This result can be interpreted in a purely diagrammatic way: it is just a statement on the growth of the number of double-line diagrams with the number of holes (or vertices) at fixed genus.

This behavior is in sharp contrast with standard perturbation theory, where the coefficients grow factorially, rather exponentially, and there is zero radius of convergence

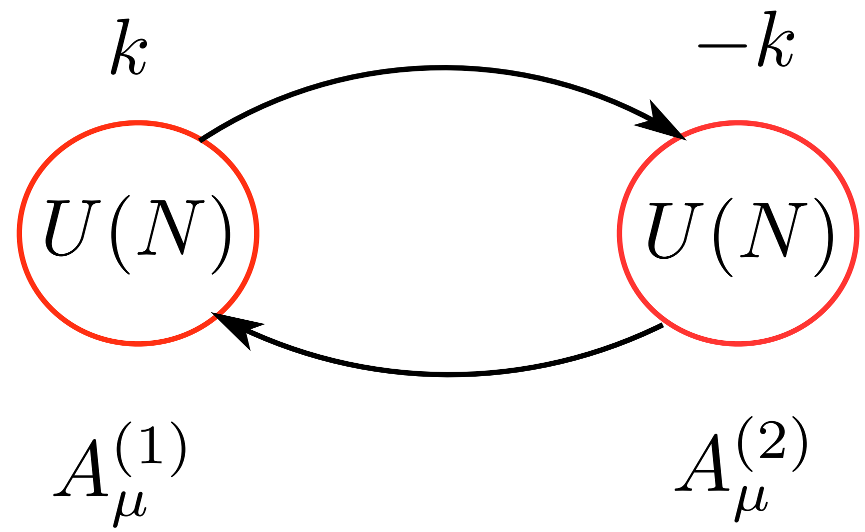
If there are no singularities on the real axis, we can analytically continue the genus g free energies outside the region of convergence, to the strong coupling region. This is the mathematical counterpart of the weak-strong coupling interpolation



Warning: this analyticity property is *not* expected to be valid for generic gauge theories, like YM, due to the presence of renormalons

A B J M theory

Basic building block: *two* Chern-Simons theories with gauge symmetry *U(N)* and opposite levels (or coupling constants)



In order to proceed, we have to supersymmetrize the model and couple both “nodes”

N=2 SUSY in 3d

We will work in Euclidean signature, and on the 3-sphere, so the SUSY transformations we will describe are slightly different from the usual ones

Conventions: $\gamma_\mu =$ Pauli matrices

$$C_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$ symplectic product $$\bar{\epsilon}\lambda = \bar{\epsilon}^\alpha C_{\alpha\beta}\lambda^\beta$$

N=2 vector multiplet (dim. red. of the *N*=*1* vector multiplet in *d*=*4*)

$$V : \quad A_\mu,\ \sigma,\ \lambda,\ \bar{\lambda},\ D$$

scalars

SUSY transformations

$$\delta = \delta_\epsilon + \delta_{\bar\epsilon}$$

$$\delta A_\mu = \frac{\mathrm{i}}{2}(\bar\epsilon\gamma_\mu\lambda - \bar\lambda\gamma_\mu\epsilon),$$
$$\delta\sigma = \frac{1}{2}(\bar\epsilon\lambda - \bar\lambda\epsilon),$$
$$\delta\lambda = -\frac{1}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu} - D\epsilon + \mathrm{i}\gamma^\mu\epsilon D_\mu\sigma + \frac{2\mathrm{i}}{3}\sigma\gamma^\mu D_\mu\epsilon$$

etc.

SUSY parameters are space-time dependent. They are required to be *Killing spinors*

$$D_\mu\epsilon = \frac{\mathrm{i}}{2r}\gamma_\mu\epsilon, \quad D_\mu\bar\epsilon = \frac{\mathrm{i}}{2r}\gamma_\mu\bar\epsilon$$

$r =$ radius of three-sphere

We can now construct a *supersymmetric extension* of CS theory

$$S_{\rm SCS} \quad = - \int_{\mathbb{S}^3} \operatorname{tr} \left(A \wedge A + \tfrac{2\mathrm{i}}{3} A^3 - \bar{\lambda}\lambda + 2D\sigma \right)$$

Exercise: check that $\delta S_{\rm SCS} = 0$

Of course, there is another supersymmetric action for the 3d vector multiplet, namely *super-Yang-Mills theory*

$$S_{\rm SYM} = \int_{\mathbb{S}^3} \operatorname{tr} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \sigma D^\mu \sigma + \frac{1}{2} \left(D + \frac{\sigma}{r} \right)^2 + \text{fermions} \right]$$

full covariant derivative
(gauge+spin connections)

Finally, we need matter supermultiplets:

N=2 chiral multiplet (dim. red. of the *N*=2 chiral multiplet in *d*=4)

$$\Phi : \quad \phi, \ \bar\phi, \ \psi, \ \bar\psi, \ F, \ \bar F$$

auxiliary

$$\begin{aligned}
\delta\phi &= \bar\epsilon\psi, \\
\delta\bar\phi &= \epsilon\bar\psi, \\
\delta\psi &= \mathrm{i}\gamma^\mu\epsilon D_\mu\phi + \epsilon\sigma\phi + \frac{2\Delta\mathrm{i}}{3}\gamma^\mu D_\mu\epsilon\phi + \bar\epsilon F
\end{aligned}$$

etc.

$\Delta =$ (anomalous) dimension of Φ

$= 1/2$ if $\mathcal{N} \geq 3$

Action of chiral multiplet coupled to *N*=2 VM for $\Delta = 1/2$

$$S = \int_{\mathbb{S}^3} \mathrm{d}^3x \sqrt{g} \left(D_\mu \bar{\phi} D^\mu \phi + \frac{3}{4r^2} \bar{\phi}\phi - \mathrm{i}\bar{\psi}\gamma^\mu D_\mu \psi + \bar{F}F + \text{couplings to VM} \right)$$

This is the conformal coupling to gravity, required for conformal invariance of the scalar field on a curved space. In general we have

$$\int \mathrm{d}^n x \sqrt{g} \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{4} \frac{n-2}{n-1} R \phi^2 \right)$$

R: scalar curvature

For non-canonical dimensions, the coupling to curvature is fixed by supersymmetry [Jafferis]

$$\frac{\Delta(2-\Delta)}{r^2} \bar{\phi}\phi$$

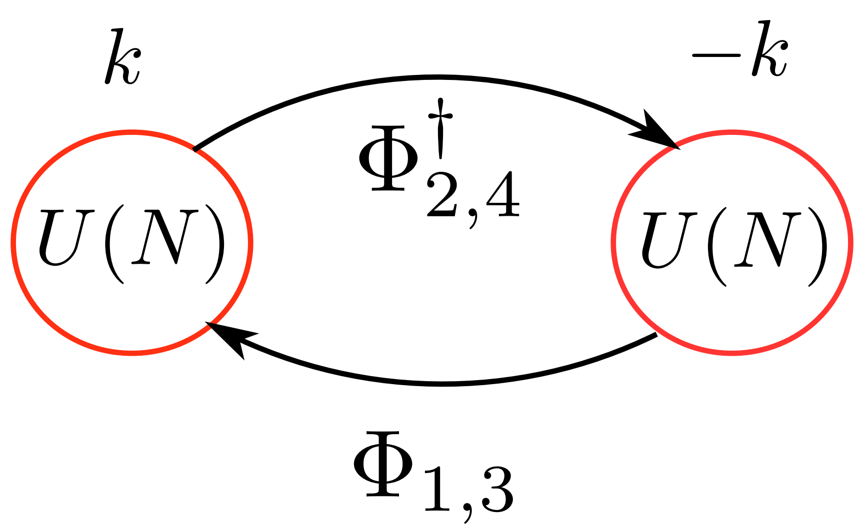
ABJM theory is then obtained as follows:

1) two *U(N)* VMs with super Chern-Simons actions and opposite couplings $V^{(1)}, \quad V^{(2)}$

2) four chiral multiplets in the bifundamental rep of *U(N)xU(N)* $\Phi_i^{a\bar{b}}, \quad i = 1, \cdots, 4$

3) a superpotential for the chiral fields

$$W = \frac{4\pi}{k} \mathrm{tr} \left(\Phi_1 \Phi_2^\dagger \Phi_3 \Phi_4^\dagger - \Phi_1 \Phi_4^\dagger \Phi_3 \Phi_2^\dagger \right),$$



Some properties of ABJM theory

1) it is a conformal field theory [Gaiotto-Yin] with *N*=6 supersymmetry (i.e. 24 supercharges) [ABJM]

notice that *k* is quantized, so it cannot run under a beta function if the theory is consistent

2) it describes the gauge theory for N M2 branes on $\mathbb{C}^4/\mathbb{Z}_k$

[building on Bagger-Lambert, Gustavsson]

3) it has a large N string/M-theory dual [Maldacena 1998]

Large N duals I: M-theory

Background: $\mathrm{AdS}_4 \times X_7$ $\qquad X_7 = \mathbb{S}^7/\mathbb{Z}_k$

Metric: $$\mathrm{d}s^2_{11} = L^2_{X_7}\left(\tfrac{1}{4}\mathrm{d}s^2_{\mathrm{AdS}_4} + \mathrm{d}s^2_{X_7}\right),$$

(common) radius

In 11d sugra there are two bosonic fields: the metric and the *C* field (which is a 3-form). We also have a flux for *C*

$$F_4 = \mathrm{d}C = \frac{3}{8} L^3_{X_7}\omega_{\mathrm{AdS}_4}$$

volume form

Exercise: show that this *Freund-Rubin background* solves the EOM of 11d sugra

Remember that *C* couples to the M2 brane, therefore Gauss law gives

charge of *N* M2 branes $\longrightarrow$ $$Q = \frac{1}{(2\pi\ell_p)^6}\int_{X_7} \star_{11} F_4$$

This leads to the following relation between the radius and the number of M2 branes

$$\frac{1}{k}\left(\frac{L}{\ell_p}\right)^6 = 32\pi^2 N$$

since at large *N* $\quad Q \simeq N$

Exercise: derive an expression for the 4d Newton's constant as a function of *N*, *k*

We then have a dictionary between M-theory and ABJM theory: the coupling k is purely geometric, and the "size of the universe" grows with N

$$L/\ell_p$$

$$L/\ell_p \simeq 1$$

Planckian sizes, strong quantum gravity effects

N small, *k* fixed

$$L/\ell_p \gg 1$$

weak curvature, classical SUGRA is a good approximation

N large, *k* fixed

"thermodynamic limit"

The natural expansion in M-theory is in powers of ℓ_p/L

This leads to the *M-theory expansion* of ABJM theory: a $1/N$ expansion at fixed k. This is *not* the 't Hooft expansion

Large N duals II: Type IIA theory

To make contact with IIA theory, we have to reduce M-theory on a circle, and for this we use the Hopf fibration of the seven-sphere [Nilsson-Pope 1984]

$$\begin{array}{ccc} \mathbb{S}^1 & \longrightarrow & \mathbb{S}^7 \\ & & \downarrow \\ & & \mathbb{CP}^3 \end{array} \quad \Rightarrow \text{target} \quad \mathrm{AdS}_4 \times \mathbb{CP}^3$$

Type IIA theory has *two* parameters: the string coupling constant, and the ratio of *L* to the string length. After Hopf reduction one finds,

$$\lambda = \frac{N}{k}$$

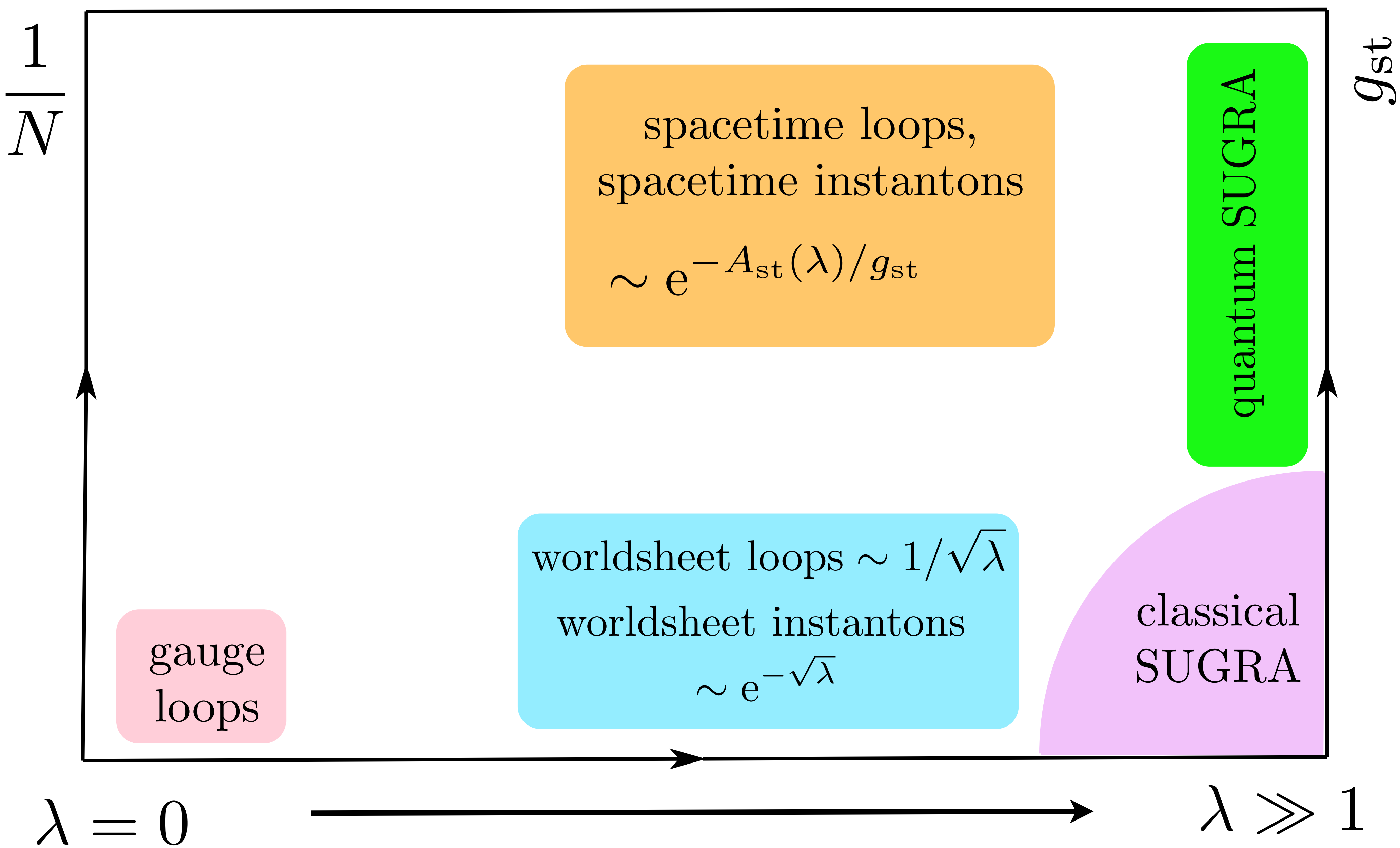
't Hooft parameter

$$\left(\frac{L}{\ell_s}\right)^4 = 32\pi^2\lambda$$

$$g_{\mathrm{st}} = \frac{1}{k}\left(\frac{L}{\ell_s}\right) \propto \frac{\lambda^{5/4}}{N}$$

weight of quantum effects in the worldsheet theory

$$S_{\mathrm{ws}} \sim \frac{L^2}{\ell_s^2} \Rightarrow \lambda \sim \frac{L^4}{\ell_s^4} \sim \frac{1}{\hbar_{\mathrm{ws}}^2}$$



weak-strong interpolation

The natural spacetime expansion in type IIA theory is the *genus expansion* (in powers of the string coupling constant) at a given curvature radius.

In the gauge theory this corresponds precisely to the 't Hooft expansion.

We conclude that there are *two possible expansions* in the gauge theory, making contact with M-theory and type IIA theory, respectively. These expansions are *a priori* different.

This is a new feature of ABJM theory which is absent from *N=4* SYM, where there is no M-theory picture

Free energies and degrees of freedom

Our goal now is to test this large *N* duality, and in particular to understand the "number of degrees of freedom" in this theory. For this, we need an observable that probes this number.

Obvious guess: look at the *thermal free energy*. To calculate this, we should consider the path integral for field configurations living in the manifold

$$M = \mathbb{R}^2 \times \mathbb{S}^1_\beta \quad \longleftarrow \text{thermal circle}$$

Problem: this setting *breaks supersymmetry*, since bosons (fermions) have (anti)periodic boundary conditions. A strong coupling calculation in the gauge theory is out of reach with the current techniques

However, the free energy on *any* manifold is sensitive to the number of degrees of freedom, since at weak coupling

$$Z(M;N) \sim (Z(M;1))^{N^2} \Rightarrow F(M) \sim \mathcal{O}(N^2)$$

We can then look at the free energy on the three-sphere. As we will show in a moment, the AdS dual predicts as well that, at strong coupling,

$$F_{\mathrm{ABJM}}(\mathbb{S}^3) \sim \mathcal{O}(N^{3/2})$$

so this is a good quantity as well to understand the change in scaling. We will then focus on this quantity, see what is the precise AdS prediction for its strong coupling limit, and develop techniques for a first principle derivation from the ABJM gauge theory

Moreover, there are very good reasons to think that this is in fact *THE* right quantity to measure the number of degrees of freedom in 3d QFTs:

F-"theorem" [Jafferis, Klebanov, Pufu, Sachdev, Safdi 2011]: the free energy on the three-sphere (with a minus sign) decreases along RG flows and it is stationary at fixed points

Z-extremization [Jafferis]: the anomalous dimensions of the matter fields, as a function of the parameters and couplings, can be obtained by extremizing the partition function (in absolute value)

AdS/CFT correspondence and strong coupling

We will need in a sense the most elementary consequence of the (Euclidean) AdS/CFT correspondence, namely equality of the partition functions [Witten1998]

$$Z_{\mathrm{ABJM}}(M; N, k) = Z_{\mathrm{M/string}}\left(\mathrm{AdS}_4 \times X_{7,6}\right)$$

Here, *M* is a three-manifold, and we choose a realization of AdS such that

$$\partial(\mathrm{AdS}_4) = M$$

Of course, a similar equality holds for other 3d CFTs and for *N=4* SYM, with the obvious changes.

In particular, the AdS/CFT correspondence predicts that the genus *g* free energy of the gauge theory, in the 't Hooft expansion, is equal to the genus expansion of the type IIA string theory:

$$F_{\mathrm{ABJM}}(\mathbb{S}^3) = F_{\mathrm{string}}(\mathrm{AdS}_4 \times \mathbb{CP}^3)$$

$$F_g^{\mathrm{string}}(\lambda) = \lambda^{3(g-1)/2} F_g^{\mathrm{ABJM}}(\lambda)$$

Let us look at the M-theory picture for concreteness, and assume that *N* is very large. In that case, gravity is weakly coupled, and the semiclassical approximation (SUGRA) should give the leading behavior:

$$Z_{\mathrm{M}}\left(\mathrm{AdS}_4 \times X_7\right) \approx \mathrm{e}^{-I(\mathrm{AdS}_4)}$$

classical gravity action evaluated on AdS, after reduction to 4d

$$I(\mathrm{AdS}_4) = -\frac{1}{16\pi G_N}\int \mathrm{d}^4x\,\sqrt{G}\,(R-2\Lambda) \qquad\qquad \Lambda = -\frac{6}{L^2}$$

This should be evaluated on-shell, on the AdS metric

$$\mathrm{d}s^2 = L^2\left(\mathrm{d}\rho^2 + \sinh^2(\rho)\mathrm{d}\Omega_3^2\right)$$

radial direction

metric on a unit radius $\mathbb{S}^3$

Problem: the result is divergent! This is easy to see if we use Einstein's equation

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \quad \Rightarrow I = \frac{3}{8\pi G_N L^2} \mathrm{vol}\left(\mathrm{AdS}_4\right)$$

To understand this divergence, we can introduce a cutoff in the radial direction

$$\begin{aligned}
\mathrm{vol}\left(\mathrm{AdS}_4;\rho_0\right) &= L^4 \mathrm{vol}(\mathbb{S}^3) \int_0^{\rho_0} \mathrm{d}\rho \,\left(\sinh\rho\right)^3 \\
&= 2\pi^2 L^4 \left[\frac{1}{12}\cosh(3\rho_0) - \frac{3}{4}\cosh(\rho_0) + \frac{2}{3}\right]
\end{aligned}$$

finite piece!

We can also use dimensional regularization to recover the same finite piece

Exercise: compute the volume in *d* dimensions and show that the analytic continuation to *d=3* produces the *same* finite piece [solution: Diaz-Dorn 0702163]

Keeping just this finite piece, and using the result for 4d Newton's constant, we obtain

$$\boxed{I = \frac{\pi L^2}{2G_N} = \frac{\pi\sqrt{2}}{3} k^{1/2} N^{3/2}}$$

The divergences appearing in this calculation are well understood: they are simply the IR duals of the UV divergences in the CFT. They can be regulated directly in the gravity theory with a technique called *holographic renormalization* [Henningson-Skenderis, Balasubramanian-Kraus]

The detailed procedure goes as follows: the full gravity action contains as well a boundary (or Gibbons-Hawking) term,

$$I = -\frac{1}{16\pi G_N}\int_S \mathrm{d}^{n+1}x\,\sqrt{G}\,(R-2\Lambda) - \frac{1}{8\pi G_N}\int_{\partial S} K|\gamma|^{1/2}\mathrm{d}^n x,$$

γ : induced metric on the boundary

K : extrinsic curvature of the boundary

Both terms diverge, but one can find a *universal* set of *boundary counterterms*, which only depend on the induced metric, and lead to a finite result:

$$I_{\mathrm{ct}} = \frac{1}{8\pi G_N}\int \mathrm{d}^n x\,\sqrt{\gamma}\left(\frac{n-1}{L} + \frac{L}{2(n-2)}R[\gamma] + \cdots\right)$$

Exercise: compute the bulk, boundary and counterterm action with a cutoff and show that all divergences cancel when the cutoff goes to infinity, leaving the finite result quoted above

[solution: lecture notes, Emparan-Johson-Myers]

We conclude that

$$F_{\mathrm{ABJM}}(\mathbb{S}^3) \approx -\frac{\pi\sqrt{2}}{3} k^{1/2} N^{3/2}$$

at large *N* and fixed *k*. Equivalently, the planar free energy should be given, at strong coupling, by

$$F_0^{\mathrm{ABJM}}(\lambda) \approx -\frac{\pi\sqrt{2}}{3} \frac{1}{\sqrt{\lambda}} \qquad \lambda \gg 1$$

ABJM theory at weak coupling

We will now sketch the computation of the one-loop planar free energy of ABJM theory

Let us consider a free matter multiplet on the three-sphere. It has one conformally coupled complex scalar and two Weyl fermions. Its one-loop partition function is given by

$$Z^{\mathrm{matter}}_{1-\mathrm{loop}}(\mathbb{S}^3) = \frac{\det\left(-\mathrm{i}\not{D}\right)}{\det \Delta_c}$$

$$\Delta_c = -\nabla_\mu \nabla^\mu + \frac{3}{4}$$ conformal Laplacian

This is also infinite, but we can regularize it with zeta-function techniques

Let *T* be a self-adjoint operator with positive eigenvalues λ_n. Its zeta function is defined as

$$\zeta_T(s) = \sum_n \lambda_n^{-s}$$

This converges for sufficiently large *s*, and defines a meromorphic function which is regular at *s=0*. A natural definition of the determinant of *T* is then

$$\det(T) = \mathrm{e}^{-\zeta_T'(0)}$$

The spectrum of differential operators on spheres is explicitly known, and their zeta functions can be computed in detail. With some work one finds,

$$Z_{1-\mathrm{loop}}^{\mathrm{matter}}(\mathbb{S}^3) = \frac{1}{\sqrt{2}}$$

Exercise (long): calculate the spectrum and determinant of the conformal Laplacian and of the Dirac operator on the three-sphere [solution: lecture notes]

The other ingredient in ABJM theory is CS theory. The one-loop calculation of its partition function is much more subtle, but one can use as a shortcut the exact answer obtained by Witten in 1988

$k \gg 1$

$$Z_{\rm CS}(\mathbb{S}^3) = \frac{1}{k^{N/2}} \prod_{j=1}^{N-1} \left(2 \sin \frac{\pi j}{k} \right)^{N-j} \approx k^{-N/2} \prod_{j=1}^{N-1} \left(\frac{2\pi j}{k} \right)^{N-j}$$

$$= (2\pi)^{\frac{1}{2}N(N-1)} k^{-N^2/2} G_2(N+1)$$

Barnes function

$$G_2(z+1) = \Gamma(z) G_2(z) \qquad G_2(1) = 1$$

We have to analyze this expression at large *N*. We can use the asymptotic expansion of Barnes function

$$\log G_2(N+1) \approx \frac{N^2}{2}\log N - \frac{3}{4}N^2, \quad N \to \infty$$

and finally $$F_{\rm CS}(\mathbb{S}^3) \approx \frac{N^2}{2}\left(\log(2\pi\lambda) - \frac{3}{2}\right) \quad N,\ k \gg 1$$

since ABJM theory =2 CS+ 4 matter multiplets in the bifundamental, we finally obtain

$$F_{\rm ABJM}(\mathbb{S}^3) \approx N^2 \left\{ \underbrace{\log(2\pi\lambda) - \frac{3}{2}}_{\text{Chern-Simons}} \underbrace{-2\log(2)}_{\text{matter}} \right\} \quad N,\ k \gg 1$$

number of degrees of freedom at weak coupling

We conclude that the planar free energy of ABJM theory on the three-sphere is a non-trivial function of the 't Hooft parameter, interpolating between the strong and weak coupling regions. Its strong coupling behavior displays the
3/2 scaling in the number of degrees of freedom

$$-\lim_{N\to\infty}\frac{1}{N^2}F_{\rm ABJM}(\mathbb{S}^3)\approx\begin{cases}-\log(2\pi\lambda)+\frac{3}{2}+2\log(2), & \lambda\to 0\\ \\ \frac{\pi\sqrt{2}}{3\sqrt{\lambda}}, & \lambda\to\infty\end{cases}$$

We now introduce a powerful technique which makes possible, in principle, to compute the interpolating function and the strong coupling behavior

Localization

[Witten 1980s]

[Nekrasov, Pestun]

Let us suppose that we have a field theory with a Grasmann symmetry $\mathcal{Q}$ such that

$$\mathcal{Q}S(\phi) = 0$$

$$\mathcal{Q}^2 = \mathcal{L}_B \quad \text{bosonic symmetry}$$

Let *V* be a Grasmann-valued functional of the fields, invariant under the bosonic symmetry. We will assume that

$$(\delta V)_B \geq 0$$

Let us consider the perturbed partition function

$$Z(t) = \int \mathcal{D}\phi \, \mathrm{e}^{-S - t\delta V}$$

It is easy to see that *Z* does not depend on the value of *t*, at least formally

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = -\int \mathcal{D}\phi \, \delta V \, \mathrm{e}^{-S-t\delta V} = -\int \mathcal{D}\phi \, \delta\left(V \mathrm{e}^{-S-t\delta V}\right) = 0$$

$Z(0)$ original quantity we want to calculate

$Z(\infty)$ "localized" on the locus in field space ϕ_c where $(\delta V)_B = 0$

Another way to think about this: calculate the path integral for *t* large by saddle point

$$Z(t) \propto \int \mathrm{d}\phi_c \left(\frac{2\pi}{(\delta V)''(\phi_c)} \right)^{1/2} \mathrm{e}^{-S(\phi_c)} \left(1 + \mathcal{O}\left(\frac{1}{t}\right)\right)$$

but $Z(0) = Z(\infty)$, therefore

$$Z(0) \propto \int \mathrm{d}\phi_c \left(\frac{2\pi}{(\delta V)''(\phi_c)} \right)^{1/2} \mathrm{e}^{-S(\phi_c)}$$

One-loop is exact!

This looks like black magic, but it leads to well-established mathematical theorems, like Duistermaat-Heckmann or Poincare-Hopf (including all details)

Localizing (super) Chern-Simons theory

We will now apply this localization technique to super Chern-Simons theory [Kapustin-Willett-Yaakov 2009]. Consider the Grassmann symmetry $\mathcal{Q}$ defined by $\delta_\epsilon = \epsilon\mathcal{Q}$. By looking at the SUSY transformations, it is easy to check that

$$\mathcal{Q}^2 = 0$$

Now we have to pick a *V*. It turns out that the super-Yang-Mills action can be written as

$$S_{\mathrm{SYM}} = \mathcal{Q}V$$

and indeed its bosonic part is positive definite.

We then look at the deformed partition function

$$Z_{\mathrm{SCS}}(t) = \int \mathcal{D}A \cdots \mathcal{D}\sigma \, \mathrm{e}^{\frac{\mathrm{ik}}{4\pi} S_{\mathrm{SCS}} - t S_{\mathrm{SYM}}}$$

The bosonic part of the super-Yang-Mills action vanishes for

$$F_{\mu\nu} = 0, \quad D_\mu \sigma = 0, \quad D + \frac{\sigma}{r} = 0$$

The first equation says that the gauge connection is flat. But on a three-sphere the only flat connection is the trivial one, so $A_\mu = 0$

The second and third equations imply that

$$\sigma = \sigma_0 = \mathrm{constant}$$

$$D = -\frac{\sigma_0}{r}$$

All bosonic fields are constant! The path integral reduces to an *ordinary (matrix) integral*!

Now we evaluate the super-Chern-Simons action on the localizing locus, to obtain

$$S_{\mathrm{SCS}} = -\frac{\mathrm{i}k}{4\pi}\int_{\mathbb{S}^3} \mathrm{d}^3x\sqrt{g}\,\mathrm{tr}\left(2D(\sigma_0)\sigma_0\right) = \frac{\mathrm{i}k}{2\pi r}\mathrm{tr}\left(\sigma_0^2\right)\mathrm{vol}(\mathbb{S}^3)$$

Finally, we have to evaluate the one-loop contribution coming from quadratic fluctuations around the localizing locus. We rescale the fields to have canonical kinetic terms for the fluctuations

$$\sigma = \sigma_0 + \frac{1}{\sqrt{t}}\sigma' \qquad\qquad A,\ \lambda \to \frac{1}{\sqrt{t}}A,\ \frac{1}{\sqrt{t}}\lambda$$

$$D = -\frac{\sigma_0}{r} + \frac{1}{\sqrt{t}}D'$$

The relevant part of the quadratic fluctuations is $(r = 1)$

$$\frac{1}{2}\int_{\mathbb{S}^3} \sqrt{g}\,\mathrm{d}^3x\,\mathrm{tr}\left(-A^\mu\nabla^2 A_\mu - [A_\mu,\sigma_0]^2 + \mathrm{i}\bar{\lambda}\gamma^\mu\nabla_\mu\lambda + \mathrm{i}\bar{\lambda}[\sigma_0,\lambda] - \frac{1}{2}\bar{\lambda}\lambda + \cdots\right)$$

One important technical point is that, in order to compute the path integral, we have to fix the gauge. A standard choice is the covariant Feynman gauge

$$\nabla^\mu A_\mu = 0$$ transverse vector field

We have to introduce Faddeev-Popov ghosts, but it can be seen that the contribution of these fields at one loop cancels exactly the contribution of the $D,\ \sigma$ fields. The integration over A_μ is restricted then to transverse vector fields.

Schematically we find $\quad (r=1)$

$$Z_{\rm SCS}(\mathbb{S}^3) \propto \int \mathrm{d}\sigma_0 \, \mathrm{e}^{\mathrm{i}k\pi \mathrm{tr}(\sigma_0^2)} \left(\det\left(-\nabla^2 + [\sigma_0, \cdot]^2\right)\right)^{-1/2} \det\left(\mathrm{i}\not{D} - \frac{1}{2} + \mathrm{i}[\sigma_0, \cdot]\right)$$

adjoint action

This is what is called a *matrix model*, since we are integrating over a Hermitian matrix σ_0. It has the residual "gauge" symmetry

$$\sigma_0 \to U\sigma_0 U^\dagger \qquad U \in U(N)$$

The most useful gauge choice in matrix models is the Abelian or diagonal gauge, in which we set

$$2\pi\sigma_0 = \mathrm{diag}\,(\mu_1, \cdots, \mu_N)$$

Finally, one has to perform the calculation of the determinants, which can be done again with zeta-function regularization, and is left as an **Exercise**. The result is

$$\prod_{i<j}\left(\frac{2\sinh(\frac{\mu_i-\mu_j}{2})}{\mu_i-\mu_j}\right)^2$$

But in going to the diagonal gauge one inherits a "Faddeev-Popov determinant," due to integrating out the off-diagonal elements of the matrix. This is simply a square Vandermonde determinant

$$\Delta^2(\mu)=\prod_{i<j}(\mu_i-\mu_j)^2$$

We finally obtain

$$g_s = \frac{2\pi \mathrm{i}}{k}$$

$$Z_{\rm SCS}\left(\mathbb{S}^3\right) = \frac{\mathrm{i}^{-\frac{N^2}{2}}}{N!} \int \prod_{i=1}^{N} \frac{\mathrm{d}\mu_i}{2\pi} \prod_{i<j} \left(2 \sinh \frac{\mu_i - \mu_j}{2}\right)^2 \mathrm{e}^{-\frac{1}{2g_s}\sum_{i=1}^{N} \mu_i^2}$$

This is the so-called *Chern-Simons matrix model* [M.M. 2002]. The overall constant is fixed by the following requirement: in super CS theory, all the fields except the gauge connection are auxiliary. Therefore, the above partition function should be *identical* to the partition function of bosonic CS theory on the three-sphere.

Exercise: use Weyl's denominator formula to calculate the above integral for finite, and check that it reproduces the result of Witten [solution: lecture notes]

Localizing Chern-Simons-matter theories

We can easily add matter in this framework [Kapustin-Willett-Yaakov 2009, Jafferis & Hama-Hosomichi-Lee 2010]. It turns out that the matter Lagrangian can be written as

$$S_{\rm matter} = \mathcal{Q} V_{\rm matter}$$

We now modify the Lagrangian by introducing a parameter

$$-t' S_{\rm matter}$$

We can restrict the calculation to the localizing locus of the gauge sector. The original calculation is for $t' = 1$, but we can use the independence w.r.t. t' to do it at $t' = \infty$. The localizing locus is

$$\phi = 0$$

We have then to compute the one-loop determinants due to fluctuations of the chiral multiplet. The quadratic terms are given by

$$\begin{aligned}\mathcal{L}_\phi =& g^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\phi + \bar{\phi}\sigma_0^2\phi + \frac{2\mathrm{i}(\Delta-1)}{r}\bar{\phi}\sigma_0\phi + \frac{\Delta(2-\Delta)}{r^2}\bar{\phi}\phi\\ \mathcal{L}_\psi =& -\mathrm{i}\bar{\psi}\gamma^\mu\partial_\mu\psi + \mathrm{i}\bar{\psi}\sigma_0\psi - \frac{\Delta-2}{r}\bar{\psi}\psi.\end{aligned}$$

We use again zeta function regularization for this computation. The final result is simple for fields with canonical dimension in self-conjugate representations:

$$\prod_\Lambda \left(2\cosh\frac{\Lambda(\mu)}{2}\right)^{-1/2}$$

weights of the representation

Example: 1 bifundamental + conjugate

$$\prod_{i,j}\left(2\cosh\left(\frac{\mu_i-\nu_j}{2}\right)\right)^{-1}$$

Example: 1 fundamental + conjugate

$$\prod_{i}\left(2\cosh\left(\frac{\mu_i}{2}\right)\right)^{-1}$$

Localizing ABJM theory

2 SCS theories+ 4 multiplets

$$\mu_i, \nu_i$$

$$Z_{\rm ABJM}(\mathbb{S}^3) = \frac{1}{N!^2} \int \prod_{i=1}^{N} \frac{{\rm d}\mu_i}{2\pi} \frac{{\rm d}\nu_i}{2\pi} \prod_{1\le i<j\le N} \left(2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right)^2$$
$$\times \prod_{1\le i<j\le N} \left(2\sinh\left(\frac{\nu_i-\nu_j}{2}\right)\right)^2 \prod_{i,j} \left(2\cosh\left(\frac{\mu_i-\nu_j}{2}\right)\right)^{-2} {\rm e}^{-\frac{1}{2g_s}\left(\sum_i(\mu_i^2-\nu_i^2)\right)}$$

This can be trivially generalized to the case in which the CS nodes have different rank. Overall factors are fixed by comparing to weak-coupling analysis. We will call this the *ABJM or KWY* [Kapustin-Willett-Yaakov] *matrix model*.

One can write down similar expressions for other N=3 CS-matter theories

Localizing *N=4* SYM

It turns out that a similar analysis can be made for *N=4* SYM on the four-sphere [Pestun 2007]. However, the result is much simpler

$$Z_{N=4\,\mathrm{SYM}}(\mathbb{S}^4) \propto \int \mathrm{d}\sigma_0 \ \exp\left(-\frac{4\pi^2 r^2}{g_{\mathrm{YM}}^2}\mathrm{tr}(\sigma_0^2)\right)$$

This is just the *Gaussian* matrix model. It only contains information about the conformal anomaly, and there is no non-trivial weak-strong coupling interpolation. The result becomes however non-trivial when one includes Wilson loops

Calculating at large *N*

Coming back to ABJM theory, we have reduced our problem drastically, from a field theory path integral to a matrix integral. Still, the latter is quite complicated and it is not obvious how to extract the large *N* physics from it.

1) *'t Hooft expansion*: use standard large *N* technology from matrix model theory [Brezin-Itzykson-Parisi-Zuber 1978...]. Exact results for the free energies at *all genera* can be obtained in this way [Drukker-M.M.-Putrov 2010] These technology is however hard to generalize to other CS-matter theories (although it has been done in some cases)

This approach captures all α' corrections, including worldhseet instantons

2) *M-theory expansion*: two techniques have been introduced to understand this limit

2.a) *density functional approach* [Herzog-Klebanov-Safdi-Pufu, Martelli-Sparks, ...]: very general and powerful. It only captures the leading large N behavior

2.b) *Fermi gas approach* [M.M.-Putrov 2011]: captures the full $1/N$ expansion, but it is more effective for $\mathcal{N} \geq 3$ theories. Density functional approach appears as the Thomas-Fermi approximation to the Fermi gas. Gives information also on non-perturbative effects in M-theory

I will focus here on the Fermi gas approach to ABJM theory, since it uses very elementary and intuitive physics

M-theory expansion: ABJM theory as a Fermi gas

Two technical ingredients:

1) Cauchy identity:

$$\frac{\prod_{i<j}\left[2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right]\left[2\sinh\left(\frac{\nu_i-\nu_j}{2}\right)\right]}{\prod_{i,j}2\cosh\left(\frac{\mu_i-\nu_j}{2}\right)}=\sum_{\sigma\in S_N}(-1)^{\epsilon(\sigma)}\prod_i\frac{1}{2\cosh\left(\frac{\mu_i-\nu_{\sigma(i)}}{2}\right)}$$

2) elementary Fourier transform:

$$\int \mathrm{d}\sigma\,\frac{\mathrm{e}^{2\pi\mathrm{i}\sigma\eta}}{\cosh(\pi\sigma)}=\frac{1}{\cosh(\pi\eta)}$$

This makes possible to write the ABJM matrix model as

$$Z_{\mathrm{ABJM}}(N) = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^{\epsilon(\sigma)} \int \mathrm{d}^N x \prod_{i=1}^{N} \rho\left(x_i, x_{\sigma(i)}\right)$$

$$\rho(x, x') = \frac{1}{2\pi k} \frac{1}{\left(2\cosh\left(\frac{x}{2}\right)\right)^{1/2} \left(2\cosh\left(\frac{x'}{2}\right)\right)^{1/2}} \frac{1}{2\cosh\left(\frac{x-x'}{2k}\right)}$$

Exercise: derive this formula [essentially in Kapustin-Willet-Yaakov 2010]

Claim: this is the canonical partition function of an ideal Fermi gas in one dimension with a non-trivial one-particle Hamiltonian

Remember that the canonical density matrix for N distinguishable particles is defined as

$$\rho_D(\{x_1, \cdots, x_N\}, \{x'_1, \cdots, x'_N\}) = \langle x_1, \cdots, x_N | \mathrm{e}^{-\hat{H}_N} | x'_1, \cdots, x'_N \rangle$$

$\beta = 1$

Hamiltonian N-particle system

If we have indistinguishable fermions, we have to use the projection operator

$$P = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^{\epsilon(\sigma)} \sigma$$

to obtain the antisymmetrized density matrix [Tomonaga, Feynman]

$$\rho(\{x_1, \cdots, x_N\}, \{x'_1, \cdots, x'_N\}) = \langle x_1, \cdots, x_N | P \mathrm{e}^{-\hat{H}_N} P | x'_1, \cdots, x'_N \rangle$$

$$= \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^{\epsilon(\sigma)} \rho_D(\{x_1, \cdots, x_N\}, \{x'_{\sigma(1)}, \cdots, x'_{\sigma(N)}\})$$

If the particles do not interact (i.e. the Hamiltonian is the sum of *N* one-particle Hamiltonians) the density matrix factorizes:

$$\rho_D(\{x_1, \cdots, x_N\}, \{x'_1, \cdots, x'_N\}) = \prod_{i=1}^{N} \rho(x_i, x'_i)$$

$$\rho(x, x') = \langle x | \mathrm{e}^{-\hat{H}} | x' \rangle$$

For an ideal Fermi gas, therefore, the partition function can be written as

$$\begin{aligned} Z(N) &= \int \mathrm{d}^N x \, \rho\left(\{x_1, \cdots, x_N\}, \{x_1, \cdots, x_N\}\right) \\ &= \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^{\epsilon(\sigma)} \int \mathrm{d}^N x \, \prod_{i=1}^{N} \rho(x_i, x_{\sigma(i)}) \end{aligned}$$

We just have to extract the one-particle Hamiltonian from our formula

$$\hat{\rho} = \mathrm{e}^{-\frac{1}{2}U(\hat{q})}\mathrm{e}^{-T(\hat{p})}\mathrm{e}^{-\frac{1}{2}U(\hat{q})}$$

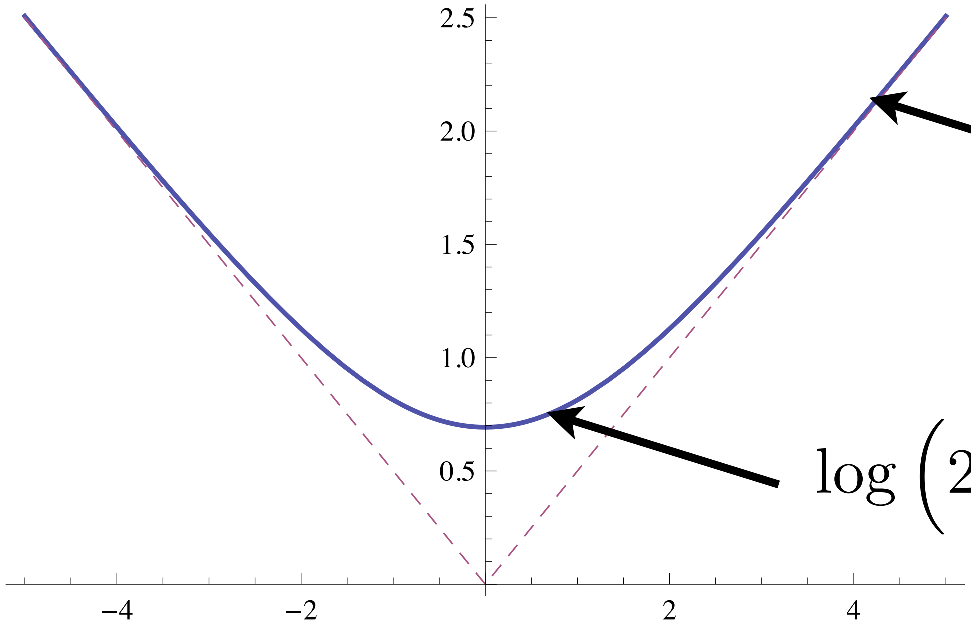
with $$U(q) = \log\left(2\cosh\frac{q}{2}\right), \quad T(p) = \log\left(2\cosh\frac{p}{2}\right)$$

$\hat{q},\ \hat{p}$ are position and momentum operators

$$[\hat{q}, \hat{p}] = \mathrm{i}\hbar, \quad \hbar = 2\pi k$$

and we find, at leading order in $\hbar$

$$H = \log\left(2\cosh\frac{p}{2}\right) + \log\left(2\cosh\frac{q}{2}\right) + \mathcal{O}(\hbar^2)$$



at large energies

$$H \approx \frac{|p|}{2} + \frac{|q|}{2}$$

an ultrarrelativistic Fermi gas in a linearly confining potential

The large *N* limit is just the thermodynamic limit. As usual in ideal quantum gases, it is more convenient to use the grand canonical ensemble

$$J(\mu) = \log\left(1 + \sum_{N=1}^{\infty} Z(N) z^N\right), \quad z = \mathrm{e}^{\mu}$$

chemical potential

Review of elementary StatMech

$$N = \frac{\partial J}{\partial \mu} \qquad F(N) \approx J(\mu(N)) - \mu(N)N, \quad N \gg 1$$

Legendre transform

Use occupation numbers to calculate *N*

density of states

$$N = \int_0^\infty \mathrm{d}E \, \frac{\rho(E)}{\mathrm{e}^{E-\mu} + 1} \approx \int_0^\mu \mathrm{d}E \, \rho(E) = n(\mu)$$

Fermi occupation number

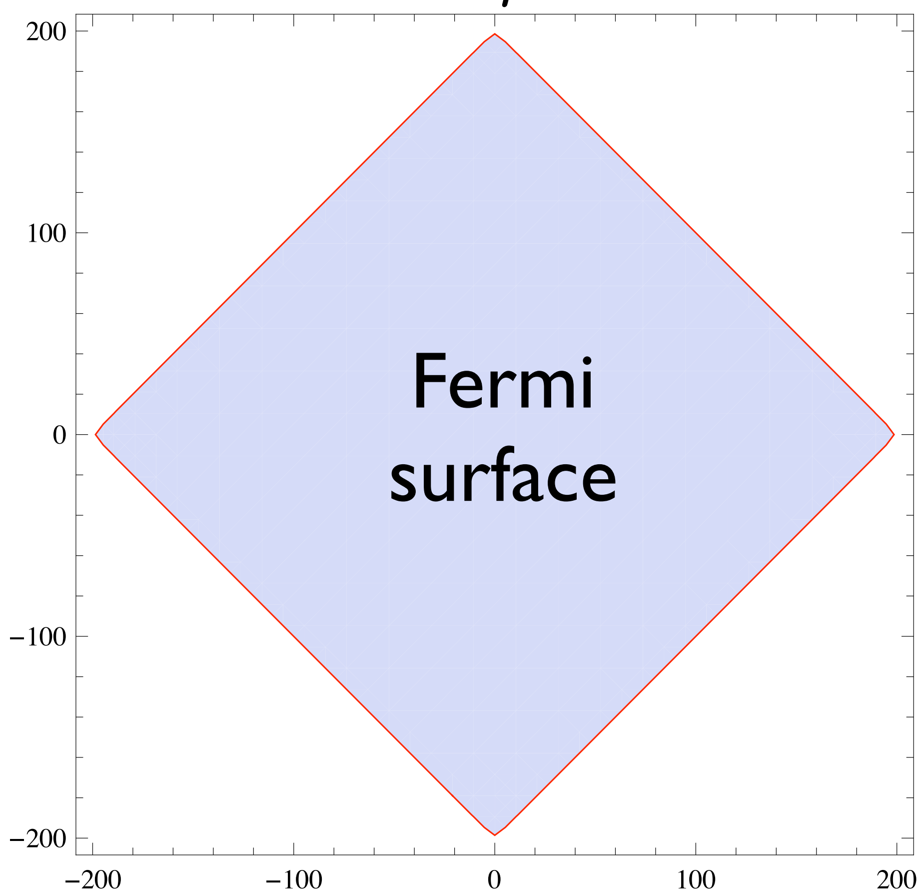
large E and fugacity (zero temperature approximation)

number of states with energy up to μ

Semiclassically,

$$n(\mu) = \frac{1}{2\pi\hbar}$$ Volume of phase space with energy $\leq \mu$ $$= \frac{1}{2\pi\hbar}$$ Area of Fermi surface

2μ



2μ

$$\frac{|p|}{2} + \frac{|q|}{2} \leq \mu$$

$$n(\mu) \approx \frac{2\mu^2}{\pi^2 k}$$

$$J(\mu) \approx \frac{2\mu^3}{3\pi^2 k}$$

$$\boxed{F(N) \approx -\frac{\sqrt{2}\pi}{3} N^{3/2} k^{1/2}}$$

YES!

This is a leading semiclassical computation (i.e. at leading order in *k*), but the thermodynamic limit is semiclassical, so it gives the right large *N* behavior

Using essentially the WKB method, it is possible to calculate systematically the corrections to the semiclassical limit

$$J(\mu) = \frac{2\mu^3}{3k\pi^2} + \mu\left(\frac{1}{3k} + \frac{k}{24}\right) + A(k) + \mathcal{O}\left(\mu^2 \mathrm{e}^{-2\mu}\right)$$

To go beyond the large *N* result, we have to remember that the Legendre transform relating the canonical and the grand-canonical potentials is just the saddle, large *N* approximation to an integral

$$Z(N) = \frac{1}{2\pi \mathrm{i}} \int \mathrm{d}\mu \, \exp\left[J(\mu) - \mu N\right]$$

Calculating corrections to the saddle leads to an expansion to *all orders* in *1/N*

$$Z_{\mathrm{ABJM}}(N) \propto \mathrm{e}^{A(k)} \mathrm{Ai}\left[\left(\frac{\pi^2 k}{2}\right)^{1/3}\left(N - \frac{1}{3k} - \frac{k}{24}\right)\right] + \mathcal{O}\left(\mathrm{e}^{-\sqrt{Nk}}\right)$$

Airy function

originally [Fuji, Hirano, Moriyama]

‘t Hooft expansion and interpolating function

The ABJM matrix model can be used to study the partition function of the gauge theory in the ‘t Hooft expansion, in a much simpler way (we have now a matrix integral!). At weak ‘t Hooft coupling we can calculate, in perturbation theory around the Gaussian model,

$$F_0(\lambda) = \log(2\pi\lambda) - \frac{3}{2} - 2\log 2 - \underbrace{\frac{\pi^2\lambda^2}{9}}_{\text{two loops}} + \underbrace{\frac{283\pi^4\lambda^4}{5400}}_{\text{three loops}} - \cdots$$

As we explained at the beginning of these lectures, we have to resum this expansion in the matrix model in order to understand the strong coupling limit.

The resummation of this expansion in matrix models was pioneered by [Brezin-Itzykson-Parisi-Zuber 1978]. I will now sketch their method.

Let us consider the partition function of a matrix model with one single set of eigenvalues

$$Z = \frac{1}{N!}\frac{1}{(2\pi)^N}\int \prod_{i=1}^{N} \mathrm{d}\lambda_i \, \underbrace{\Delta^2(\lambda)}_{\substack{\text{Vandermonde} \\ \text{(two-body)}}} \mathrm{e}^{-\frac{1}{g_s}\sum_{i=1}^{N} \overbrace{V(\lambda_i)}^{}}$$

potential (one-body)

In the 't Hooft expansion, we fix $t = g_s N$

including the Vandermonde term in an "effective" action for the eigenvalues, we find

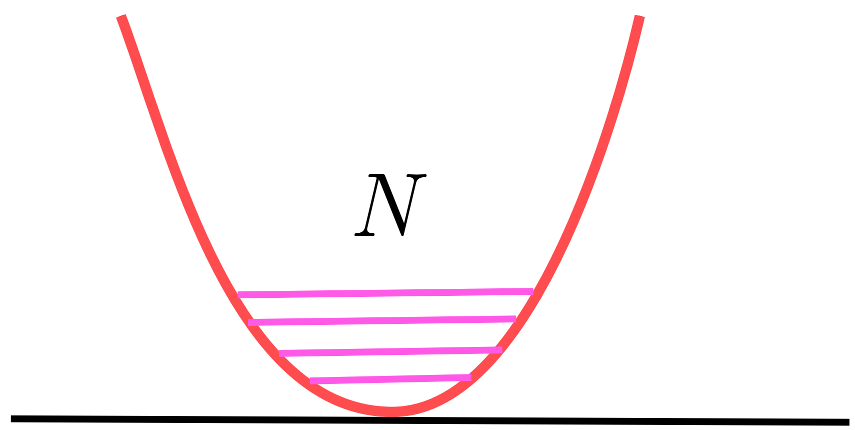
$$Z = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{\mathrm{d}\lambda_i}{2\pi} \mathrm{e}^{g_s^{-2} S_{\mathrm{eff}}(\lambda)}$$

$$S_{\mathrm{eff}}(\lambda) = -\frac{t}{N} \sum_{i=1}^{N} V(\lambda_i) + \frac{2t^2}{N^2} \sum_{i<j} \log |\lambda_i - \lambda_j|$$

attractive potential

"eigenvalue repulsion"

In the 't Hooft regime of a matrix integral, there are *two competing effects*: eigenvalues want to be at the minimum of the potential, but since they repel each other, they spread out over a finite interval



$$V(x) = \frac{x^2}{2}$$

The equilibrium positions for the eigenvalues satisfy the saddle-point equation

$$\frac{1}{2t} V'(\lambda_i) = \frac{1}{N} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}$$

When N is large, the eigenvalues become *dense* in a *finite interval*, and they are described by a density of eigenvalues

$$\rho(\lambda) = \frac{1}{N}\sum_{i=1}^{N} \delta(\lambda - \lambda_i) \underset{N\to\infty}{\longrightarrow} \rho_0(\lambda)$$

$$\int \mathrm{d}\lambda\, \rho_0(\lambda) = 1$$ normalization

In terms of this density of eigenvalues,

$$F_0(t) = -\frac{1}{t}\int_{\mathcal{C}} \mathrm{d}\lambda\, \rho_0(\lambda) V(\lambda) + \int_{\mathcal{C}\times\mathcal{C}} \mathrm{d}\lambda\, \mathrm{d}\lambda'\, \rho_0(\lambda)\rho_0(\lambda') \log|\lambda - \lambda'|$$

support of the density

The density can be obtained from the large *N* limit of the equilibrium equation, which is a singular integral equation

$$\frac{1}{2t}V'(\lambda) = \mathrm{P}\int_{\mathcal{C}} \mathrm{d}\lambda' \frac{\rho_0(\lambda')}{\lambda - \lambda'}$$

principal part

For the simplest case (the Gaussian matrix model) one finds, for example,

$$\rho_0(\lambda) = \frac{1}{2\pi t}\sqrt{4t - \lambda^2}$$

which means that at large *N* the eigenvalues "condense" along the cut

$$(-2\sqrt{t}, 2\sqrt{t})$$

The case of the original ABJM matrix integral is a little different, since there are *two* sets of eigenvalues. However, the above method can be adapted. The saddle-point equations are of the form:

$$\begin{aligned}\frac{\mu_i}{g_s} &= \quad \sum_{j\neq i}^{N_1} \coth \frac{\mu_i-\mu_j}{2} - \sum_{a=1}^{N_2} \tanh \frac{\mu_i-\nu_a}{2}, \\ -\frac{\nu_a}{g_s} &= \quad \sum_{b\neq a}^{N_2} \coth \frac{\nu_a-\nu_b}{2} - \sum_{i=1}^{N_1} \tanh \frac{\nu_a-\mu_i}{2}\end{aligned}$$

However, it is more convenient to write the equations in the more symmetric form $\quad t_i = g_s N_i$

$$\begin{aligned}\mu_i &= \quad \frac{t_1}{N_1} \sum_{j\neq i}^{N_1} \coth \frac{\mu_i-\mu_j}{2} + \frac{t_2}{N_2} \sum_{a=1}^{N_2} \tanh \frac{\mu_i-\nu_a}{2}, \\ \nu_a &= \quad \frac{t_2}{N_2} \sum_{b\neq a}^{N_2} \coth \frac{\nu_a-\nu_b}{2} + \frac{t_1}{N_1} \sum_{i=1}^{N_1} \tanh \frac{\nu_a-\mu_i}{2},\end{aligned}$$

The original situation can be recovered at the end of the day by setting $t_2 \to -t_2$

The resulting equations were analyzed in the context of a closely related model, the so-called *lens space matrix model* which appears in the analysis of pure bosonic CS theory on a lens space *L(2,1)* [M.M., Aganagic-Klemm-M.-Vafa]

$$Z_{\mathrm{CS}}(L(2,1)) = \frac{1}{N_1! N_2!} \int \prod_{i=1}^{N_1} \frac{\mathrm{d}\mu_i}{2\pi} \prod_{a=1}^{N_2} \frac{\mathrm{d}\nu_a}{2\pi} \prod_{1\le i<j\le N_1} \left(2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right)^2$$

$$\times \prod_{1\le a<b\le N_2} \left(2\sinh\left(\frac{\nu_a-\nu_b}{2}\right)\right)^2 \prod_{i,a} \left(2\cosh\left(\frac{\mu_i-\nu_a}{2}\right)\right)^2 \mathrm{e}^{-\frac{1}{2g_s}\left(\sum_i \mu_i^2 + \sum_a \nu_a^2\right)}$$

With some work, one can calcuate the densities of eigenvalues for this model

$$\rho_1(\mu) = \frac{1}{\pi t_1} \tan^{-1}\left[\sqrt{\frac{\alpha - 2\cosh\mu}{\beta + 2\cosh\mu}}\right] \qquad \rho_2(\nu) = \frac{1}{\pi t_2} \tan^{-1}\left[\sqrt{\frac{\beta - 2\cosh\nu}{\alpha + 2\cosh\nu}}\right]$$

The parameters α, β are fixed by normalization

To recover ABJM theory, we must require $t_1 = -t_2$

and both purely imaginary. This fixes

$$\alpha = 2 + \mathrm{i}\kappa, \qquad \beta = 2 - \mathrm{i}\kappa, \qquad \kappa \text{ real}$$

One eventually finds from the above equations

$$\lambda(\kappa) = \frac{\kappa}{8\pi}\ {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; -\frac{\kappa^2}{16}\right)$$

$$\partial_\lambda F_0(\lambda) = \frac{\kappa}{4} G^{2,3}_{3,3}\left(\begin{array}{ccc} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ 0, & 0, & -\frac{1}{2} \end{array} \middle| -\frac{\kappa^2}{16}\right) + 4\pi^3 \mathrm{i}\lambda(\kappa)$$

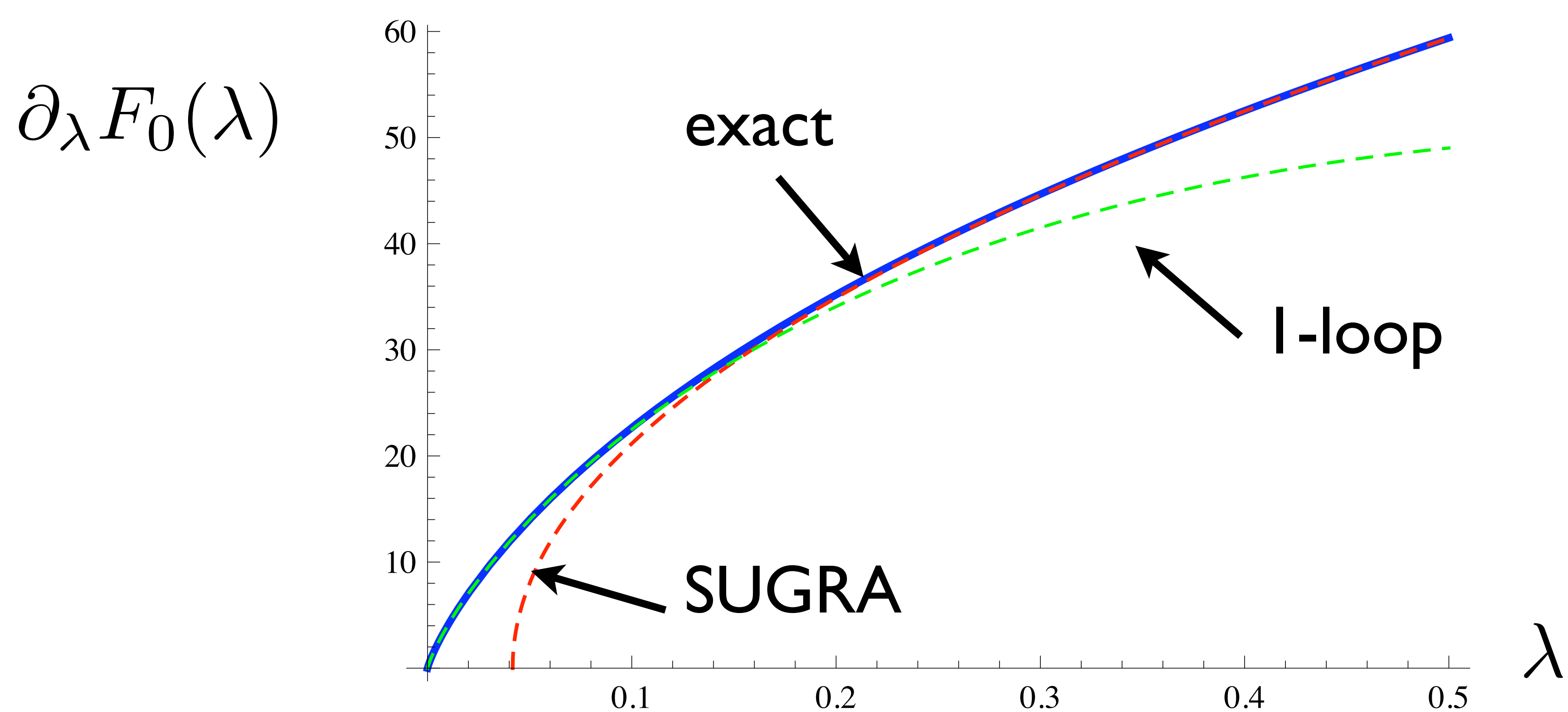
here we used a slightly different normalization for the free energy:

$$F_{\mathrm{ABJM}}(\mathbb{S}^3) = g_s^{-2} F_0(\lambda) + \cdot$$

The analytic structure of the planar free energy is as expected: there is a finite radius of convergence around the origin, controlled by the singularity at

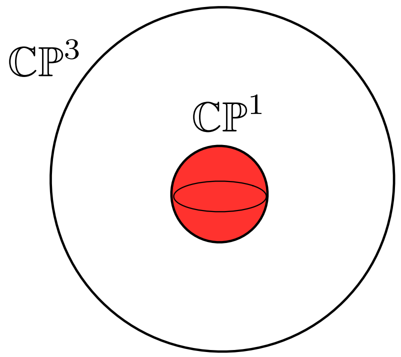
$$\kappa^2 = -16$$

which leads to branch cut singularities along the imaginary axis in the λ plane. Therefore, analytic continuation to large 't Hooft coupling is possible!



A detailed analysis of this planar solution shows that its strong coupling limit is given by

$$F_0 = -\frac{\sqrt{2}\pi}{3} k^{1/2} \left(N - \frac{k}{24} \right)^{3/2} + \mathcal{O}\left(\mathrm{e}^{-\sqrt{\frac{N}{k}}} \right)$$



worldsheet instantons! $\sim \mathcal{O}\left(\mathrm{e}^{-L^2/\alpha'} \right)$

In contrast, in the M-theory expansion we obtained

$$F = -\frac{\sqrt{2}\pi}{3} k^{1/2} \left(N - \frac{1}{3k} - \frac{k}{24} \right)^{3/2} + \mathcal{O}(1/N) + \mathcal{O}\left(\mathrm{e}^{-\sqrt{kN}} \right)$$

$$\sim \mathcal{O}\left(\mathrm{e}^{-L^3/\ell_P^3} \right)$$

in type IIA, genus one correction

membranes