CERN LECTURES

* Loop operators in Q.F.T.

W[C] ↖ loop

RATHER than local operators $\mathcal{O}(x)$

Q: Why? . nice observables

. They arise in many physical situations:

- Propagation of a particle in the presence of a gauge field/e.m
- Useful to study confinement
- construction of gauge invariant operators

. Extra motivations: . Relation to scattering amplitudes for a particular theory

. In some cases, they can be computed exactly!

Q₂: How? to compute them: . Good old perturbation theory ✓

- AdS/CFT ✓
- Strong coupling lattice computations ×
- In SUSY theories, localization techniques (M.M. talks)

PLAN OF
THE TALKS

* Focus in the physical situations mentioned above and construct/rediscover the W.L. (TODAY)

* WORK some examples (TODAY)

* W.L. in N=4 SYM, ADS/CFT } 3nd + 4th

* CIRCULAR W.L

1st physical sit: propagation of a particle from X to Y

propagator $G(X,Y) = \langle \phi(X)\phi(Y)\rangle = \langle X| \frac{i}{P^2+m^2} |Y\rangle$

- We want to write this as $\sum_{\text{HISTORIES/PATHS}}$ ← World-line formalism / 1st quant. formalism

$$G(X,Y) \overset{\text{TRICK}}{=} \int_0^\infty dT \langle X| e^{-iT(\hat{P}^2+m^2)} |Y\rangle \quad \Big| \quad \frac{1}{\alpha} = \int_0^\infty e^{-T\alpha} dT$$

- We will follow the path integral derivation in Q.M.

* Divide

in very small pieces

ε, N OF THEM, Nε = T

$$G(X,Y) = \int_0^\infty dT \langle X| e^{-i\varepsilon(\hat{P}^2+m^2)} e^{-i\varepsilon(\hat{P}^2+m^2)} \cdots e^{-i\varepsilon(\hat{P}^2+m^2)} |Y\rangle$$

- Insert many identities $1 = \int dx\, |x\rangle\langle x|$

$$G(X,Y) = \int_0^\infty dT \int_{-\infty}^{\infty} \cancel{dx} \langle X| e^{-i\varepsilon(\hat{P}^2+m^2)} |X_1\rangle\langle X_1| e^{-i\varepsilon(\hat{P}^2+m^2)} |X_2\rangle \cdots \langle X_{N-1}| e^{-i\varepsilon(\hat{P}^2+m^2)} |Y\rangle$$

NOW $\langle X_j| e^{-i\varepsilon(\hat{P}^2+m^2)} |X_{j+1}\rangle = \int dP_j \langle X_j|P_j\rangle \langle P_j| e^{-i\varepsilon(\hat{P}^2+m^2)} |X_{j+1}\rangle$

$\langle X|P\rangle = e^{iPX}$

WHEN THE OPERATOR ACTS ON $\langle P_j|$, we get a $\# \cdot \langle P_j|$

$$= \int dP_j \, e^{-i\varepsilon(P_j^2+m^2)} \langle X_j|P_j\rangle\langle P_j|X_{j+1}\rangle = \int dP_j \, e^{-i\varepsilon(P_j^2+m^2)+iP_j(X_j-X_{j+1})}$$

$$= e^{-i\varepsilon m^2 + \frac{i\varepsilon}{4}\left(\frac{X_j-X_{j+1}}{\varepsilon}\right)^2}$$

$\longrightarrow$ goes to the derivative in the cont. limit

$\downarrow N\to\infty, \varepsilon\to 0$

$$G(X,Y) = \int_0^\infty dT \int_{\substack{X(0)=X \\ X(T)=Y}} \mathcal{D}X \quad \exp\left(i\int_0^T dt\left(\frac{\dot{X}(t)^2}{4} - m^2\right)\right)$$

Integration over paths

Sum over paths weighted by this action

. We broke $\underset{0}{\vdash}\!\!\!-\!\!\!\underset{T}{\dashv}$ in equal pieces, why?! we could rescale our ε by some metric: $\varepsilon \to \varepsilon\, e(t)$, with $\int_0^T e(t)\,dt = T$

Easy to see that we would get $\Big\{$ $G(X,Y) = \int_0^\infty dT \int \mathcal{D}X \, \exp\left(i\int_0^T dt\left(\frac{\dot{X}^2}{4e} - m^2 e\right)\right)$

Correct prescription: . Integrate over all these metrics

. $e(t)$ has no propagating d.o.f., we can integrate it out using its e.o.m.

$$\frac{\delta S}{\delta \varepsilon(t)} = 0 \Rightarrow -\frac{\dot{X}^2}{4e(t)^2} - M^2 = 0 \Rightarrow e(t) = \frac{i}{2}\frac{\sqrt{\dot{X}^2}}{m}$$

$$G(X,Y) = \int \underset{\substack{\uparrow \\ \text{paths: } dT\,\mathcal{D}X}}{\mathcal{D}(X)} \; e^{-m \underbrace{\int_0^T dt \sqrt{\dot{x}^2}}}$$

$\hookrightarrow$ Nothing but the length of the path!

Q: Same problem in the presence of a gauge field!

We know what to do
$$\begin{cases} \partial_\mu \to D_\mu = \partial_\mu + ieA_\mu \\ \hat{P}_\mu \to \hat{P}_\mu + eA_\mu \end{cases}$$

for now, focus in on <u>abelian</u> theory

NOW:

$$\int dP_j \langle X_j | P_j \rangle \langle P_j | \, e^{-i\varepsilon\left((\hat{P} + \underset{\substack{\uparrow \\ \text{EXTRA}}}{eA})^2 + m^2 \right)} | X_{j+1} \rangle =$$

$$= e^{-i\varepsilon m^2 + \frac{i\varepsilon}{4}\left(\frac{x_j - x_{j+1}}{\varepsilon}\right)^2 + ie\varepsilon A_\mu(x_{j+1})\left(\frac{-x_j^\mu + x_{j+1}^\mu}{\varepsilon}\right)}$$

← GAUGE FIELD COUPLES TO THE DERIVATIVE

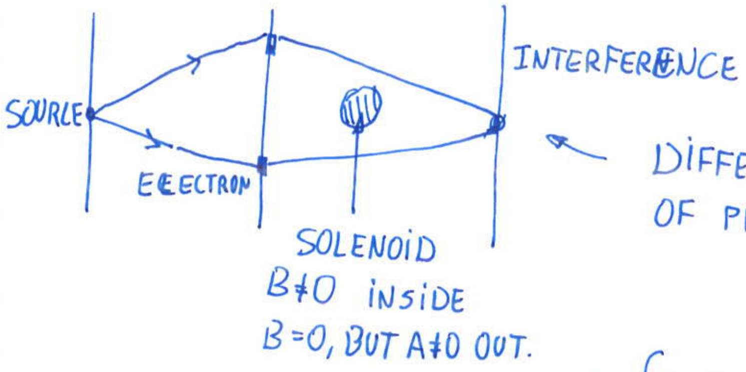
$$\Rightarrow G(X,Y) = \int dT \int \mathcal{D}X \; \underbrace{e^{-m\int_0^T dt \sqrt{\dot{x}^2}}}_{e^{-S}} \; \underbrace{e^{ie\int_X^Y A_\mu \cdot \dot{X}^\mu dt}}$$

This line integral along the path → Building block of the W.L.

↗

phase acquired by a particle going from X to Y in the presence of a (external) g.f./em. field

OPTIONAL: you can measure this! AHARONOV-BOHM effect



DIFFERENCE OF PHASE $= ie\int_{\text{ABOVE}} A_\mu dx^\mu - ie\int_{\text{BELOW}} A_\mu dx^\mu$

$$= ie\oint_{\text{CONTOUR}} A_\mu dx^\mu \underset{\uparrow\,\text{STOKES}}{=} \int_{\Sigma} (\overbrace{\nabla\wedge A}^{B})\cdot ds = \Phi$$

flux of $\vec{B}$ thru the solenoid!

Σ ← SPANS THE CONTOUR

Finally: What if the G.T. is non abelian?! A is a matrix

Before: $e^{i\varepsilon e A(x_0)\cdot\dot{X}_0}\, e^{i\varepsilon e A(x_1)\cdot\dot{X}_1} \dots e^{i\varepsilon e A(x_N)\cdot\dot{X}_{N-1}} = e^{ie\int A_\mu dx^\mu}$

↓

Now $(e^{i\varepsilon e A\cdot\dot{X}})_{a_0 a_1}(\quad)_{a_1 a_2}\cdots(\quad)_{a_{N-1}a_N} \neq e^{ie\int A_\mu dx^\mu}$

↑ matrix indices

↑ because $e^A e^B \neq e^{A+B}$

⇓ BY DEFINITION, IN $\varepsilon \to 0$ LIMIT

$$\mathcal{P}\exp\left(ie\int A_\mu dx^\mu\right)$$

Equivalent to other definitions you may have found

$$\mathcal{P}\exp\left(\int_0^T \overset{\text{matrix}}{\downarrow} M(t)\,dt\right) = I + \int_0^T M(t)\,dt + \int_0^T dt_1 \int_{t_1}^T dt_2\, \overset{\substack{\text{matrix}\\ \text{multiplic.}\\ \downarrow}}{M(t_1)\,M(t_2)}$$

$+ \dots$

↗

EX 1 ?

• Sometimes, the trace is taken, so contracting 1st & last indices:

$$W.L.\left(\int_x^y\right) = \frac{1}{N}\, \mathrm{tr}\, \mathcal{P} \exp\left(i \overset{\text{NON ABELIAN G.T. COUPLING CONSTANT}}{g} \int A_\mu dx^\mu \right)$$

$\uparrow$ SO that the expansion starts with 1

. Motivation to study W.L! $W(C) = \frac{1}{N}\, \mathrm{tr}\left(\mathcal{P} \exp\left(i g \oint_C A_\mu dx^\mu \right)\right)$

(BREAK ??)

\- - - - - -

2nd motivation: Nice gauge invariant non local operators!

<u>REMINDER</u>: In a non-abelian G.T. with fields $\underset{\substack{\uparrow\\ \text{matrix ind.}}}{\Phi_{ab}}$ in the adjoint

(e.g. N=4 SYM)

• Gauge transformations: $\Phi_{ab}(x) \to U_{aa'}(x)\, \Phi_{a'b'}(x)\, U^{-1}_{b'b}(x)$

• So, gauge inv. operator can be constructed taking the trace

$$\mathcal{O} = \mathrm{tr}\, \Phi(x) \Phi(x) \quad \to \text{ gauge invariant \underline{local} operator}$$

• If you have $\Phi(x)\Phi(y)$, the trace doesn't help!

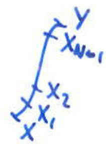
Q: How do W. Lines transform under G.T ?!

SMALL PATH

$$= \overset{\text{CONVENIENT NOT}}{\widetilde{W(x+\varepsilon,x)}} = \exp\left(i g\, \varepsilon^\mu \overset{\substack{x\\ \downarrow}}{\underset{\text{MATRIX}}{A_\mu}} \right) \approx \overset{\text{identity matrix}}{1} + i g\, \varepsilon^\mu A_\mu$$

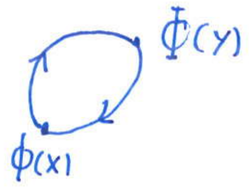
$$\xrightarrow{G.T} 1 + i g \varepsilon^\mu \left(U A_\mu U^{-1} - \frac{i}{g} (\partial_\mu U) U^{-1} \right) = U(x+\varepsilon)\, W(x+\varepsilon,x)\, U^{-1}(x)$$

• Now consider a big path!



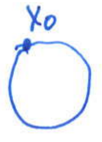
$$W(Y,X) = W(Y,X_{N-1})\,W(X_{N-1},X_{N-2})\ldots \overset{\text{SMALL}}{\overset{\downarrow}{W(X_1,X)}} \xrightarrow{G.T} U(Y)\,W(Y,X)\,U^{-1}(X)$$

So $\mathrm{tr}\left(\Phi(x)\,W(x,y)\,\Phi(y)\,W(y,x)\right)$



is a gauge invariant <u>non-local</u> operator!

Particular case, just a W.L $\quad \mathrm{tr}\, W(x_0,x_0)$

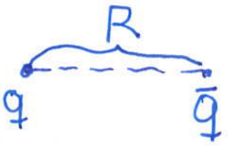


← x_0 is NOT IMPORTANT if we take the trace

So we have a nice gauge invariant operator!

3rd motivation: They are good to understand confinement

Idea: compute the force between 2 charges



- FORCE DECAYS → NO CONF
- " INCREASES LINEARLY → CONF.

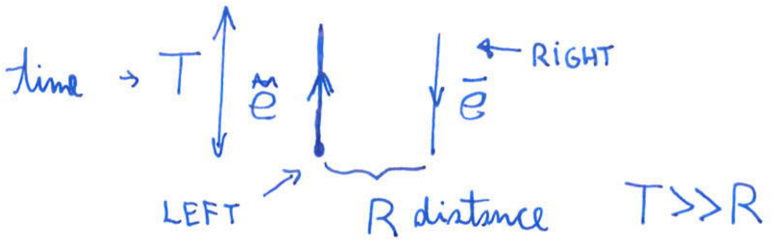
A W.L. can compute such force!

* Consider the path integral $Z = \int \mathcal{D}A\, e^{-S[A]} \rightarrow \sum_{\text{STATES}} \rightarrow e^{-E_0 T}$

FOR large times, only the ground state survives

for large times →

- Now add 2 static charges to the path integral



Then, we expect $Z \to e^{-(E_0 + \underbrace{V(R)}_{\text{potential we are interested in}})T}$

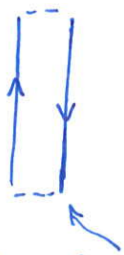
* To "add" two static charges we simply add sources to the action

$$S[A] \to S[A] + i\int dt\, d^3x\; J_\nu(t,x) A^\nu(t,x)$$

- Static charges separated a dist. R $\Rightarrow$ $J^\nu = \delta(\vec{x}-0)\,\delta^\nu_0\, e - \delta(\vec{x}-R)\,\overset{\text{Static}}{\overset{\downarrow}{\delta^\nu_0}}\, e$

$$\Rightarrow S[A] \to S[A] + ie\int A_\nu(x_L(t)) \cdot \dot{X}^\nu_L(t)\, dt + ie\int A_\nu(x_R(t)) \cdot \dot{X}^\nu_R\, dt$$

- e.g. $X^\nu_L = (t,0,0,0)$
  $X^\nu_R = (-t,0,0,R)$



- Now, if T is really large, we can close the W.L! this will not change V(R)

$$\Rightarrow S[A] \to S[A] + ie\oint_{\square} A_\nu dx^\nu$$

$\square$ ← RECTANGULAR Loop!

$$e^{-V(R)T} = \frac{1}{Z}\int \mathcal{D}A\, e^{-S[A]}\, \underset{\overset{\uparrow}{\text{RECTANGLE}}}{W(\square)} \equiv \langle W(\square)\rangle$$

Expectation value of rect. W.L. !

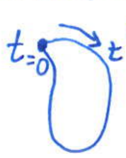
- EXAMPLE: Pure QED (abelian gauge theory)

$$S[A] = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu} \quad \text{(NO FERMIONS)}$$

In grsl:

$$\langle W(\mathcal{D})\rangle = \langle \exp\left(ie\oint A_\mu(x(t))\,\dot{X}^\mu(t)\,dt\right)\rangle$$

a point along the loop



perturbation theory
powers of e

$$= 1 + ie\underset{\substack{\downarrow\\ 0}}{\oint\langle A_\mu(x)\dot{X}^\mu dt\rangle} - \frac{e^2}{2}\oint dt_1\oint dt_2\,\langle A_\mu(X(t_1))A_\nu(X(t_2))\rangle\,\dot{X}^\mu(t_1)\,\dot{X}^\nu(t_2)$$

$$+ \dots$$

$$\langle A_\mu(x)A_\nu(y)\rangle = \frac{1}{4\pi^2}\,\frac{1}{|x-y|^2}\,\delta_{\mu\nu}$$

nothing but the propagator in position space! (Euclidean)

So $\langle W(\mathcal{D})\rangle = 1 - \frac{e^2}{2}\oint dt_1\oint dt_2\,\frac{1}{4\pi^2}\,\frac{\dot{X}(t_1)\cdot\dot{X}(t_2)}{|X(t_1)-X(t_2)|^2} + \dots$

- DIAGRAMATICALLY:

$\langle W(\mathcal{D})\rangle = 1 - \frac{e^2}{2}$  $+ \frac{e^4}{4!}\times 3\times$  $+\dots$

GLUON FROM ONE POINT TO ANOTHER

OF WAYS OF WICK CONTRACTING 4 FIELDS

but in pure QED  = ( $)^2$!

So $\langle W_0 \rangle = \exp\left(-\frac{e^2}{2}\right.$  $\left.\right) = \exp\left(-\frac{e^2}{2}\iint \frac{dt_1 dt_2}{4\pi^2}\frac{\dot{X}(t_1)\cdot\dot{X}(t_2)}{|X(t_1)-X(t_2)|^2}\right)$

↑ TO all loops!

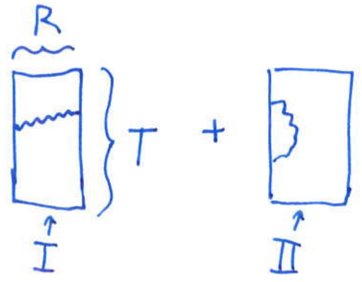
NOTE: Not at all true for non-abelian theories, e.g.  DIAGRAMS, as $F_{\mu\nu} \sim \partial A - \partial A + [A,A]$

--- (BREAK ?)

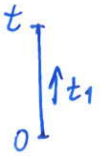
- Now, let's try the rectangular W.L. !

$\langle W(\square) \rangle \simeq e^{-V(R)T}$ ← for very large T

Q: Which diagrams contribute linearly in T for large T ?!



, that's all



$X(t_1) = (t_1, 0, 0, 0)$

$\dot{X} = (1, 0, 0, 0)$

BECAUSE WE INTEGRATE ALONG THE WHOLE LOOP

$$\mathrm{I} = -\frac{e^2}{2}\cdot 2\int_0^T dt_1 \int_0^T \frac{dt_2}{4\pi^2}\frac{1}{(t_1-t_2)^2+R^2} \underset{\text{large } T}{=} -\frac{e^2}{8\pi^2}\left(\pi\frac{T}{R}+\dots\right)\times 2$$

$$\mathrm{II} = -\frac{e^2}{2}\int_0^T dt_1 \int_0^T \frac{dt_2}{4\pi^2}\frac{1}{(t_1-t_2)^2}$$

PROBLEM: II is divergent! We need to use a UV cut-off "a"

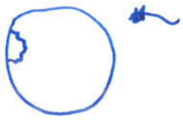
↙ Short dist.

$$\mathrm{II}_{REG} = -\frac{e^2}{8\pi^2}\iint dt_1 dt_2 \frac{1}{(t_1-t_2)+a^2} = -\frac{e^2}{8\pi^2}\left(\frac{\pi T}{a}+\dots\right)\cdot 2$$

↑

CUT-OFF

Gral statement: for any loop



this divergence is always there! → Intuitively, proportional to the length

- We need to introduce a cut-off "a"

$$I_{REG} = \iint \frac{\dot{X}(s) \cdot \dot{X}(s+t)\, ds\, dt}{(X(s+t)-X(s))^2 + a^2} \simeq \int ds\, \dot{X}^2(s) \int \frac{dt}{\dot{X}(s)^2 t^2 + a^2} \simeq \frac{\pi}{2a} \underbrace{\int ds \sqrt{\dot{x}^2(s)}}_{\text{length of the loop}} + \text{FIN}$$

CONVENIENT VARIABLES $t = t_1 - t_2$

POWERS OF t

- So the divergent piece is proportional to the length of the W.L.
- Physical interpretation: Remember the prop. of a part. on a gauge field

$$G(X,Y) = \int \mathcal{D}X\, e^{-m\int_0^T ds \sqrt{\dot{x}^2}}\, e^{ie\int_X^Y A_\mu dX^\mu}$$

- So our divergence is a correction to the mass of the particle!
- Defining a new mass → finite W.L → drop it!

Thence $\langle W_O \rangle = \underset{\substack{\uparrow \\ \text{divergent multiplicative factor}}}{e^{-c\frac{L}{a}}} \langle W_O \rangle_{REN}$

$$\Rightarrow \langle W_\square \rangle = e^{-\frac{e^2}{4\pi^2}\frac{T}{R}} = e^{-T\,V(R)} \Rightarrow V(R) = \frac{e^2}{4\pi R}$$

Coulomb's law!

Wilson loops in N=4 SYM

- N=4 SYM:
  - Most symmetric gauge theory in 4d (TOY MODEL FOR QCD) BUT STILL VERY RICH AND WE HAVE ADS/CFT
  - SU(N) gauge group → LAGRANGIAN IS FIXED
  - Observables depend only on N, g_{YM} : $A(N, g_{YM})$
  - We will consider the planar limit, $N >> 1$, $\lambda = g^2 N$ fixed → $A(\lambda)$
  - Only massless fields, in the adjoint of SU(N)
- Can be obtained by dimensional reduction of N=1 in 10D → 4D

$$\mathcal{L} = tr\left(-\frac{1}{4} F_{MN} F^{MN} + i g \bar{\Psi} \Gamma^N D_N \Psi\right)$$

with all indices :

$$\bar{\Psi}^A_{ab} \Gamma^M_{AB} \left(\partial_M \Psi^B + i \left[A_M, \Psi^B\right]\right)_{ba}$$

N×N matrix

Color indices = 1,…,N

Spinor indices A,B = 1,…,16

| M = 0,…,9 SPACE TIME INDICES

DIM.RED: . The fields depend only on x_0, x_1, x_2, x_3

- $A_M \to (A_\mu, \Phi_i)$ $\mu = 0,1,2,3$ $i = 1,..,6$

$$F_{MN} = \partial_M A_N - \partial_N A_M + g\left[A_M, A_N\right] \Rightarrow F_{\mu i} = \partial_\mu \Phi_i + g\left[A_\mu, \Phi_i\right] = D_\mu \phi_i$$

$$\mathcal{L}_{N=4} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} D_\mu \Phi_i D^\mu \Phi^i - \frac{1}{4} g^2 \left[\Phi_i, \Phi_j\right]\left[\Phi^i, \Phi^j\right] + \text{fermions}$$

Q: What is the natural W.L in $\mathcal{N}=4$ SYM?

A: Start from the usual W.L. in 10D and go down to 4D!

$$W = \mathrm{tr}\,\mathcal{P}\exp\left(ig\oint A_M\dot{X}^M\right) \to \mathrm{tr}\,\mathcal{P}\exp\left(ig\oint\left(A_\mu\dot{X}^\mu + \underset{\uparrow}{\Phi_i}\,\dot{y}^i\right)\right)$$

EXTRA COUPLING TO THE
SCALARS

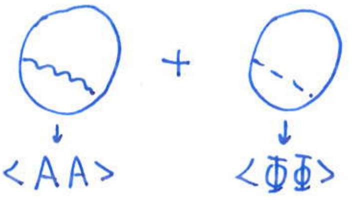
* Remember the divergent piece was prop to $\dot{X}^M\dot{X}^M = |\dot{X}|^2 + |\dot{y}|^2$
* To cancel this we can choose $y^i = i|\dot{X}|\theta^i$ with $\theta^i\theta_i = 1$

e.g $\theta^i = (1,0,0,0,0,0)$

SO:

$$W_{\mathcal{N}=4} = \frac{1}{N}\,\mathrm{tr}\,\mathcal{P}\exp\left(ig\oint\left(A_\mu\dot{X}^\mu + i|\dot{X}|\underbrace{\Phi_i\theta^i}_{\Phi_1 \text{ with our choice}}\right)dt\right)$$

REASON FOR
FINETENESS



$$\dot{X}(t_1)\cdot\dot{X}(t_2) - |\dot{X}(t_1)||\dot{X}(t_2)| \to \text{finite as } t_1 - t_2$$

BONUSES:
* The WL is locally SUSY → exactly computable in some cases!
* We can compute these $\langle W\rangle$ at strong coupling by the AdS/CFT correspondence!

AdS/CFT

$$4d\ \mathcal{N}=4\ \text{SYM}\ \longleftrightarrow\ \text{Type IIB S.T on } AdS_5\times S^5$$

$$(g_{YM}, N) \qquad\qquad (g_s, R)$$

AdS/CFT dictionary $\sqrt{\lambda}\equiv\sqrt{g^2N}=R^2/\alpha'$ $\qquad \frac{1}{N}=g_s$ → In the planar limit, strings don't split

↑

When λ is large, S.T. lives in a weakly curved background: we can use it to make computations around $\lambda=\infty$!

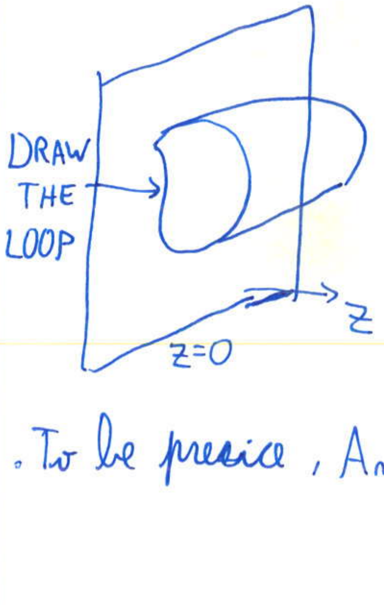
So $F(\lambda)=F_0+\lambda F_1+\lambda^2F_2+\dots$ $\qquad F(\lambda)=\sqrt{\lambda}\,\tilde{F}_0+\tilde{F}_1+\frac{1}{\sqrt{\lambda}}\tilde{F}_2+\dots$

↑ perturbation theory $\qquad$ ↑ S.T.

Nice feature: To compute the leading term, just some geometrical comp. in AdS!!

W.L. at strong coupling

. AdS in Poincaré $\quad ds^2=\frac{dx_4^2+dz^2}{z^2} \qquad z=0$ BRY

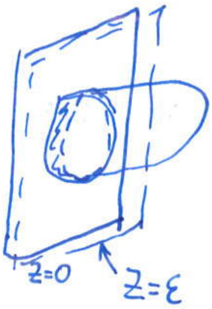


← CONSIDER THE MINIMAL SURFACE

$$\langle W\rangle = e^{-\frac{\sqrt{\lambda}}{2\pi}A_{min}}$$

. To be precise, A_{min} diverges, because of the $\frac{1}{z^2}$ factor!

- Put a cut-off at $Z=\varepsilon$



→ Substract a piece proportional to the length

$$\langle W \rangle = e^{-\frac{\sqrt{\lambda}}{2\pi}\left(A_{min} - \frac{l}{\varepsilon}\#\right)}$$

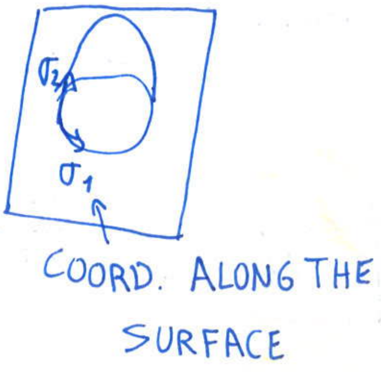
↑ length x some #

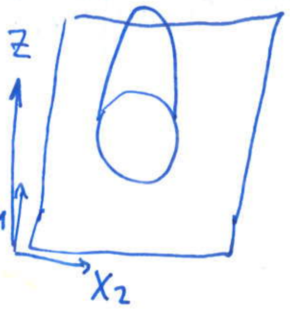
— — — — — —

- Computing W.L. in $N=4$ SYM

* Strong coupling → minimal surface → how to find minimal surfaces?

Minimize the N.G. action $$S_{NG} = \int d^2\sigma \sqrt{\det_{a,b} g_{\mu\nu}\,\partial_a X^\mu \partial_b X^\nu}$$





Choose: $X_1=\sigma_1$

$X_2=\sigma_2$

$Z=Z(X_1,X_2)$

} parametrize the surface by its projection to the (X_1,X_2) plane

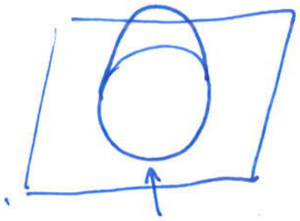
Remember $ds^2 = \frac{1}{z^2}(dz^2+dx_1^2+dx_2^2) \Rightarrow S_{NG} = \int \frac{1}{z^2}\sqrt{1+(\partial_1 z)^2+(\partial_2 z)^2}\;dx_1 dx_2$

↓

NEED TO SOLVE EQUATIONS FROM HERE

+ b.c. $Z(X_1,X_2)=0$ At the loop

e.g. Circular W.L.



$X_1^2+X_2^2=R^2$

$$\text{EOM} + \text{b.c} = Z = \sqrt{R^2-X_1^2-X_2^2}$$ minimal surface!

COMPUTE THE AREA: Plug into NG, integrate

$$A_{min} = S_{NG}(Z_{cl}) = \int_{\text{CIRCLE}} dx_1 dx_2 \frac{R}{(R^2-x_1^2-x_2^2)^{3/2}} = 2\pi \int_0^R r\,dr \frac{R}{(R^2-r^2)^{3/2}}$$ diverges as $r \to R$

Use a cut-off: $Z \geq \varepsilon \Rightarrow r \in (0, R - \varepsilon^2/2R)$

$$A_{min} = \frac{2\pi R}{\varepsilon} - 2\pi + \mathcal{O}(\varepsilon)$$

nice! divergent piece is prop. to length

$$\boxed{\langle W \rangle_0 = e^{-\frac{\sqrt{\lambda}}{2\pi}(-2\pi)} = e^{\sqrt{\lambda}}}$$

nice prediction at strong coupling!

Now let's compute the circular W.L at ~~strong~~ weak coupling!

$$\langle W_0 \rangle = \frac{1}{N} \langle \mathrm{tr}\, \mathcal{P} \exp\left(ig \oint (A_\mu \dot{X}^\mu + i|\dot{X}|\, \Phi_1)\, dt \right) \rangle$$

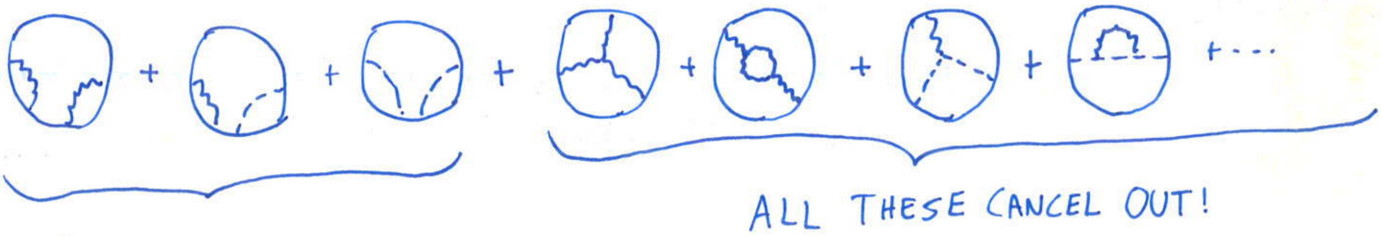
$= 1 +$  $+$  $+ \cdots$

$$= 1 - \frac{g^2 N}{8\pi^2} \int_0^{2\pi} dt_1 \int_{t_1}^{2\pi} dt_2 \underbrace{\frac{\dot{X}(t_1)\cdot\dot{X}(t_2) - |\dot{X}(t_1)||\dot{X}(t_2)|}{|X(t_1) - X(t_2)|^2}} + \cdots$$

For the circle: $\left.\begin{array}{l} X(t) = R(\cos t, \sin t, 0, 0) \\ \dot{X}(t) = R(-\sin t, \cos t, 0, 0) \end{array}\right\} = -\frac{1}{2}$ just a constant!

$$\langle W_0 \rangle = 1 + \frac{\lambda}{16\pi^2} \cdot \underbrace{\frac{(2\pi)^2}{2!}}_{\text{FROM INTEGRATION}} + \cdots$$

At 2 loops, many diagrams:



ONLY THIS DIAGRAMS SURVIVE!

ALL THESE CANCEL OUT!

↓

Very easy to compute, as full propagator $= -\frac{1}{2}$

@ 2LOOPS:

$$\langle W\rangle = 1+\frac{\lambda}{16\pi^2}\frac{(2\pi)^2}{2!}+\underbrace{\left(\frac{\lambda}{16\pi^2}\right)^2}_{\text{FROM PROP.}}\underset{\substack{\uparrow\\ \text{FROM INTEGRATION}}}{\frac{(2\pi)^4}{4!}}\cdot \underset{\nwarrow}{C_2}+\dots$$

OF WICK CONTRACTING $\langle A(t_1)A(t_2)A(t_3)A(t_4)\rangle$ in a planar way

$\langle \overbrace{1\,2}\,\overbrace{3\,4}\rangle$ or $\langle \overbrace{1\,\overbrace{2\,3}\,4}\rangle \Rightarrow C_2 = 2$

BOLD ASSUMPTION: Only ladder diagrams contribute!

$$\langle W\rangle = 1+\frac{\lambda}{16\pi^2}\frac{(2\pi)^2}{2!}\underset{\substack{\uparrow\\ 1}}{C_1}+\left(\frac{\lambda}{16\pi^2}\right)^2\frac{(2\pi)^4}{4!}\underset{\substack{\uparrow\\ 2}}{C_2}+\left(\frac{\lambda}{16\pi^2}\right)^3\frac{(2\pi)^6}{6!}C_3+\dots$$

C_3 : CONTRACT
$$\left.\begin{matrix}\langle \overbrace{1\,\overbrace{2\,3}\,\overbrace{4\,5}\,6}\rangle\\ \langle \overbrace{1\,2}\,\overbrace{3\,4}\,\overbrace{5\,6}\rangle\\ \vdots\end{matrix}\right\} C_3 = 5$$

you can show $C_n = \frac{(2n)!}{n!\,(n+1)!}$ Catalan numbers!

Prediction after bold assumption :
$$\langle W\rangle = \sum_{n=0}^{\infty}\left(\frac{\lambda}{4}\right)^n\frac{C_n}{(2n)!} = \frac{2\,I_1(\sqrt{\lambda})}{\sqrt{\lambda}}$$
to all values of λ !!

CHECK: What about $\lambda \gg 1$

$\frac{2\,I_1(\sqrt{\lambda})}{\sqrt{\lambda}} \sim e^{\sqrt{\lambda}}$ in perfect agreement with AdS/CFT !! HIGHLY NON TRIVIAL!

* That this is correct was proven by localization!