

# Resummation or exclusive exponentiation

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(A) Monte Carlo programs such as `Bhlumi` for luminosity measurements with Bhabha scattering or `KKMC` for lepton pair productions, were demonstrated thanks to associated efforts that the precision for observables with complex cuts of sub permille level (even for 0.043% Opal luminosity measurement) were possible.

(B) Main building aspects enabling such precision:

1. Phase space (issues of iterations)
2. Matrix Elements and spin (issues of iterations)
3. Tests

(C) `KKMC` Monte Carlo solutions as main example, `photos` for extension hints?  
`KKMC` is for  $e^+e^- \rightarrow l^+l^-(n\gamma)$  and `photos` for radiative corrections in decays.

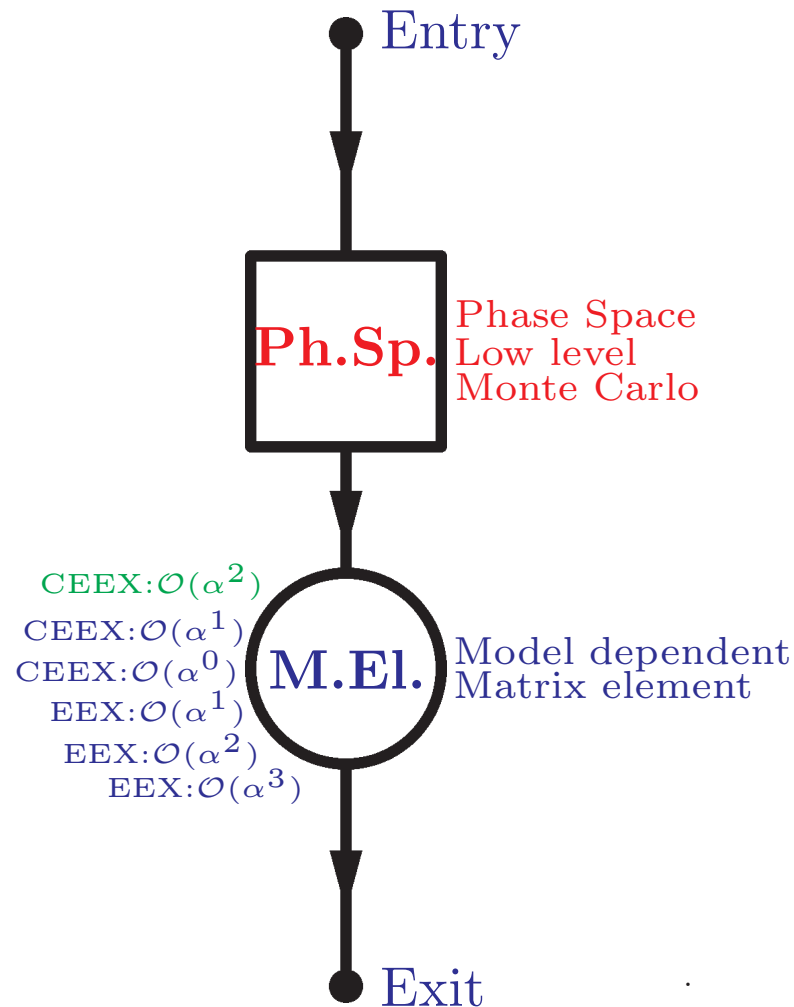
(D) My aim is to underline difficulties, why it worked for QED of solvable classical limit, but not so elsewhere.

(E) Can my talk be of interest for resummation community, can it be presentable?

## Resummation or exclusive exponentiation

- I will be talking about QED, but may be that may be of inspiration elsewhere.
- We had to through matching of second order matrix elements with resummation of multi-photon effects.
- That came with the price. Matrix elements had to be recalculated accordingly.
- Helpful in systematization was concept of group layers. Specifically Lorentz group layers with respect of its rotation sub-group.
- Amplitudes could be divided into parts more conveniently, than counting powers of “N”.
- (i) Look for the most singular part, (ii) identify what is needed to complete gauge invariant part. (iii) Take away this group of terms and (iv) loop to point (i) or finish.
- **Am I able to build around that useful talk? I am full of worries.**

KKMC or `photos` rigorous “matrix element  $\times$  full phase space” implementation



- Phase-space Monte Carlo simulator is a module producing “raw events” (including importance sampling for possible intermediate resonances/singularities)
- Library of Matrix Elements; input for “model weight”; independent module
- KKMC for  $e^+e^- \rightarrow \tau^+\tau^-n\gamma$  and `photos` for radiative corrections in decays are non-Markovian algorithms, photons are generated independently first, phase space constraints are added later, thanks to conformal symmetry of eikonal QED part KKMC or iteratively, Kinoshita-Lee-Nauenberg theorem, for `photos`.
- KKMC handle initial state radiation, `photos` massive states emission too.

$$Lips_{n+1} \rightarrow Lips_n$$

Orthodox Lorentz-invariant phase space (*Lips*) is in use in photos!

$$\begin{aligned} dLips_{n+1}(P) &= \\ &= \frac{d^3 k_1}{2k_1^0 (2\pi)^3} \cdots \frac{d^3 k_n}{2k_n^0 (2\pi)^3} \frac{d^3 q}{2q^0 (2\pi)^3} (2\pi)^4 \delta^4 \left( P - \sum_1^n k_i - q \right) \\ &= d^4 p \delta^4 (P - p - q) \frac{d^3 q}{2q^0 (2\pi)^3} \frac{d^3 k_1}{2k_1^0 (2\pi)^3} \cdots \frac{d^3 k_n}{2k_n^0 (2\pi)^3} (2\pi)^4 \delta^4 \left( p - \sum_1^n k_i \right) \\ &= d^4 p \delta^4 (P - p - q) \frac{d^3 q}{2q^0 (2\pi)^3} dLips_n(p \rightarrow k_1 \dots k_n). \end{aligned}$$

Introduce factor equal 1:  $d^4 p$  of four-vector  $p$ , times  $\delta^4(p - \sum_1^n k_i)$ , and another factor equal 1, integration variable  $dM_1$  times  $\delta(p^2 - M_1^2)$ .

## Phase Space Formula of KKMC

$$dLips_{n+1}(P \rightarrow q_1 \dots q_n, k_1) = dLips_n^{+1 \text{ tangent}} \times W_n^{n+1}; \quad W_n^{n+1} = k_\gamma$$

$$dLips_n^{+1 \text{ tangent}} = dk_\gamma d \cos \theta d\phi \times dLips_n(p \rightarrow \bar{q}_1 \dots \bar{q}_n),$$

$$dLips_n^{+1} = \lambda^2 dk_\gamma k_\gamma d \cos \theta d\phi \times dLips_n(p \rightarrow \bar{q}_1 \dots \bar{q}_n). \quad (1)$$

1. Factor  $\lambda$  of photon energy re-scaling lead to simple Jacobian. In this way relation between  $P$  and  $p$  can be fixed. This enables separation of line-shape variable generation from generation of bremsstrahlung photons.

2. This extends to multiple photon generation:

$$dLips_n^{+l} = \lambda^{2l} \frac{1}{l!} \left( \prod_{i=1}^l (dk_{\gamma_i} k_{\gamma_i}) d \cos \theta d\phi \times \right) dLips_n(p \rightarrow \bar{q}_1 \dots \bar{q}_n) \quad (2)$$

3.

$$dLips_n^{+exp} = \sum_{l=0}^{\infty} \lambda^{2l} \frac{1}{l!} \left( \prod_{i=1}^l (dk_{\gamma_i} k_{\gamma_i}) d \cos \theta d\phi \times \right) dLips_n(p \rightarrow \bar{q}_1 \dots \bar{q}_n) \quad (3)$$

4. We can generate each photon variables independently from other
5. once variables generated energy momentum conservation can be assure line shape variable ( $s = p^2 / P^2$ ) generated independently by simple rescaling (factor  $\lambda^{2l}$ ).
6. Of course the above formulas are incomplete, the matrix elements need to be added, that will be covered in part two of my talks
7. Recalling work fine for upper limit of phase space sectors, for the lower one extra effort is need, but there matrix elements are following eikonal form.

## *Phase Space Formula of photos*

$$dLips_{n+1}(P \rightarrow k_1 \dots k_n, k_{n+1}) = dLips_n^{+1 \text{ tangent}} \times W_n^{n+1},$$

$$dLips_n^{+1 \text{ tangent}} = dk_\gamma d \cos \theta d\phi \times dLips_n(P \rightarrow \bar{k}_1 \dots \bar{k}_n),$$

$$\{k_1, \dots, k_{n+1}\} = \mathbf{T}(k_\gamma, \theta, \phi, \{\bar{k}_1, \dots, \bar{k}_n\}). \quad (4)$$

1. If  $dLips_n(P)$  was exact, then this formula is exact parametrization of  $dLips_{n+1}(P)$
2. Practical implementation: Take the configurations from n-body phase space.
3. Turn it back into some coordinate variables.
4. construct new kinematical configuration from all variables.
5. **Forget about temporary  $k_\gamma \theta \phi$ . Only weight  $W_n^{n+1}$  and four vectors count.**
6. Simultaneous use of several  $\mathbf{T}$  is possible and necessary/convenient if more than one charge is present in final state.
7. Choice for  $\mathbf{T}$  construction depend on matrix element: must tangent at singularities, see next slide.

## *Phase Space: (main formula)*

If we choose

$$G_n : M_{2\dots n}^2, \theta_1, \phi_1, M_{3\dots n}^2, \theta_2, \phi_2, \dots, \theta_{n-1}, \phi_{n-1} \rightarrow \bar{k}_1 \dots \bar{k}_n \quad (5)$$

and

$$G_{n+1} : k_\gamma, \theta, \phi, M_{2\dots n}^2, \theta_1, \phi_1, M_{3\dots n}^2, \theta_2, \phi_2, \dots, \theta_{n-1}, \phi_{n-1} \rightarrow k_1 \dots k_n, k_{n+1} \quad (6)$$

then

$$\mathbf{T} = G_{n+1}(k_\gamma, \theta, \phi, G_n^{-1}(\bar{k}_1, \dots, \bar{k}_n)). \quad (7)$$

The ratio of the Jacobians form the phase space weight  $W_n^{n+1}$  for the transformation. Such solution is universal and valid for any choice of  $G$ 's. However,  $G_{n+1}$  and  $G_n$  has to match matrix element, otherwise algorithm will be inefficient (factor  $10^{10}$  ...).

In case of photos  $G_n$ 's

$$W_n^{n+1} = k_\gamma \frac{1}{2(2\pi)^3} \times \frac{\lambda^{1/2}(1, m_1^2/M_{1\dots n}^2, M_{2\dots n}^2/M_{1\dots n}^2)}{\lambda^{1/2}(1, m_1^2/M^2, M_{2\dots n}^2/M^2)}, \quad (8)$$



*Phase Space: (multiply iterated)*

By iteration, we can generalize formula (4) and add  $l$  particles:

$$\begin{aligned}
 dLips_{n+l}(P \rightarrow k_1 \dots k_n, k_{n+1} \dots k_{n+l}) &= \frac{1}{l!} \prod_{i=1}^l \left[ dk_{\gamma_i} d \cos \theta_{\gamma_i} d\phi_{\gamma_i} W_{n+i-1}^{n+i} \right] \\
 &\times dLips_n(P \rightarrow \bar{k}_1 \dots \bar{k}_n), \\
 \{k_1, \dots, k_{n+l}\} &= \mathbf{T}(k_{\gamma_l}, \theta_{\gamma_l}, \phi_{\gamma_l}, \mathbf{T}(\dots, \mathbf{T}(k_{\gamma_1}, \theta_{\gamma_1}, \phi_{\gamma_1}, \{\bar{k}_1, \dots, \bar{k}_n\}) \dots).
 \end{aligned} \tag{9}$$

Note that variables  $k_{\gamma_m}, \theta_{\gamma_m}, \phi_{\gamma_m}$  are used at a time of the  $m$ -th step of iteration only, and are not needed elsewhere in construction of the physical phase space; the same is true for invariants and angles  $M_{2\dots n}^2, \theta_1, \phi_1, \dots, \theta_{n-1}, \phi_{n-1} \rightarrow \bar{k}_1 \dots \bar{k}_n$  of (5,6), which are also redefined at each step of the iteration. Also intermediate steps require explicit construction of temporary  $\bar{k}'_1 \dots \bar{k}'_n \dots \bar{k}'_{n+m}$ , statistical factor  $\frac{1}{l!}$  added.

We have got **exact distribution of weighted** events over  $n + l$  body phase space.

## *Phase Space Formula: multichannels.*

Often MC algorithm has to be split into branches. In the most general case, when  $n$  different parametrisations of the phase space with different orderings of particles are in use, the cross section can be written as follows:

$$d\Gamma_X = \sum_{\lambda=1}^n \int_0^1 \prod_{i=1}^m dx_i P_\lambda \left[ \sum_{\delta=1}^n P_\delta J_\delta^{-1}(q_1(\lambda, x_i), \dots, q_k(\lambda, x_i)) \right]^{-1} \times |M|^2.$$

In the above formula the four-momenta  $q_i(\lambda, x_i)$  are calculated from the random numbers  $x_i$  according to the parametrization of the phase space of type  $\lambda$ . The Jacobians  $J_\delta$  have to be calculated for all parametrisations of the phase space at the point  $q_i$ ;  $P_\lambda$  denotes the probability of choosing the parametrization of type  $\lambda$  in the generation,  $\lambda$  thus takes<sup>a</sup> a role of an additional discrete variable in the generation. Numerical values of probabilities  $P_\lambda$  do not affect the final distributions, but only the efficiency of the generation.

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<sup>a</sup>But not  $\delta$ .

## *Phase Space parametrizations $G_n$ for complex singularity structure*

- Several  $G_{n+1}$  can be used simultaneously (branching of the generation algorithm).
- Each  $G_{n+1}$  can be used presample distinct singularities. See slides 30+.
- Price:  $W_n^{n+1}$  is more complicated but remain exact.  $\mathbf{T} = G_{n+1}(k_\gamma, \theta, \phi, G_n^{-1}(\dots))$
- **HOWEVER:** We have observed that while matching Jacobians for the two branches related to collinear singularity of photons along direction of  $l^+$  and  $l^+$  (in  $Z$  decay) approximation must be used if more than one photon is present in final state. Otherwise solution become inconsistent. Phase space is not iterative, whereas matrix element for multi-photon state is obtained by iteration.
- **AVOID INCONSISTENCY:** in expanding manifold curvature: must be the same for phase space and Matrix Element. Phase space is manifold, Matrix element squared – bi-linear form on it. Truncation of perturbative expansion or iterative solutions mean truncation in powers of Ricci tensor, this has to be consistent. **Multi-channel phase space is not iterative, single branch algorithm we explained before is that is OK for expansion and exact phase space remain. I have learned that hard way.**

*Phase Space: (multiply iterated)*

We have generalized formula phase space formula to case of  $l$  particles added:

$$\begin{aligned}
 dLips_{n+l}(P \rightarrow k_1 \dots k_n, k_{n+1} \dots k_{n+l}) &= \frac{1}{l!} \prod_{i=1}^l \left[ dk_{\gamma_i} d \cos \theta_{\gamma_i} d\phi_{\gamma_i} W_{n+i-1}^{n+i} \right] \\
 &\times dLips_n(P \rightarrow \bar{k}_1 \dots \bar{k}_n), \\
 \{k_1, \dots, k_{n+l}\} &= \mathbf{T}(k_{\gamma_l}, \theta_{\gamma_l}, \phi_{\gamma_l}, \mathbf{T}(\dots, \mathbf{T}(k_{\gamma_1}, \theta_{\gamma_1}, \phi_{\gamma_1}, \{\bar{k}_1, \dots, \bar{k}_n\}) \dots).
 \end{aligned} \tag{10}$$

*Now we have to start talking about matrix elements:* Our relation between  $n$  and  $n+l$  body phase space is motivated by cancellation of infrared singularities. It provides kind of **triangulation**. Measure defining distance between points from manifolds of distinct no. of particles. Such phase space points are close if they differ by presence of soft photons only.

Experimental user attention necessary. Can 1 GeV photon be ignored or only 0.1 MeV one.

We will move now from **exact distribution** of **weighted** events over  $n + l$  body phase space to case where  $l$  is parameter too, but all remain exact!

## Crude $\mathcal{D}$ istribution and crude matrix element

If we add **arbitrary** factors  $f(k_{\gamma_i}, \theta_{\gamma_i}, \phi_{\gamma_i})$  and sum over  $l$  we obtain:

$$\sum_{l=0} \exp(-F) \frac{1}{l!} \prod_{i=1}^l f(k_{\gamma_i}, \theta_{\gamma_i}, \phi_{\gamma_i}) dLips_{n+l}(P \rightarrow k_1 \dots k_n, k_{n+1} \dots k_{n+l}) =$$

$$\sum_{l=0} \exp(-F) \frac{1}{l!} \prod_{i=1}^l \left[ f(k_{\gamma_i}, \theta_{\gamma_i}, \phi_{\gamma_i}) dk_{\gamma_i} d \cos \theta_{\gamma_i} d\phi_{\gamma_i} W_{n+i-1}^{n+i} \right] \times$$

$$dLips_n(P \rightarrow \bar{k}_1 \dots \bar{k}_n), \tag{11}$$

$$\{k_1, \dots, k_{n+l}\} = \mathbf{T}(k_{\gamma_l}, \theta_{\gamma_l}, \phi_{\gamma_l}, \mathbf{T}(\dots, \mathbf{T}(k_{\gamma_1}, \theta_{\gamma_1}, \phi_{\gamma_1}, \{\bar{k}_1, \dots, \bar{k}_n\}) \dots),$$

$$F = \int_{k_{min}}^{k_{max}} dk_{\gamma} d \cos \theta_{\gamma} d\phi_{\gamma} f(k_{\gamma}, \theta_{\gamma}, \phi_{\gamma}). \leftarrow \text{KLN good start}$$

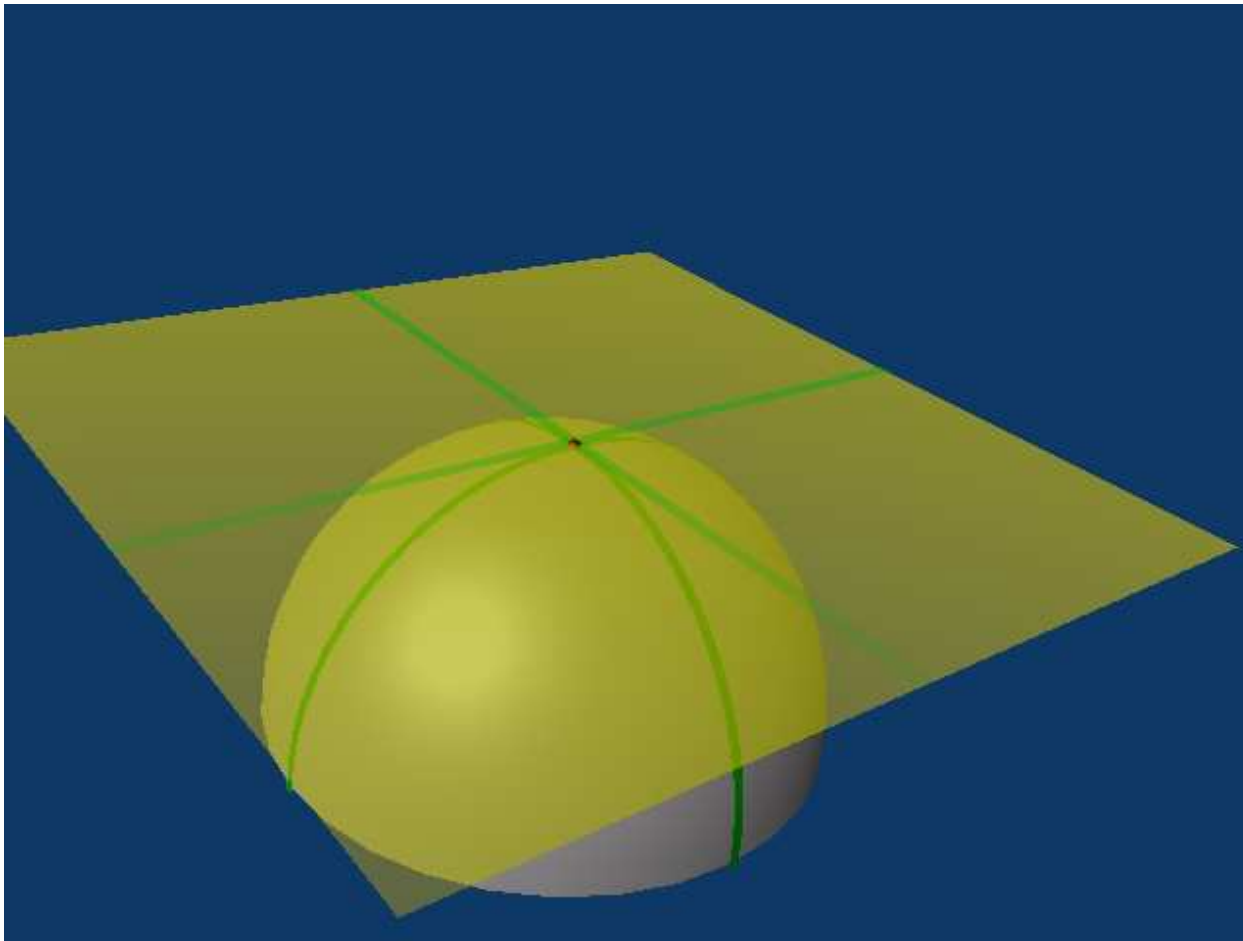
- The **olive** parts of rhs. give crude distribution over tangent space, orthogonal  $(k_i, \theta_i, \phi_i)$ .

We restrict phase space by  $k_{min}$  (typically  $10^{-6}$ )  $k_{max}$  arbitrary (boundary by  $W_{n+i-1}^{n+i}$ ).

## *Heuristic CW complexes*

We define our crude distribution over yellow space (surface=1) (represented by sum of: red point, green lines and flat yellow square). Later we do projections into physics space, using  $\mathbf{T}$  and matrix elements.

*NOTE: in KKMC YFS exclusive exponentiation – conformal symmetry is used instead.*



## Both for KKMC and `photos` algorithms are non Markovian

1. For `photos` fixed order algorithm is preserved for tests. It is good for comparisons with fixed order orthodox calculations, for KKMC such option is abandoned. It was present in its predecessor KORALZ.
2. For fixed order MC binomial distribution for number of photons candidates has to be used. To regulate infrared singularity, soft photon region need to be integrated out and combined with virtual correction. That leads to technical approximation, photons below threshold value are not generated. Alternatively negative weight events can be used. Both solutions are not good for experiments.
3. If one goes to second order, things are getting worse for technical approximation.
4. Here exponentiation helps, primary distribution of photon number is poissonian and any value can be used for minimal energy of the photon to be generated.
5. That is all I can say before entering discussions of matrix elements.
6. May be the only comment can be that in case of KKMC phase space constraints are introduced in one step  $\lambda$  factor rescaling of photon momenta. In case of `photos` phase space constraints are introduced iteratively.

1. Both KKMC and `photos` generation starts from non-markovian generation of photon candidates accordingly to poissonian distribution, number of photons and independently each photon energy,  $\theta$  and  $\phi$ .
2. Solution of KKMC phase space, is prepared for use with second order or higher matrix elements. This is thanks to conformal symmetry of multiphoton phase space. This enables independent generation of line-shape and beamstrahlung.
3. Solution of `photos` could have been extended to generation of additional massive particles (lepton pair). Kinoshita-Lee-Nauenberg theorem is used to relate phase-space slots of given multiplicity. It is less convenient for higher order matrix element implementation, because of iterative nature of phase space, jacobians and boundaries, implementation.
4. In both cases extensions to QCD higher orders, etc are possible, but there is a question of man power.
5. Training takes time and other domains value expertise higher...



## Phase-space alone makes no sense:

1. Matrix elements and spin degrees of freedom must match that. For spin, language of matrix elements is more convenient than work with distributions
2. Especially important is matching enhancements of matrix elements; collinear and soft, with phase space parametrisations and pre-samplers.
3. I will drop virtual corrections and mixed real-virtual ones. Nothing about complex masses, how they affect parametric ambiguities → non-analytic nature of dispersion relations. That is valid at one loop level, what is beyond?
4. Kleiss-Stirling formalism for spin amplitudes. We had to revise reference frames, common definition independently of number of particles (photons) in final state: S. Jadach, B.F.L. Ward, Z. Was, *Global positioning of spin GPS scheme for half spin massive spinors* Eur.Phys.J.C 22 (2001) 423. For `photos` (working on distributions) reference frame orientation was essential too. **Matrix elements required re-do:** divide them in parts corresponding to: crude level distribution build from eikonal parts and parts which could have been identified in higher order amplitudes.

### Matrix Element for $Z$ decay:

- Our discussion of double emission amplitudes was started from the single photon one, careful choice of current  $J$ , it must be nearly like at Born level, but match what is in higher order amplitudes too.
- The same is true for amplitudes of other processes. We have to check if they are similar to this for  $Z$  decay. Only if their structure match the pattern, we can expect that of multi-emission will work. Classical limit of QED helps.

$$I = I^A + I^B + I^C$$

$$I = \mathcal{J} \left[ \left( \frac{p \cdot e_1}{p \cdot k_1} - \frac{q \cdot e_1}{q \cdot k_1} \right) \right] - \left[ \frac{1}{2} \frac{\not{\epsilon}_1 \not{k}_1}{p \cdot k_1} \right] \mathcal{J} + \mathcal{J} \left[ \frac{1}{2} \frac{\not{\epsilon}_1 \not{k}_1}{q \cdot k_1} \right]$$

three gauge invariant parts,  $I^A$  is eikonal;  $I^B$ ,  $I^C$  carry collinear contrib from  $p$  and  $q$

## Scalar QED: $B \rightarrow K\pi$ decays – pure $I^A$

- The one-loop QED correction to the decay width can be represented as the sum of the Born contribution with the contributions due to virtual loop diagrams and soft and hard photon emissions.

$$d\Gamma^{\text{Total}} = d\Gamma^{\text{Born}} \left\{ 1 + \frac{\alpha}{\pi} \left[ \delta^{\text{Soft}}(m_\gamma, \omega) + \delta^{\text{Virt}}(m_\gamma, \mu_{UV}) \right] \right\} + d\Gamma^{\text{Hard}}(\omega)$$

- where for **Neutral meson decay channels**, hard photon contribution:

$$d\Gamma^{\text{Hard}} = |A^{\text{Born}}|^2 4\pi\alpha \left( q_1 \frac{k_1 \cdot \epsilon}{k_1 \cdot k_\gamma} - q_2 \frac{k_2 \cdot \epsilon}{k_2 \cdot k_\gamma} \right)^2 dLips_3(P \rightarrow k_1, k_2, k_\gamma)$$

- for **Charged meson decay channels**, hard photon contribution:

$$d\Gamma^{\text{Hard}} = |A^{\text{Born}}|^2 4\pi\alpha \left( q_1 \frac{k_1 \cdot \epsilon}{k_1 \cdot k_\gamma} - q \frac{P \cdot \epsilon}{P \cdot k_\gamma} \right)^2 dLips_3(P \rightarrow k_1, k_2, k_\gamma)$$

## Scalar QED for $\gamma^* \rightarrow \pi^+ \pi^- \gamma$ : $I^A$ and non-leading

- This case is different, because of spin structure. One can not make spin of initial state out of internal spin of outgoing particles.

$$H^\mu = \frac{e^2 F_{2\pi}(p^2)}{p^2} \left\{ (q_1 + k - q_2)^\mu \frac{q_1 \cdot \epsilon^*}{q_1 \cdot k} + (q_2 + k - q_1)^\mu \frac{q_2 \cdot \epsilon^*}{q_2 \cdot k} - 2\epsilon^{*\mu} \right\}$$

- As in case of  $Z$  decay one can separate spin amplitude into gauge invariant parts ( $C = \frac{e^2 F_{2\pi}(p^2)}{p^2}$ ):

$$H_I^\mu = C (q_1 - q_2)^\mu \left( \frac{q_1 \cdot \epsilon^*}{q_1 \cdot k} - \frac{q_2 \cdot \epsilon^*}{q_2 \cdot k} \right), H_{II}^\mu = C \left( k^\mu \left( \frac{q_1 \cdot \epsilon^*}{q_1 \cdot k} + \frac{q_2 \cdot \epsilon^*}{q_2 \cdot k} \right) - 2\epsilon^{*\mu} \right), \quad (12)$$

- This can be improved with the following change:

$$H_{I'}^\mu = C \left( (q_1 - q_2)^\mu + k^\mu \frac{q_2 \cdot k - q_1 \cdot k}{q_2 \cdot k + q_1 \cdot k} \right) \left( \frac{q_1 \cdot \epsilon^*}{q_1 \cdot k} - \frac{q_2 \cdot \epsilon^*}{q_2 \cdot k} \right), \quad (13)$$

$$H_{II'}^\mu = C \left( \frac{k^\mu}{q_2 \cdot k + q_1 \cdot k} (q_1 \cdot \epsilon^* + q_2 \cdot \epsilon^*) - \epsilon^{*\mu} \right). \quad (14)$$

- In the second case non-eikonal term is free of collinear logarithm, but is non trivial and contributes 0.2 % to total rate, thus can be numerically studied!

*QED for  $W \rightarrow l\nu_l\gamma$ :  $I^A$ ,  $I^B$  and non-leading*

$$\begin{aligned}
 M_{\lambda, \lambda_\nu, \lambda_l}^\sigma(k, Q, p_\nu, p_l) &= \left[ \frac{Q_l}{2k \cdot p_l} b_\sigma(k, p_l) - \frac{Q_W}{2k \cdot Q} (b_\sigma(k, p_l) + b_\sigma(k, p_\nu)) \right] B_{\lambda_l, \lambda_\nu}^\lambda(p_l, Q, p_\nu) \\
 &+ \frac{Q_l}{2k \cdot p_l} \sum_{\rho=\pm} U_{\lambda_l, \rho}^\sigma(p_l, m_l, k, 0, k, 0) B_{\rho, -\lambda_\nu}^\lambda(k, Q, p_\nu) \\
 &- \frac{Q_W}{2k \cdot Q} \sum_{\rho=\pm} \left( B_{\lambda_l, -\rho}^\lambda(p_l, Q, k) U_{-\rho, -\lambda_\nu}^\sigma(k, 0, k, 0, p_\nu, 0) \right. \\
 &\quad \left. + U_{\lambda_l, \rho}^\sigma(p_l, m_l, k, 0, k, 0) B_{\rho, -\lambda_\nu}^\lambda(k, Q, p_\nu) \right), \tag{15}
 \end{aligned}$$

$$\begin{aligned}
 B_{\lambda_1, \lambda_2}^\lambda(p_1, Q, p_2) &\equiv \frac{g}{2\sqrt{2}} \bar{u}(p_1, \lambda_1) \hat{\epsilon}_W^\lambda(Q) (1 + \gamma_5) v(p_2, \lambda_2), \\
 U_{\lambda_1, \lambda_2}^\sigma(p_1, m_1, k, 0, p_2, m_2) &\equiv \bar{u}(p_1, \lambda_1) \hat{\epsilon}_\gamma^\sigma(k) u(p_2, \lambda_2), \\
 \delta_{\lambda_1 \lambda_2} b_\sigma(k, p) &\equiv U_{\lambda_1, \lambda_2}^\sigma(p, m, k, 0, p, m), \tag{16}
 \end{aligned}$$

$Q_l$  and  $Q_W$  are the electric charges of the fermion  $l$  and the  $W$  boson, respectively, in units of the positron charge,  $\epsilon_\gamma^\sigma(k)$  and  $\epsilon_W^\lambda(Q)$  denote respectively the polarization vectors of the photon and the  $W$  boson. An expression of the function  $U_{\lambda_1, \lambda_2}^\sigma$  in terms of the massless spinors.

*Matrix Element (anything useful seen?):*

- We have seen that in all cases terms  $I^A, I^B, I^C$  appear
- These are the only ones which carry soft or collinear contributions
- That is why universal weight of `photos` could be defined.
- That is also why the solution defined from  $Z$  amplitudes work for other processes as well
- part of reliability proof. Note that it is for spin amplitudes level, thus such corrections do not necessarily break spin correlations. → extra slides
- Tests confirm that ME complete kernel (process dependent) is not necessary even for sub-permille precision level → good for users.

**Formula of photos** if we identify factors  $f(k_{\gamma_i}, \theta_{\gamma_i}, \phi_{\gamma_i})$  with  $I^A$ -s of previous slides. **Still questions of ME not explained** see following slides ...

$$\sum_{l=0} \exp(-F) \frac{1}{l!} \prod_{i=1}^l f(k_{\gamma_i}, \theta_{\gamma_i}, \phi_{\gamma_i}) dLips_{n+l}(P \rightarrow k_1 \dots k_n, k_{n+1} \dots k_{n+l}) =$$

$$\sum_{l=0} \exp(-F) \frac{1}{l!} \prod_{i=1}^l \left[ f(k_{\gamma_i}, \theta_{\gamma_i}, \phi_{\gamma_i}) dk_{\gamma_i} d \cos \theta_{\gamma_i} d\phi_{\gamma_i} W_{n+i-1}^{n+i} \right] \times$$

$$dLips_n(P \rightarrow \bar{k}_1 \dots \bar{k}_n) |M_B(P \rightarrow \bar{k}_1 \dots \bar{k}_n)|^2, \quad (17)$$

$$\{k_1, \dots, k_{n+l}\} = \mathbf{T}(k_{\gamma_l}, \theta_{\gamma_l}, \phi_{\gamma_l}, \mathbf{T}(\dots, \mathbf{T}(k_{\gamma_1}, \theta_{\gamma_1}, \phi_{\gamma_1}, \{\bar{k}_1, \dots, \bar{k}_n\}) \dots),$$

$$F = \int_{k_{min}}^{k_{max}} dk_{\gamma} d \cos \theta_{\gamma} d\phi_{\gamma} f(k_{\gamma}, \theta_{\gamma}, \phi_{\gamma}). \leftarrow \text{KLN good start}$$

- The olive parts of rhs. give crude distribution over tangent space, The whole formula is of the lowest order in exponentiation scheme

**Eikonal parts of ME, must be valid all over phase-space!**

WE define the complete set of spin amplitudes for emission of  $n$  photons in  $\mathcal{O}(\alpha^r)_{\text{CEEX}}$ ,  $r = 0, 1, 2$  as follows:

$$\mathfrak{M}_n^{(0)} \left( \begin{matrix} p & k_1 & \dots & k_n \\ \lambda & \sigma_1 & \dots & \sigma_n \end{matrix} \right) = \sum_{\{\wp\}} \prod_{i=1}^n \mathfrak{s}_{[i]}^{\{\wp_i\}} \beta_0^{(0)} \left( \begin{matrix} p \\ \lambda \end{matrix}; X_\wp \right), \quad (1)$$

$$\mathfrak{M}_n^{(1)} \left( \begin{matrix} p & k_1 & \dots & k_n \\ \lambda & \sigma_1 & \dots & \sigma_n \end{matrix} \right) = \sum_{\{\wp\}} \prod_{i=1}^n \mathfrak{s}_{[i]}^{\{\wp_i\}} \left\{ \beta_0^{(1)} \left( \begin{matrix} p \\ \lambda \end{matrix}; X_\wp \right) + \sum_{j=1}^n \frac{\beta_{1\{\wp_j\}}^{(1)} \left( \begin{matrix} p & k_j \\ \lambda & \sigma_j \end{matrix}; X_\wp \right)}{\mathfrak{s}_{[j]}^{\{\wp_j\}}} \right\}, \quad (2)$$

$$\begin{aligned} \mathfrak{M}_n^{(2)} \left( \begin{matrix} p & k_1 & \dots & k_n \\ \lambda & \sigma_1 & \dots & \sigma_n \end{matrix} \right) &= \\ &= \sum_{\{\wp\}} \prod_{i=1}^n \mathfrak{s}_{[i]}^{\{\wp_i\}} \left\{ \beta_0^{(2)} \left( \begin{matrix} p \\ \lambda \end{matrix}; X_\wp \right) + \sum_{j=1}^n \frac{\beta_{1\{\wp_j\}}^{(2)} \left( \begin{matrix} p & k_j \\ \lambda & \sigma_j \end{matrix}; X_\wp \right)}{\mathfrak{s}_{[j]}^{\{\wp_j\}}} + \sum_{1 \leq j < l \leq n} \frac{\beta_{2\{\wp_j \wp_l\}}^{(2)} \left( \begin{matrix} p & k_j & k_l \\ \lambda & \sigma_j & \sigma_l \end{matrix}; X_\wp \right)}{\mathfrak{s}_{[j]}^{\{\wp_j\}} \mathfrak{s}_{[l]}^{\{\wp_l\}}} \right\}. \quad (3) \end{aligned}$$

The *coherent* sum is taken over set  $\{\wp\}$  of all  $2^n$  partitions – the partition  $\wp$  is defined as a vector  $(\wp_1, \wp_2, \dots, \wp_n)$ ;  $\wp_i = 1$  for an ISR and  $\wp_i = 0$  for an FSR photon. The set of all partitions is explicitly the following:

$$\{\wp\} = \{(0, 0, 0, \dots, 0), (1, 0, 0, \dots, 0), (0, 1, 0, \dots, 0), (1, 1, 0, \dots, 0), \dots (1, 1, 1, \dots, 1)\}.$$

The  $s$ -channel four-momentum in the (possibly) resonant  $s$ -channel propagator is

$$X_\wp = p_a + p_b - \sum_{i=1}^n \wp_i k_i.$$

At  $\mathcal{O}(\alpha^r)$  we have to provide functions  $\beta_k^{(r)}$ ,  $k = 0, 1, \dots, r$ , from Feynman diagrams, which are infrared-finite by construction [yfs:1961](#). Their actual precise definitions can be found in other refs. on KKMC. Here we shall define only



the most essential ingredients. The lowest-order  $\beta_0^{(0)}$  are just Born spin amplitudes times a certain kinematical factor

$$\beta_0^{(0)}(p_\lambda; X) = \mathfrak{B}(p_\lambda; X) \frac{X^2}{(p_c + p_d)^2}. \quad (21)$$

The Born spin amplitudes  $\mathfrak{B}(p_\lambda; X)$  and other spin amplitudes are calculated using the spinor technique of Kleiss and Stirling (KS) reformulated a bit.. Soft factors  $\mathfrak{s}_{[i]}^{(\omega)}$ ,  $\omega = 0, 1$ , are complex numbers, see ref. for exact definitions; here we only need to know their absolute values

$$\left| \mathfrak{s}_{[i]}^{(1)} \right|^2 = -\frac{e^2 Q_e^2}{2} \left( \frac{p_a}{k_i p_a} - \frac{p_b}{k_i p_b} \right)^2, \quad \left| \mathfrak{s}_{[i]}^{(0)} \right|^2 = -\frac{e^2 Q_f^2}{2} \left( \frac{p_c}{k_i p_c} - \frac{p_d}{k_i p_d} \right)^2. \quad (22)$$

The factor  $\bar{\Theta}(\Omega)$  defines the infrared (IR) integration limits for real photons. More precisely for a single photon, complementary domain  $\Omega$  includes the IR divergence point  $k = 0$ , which is *excluded* from the MC phase space, we define a characteristic function  $\Theta(\Omega, k) = 1$  for  $k \in \Omega$  and  $\Theta(\Omega, k) = 0$  for  $k \notin \Omega$ . The characteristic function for the phase space included in the integration is  $\bar{\Theta}(\Omega, k) = 1 - \Theta(\Omega, k)$ . The characteristic function for *all* photons in the MC phase space is

$$\bar{\Theta}(\Omega) = \prod_{i=1}^n \bar{\Theta}(\Omega, k_i). \quad (23)$$

In the present program we opt for an  $\Omega$  traditionally defined by the photon energy cut condition  $k^0 < E_{\min}$ .

Consequently, the YFS form factor reads

$$\begin{aligned}
 Y(\Omega; p_a, \dots, p_d) &= Q_e^2 Y_\Omega(p_a, p_b) + Q_f^2 Y_\Omega(p_c, p_d) \\
 &+ Q_e Q_f Y_\Omega(p_a, p_c) + Q_e Q_f Y_\Omega(p_b, p_d) - Q_e Q_f Y_\Omega(p_a, p_d) - Q_e Q_f Y_\Omega(p_b, p_c),
 \end{aligned}
 \tag{24}$$

where

$$\begin{aligned}
 Y_\Omega(p_1, p_2) &\equiv 2\alpha \tilde{B}(\Omega, p_1, p_2) + 2\alpha \Re B(p_1, p_2) \\
 &\equiv -2\alpha \frac{1}{8\pi^2} \int \frac{d^3 k}{k^0} \Theta(\Omega; k) \left( \frac{p_1}{kp_1} - \frac{p_2}{kp_2} \right)^2 \\
 &+ 2\alpha \Re \int \frac{d^4 k}{k^2} \frac{i}{(2\pi)^3} \left( \frac{2p_1 + k}{2kp_1 + k^2} - \frac{2p_2 - k}{2kp_2 - k^2} \right)^2
 \end{aligned}
 \tag{25}$$

Basic formula is the same (OK it is similar only but differ in essential results)

1. for phase space limits iterative solution instead of re-scaling available thanks to conformal symmetry is used.
2. Instead of expansion of matrix elements into  $\beta_i^{(1)}$  and sum over partition, products of  $(s_i + \beta_i^{(1)})$  of each photon is used.
3. surprisingly this does not destroy second order parts, but help include the dominant parts of that.
4.  $\beta_i^{(1)}$  as used in `photos` consist of process independent part, (after integration it gives LL parts), but also process dependent parts (for some decays only).
5. I should say more about virtual corrections.
6. for emission kernels they can be simply coded as form-factors.
7. Far more fundamental is what to do with loop corrections beyond KLM level. They need to be included in rates of events prior `photos`.
8. for more than two body substantial work may be needed for high precision level.

Higher orders are installed in KKMC only.

1. That required work on spin amplitudes, it was not straightforward.
2. How to write fixed order amplitudes into parts, the ones which could have been obtained from from lower orders and the one which could have been used for higher order ones.
3. For  $e^+e^- \rightarrow \tau^+\tau^-\gamma\gamma + \dots$  that was already complicated, need to separation into initial state and final state amplitudes appeared.
4. Fortunately interferences were easy to introduce.
5. for  $e^+e^- \rightarrow \bar{\nu}_e\nu_e\gamma\gamma + \dots$  things became more complicated. Expansion around contact interaction was necessary to use and QED was not anymore pure. Care about charged higgs ghosts and their contributions was necessary.

- One of the necessary steps was to verify, that once `photos` activated, the lepton spectra will be reproduced as far as the LL corrections to required order.
- Formal solution of QED evolution equation can be written as:

$$D(x, \beta_{ch}) = \delta(1-x) + \beta_{ch} P(x) + \frac{1}{2!} \beta_{ch}^2 \{P \times P\}(x) + \frac{1}{3!} \beta_{ch}^3 \{P \times P \times P\}(x) + \dots \quad (26)$$

where  $P(x) = \delta(1-x)(\ln \varepsilon + 3/4) + \Theta(1-x-\varepsilon) \frac{1}{x} (1+x^2)/(1-x)$   
 and  $\{P \times P\}(x) = \int_0^1 dx_1 \int_0^1 dx_2 \delta(x - x_1 x_2) P(x_1) P(x_2)$ .

- In the LL contributing regions, phase space Jacobians of `photos` trivialize (CPC 1994). The solution above is reproduced by `photos` in a straightforward manner, for each of the outgoing charged lines.
- But it is only a limit! `photos` **treat phase space exactly and covers all corners.**
- In a similar way (simplifying phase space Jacobians and dropping parts of ME) one can get convinced that distribution of soft photons is as should be for exclusive exponentiation.

- Symmetries helped for phase-space and separation of spin amplitudes into parts....
- ... useful for fully differential and valid all over the phase space predictions.
- That is important for Monte Carlo and calculations of multidimensional distributions.
- It exposes parts of GSW predictions from sub-theories like QED (eikonal QED).
- Those are parts which need to be taken to higher orders.
- Calculation of negligible terms may be then avoided.
- I talked little about loop corrections, they are important, but less of technical importance for Monte Carlo construction.
- I was not talking about amplitudes parts similarities between different processes.
- **Disadvantage:** pressure on automated methods and higher orders validity proofs.
- What future will decide? Keep these methods in tool-box.
- **Do not drop the topic out.** Even as its use bring sometimes pain, but some fun too.
- This is (nearly) the first time I present such selection like today.
- I am not sure if it can be useful. My apologies, thanks for listening.

- Do these consideration make any sense from perspective of strong interactions?
- In QCD nothing like infrared solvable region exist.
- Hadrons are complex  $\rightarrow$  underlying events, multiparton interactions, need to be taken into account in experiments data analyzes.
- But ....
- Spin amplitudes were used at early steps of factorization work (I was told by Boris Ermolayev from his experience of early Leningrad winter schools).
- I've made some attempts of identification spin amplitude parts, following symmetry of Lorentz group. Surprisingly parts corresponding to DGLAP etc. appeared naturally: A. van Hameren, Z. Was Eur.Phys.J.C 61 (2009) 33
- Also Stanislaw Jadach attempts on NLO parton shower approaches are in place to mention.
- Anyway, that was my motivation to prepare the talk.
- May be it will be of a help for somebody?
- Thank you for listening.

**... but of importance.**

In KKMC adjusted variant of Kleiss Stirling spin amplitudes formalism was used.  
Routine relating rest frames of  $\tau$ 's with lab-frame, essential.

**Essential details** `photos` **preserve spin correlations**

**Note that the spin carried out by photon is in most cases zero, orbital and spin of photon cancel out completely in soft and collinear limit.**

**only in ultra-hard ultra collinear configurations of  $\sim 1/3\alpha/\pi$  weight it is not the case.**

`photos` was developed keeping that in mind. **PURPOSE:** not to destroy spin correlations of the consecutive decays.



1. we are using **STANDARD and FORMAL** parametrizations of Lorentz group. One can express it with the help of consecutive boosts and rotations.
2. Convenient for Monte Carlo event construction!
3. For the definition of coordinate system in the  $P$ -rest frame the  $\hat{x}$  and  $\hat{y}$  axes of the laboratory frame boosted to the rest frame of  $P$  can be used. The orthogonal right-handed system can be constructed with their help in a standard way.
4. We choose polar angles  $\theta_1$  and  $\phi_1$  defining the orientation of the four momentum  $\bar{k}_2$  in the rest frame of  $P$ . In that frame  $\bar{k}_1$  and  $\bar{k}_2$  are back to back<sup>a</sup>, see fig. (1).
5. The previous two points would complete the definition of the two-body phase space, if both  $\bar{k}_1$  and  $\bar{k}_2$  had no measurable spin degrees of freedom visualizing themselves e.g. through correlations of the secondary decay products' momenta. Otherwise we need to know an additional angle  $\phi_X$  to complete the set of Euler angles defining the relative orientation of the axes of the  $P$  rest-frame system with the coordinate system used in the rest-frame of  $\bar{k}_2$  (and possibly also of  $\bar{k}_1$ ), see fig. (2).

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<sup>a</sup>In the case of phase space construction for multi-body decays  $\bar{k}_2$  should read as a state representing the sum of all decay products of  $P$  but  $\bar{k}_1$ .

6. If both rest-frames of  $\bar{k}_1$  and  $\bar{k}_2$  are of interest, their coordinate systems are oriented with respect to  $P$  with the help of  $\theta_1, \phi_1, \phi_X$ . We assume that the coordinate systems of  $\bar{k}_1$  and  $\bar{k}_2$  are connected by a boost along the  $\bar{k}_2$  direction, and in fact share axes:  $z' \uparrow\downarrow z'', x' \uparrow\uparrow x'', y' \uparrow\downarrow y''$ .
7. For the three-body phase space: We take the photon energy  $k_\gamma$  in  $P$  rest frame. We calculate: photon,  $k_1$  and  $k_2$  energies, all in  $k_1 + k_2$  frame.
8. We use the angles  $\theta, \phi$ , in the rest-frame of the  $k_1 + k_2$  pair: angle  $\theta$  is an angle between the photon and  $k_1$  direction (i.e.  $-z''$ ). Angle  $\phi$  defines the photon azimuthal angle around  $z''$ , with respect to  $x''$  axis (of the  $k_2$  rest-frame), see fig. (3).
9. If all  $k_1, k_2$  and  $k_1 + k_2$  rest-frames exist, then the  $x$ -axes for the three frames are chosen to coincide. It is OK, all frames connected by boosts along  $z''$  see fig. (3).
10. To define orientation of  $k_2$  in  $P$  rest-frame coordinate system, and to complete construction of the whole event, we will re-use Euler angles of  $\bar{k}_2$ :  $\phi_X, \theta_1$  and  $\phi_1$  (see figs. 4 and 5), defined again of course in the rest frame of  $P$ .

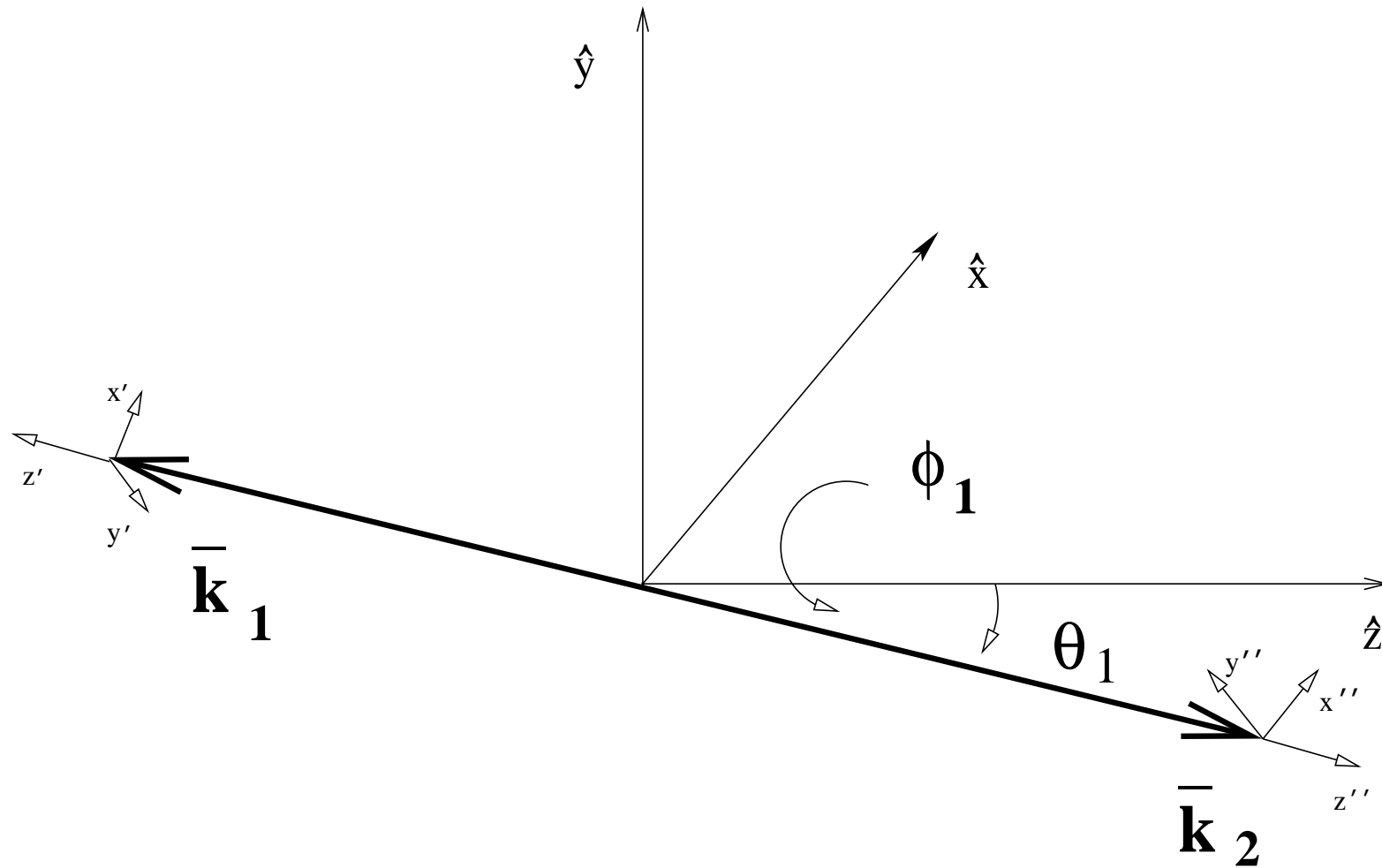


Figure 1: The angles  $\theta_1$ ,  $\phi_1$  defined in the rest-frame of  $P$  and used in parametrization of two-body phase-space.

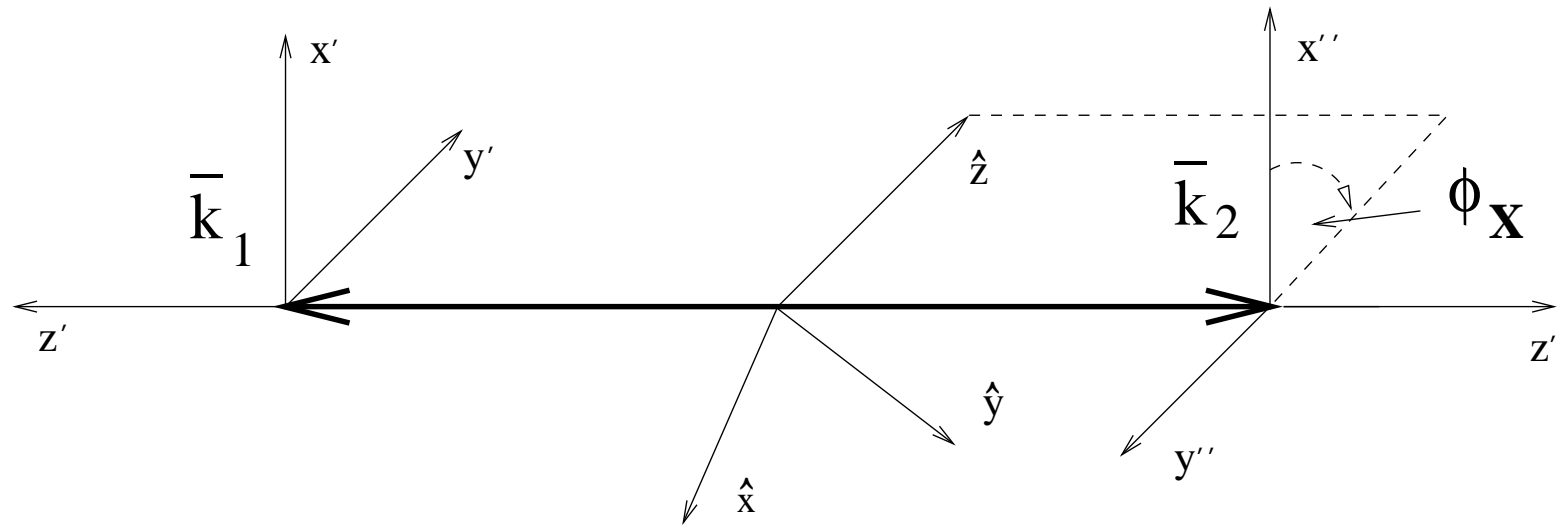


Figure 2: Angle  $\phi_X$  is also defined in the rest-frame of  $P$  as an angle between (oriented) planes spanned on: (i)  $\bar{k}_1$  and  $\hat{z}$ -axis of the  $P$  rest-frame system, and (ii)  $\bar{k}_1$  and  $x''$ -axis of the  $\bar{k}_2$  rest frame. It completes definition of the phase-space variables if internal orientation of  $\bar{k}_1$  system is of interest. In fact, Euler angle  $\phi_X$  is inherited from unspecified in details, parametrization of phase space used to describe possible future decay of  $\bar{k}_2$  (or  $\bar{k}_1$ ).

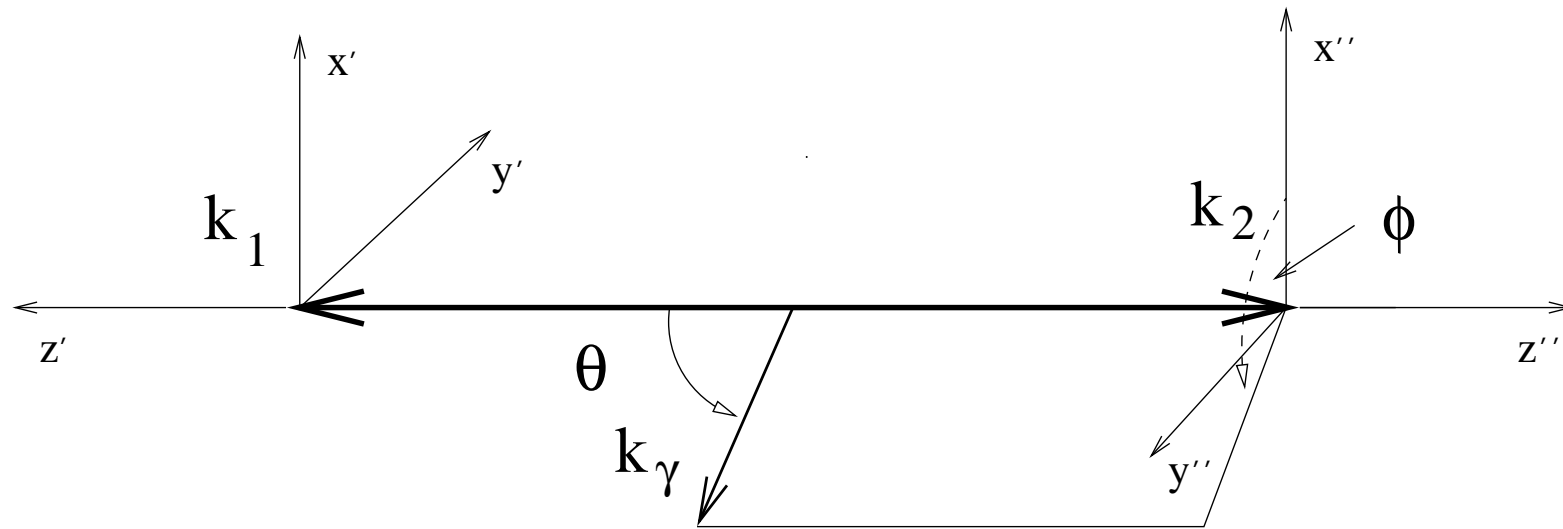


Figure 3: The angles  $\theta$ ,  $\phi$  are used to construct the four-momentum of  $k_\gamma$  in the rest-frame of  $k_1 + k_2$  pair (itself not yet oriented with respect to  $P$  rest-frame). To calculate energies of  $k_1$ ,  $k_2$  and photon, it is enough to know  $m_1$ ,  $m_2$ ,  $M$  and photon energy  $k_\gamma$  of the  $P$  rest-frame.

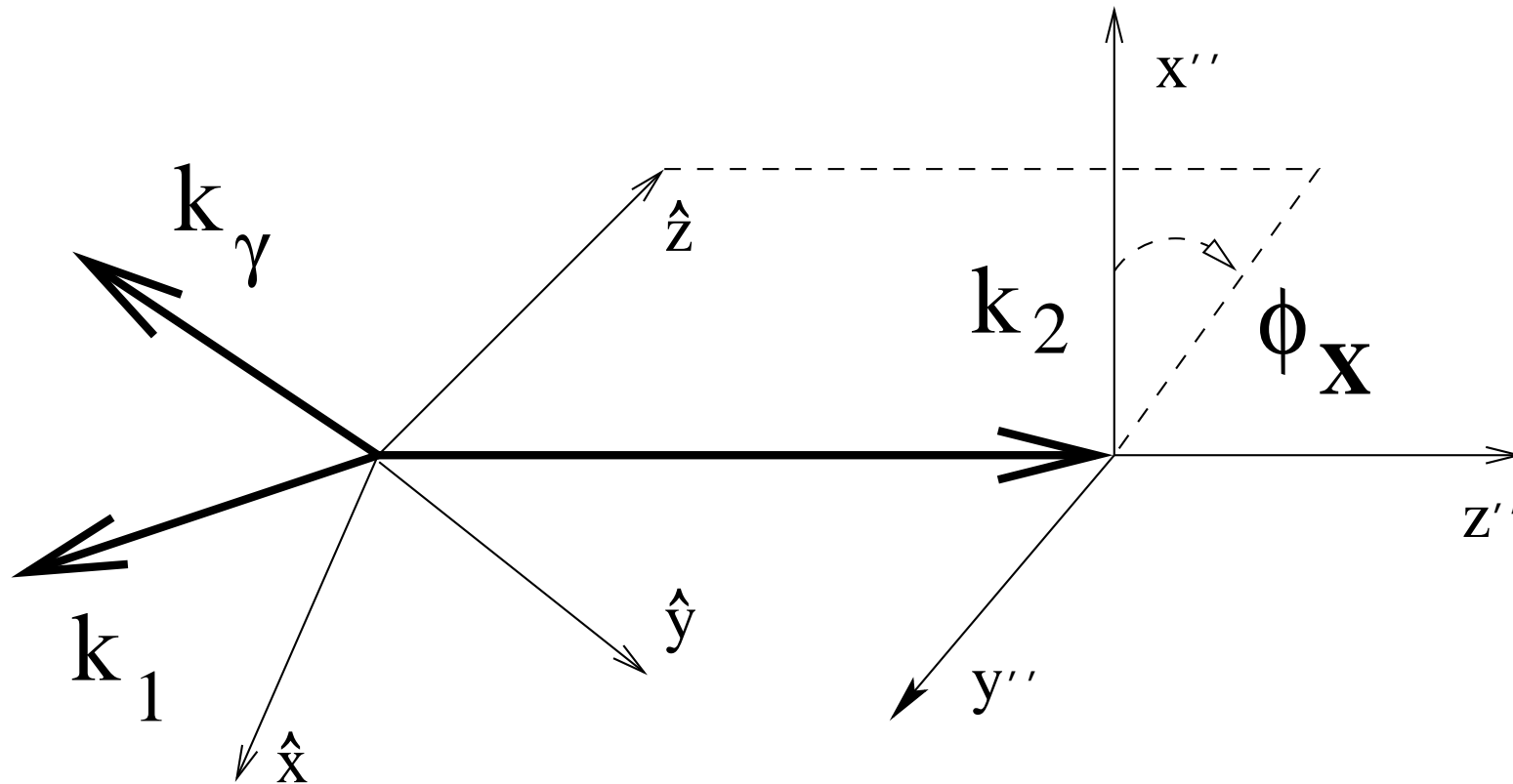


Figure 4: Use of angle  $\phi_x$  in defining orientation of  $k_1$ ,  $k_2$  and photon in the rest-frame of  $P$ . At this step only the plane spanned on  $P$  frame axis  $\hat{z}$  and  $k_2$  is oriented with respect to  $k_2 \times x''$  plane.

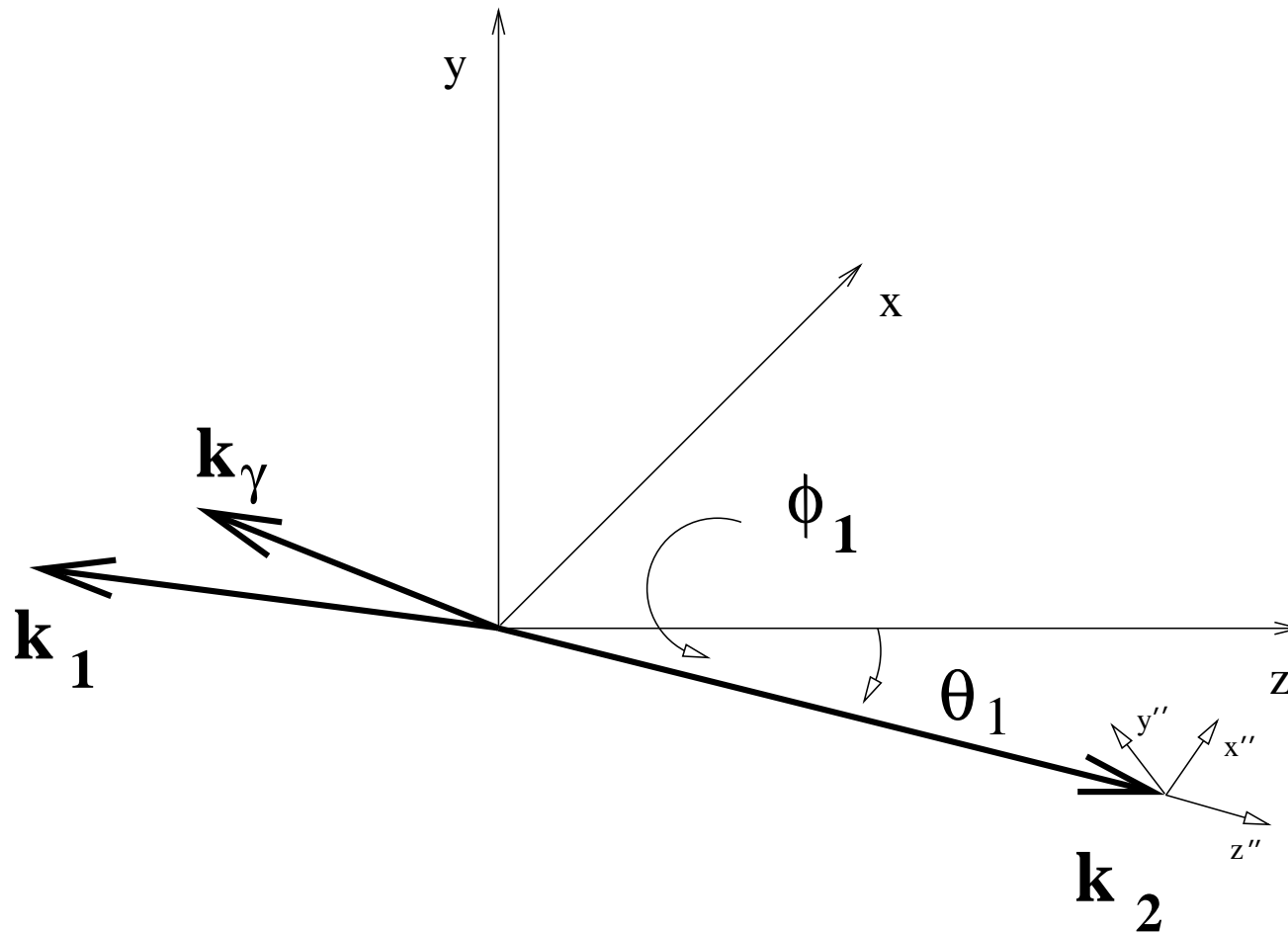


Figure 5: Final step in event construction. Angles  $\theta_1$ ,  $\phi_1$  are used. The final orientation of  $k_2$  coincide with this of  $\bar{k}_2$ .