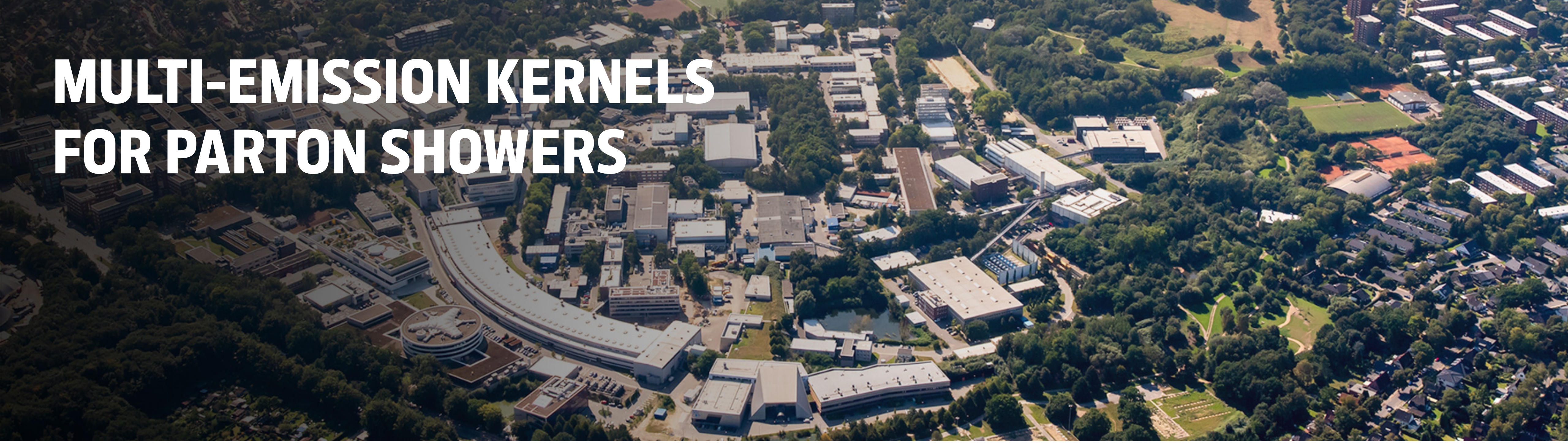

MULTI-EMISSION KERNELS FOR PARTON SHOWERS

Maximilian Löschner, DESY Theory Group

Parton Showers and Resummation, Graz, 2 July 2024

HELMHOLTZ

Complexity of Collider Events

Disentangle short and long distance physics

- Processes with colored partons result in **high multiplicity events**

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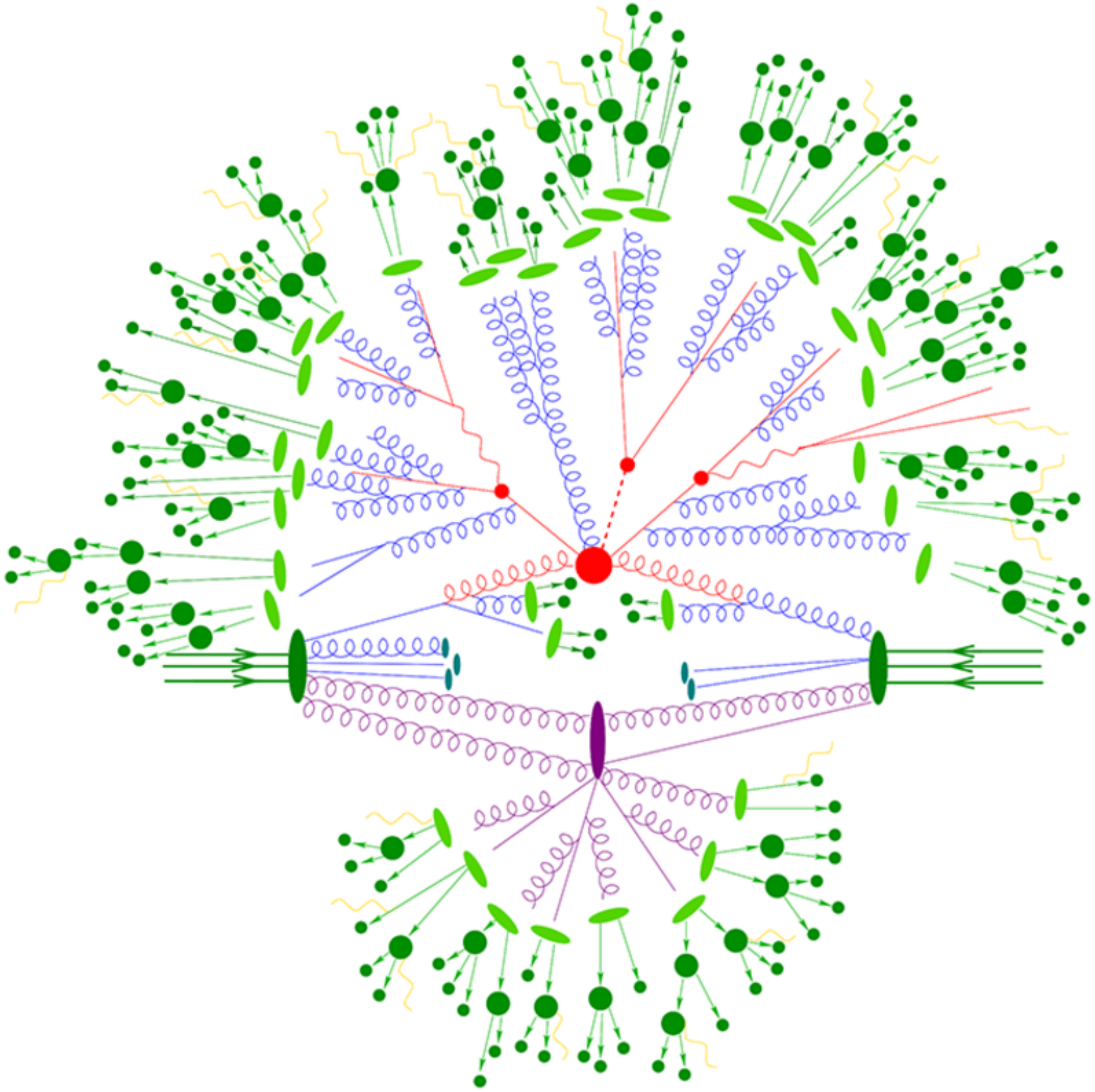

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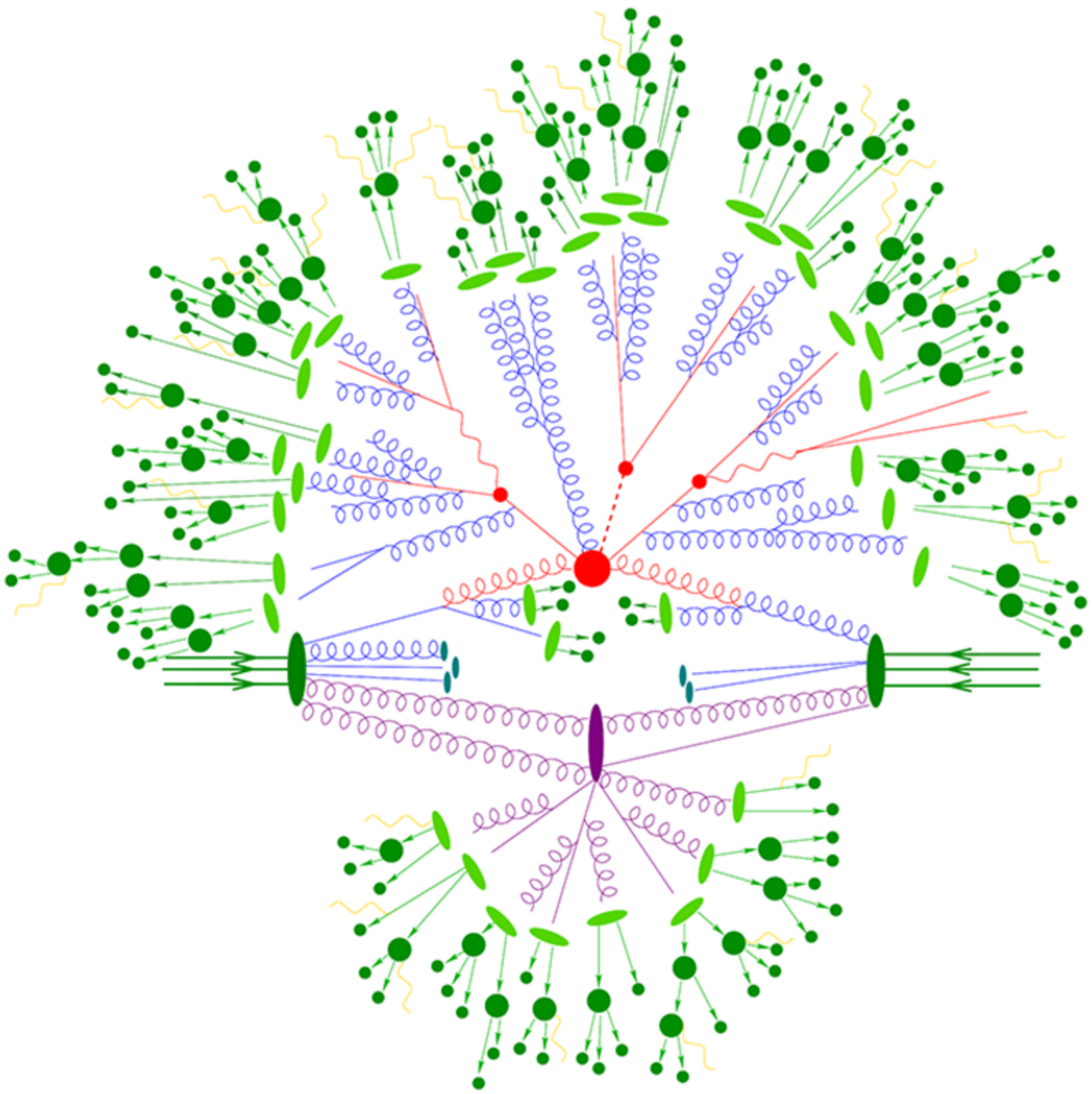
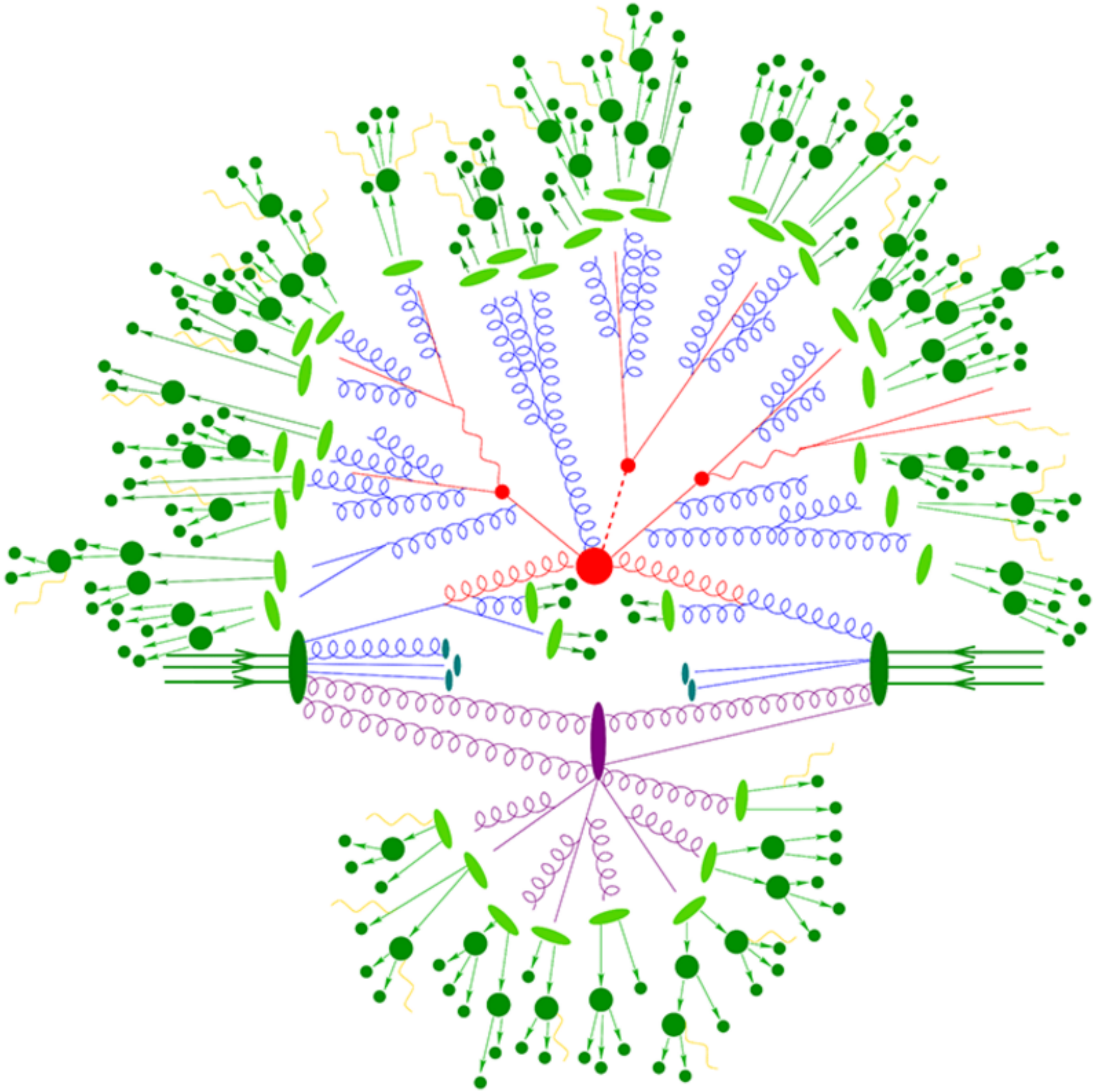


Complexity of Collider Events

Disentangle short and long distance physics

- Processes with colored partons result in **high multiplicity events**
- **Impossible to do in fixed order**
- Luckily, we can disentangle process into different energy regimes (factorization):
  - **Hard interaction** $Q \approx \mathcal{O}(100\,\text{GeV to TeV})$
  - **Parton Shower** $Q \rightarrow \mu \approx \mathcal{O}(1\,\text{GeV})$
  - **Hadronization** $\mu \rightarrow \Lambda < \mathcal{O}(1\,\text{GeV})$



Quest for precision

...and current status

[CMS at Higgs working group — ’21]

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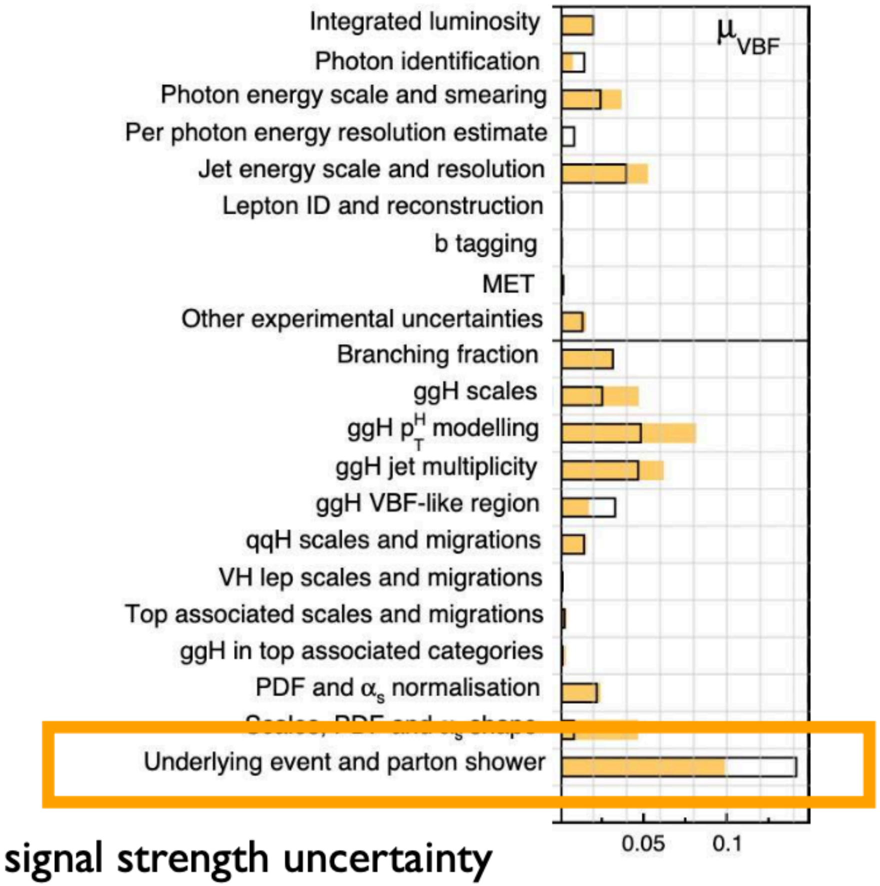

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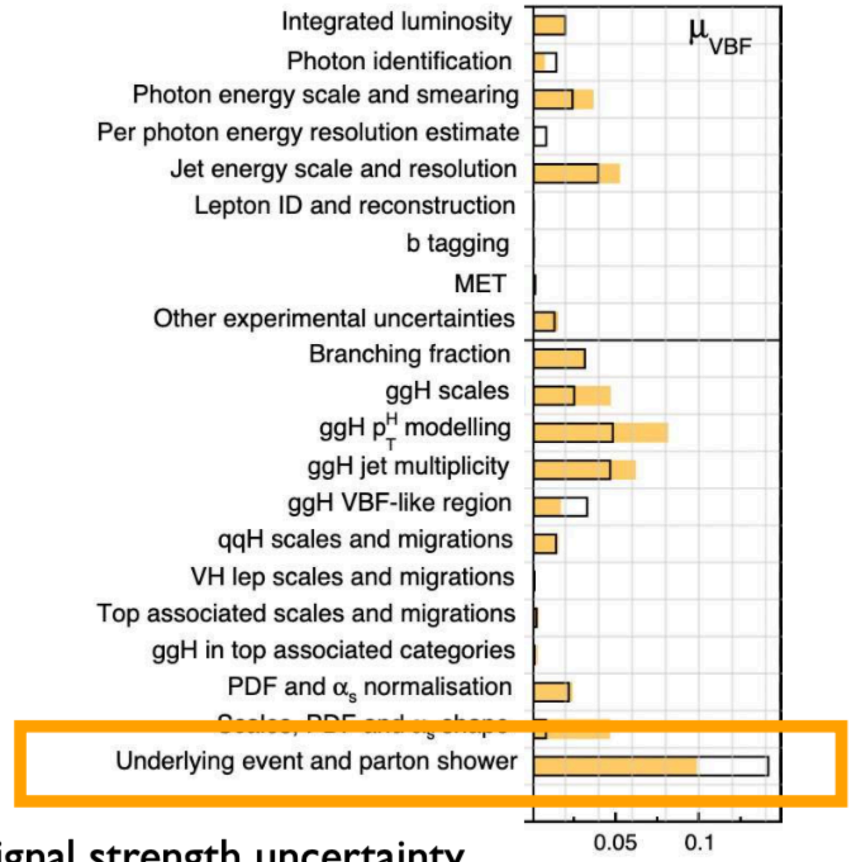

signal strength uncertainty

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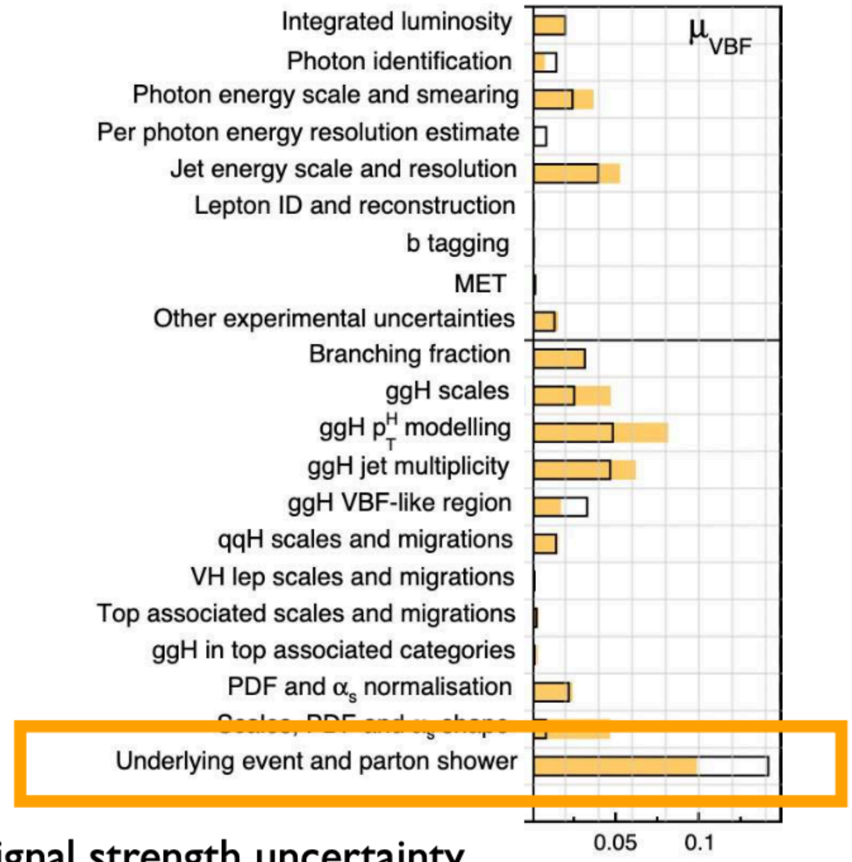

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- Lack of systematic expansion:
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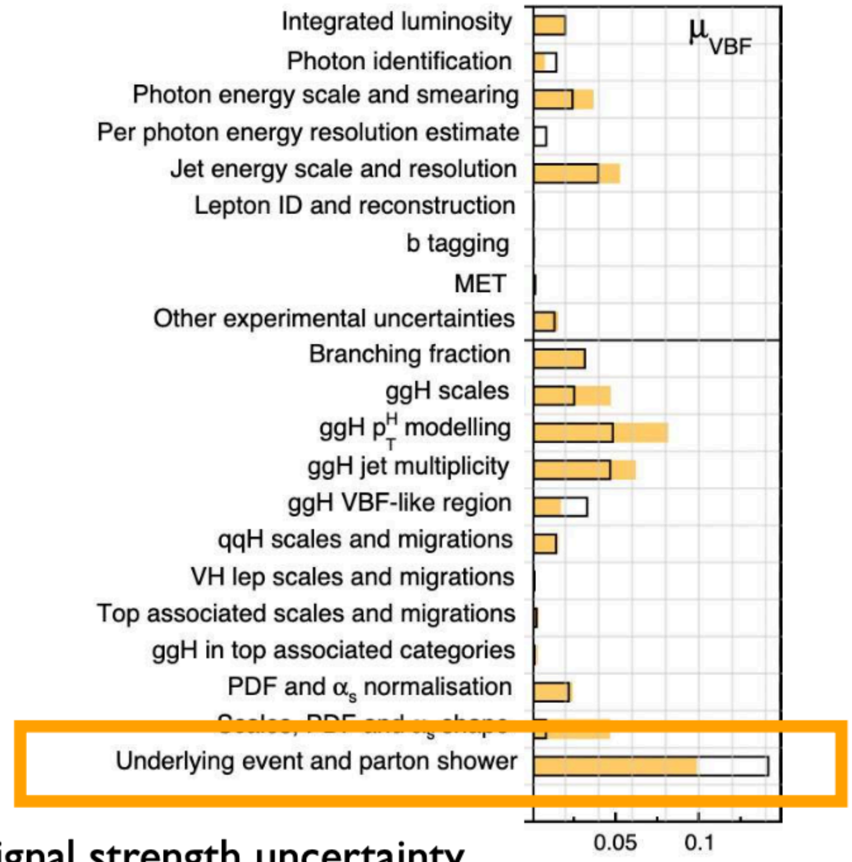
- **Unfortunate reality:** estimation of PS error often just from comparing different MC generators
- Although we know:
  different types of showers give different levels of accuracy depending on observable: **global vs. non-global**

  ➡ **"Ultimate" goal:** formal tools to show accuracy of PS, eventually **Next-to-Leading-Log @ Next-to-Leading-Colour accurate shower** for all global and non-global observables



signal strength uncertainty

Parton Shower Activity

Progress in improving the PS accuracy

- **Assessing the logarithmic accuracy of a shower**
  Herwig [1904.11866, 2107.04051], Deductor [2011.04777], Forshaw, Holguin, Plätzer [2003.06400]
  PanScales [1805.09327, 2002.11114], Alaric [2110.05964], …
- **Triple collinear / double soft splittings**
  Dulat, Höche, Krauss, Gellersen, Prestel [1705.00982, 1705.00742, 1805.03757, 2110.05964]
  Li & Skands [1611.00013], Löschner, Plätzer, Simpson Dore [2112.14454], …
- Matching to fixed-order **see Alexander's talk**
  NLO; i.e. Frixione & Webber [0204244], Nason [0409146], …
  NNLO; i.e. UNNLOPS [1407.3773], MiNNLOps [1908.06987], Vincia [2108.07133], …
  NNNLO; Prestel [2106.03206], Bertone, Prestel [2202.01082]
- Colour (and spin) correlations **see Simon's talk**
  Forshaw, Holguin, Plätzer, Sjödahl [1201.0260, 1808.00332, 1905.08686, 2007.09648, 2011.15087]
  Deductor [0706.0017, 1401.6364, 1501.00778, 1902.02105], Herwig [1807.01955], Plätzer & Ruffa [2012.15215]
  PanScales [2011.10054, 2103.16526, 2111.01161], …
- Electroweak corrections
  Vincia [2002.09248, 2108.10786], Pythia [1401.5238], Herwig [2108.10817], …

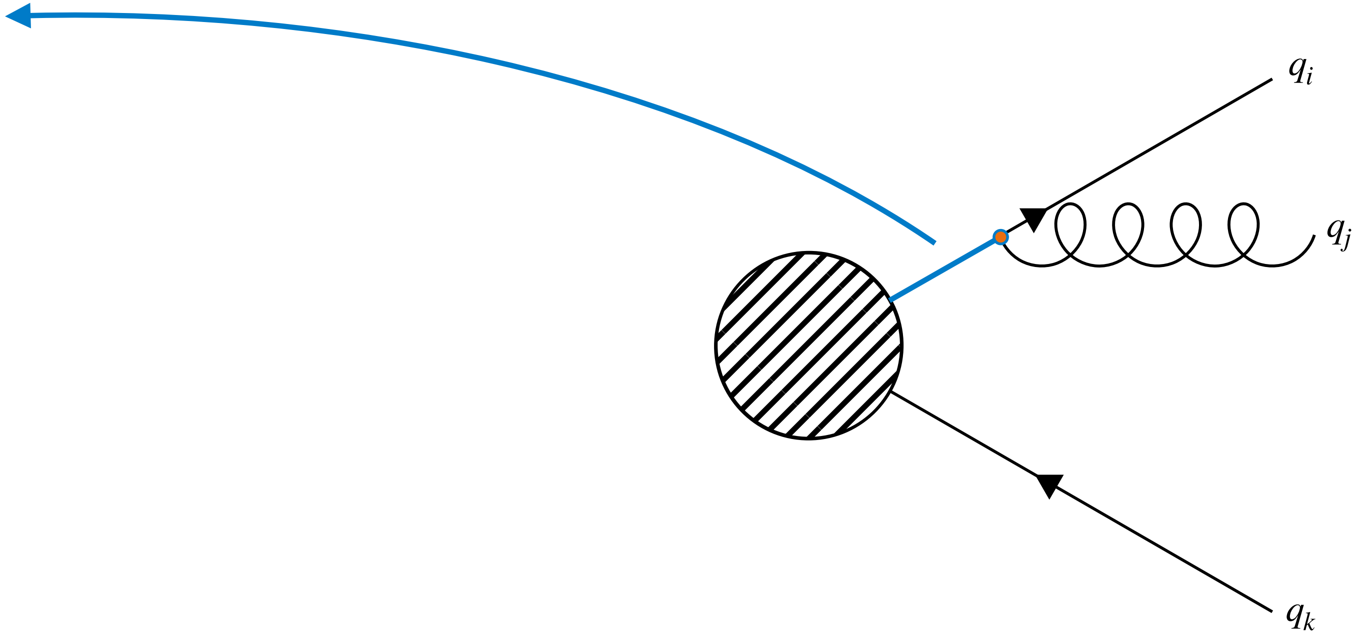
Super-active field of research:
taken from Melissa van Bleekveld's talk at the CERN workshop on parton showers for future colliders.

Building blocks of parton showers

Soft and collinear factorization

- Leading contributions from emissions in **soft and collinear regions**:

$$\frac{1}{(q_i + q_j)^2} = \frac{1}{2q_i^0 q_j^0 (1 - \cos\theta_{ij})}$$



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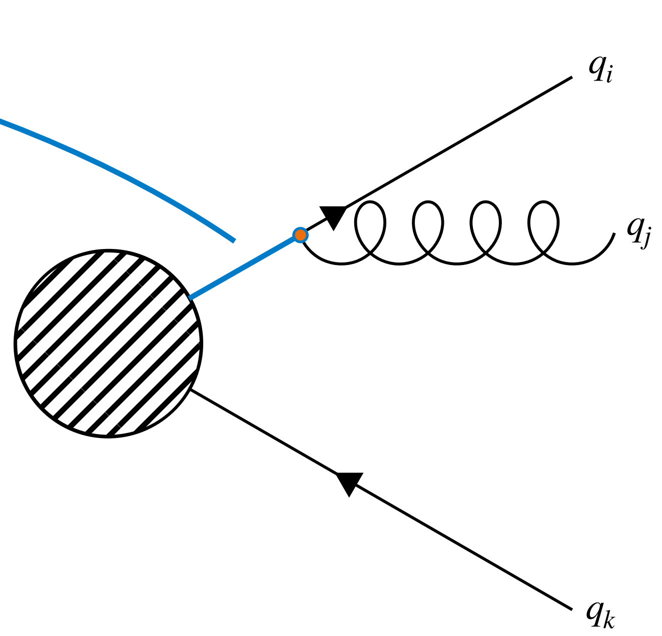

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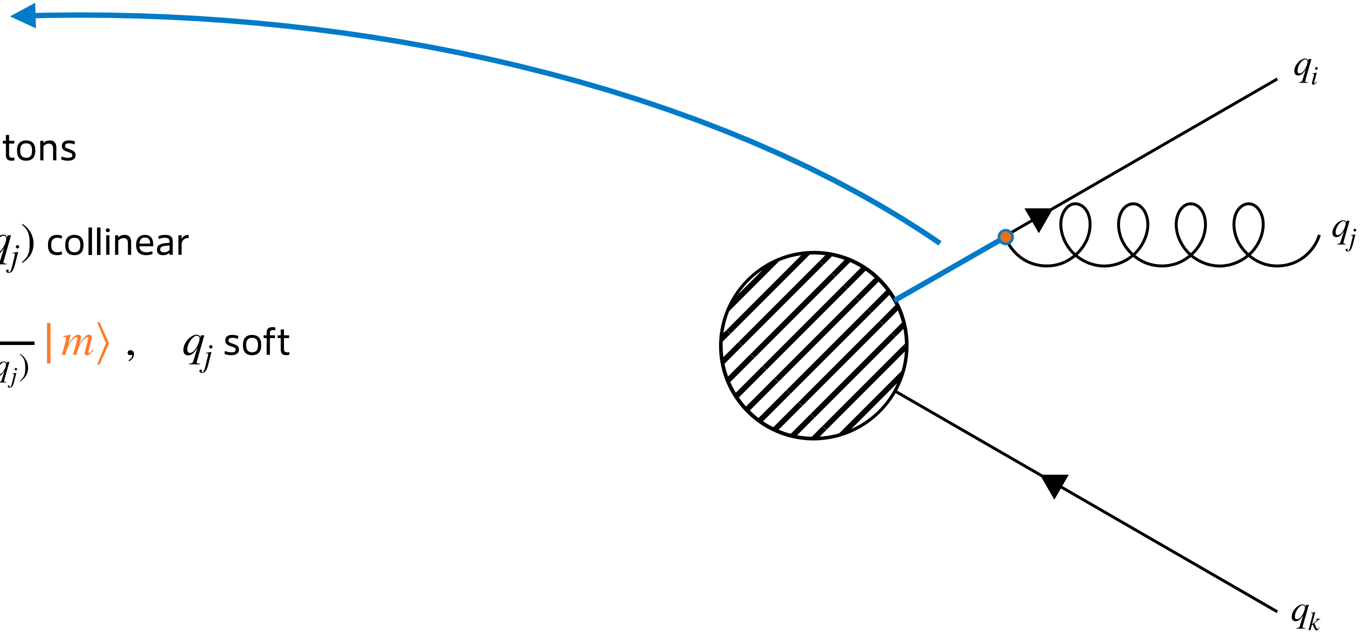
$$\langle m+1 \,|\, m+1 \rangle \simeq \begin{cases} 4\pi\mu^{2\varepsilon}\alpha_S \langle m \,|\, \hat{P}^{(ij)} \frac{1}{q_i \cdot q_j} \,|\, m \rangle \,, & (q_i, q_j) \text{ collinear} \\ -8\pi\mu^{2\varepsilon}\alpha_S \sum_{k,i} \langle m \,|\, \mathbf{T}_i \cdot \mathbf{T}_k \frac{q_i \cdot q_k}{(q_i \cdot q_j)(q_k \cdot q_j)} \,|\, m \rangle \,, & q_j \text{ soft} \end{cases}$$



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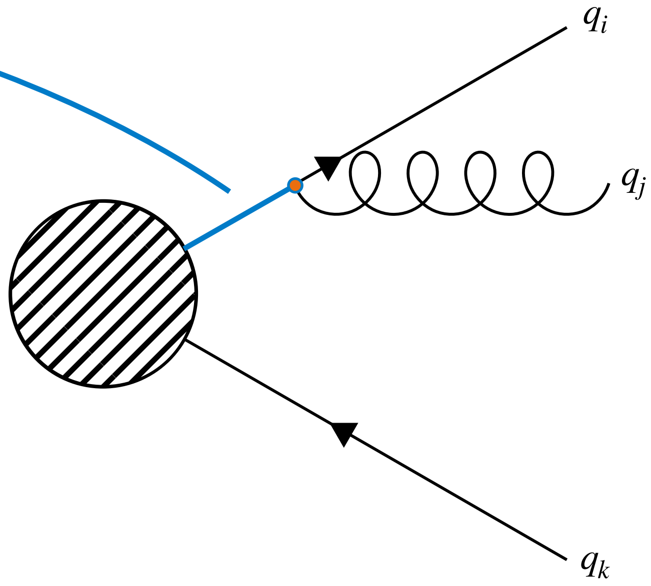
- Example for **splitting function**:

$$\langle s | \hat{P}_{qq}(z) | s' \rangle = \delta_{s,s'} C_F \left[\frac{1+z^2}{1-z} - \varepsilon(1-z) \right]$$



$$q_i^\mu = z p^\mu + k_\perp^\mu - \frac{k_\perp^2}{z} \frac{n^\mu}{2p \cdot n}$$

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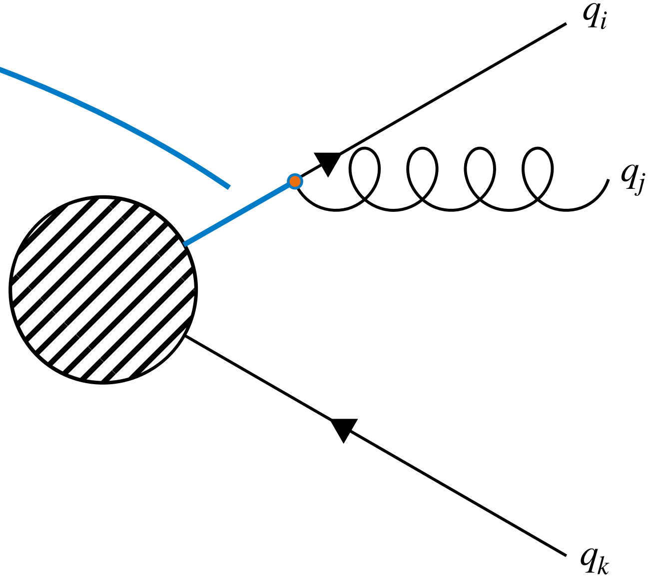

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- Use this to define **Sudakov form factor/splitting kernel** $\Delta(t_0,t)$ in PS emission probability :

$$\mathrm{d}P(\text{1st emission at }t) = \frac{\mathrm{d}t}{t}\int_{z_-}^{z_+}\frac{\alpha(z,t)}{2\pi}\hat{P}(z,t)\mathrm{d}z \times \exp[-\Delta(t_0,t)]$$

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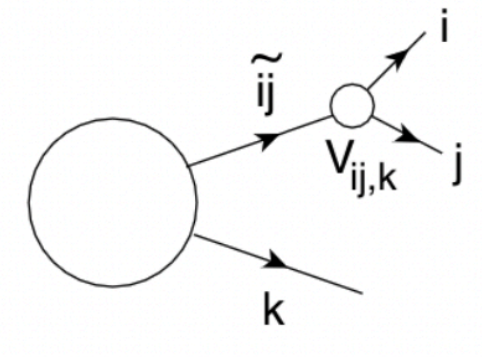
Splitting kernel example: Catani-Seymour

- Can capture soft and collinear limits simultaneously using **Catani-Seymour-dipole** kernel (e.g. by inference of large N_C -limit to construct kernel)

$$\mathscr{D}_{ij,k}(p_1,\ldots,p_{m+1}) = -\frac{1}{2q_i\cdot q_j}\,{}_m\langle\Psi|\frac{\mathbf{T}_{ij}\cdot\mathbf{T}_k}{\mathbf{T}_{ij}^2}\mathbf{V}_{ij,k}|\Psi\rangle_m, \quad |\Psi\rangle = |1,\ldots,\widetilde{ij},\ldots,\tilde{k},\ldots,m+1\rangle$$



[Catani, Seymour '97]

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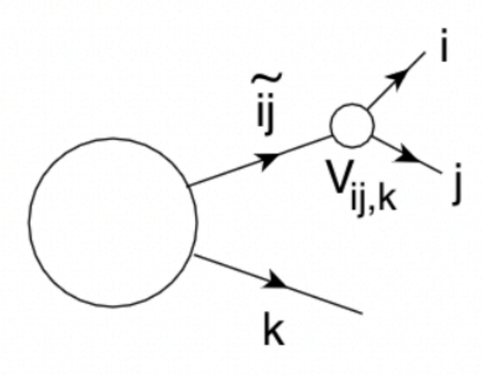
and e.g. $\langle s|\mathbf{V}_{ij,k}|s'\rangle = 8\pi\mu^{2\varepsilon}\alpha_S C_F\left[\frac{2}{1-\tilde{z}_i(1-y_{ij,k})} - (1+\tilde{z}_i) - \varepsilon(1-\tilde{z}_i)\right]\delta_{ss'}$



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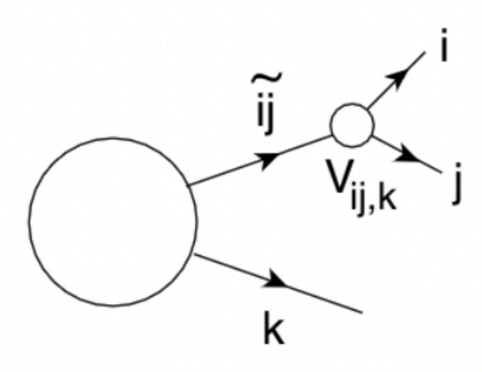

[Catani, Seymour '97]

- **Reproduce both limits with smooth interpolation.**

$$\frac{1}{q_i\cdot q_j}\mathbf{V}_{ij,k} \to \begin{cases} 8\pi\mu^{2\varepsilon}\alpha_S\frac{1}{q_i\cdot q_j}\hat{P}^{(ij)}\,, & (q_i,q_j)\text{ collinear} \\ 16\pi\mu^{2\varepsilon}\alpha_S\frac{1}{q_i\cdot q_j}\mathbf{T}_{ij}^2\frac{q_i\cdot q_k}{(q_i+q_k)\cdot q_j}\,, & q_j\text{ soft, non-singular in }(j\parallel k)\end{cases}$$

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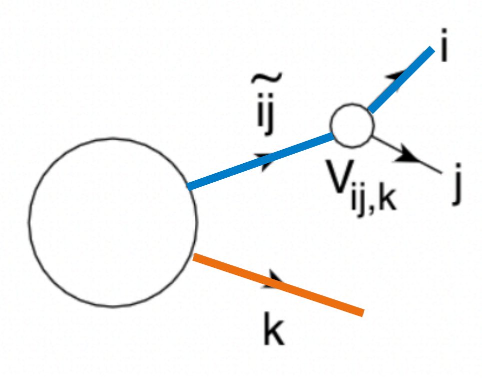

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- Soft result is **partitioned**:

$$\frac{q_i\cdot q_k}{(q_i\cdot q_j)(q_k\cdot q_j)} = \frac{q_k\cdot q_j + q_i\cdot q_j}{q_k\cdot q_j + q_i\cdot q_j}\frac{q_i\cdot q_k}{(q_i\cdot q_j)(q_k\cdot q_j)} = \frac{q_i\cdot q_k}{q_i\cdot q_j(q_i+q_k)\cdot q_j} + \frac{q_i\cdot q_k}{q_k\cdot q_j(q_i+q_k)\cdot q_j}$$

Dipole Shower

Pros and Cons

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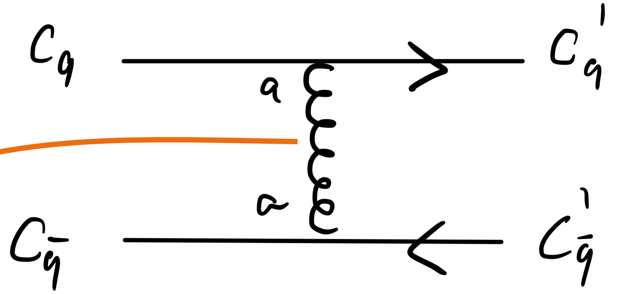
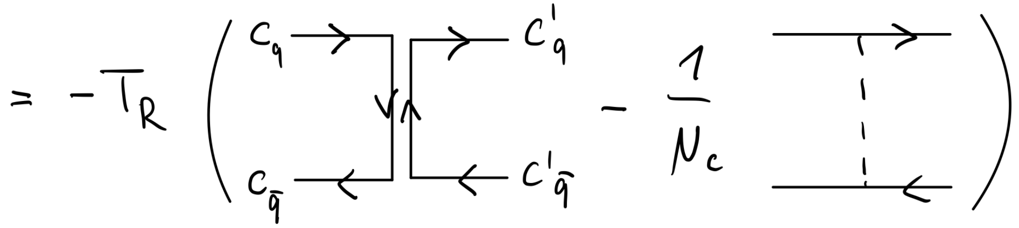


$$T_q \cdot T_{\bar q} = -(T^a)^{c_q c'_q} (T^a)_{c'_{\bar q} c_{\bar q}}$$

$$= -T_R \left[\delta^{c_q}_{c_{\bar q}} \delta^{c'_q}_{c'_{\bar q}} - \frac{1}{N_c} \delta^{c_q}_{c'_q} \delta^{c_{\bar q}}_{c'_{\bar q}} \right]$$



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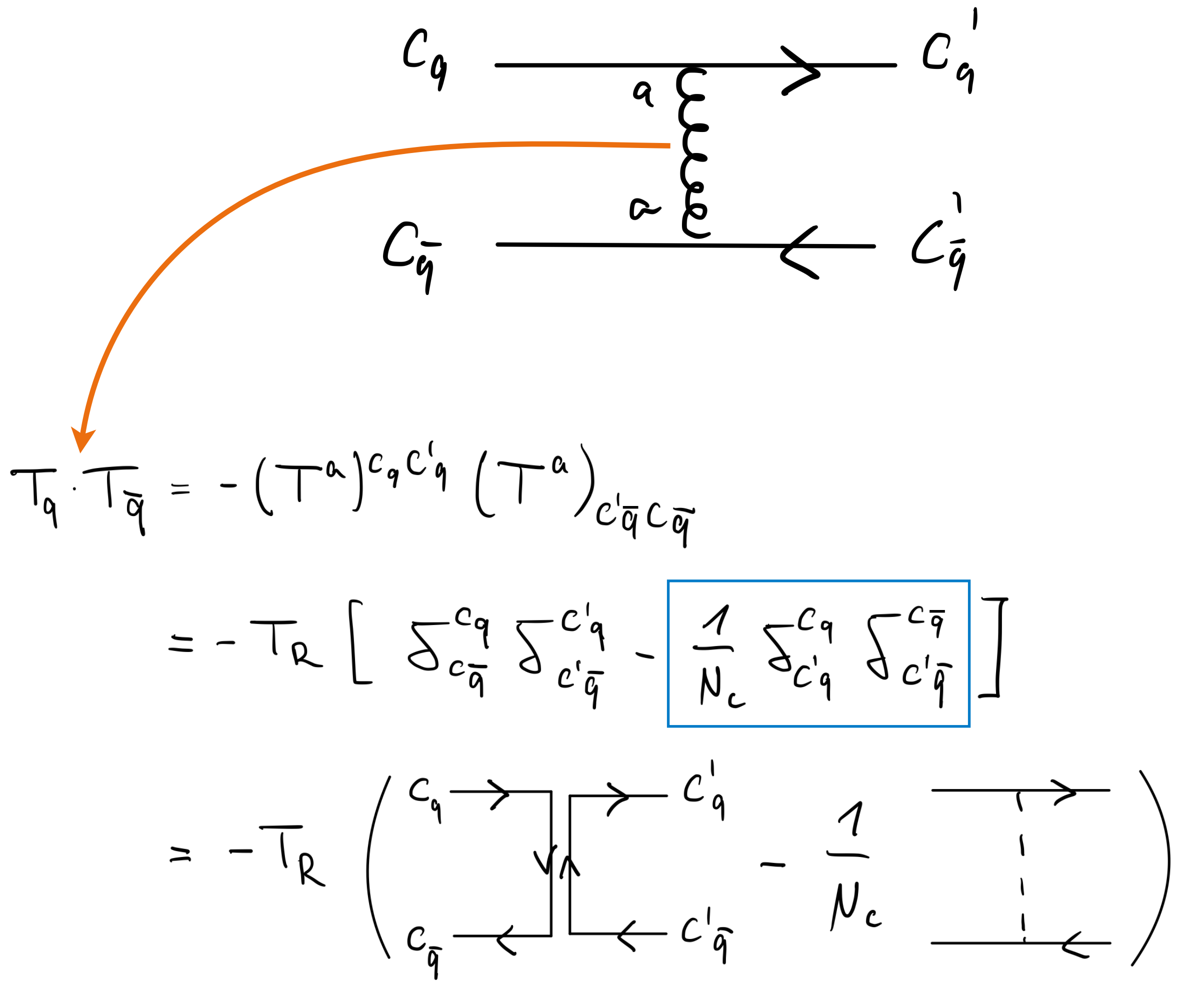
  - ➡ $1/N_c$- effects possibly become comparable to sub-leading logs, i.e. ~10% effects



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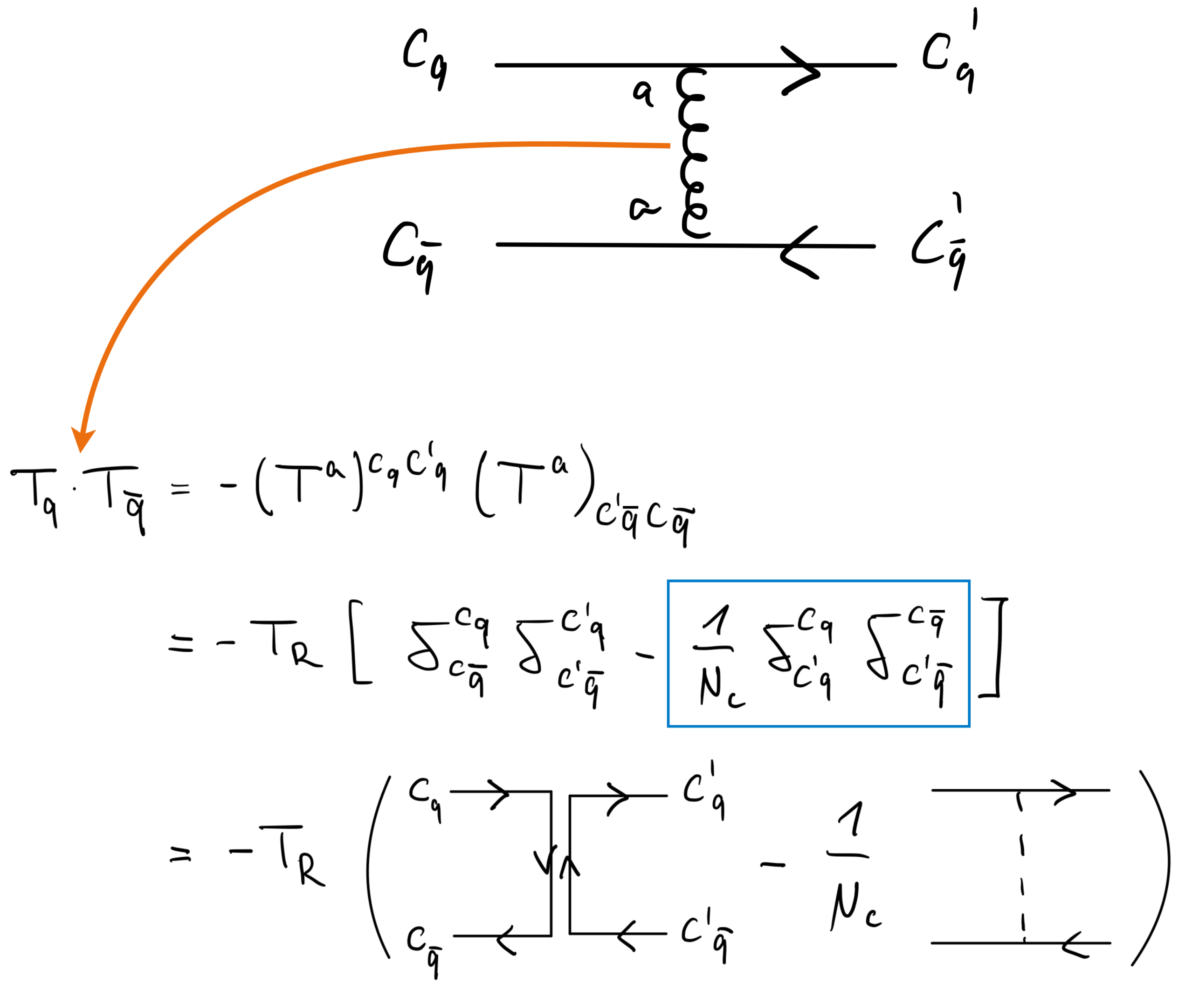
- Want: **algorithmic construction of kernel**



Towards high accuracy

- Goal: **construct (multi-)emission kernels algorithmically, inspired by Catani-Seymour dipoles,** i.e. smooth interpolation between collinear and soft:

    ➡ Organize kernels into collinear sectors

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factorized emission phase space

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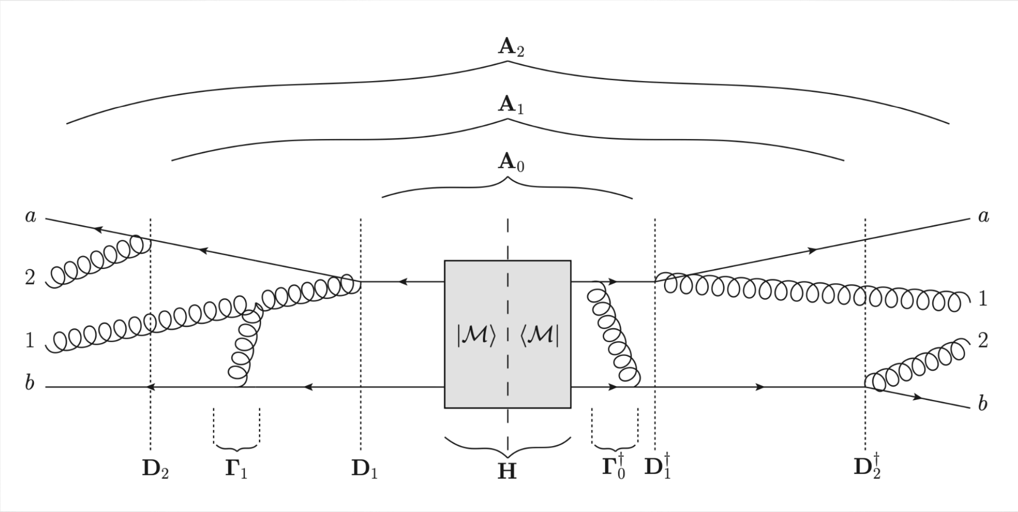
- Possibility to study the difference between **iterating the single- vs. multi-emission approximation**.



[Forshaw, Holguin, Plätzer–JHEP 09 (2020) 014]

Multi-Emission Kernels

Results

[S. Dore, ML, S. Plätzer; arXiv:2112.14454]

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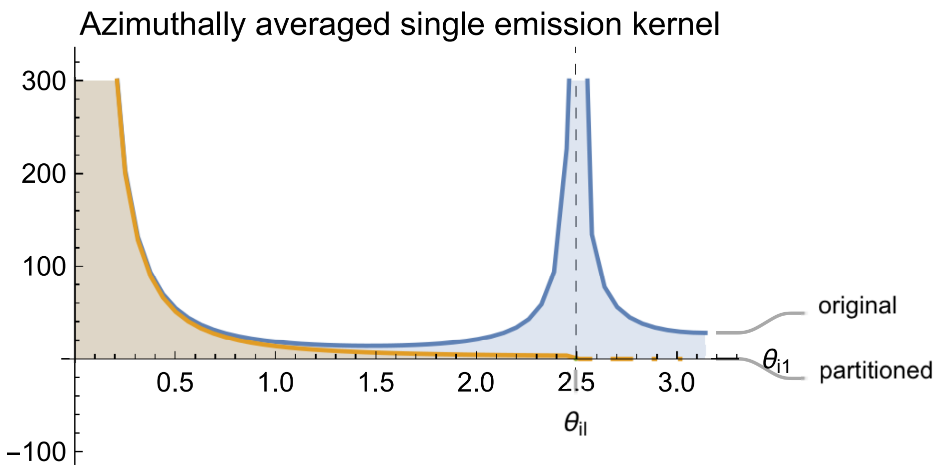

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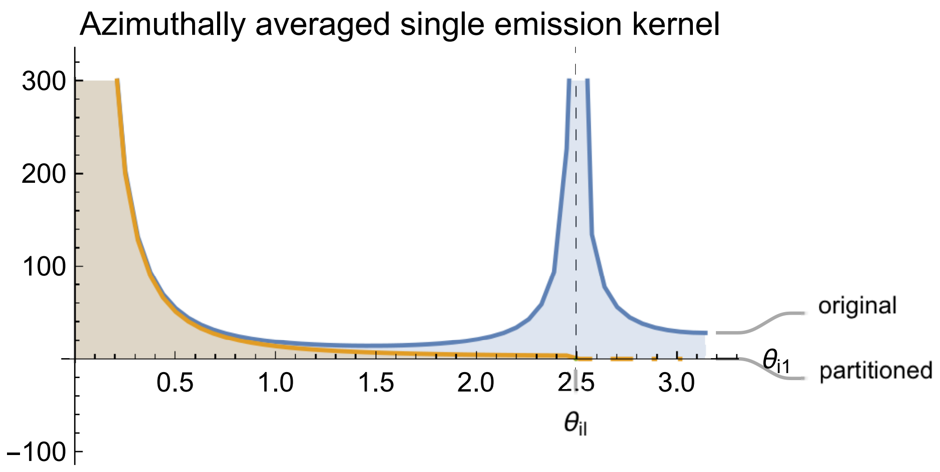
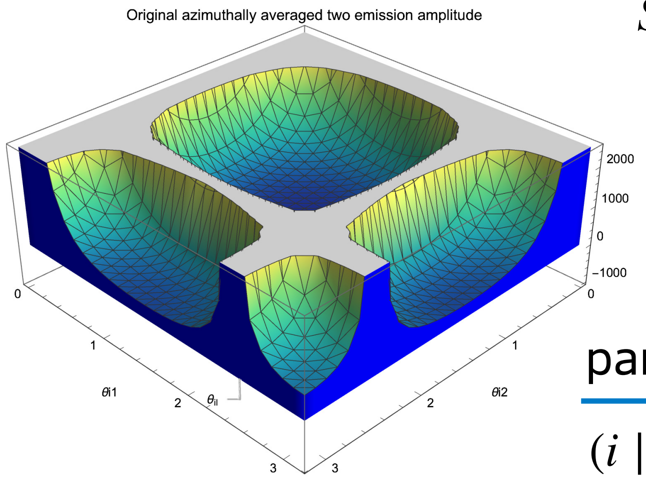
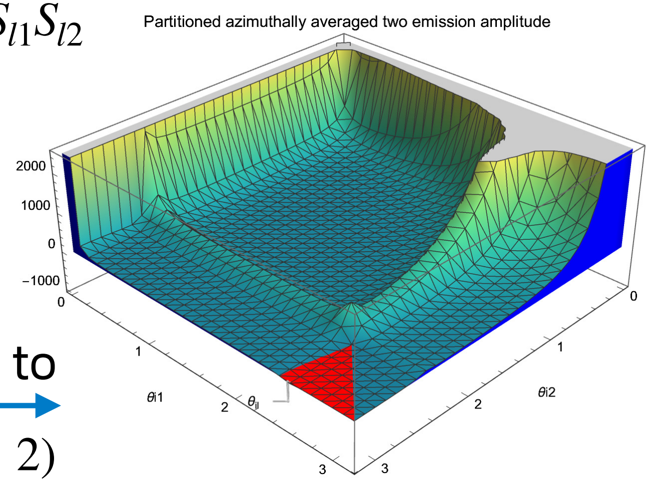


$$\mathscr{A} \propto \frac{1}{S_{i1}S_{12}}\frac{1}{S_{l1}S_{l2}}:$$



partition to

$(i \parallel 1)(l \parallel 2)$



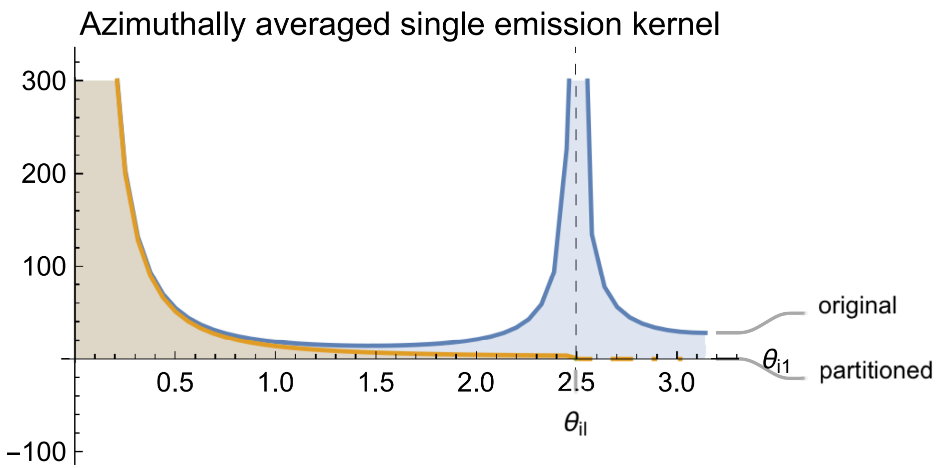
Multi-Emission Kernels

Results

[S. Dore, ML, S. Plätzer; arXiv:2112.14454]

- **Partitioning algorithms**
  - ➡ two options: fractional and subtractive
  - ➡ spread soft contributions over kernels
- **Momentum mapping**
  - ➡ Parameterization of how collinear limit is approached and transverse recoil is spread for multiple emissions

$$\mathscr{A} \propto \frac{1}{S_{i1}S_{l1}}:$$



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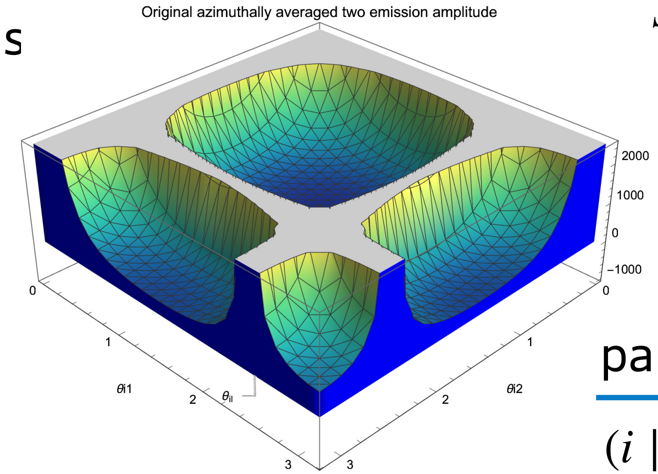

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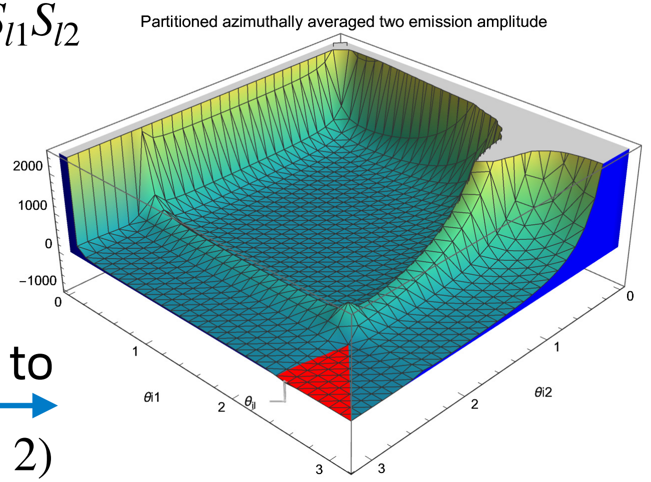

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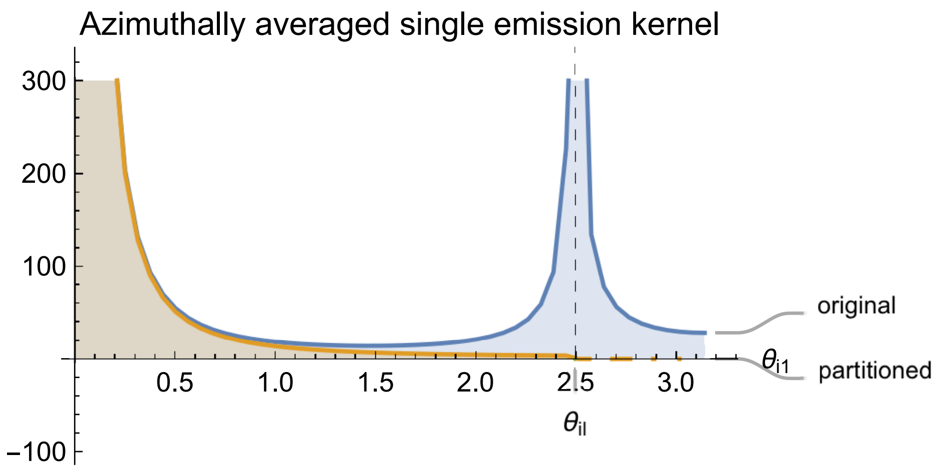
- **Amplitude level power counting**
  - ➡ extract leading soft/collinear contributions

$$\mathscr{A} \propto \frac{1}{S_{i1}S_{l1}}:$$



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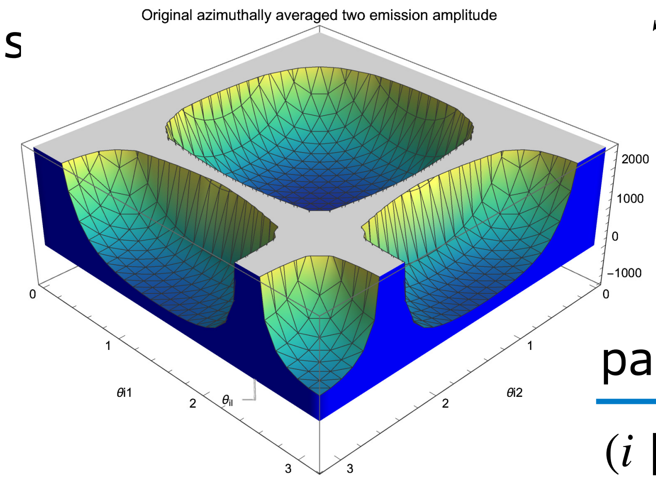

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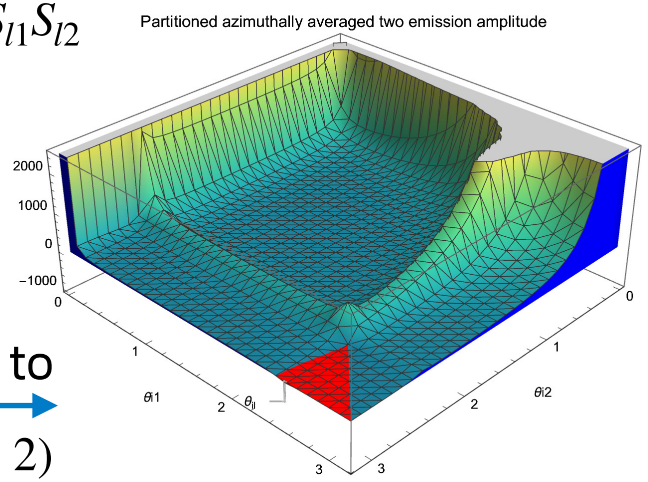
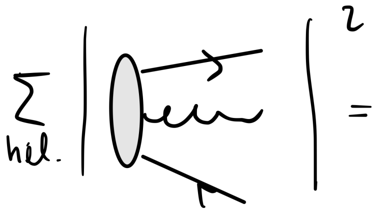


Squaring amplitudes

Uniform power counting



Want to know which amplitudes are relevant for soft/collinear limits when squaring:

➡ Determine squared amp (i.e. diff. xsec), but keep control at amplitude level

Squaring amplitudes

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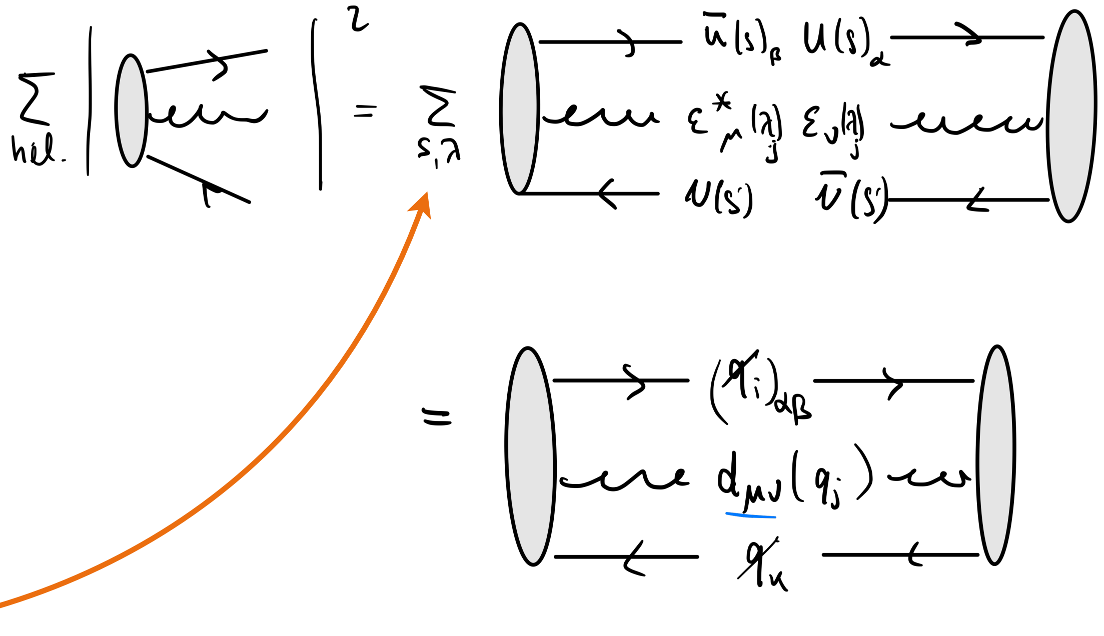

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Squaring amplitudes

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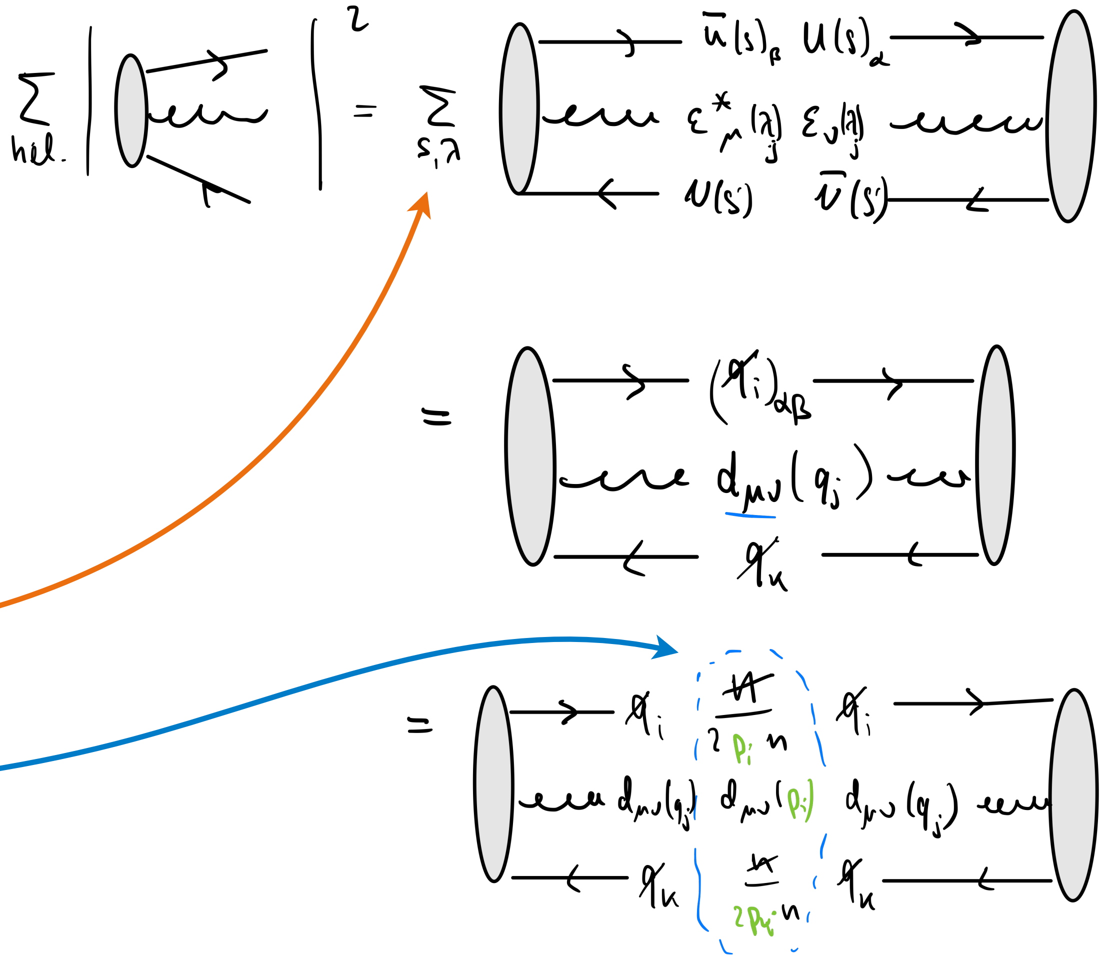

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2. Introduce **projectors** to disentangle amplitude and conjugate amplitude

$$d_{\mu\nu}(q) = -\eta^{\mu\nu} + \frac{n^{\mu}q^{\nu} + n^{\nu}q^{\mu}}{n \cdot q}$$

Squaring amplitudes

Uniform power counting

Want to know which amplitudes are relevant for soft/collinear limits when squaring:

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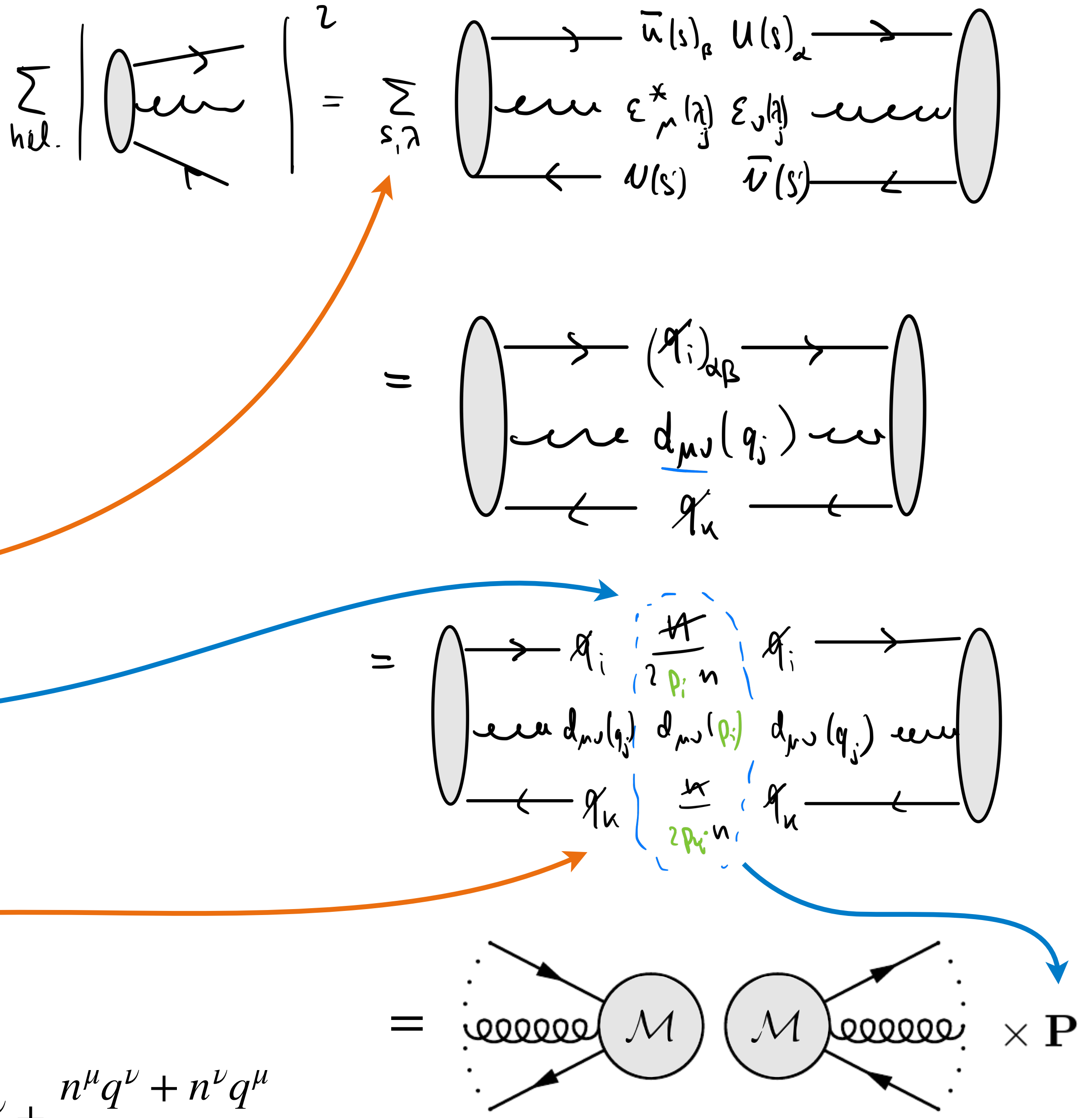
1. Carry out **spin/helicity sums** to replace spinors/ polarization vectors
2. Introduce **projectors** to disentangle amplitude and conjugate amplitude
3. Can now study soft/collinear scaling of **internal and external lines on same footing at amplitude level**



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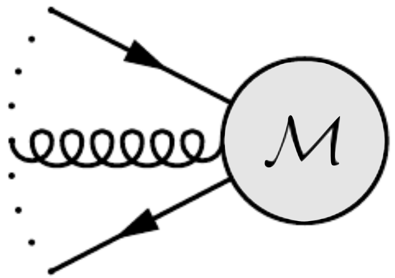
Power Counting

Sudakov decomposition



- Decompose momenta into forward, backward and transverse direction:

$$q_i^\mu = z_i\, p_i{}^\mu + y_i n^\mu + k_{\perp,i}{}^\mu$$

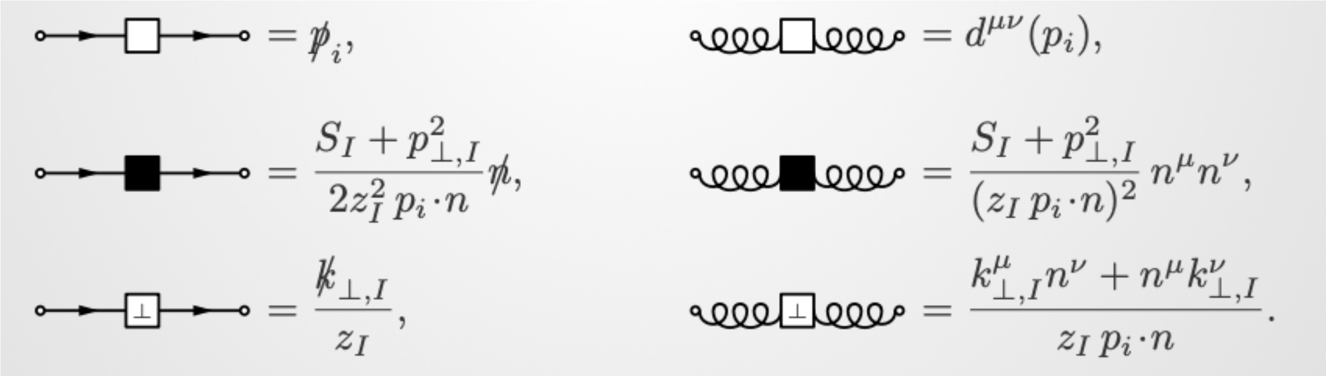
Power Counting

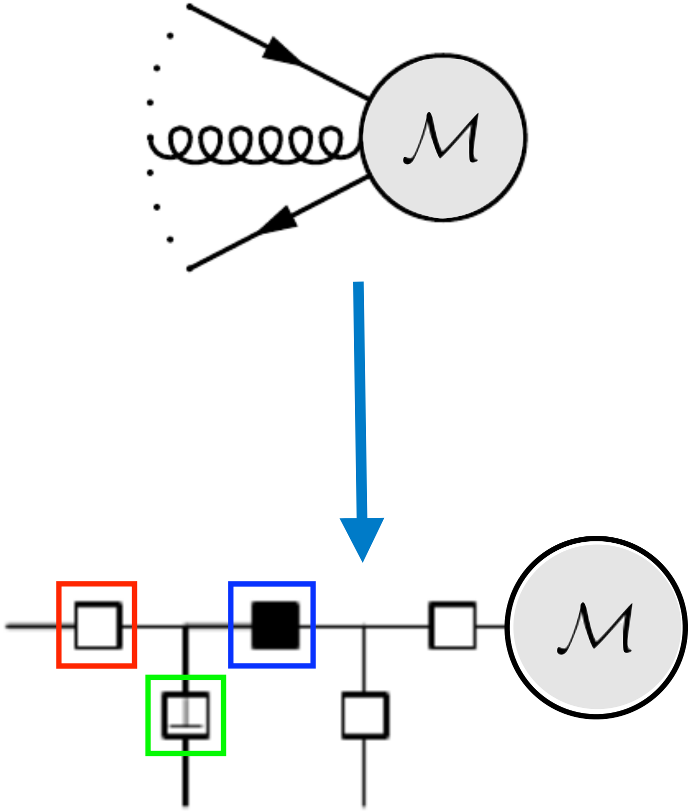
Sudakov decomposition

- Decompose momenta into forward, backward and transverse direction:

$$q_i^\mu = z_i \, p_i^\mu + y_i n^\mu + k_{\perp,i}^{\ \mu}$$

- Can decompose quark and gluon lines on same footing leading to **effective Feynman rules**:





Power Counting

Soft and collinear scaling

- Create **table of soft and collinear scaling** of lines

Scaling of hard lines:

	h	h+c	h+s	h+c+s
	λ	λ	λ	λ (bal.)
	0	λ	λ	λ (unbal.)
	λ^2	λ^2	λ	λ (bal.)
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Scaling of emissions:

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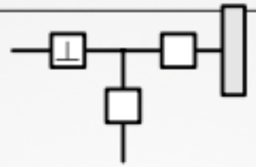
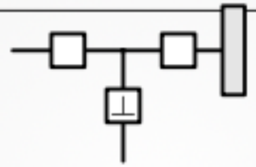
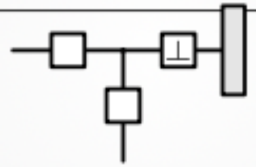
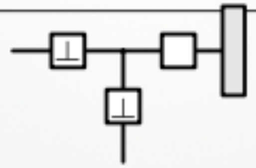
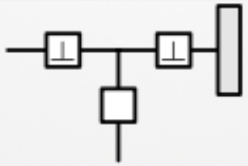
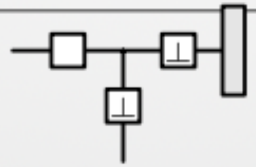
	λ	λ
	λ	1
	λ	λ
	λ^2	λ
	λ^2	λ^2
	λ^2	λ

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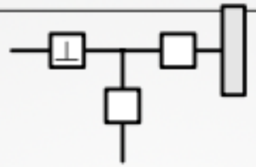
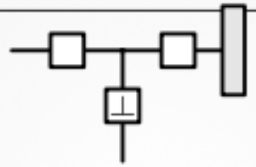
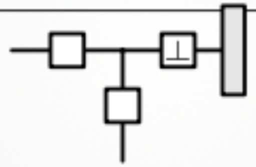
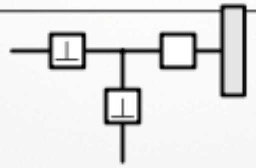
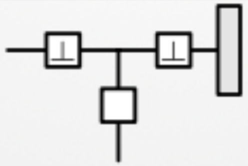
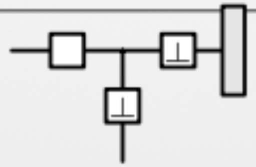
Scaling of emissions:

	s	c	s+c
	1	λ	λ
	1	λ^2	λ

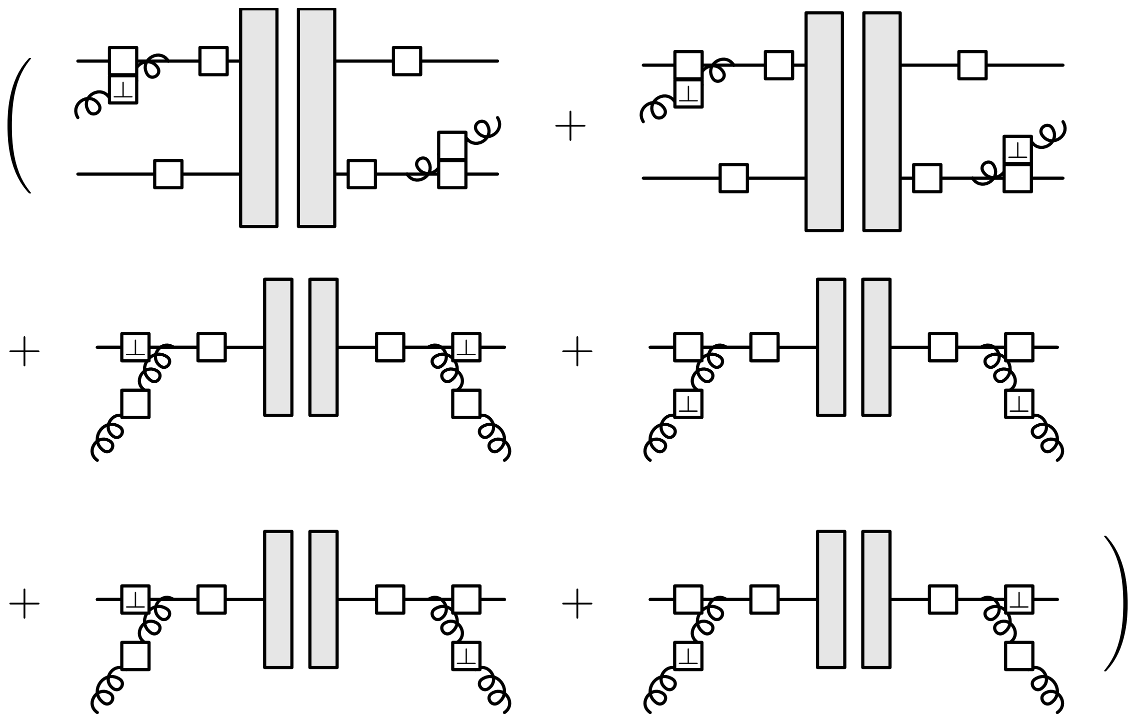
- Determine **leading amplitudes**
  (table shows scaling of amplitude numerator)

	C	S
	λ	λ
	λ	1
	λ	λ
	λ^2	λ
	λ^2	λ^2
	λ^2	λ

One emission Kernel

- One emission kernel as **partitioned collection of leading amplitudes**

$$\mathbb{U}_{(ij)} = \mathbb{P}_{(ij)}$$



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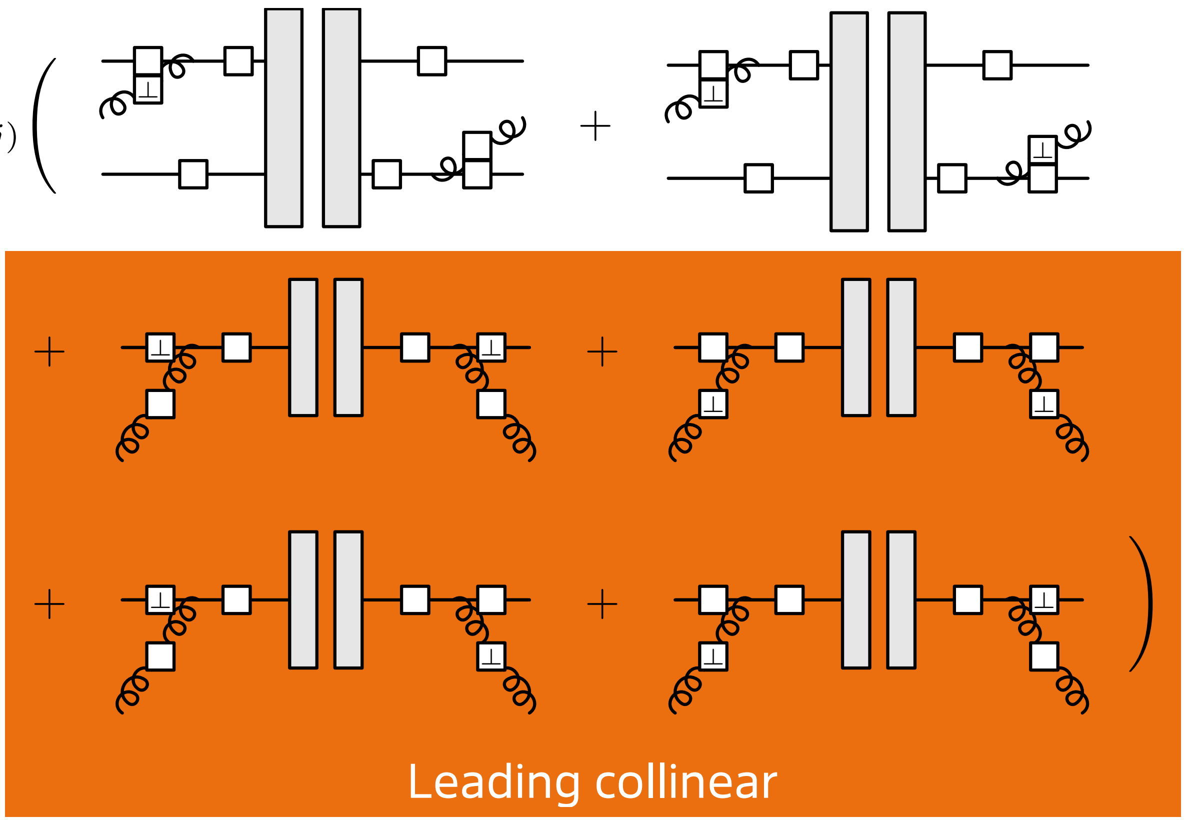

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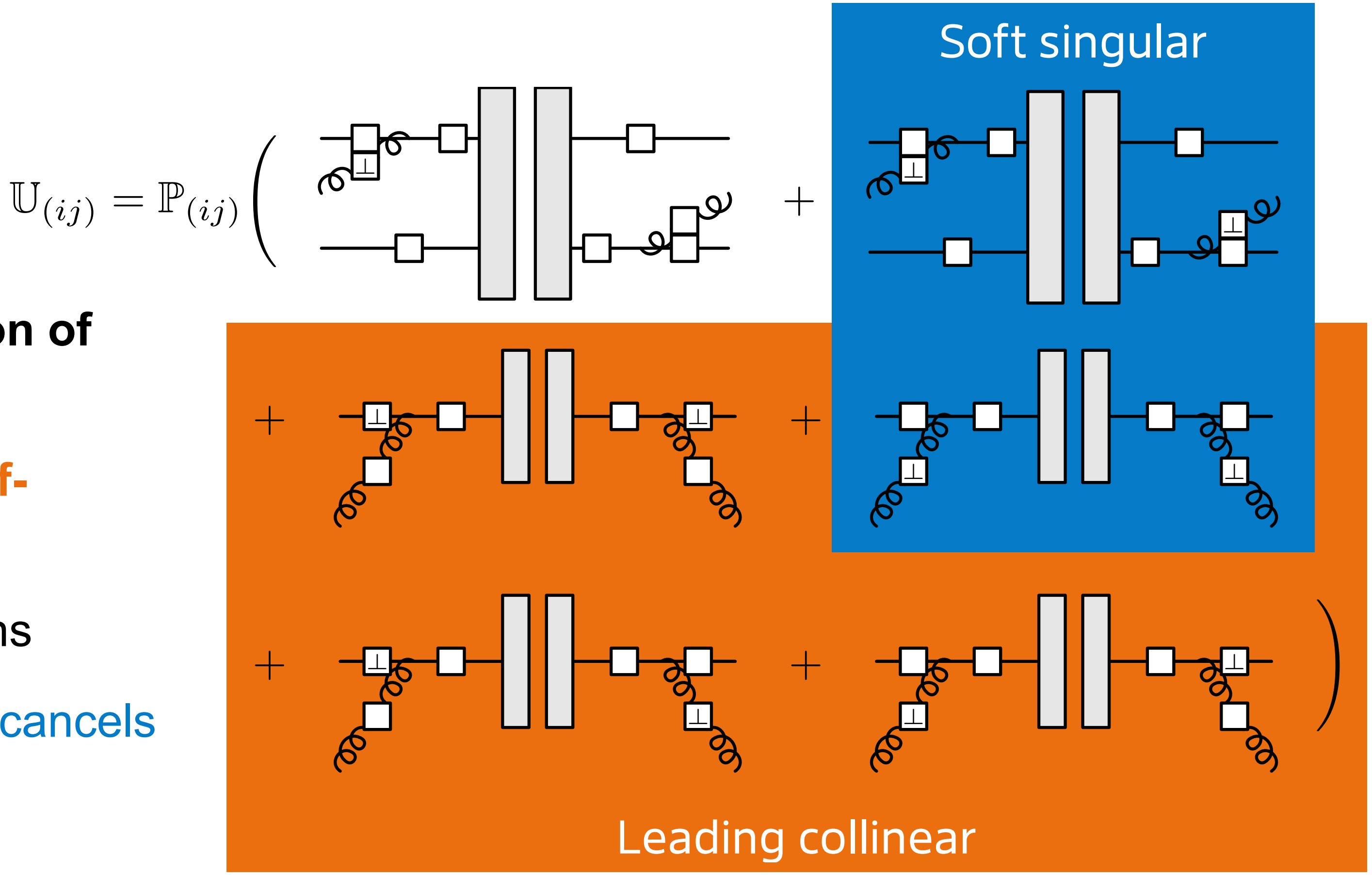
- Lightcone gauge: **leading collinear from self-energy**-like diagrams

One emission Kernel



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- Cross-talk between soft divergent contributions
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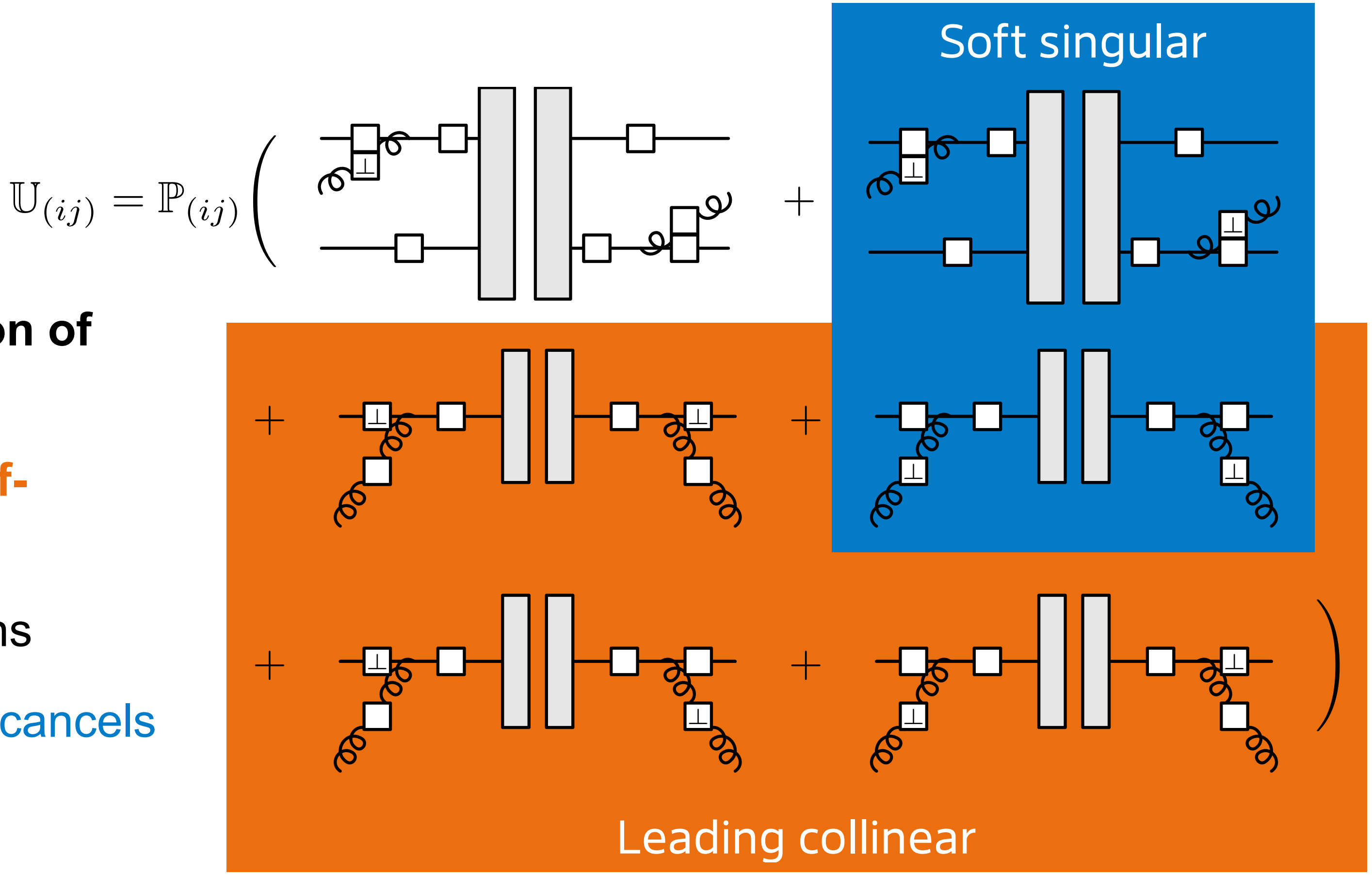
$$\langle s \,|\, \hat{P}_{qq}(z) \,|\, s' \rangle = \delta_{s,s'} C_F \left[(d-2)(1-z) + \frac{4z^2}{1-z} + 4z \right]$$

One emission Kernel



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  - **Soft singular** term from splitting function cancels
  - **Eikonal remains**
- **Kernel depends on momentum mapping,** e.g. transverse recoil distribution

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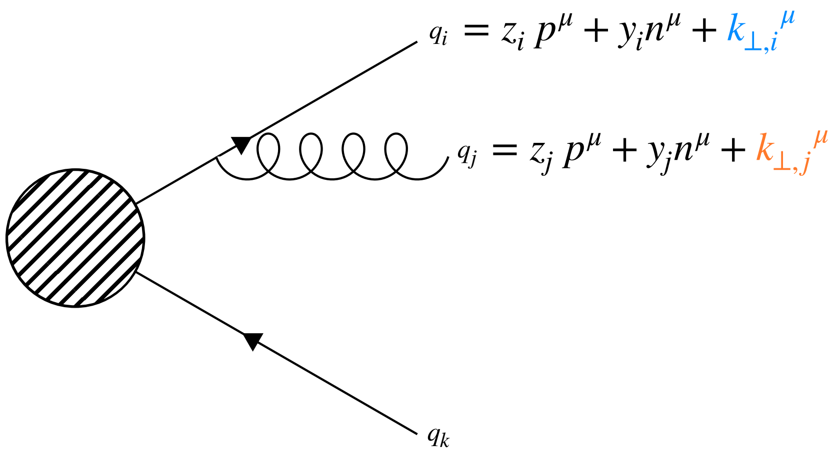
One emission Kernel

Dependence on transverse recoil

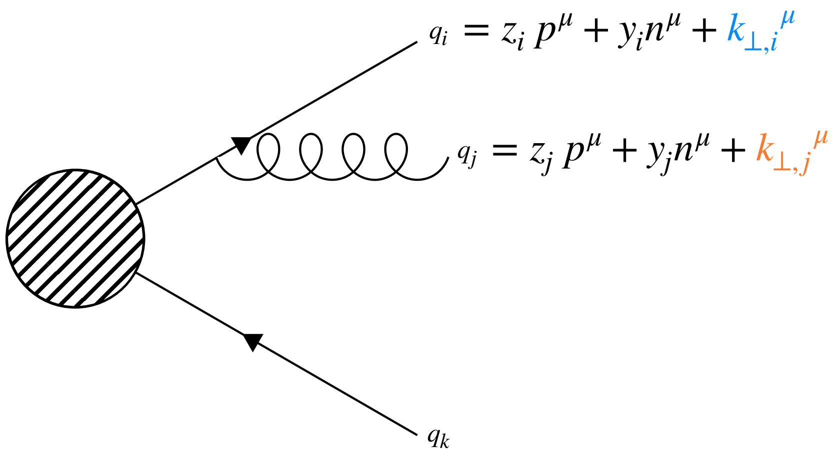


- Assignment of transverse recoil is not unique

  ➡ Choice needs to be translated to PS

One emission Kernel

Dependence on transverse recoil



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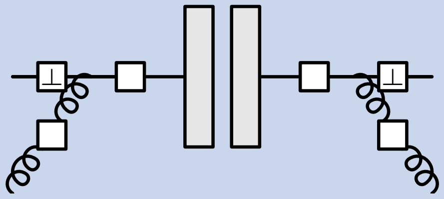
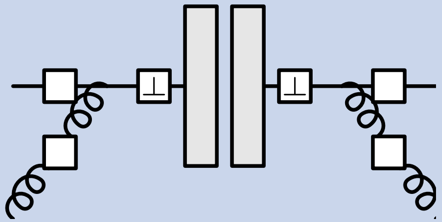
- **Form of our kernel changes** for different choices of transverse recoil!

Balanced	Unbalanced
$k_{\perp,i}{}^\mu + k_{\perp,j}{}^\mu = 0$	$k_{\perp,i}{}^\mu + k_{\perp,j}{}^\mu \neq 0$
	

One emission Kernel

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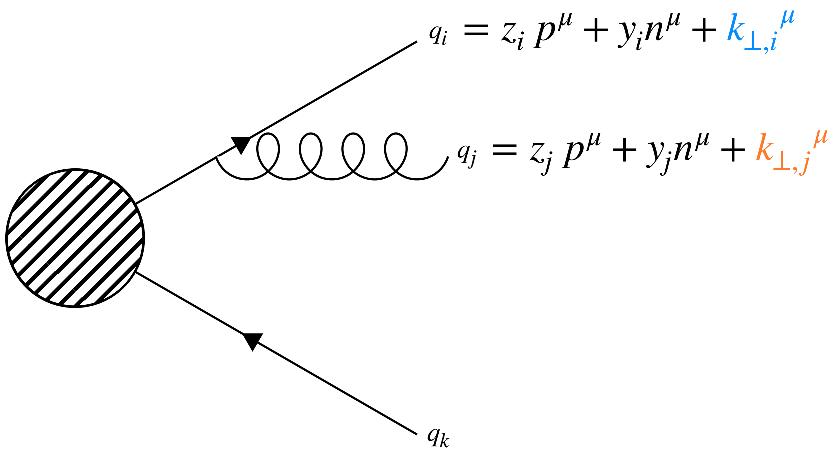

- Assignment of transverse recoil is not unique
  - ➡ Choice needs to be translated to PS
- **Form of our kernel changes** for different choices of transverse recoil!
- Still leads to the **same splitting functions** etc.

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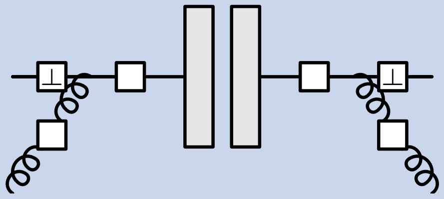
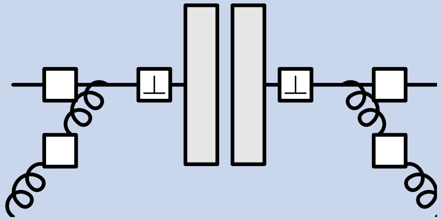
$k_{\perp,i}{}^\mu + k_{\perp,j}{}^\mu = 0$	$k_{\perp,i}{}^\mu + k_{\perp,j}{}^\mu \neq 0$
	

Example multi emission result

Triple collinear limit

Reproduction of the **triple collinear limit** using our power counting when tracing:

Example multi emission result

Triple collinear limit

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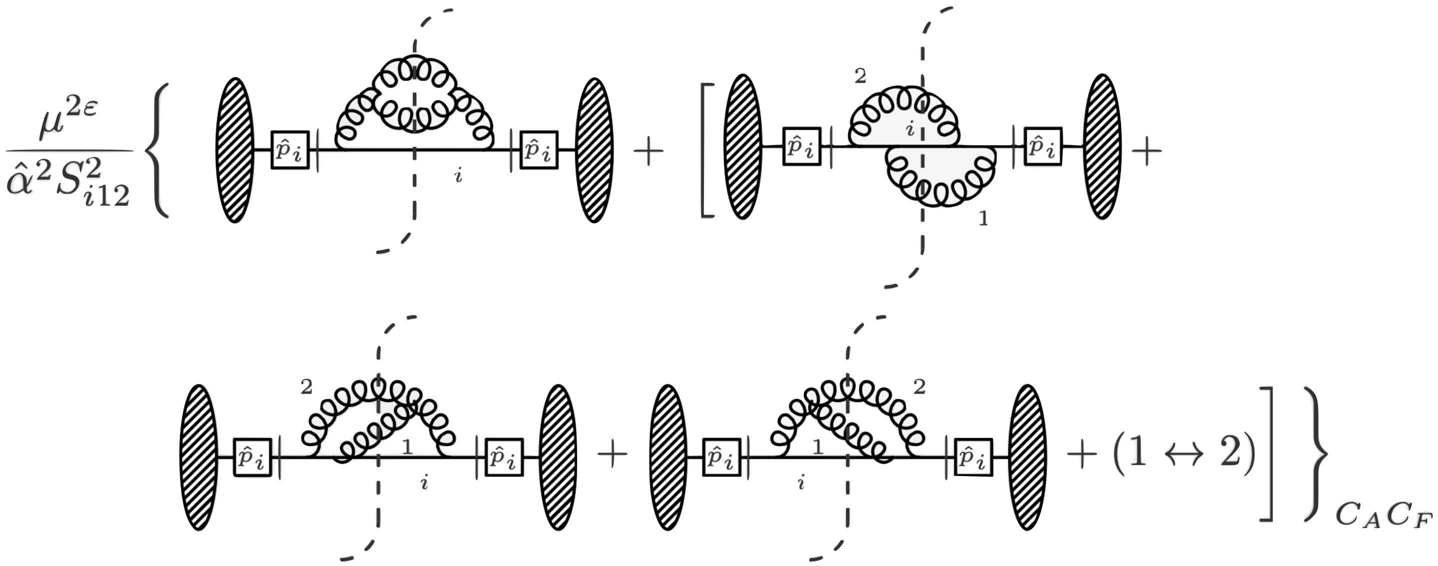

$$= \left(\frac{8\pi\alpha_S}{\hat{\alpha} S_{i12}}\mu^{\varepsilon}\right)^2 C_A C_F \, \langle \hat{P}_{ggq}^{\text{(non-Ab)}} \rangle_{\hat{\not{p}}_i} + \mathcal{O}\left(\beta_{il}^{-3/2}\right).$$

Example multi emission result

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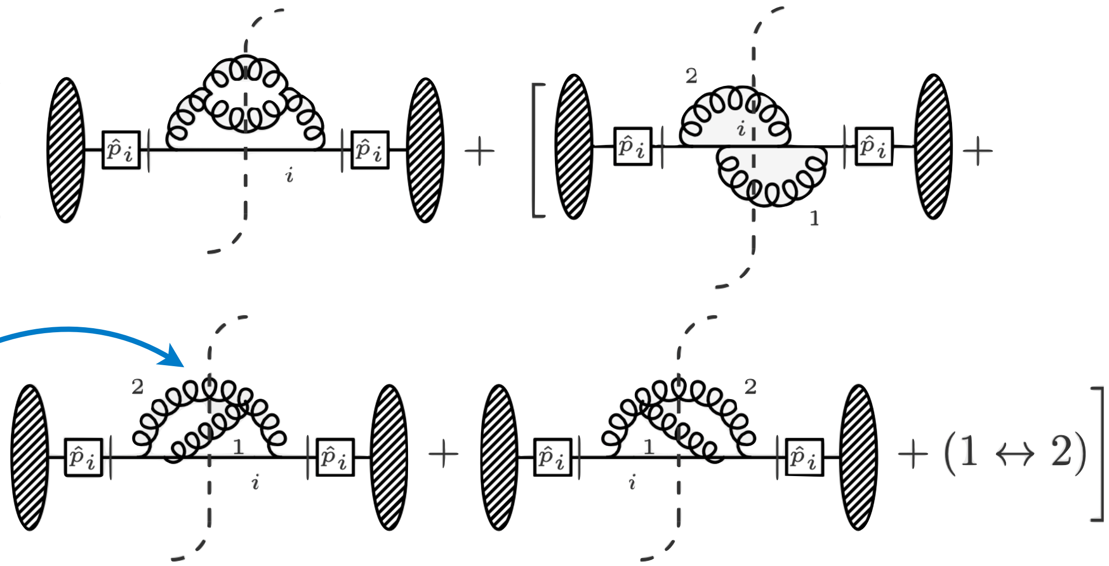
Reproduction of the **triple collinear limit** using our power counting when tracing:

$$\frac{\mu^{2\varepsilon}}{\hat{\alpha}^2 S_{i12}^2}\Bigg\{$$



$$+\,(1 \leftrightarrow 2)\Bigg]\Bigg\}_{C_A C_F}$$

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➡ "Non-iterated" topologies included

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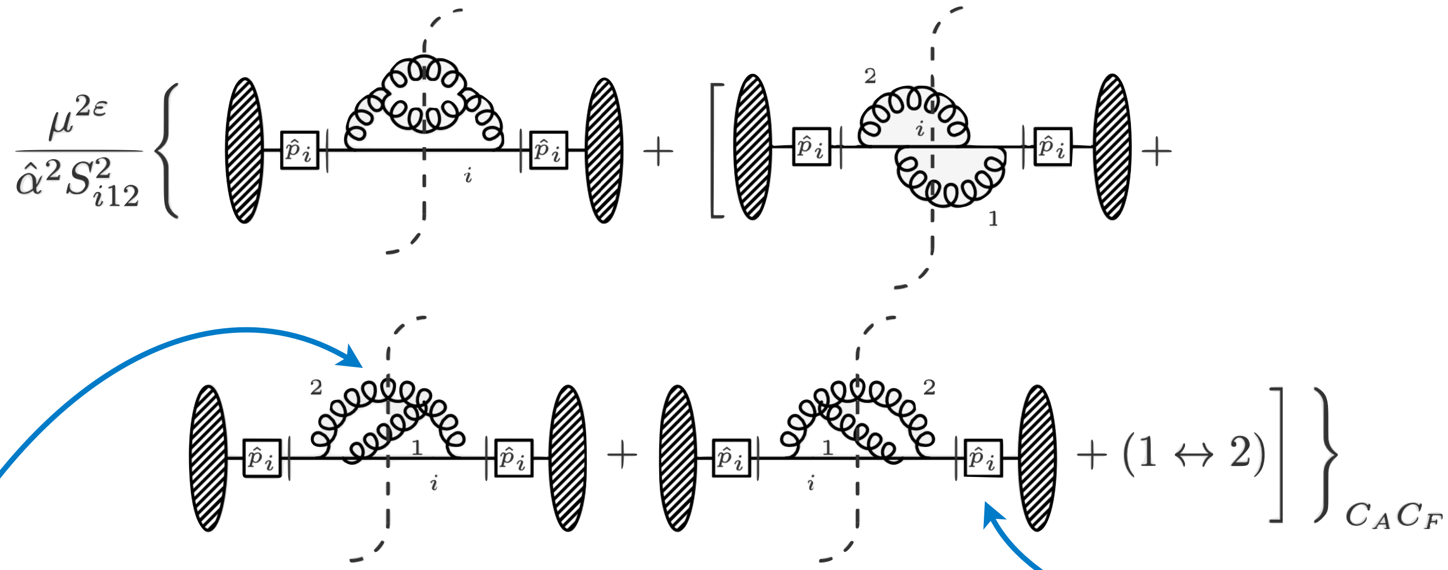

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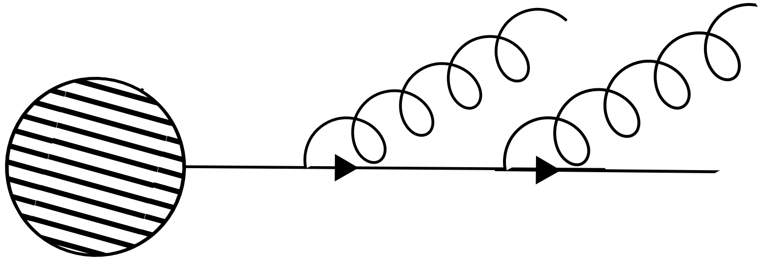
➡ "Non-iterated" topologies included

➡ Exhibition of factorization for two collinear gluons

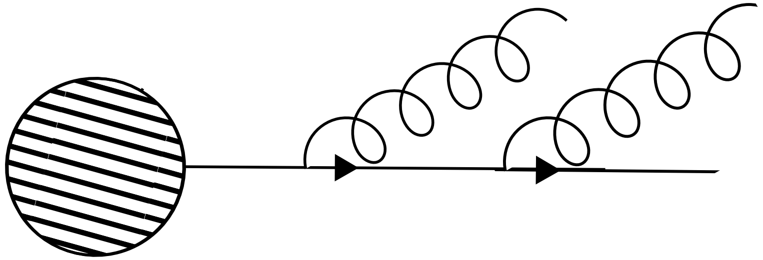
Example multi emission Result

Double soft decomposition



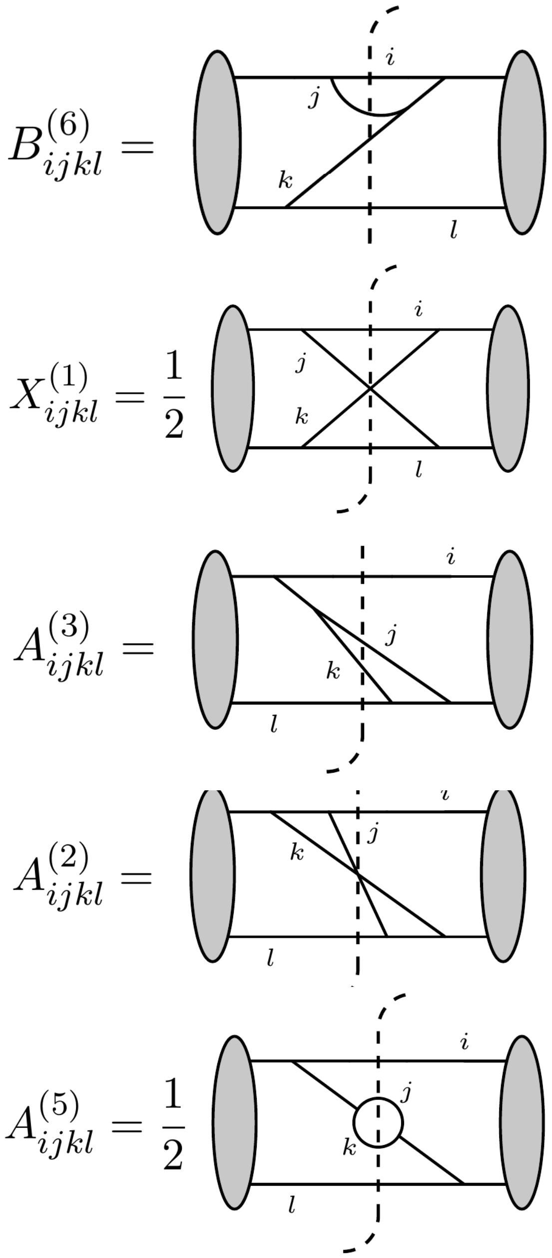
Example multi emission Result

Double soft decomposition



- Decomposition of the **two emission soft gluon current** squared:

$$\begin{aligned}\mathcal{S}_{ij}(q_1,q_2) = \; & \mathcal{P}\left(B^{(6)}_{l21i}\right) \times 2S_{ij}(S_{j1}+2S_{j2})N_{j12} \\ & -\mathcal{P}\left(X^{(1)}_{i12j}\right) \times S^2_{ij} \\ & +\mathcal{P}\left(A^{(3)}_{i12j}\right) \times 2S_{ij}(S_{j1}-3S_{j2})N_{i12}N_{j12} \\ & -\mathcal{P}\left(A^{(2)}_{i12j}\right) \times 2S^2_{ij}N_{i12}N_{j12} \\ & +\mathcal{P}\left(A^{(5)}_{i12j}\right) \times 2(1-\varepsilon)(S_{i1}S_{j2}+S_{i2}S_{j1})N_{i12}N_{j12} + (1\leftrightarrow 2).\end{aligned}$$



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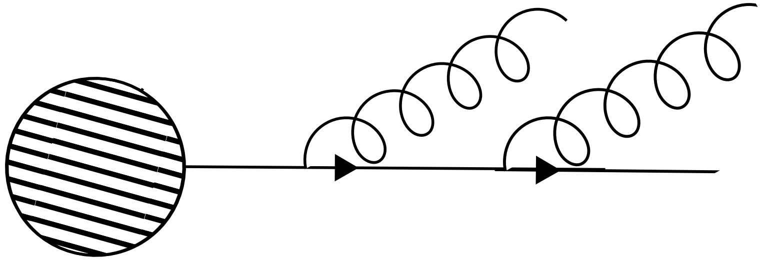

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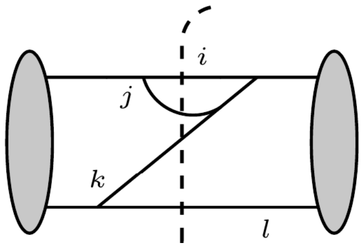
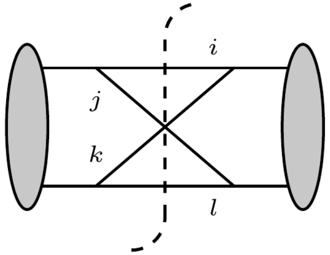
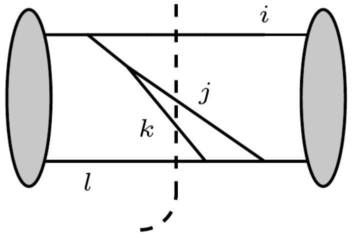
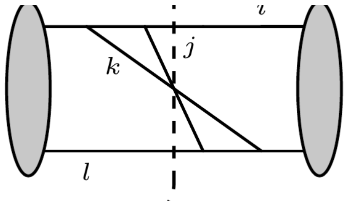
& -\mathcal{P}\left(X^{(1)}_{i12j}\right) \times S^2_{ij} \\
& + \mathcal{P}\left(A^{(3)}_{i12j}\right) \times 2S_{ij}(S_{j1} - 3S_{j2})N_{i12}N_{j12} \\
& -\mathcal{P}\left(A^{(2)}_{i12j}\right) \times 2S^2_{ij}N_{i12}N_{j12} \\
& + \mathcal{P}\left(A^{(5)}_{i12j}\right) \times 2(1-\varepsilon)(S_{i1}S_{j2} + S_{i2}S_{j1})N_{i12}N_{j12} + (1 \leftrightarrow 2).
\end{aligned}
$$

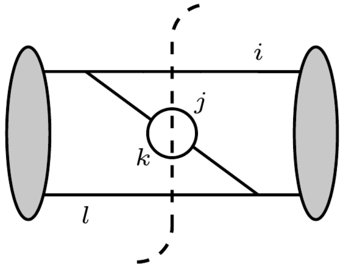
- Can **partition this topology-wise** using our partitioning algorithms

$B^{(6)}_{ijkl} =$ 

$X^{(1)}_{ijkl} = \frac{1}{2}$ 

$A^{(3)}_{ijkl} =$ 

$A^{(2)}_{ijkl} =$ 

$A^{(5)}_{ijkl} = \frac{1}{2}$ 

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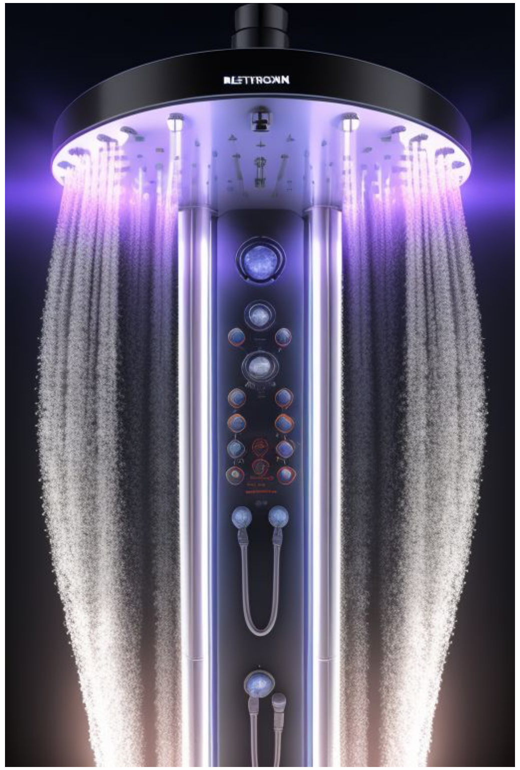

AI imagination of a "future parton shower"

Thank you!

Contact

Deutsches Elektronen-
Synchrotron DESY

www.desy.de

Maximilian Löschner
Theory Group
maximilian.loeschner@desy.de

Backup

Why Parton Showers?

...and hadronization models

[Ellis, Stirling, Webber; “Pink Book”; 1996]

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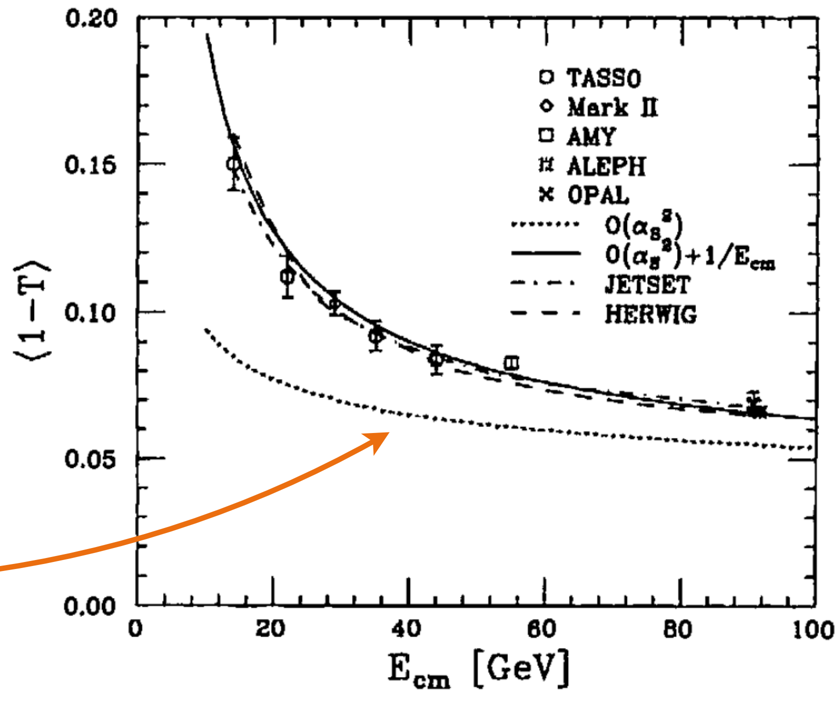

Fig. 5.13. ⟨1 − T⟩, where T is the thrust, in e^+e^- annihilation.

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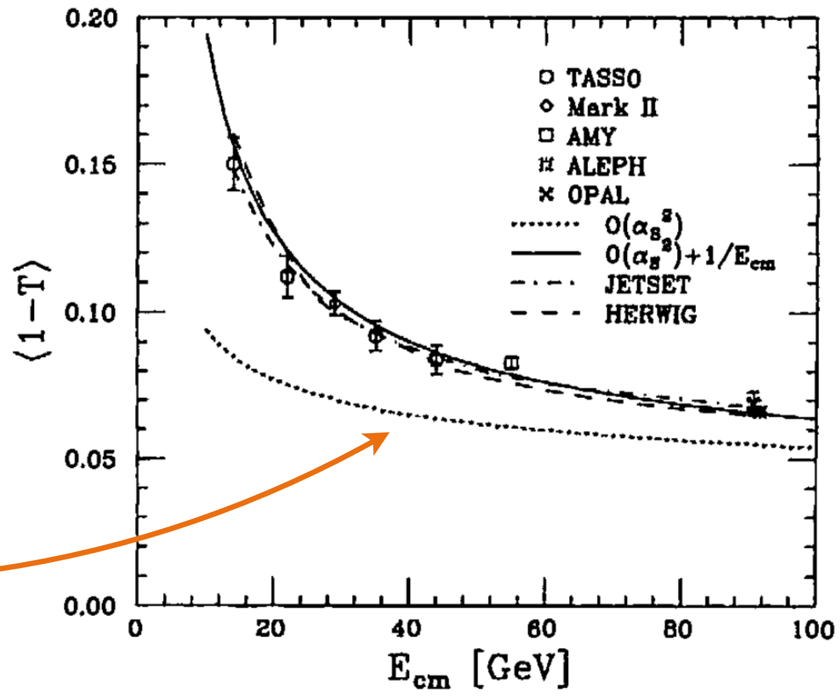

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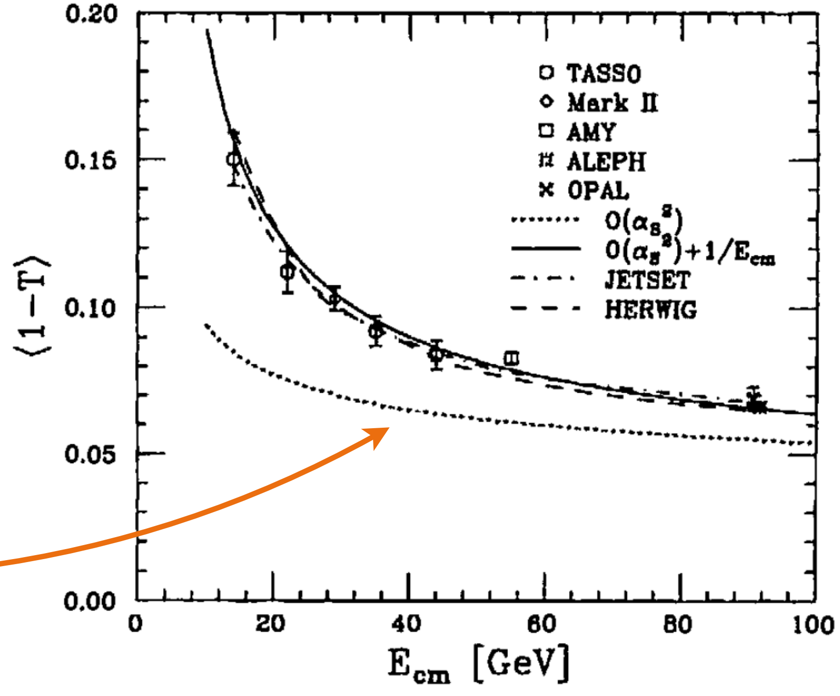
- Emission of partons and subsequent hadronization can not be neglected
- May even **change qualitative behavior** of distributions



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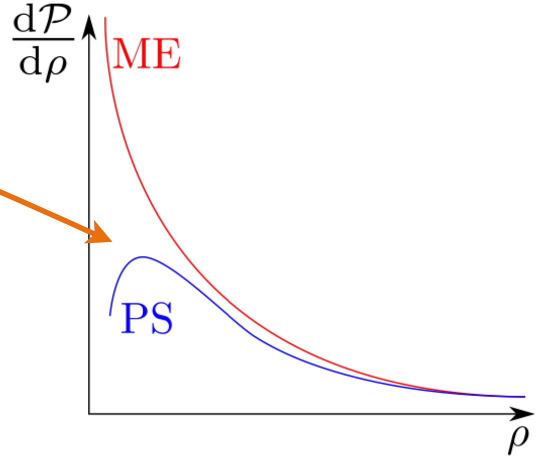
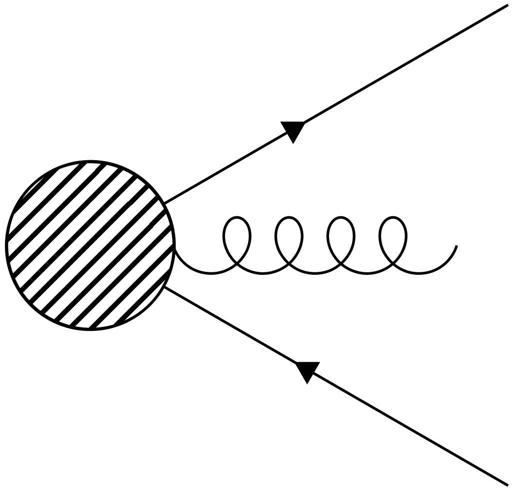


Helpful Facts & Tools



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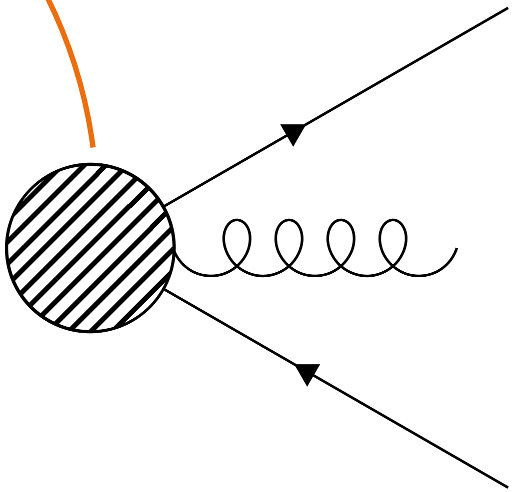

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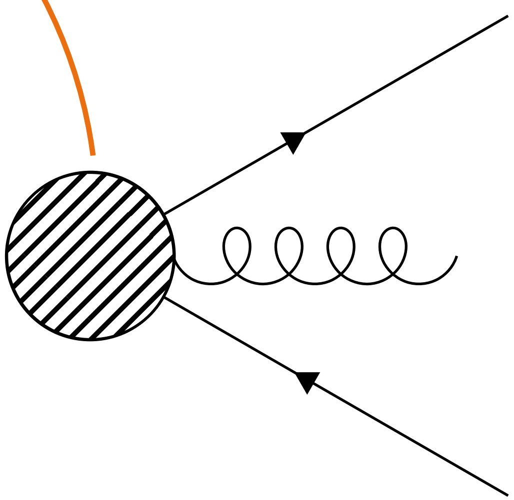

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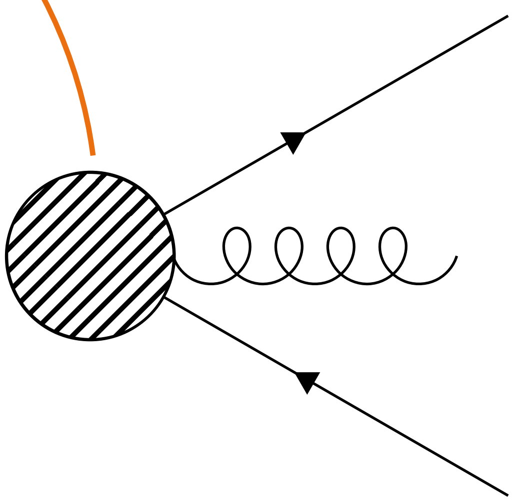
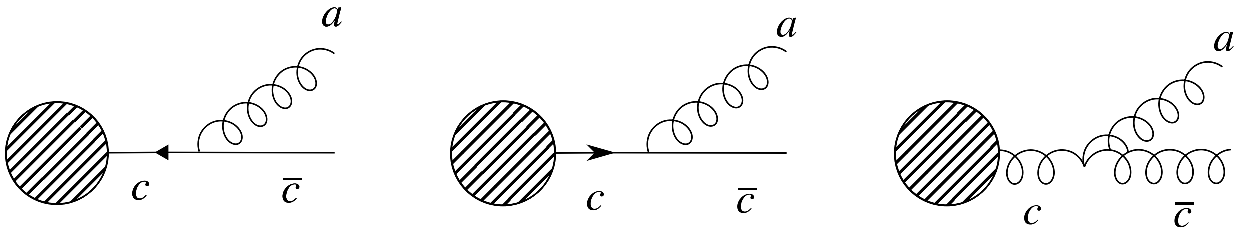


- Color charge operators: **unified language for different partons types**



$$\mathbf{T}_i = -\, T^a_{c_i \bar{c}_i} \qquad \mathbf{T}_i = T^a_{\bar{c}_i c_i} \qquad \mathbf{T}_i = f^{a\bar{c}c}$$

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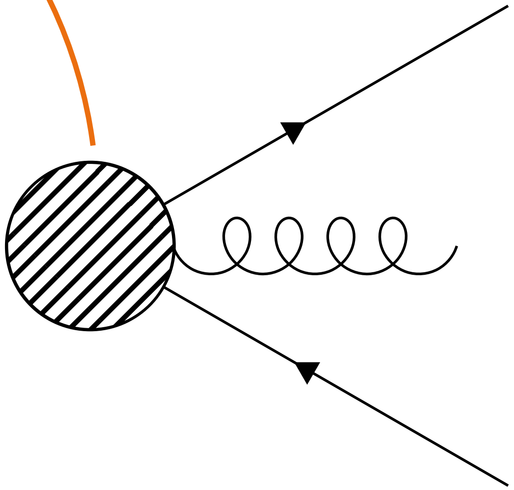

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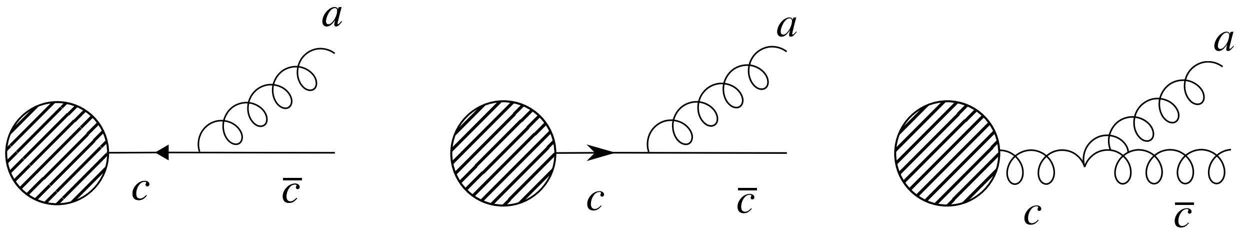

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- Work in **lightcone gauge:** more complicated gluon propagator, but decouple ghost contributions (only physical polarization propagate)

$$\Delta^{\mu\nu}(k) = \frac{i d^{\mu\nu}(k)}{k^2 + i\epsilon},$$

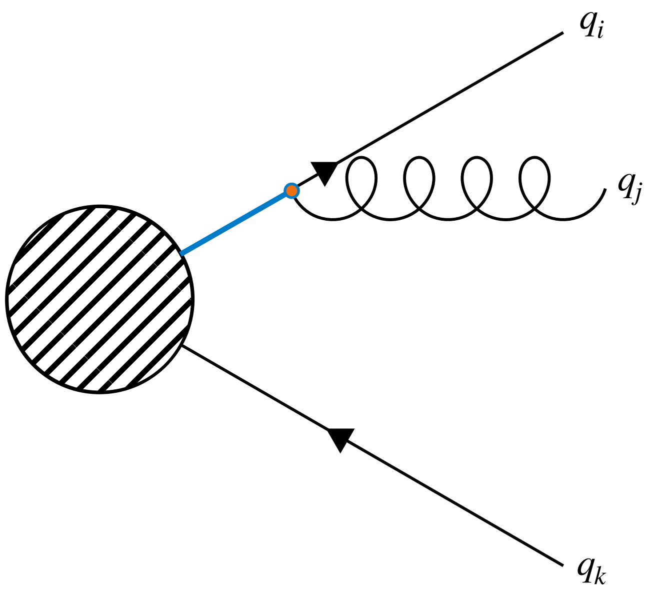
$$d^{\mu\nu}(k) = -\,\eta^{\mu\nu} + \frac{k^\mu n^\nu + n^\mu k^\nu}{n \cdot k}, n^2 = 0$$

Building blocks of parton showers

Attaching emissions

- Additional propagator factors from attaching emissions lead to enhanced soft and collinear regions in phase space:

$$\frac{1}{(q_i + q_j)^2} = \frac{1}{2q_i^0 q_j^0 (1 - \cos\theta_{ij})}$$



$$\mathrm{d}P(\text{1st emission at } t) = \frac{\mathrm{d}t}{t} \int_{z_-}^{z_+} \frac{\alpha(z,t)}{2\pi} \hat{P}(z,t)\mathrm{d}z \times \exp[-\Delta(t_0,t)]$$

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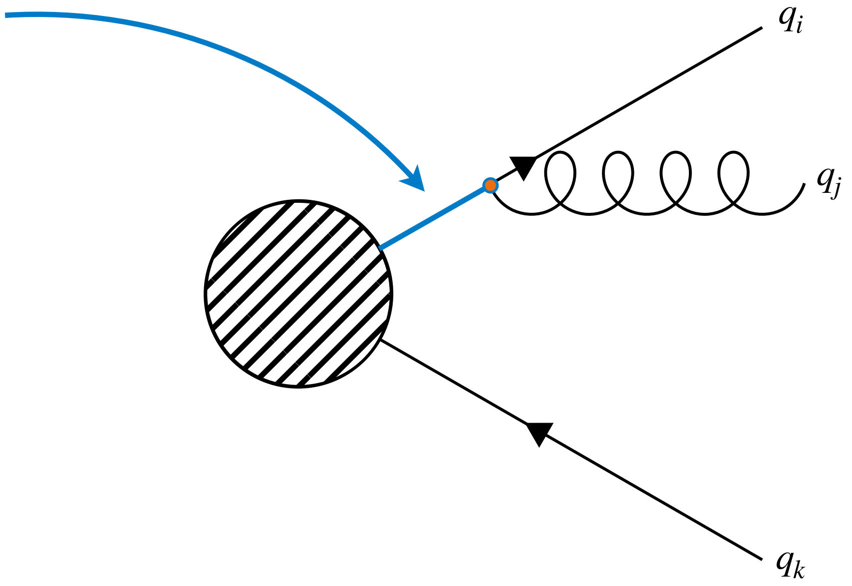

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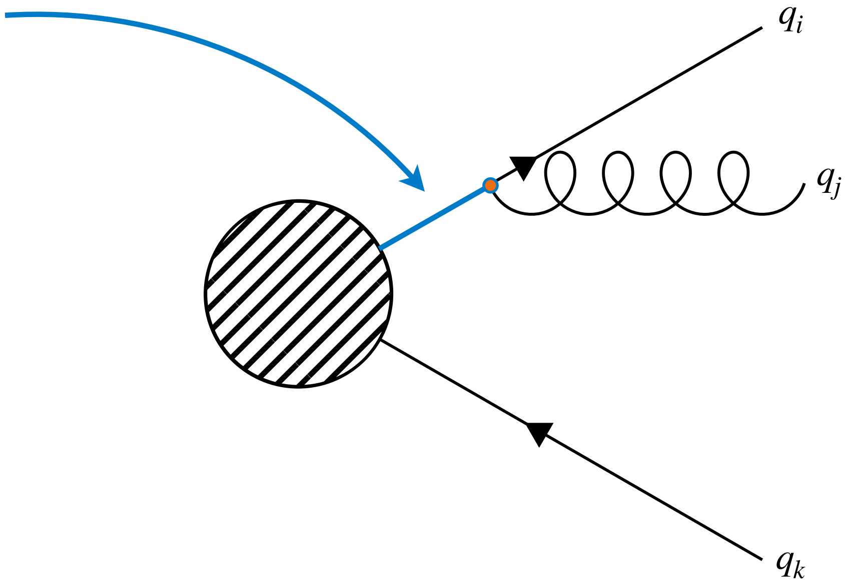

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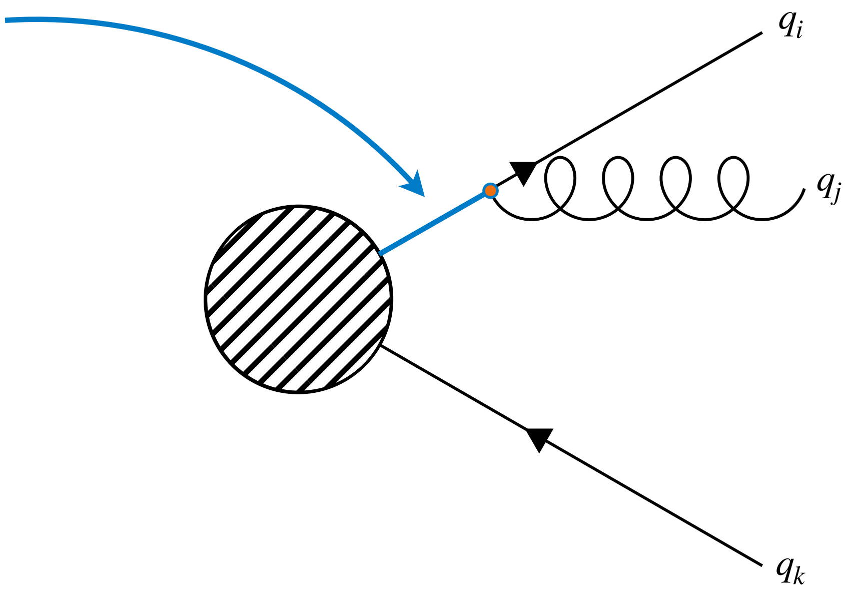

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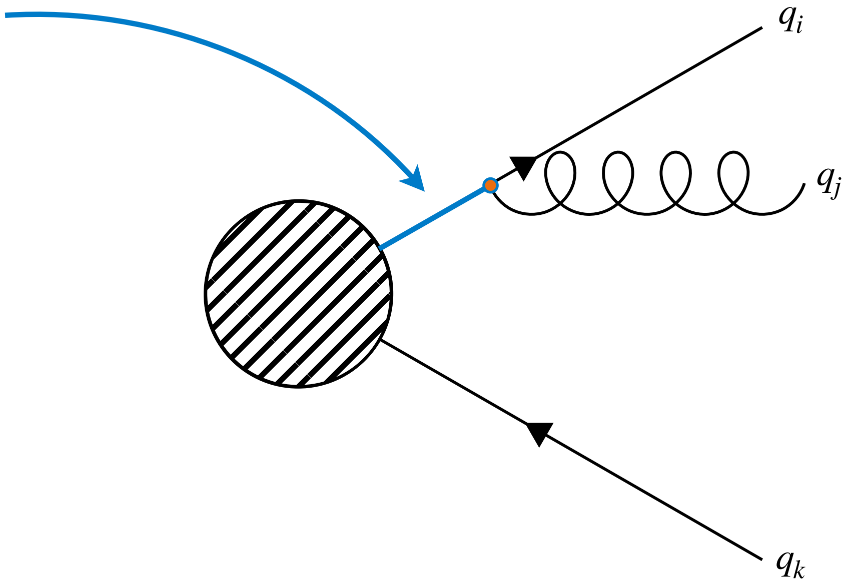

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- **Job of the parton shower: reproduce this behavior by generating emissions according to an appropriate probability distribution**

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Multiple emissions

QED example

- Single collinear photon emission gives **factorized result**:

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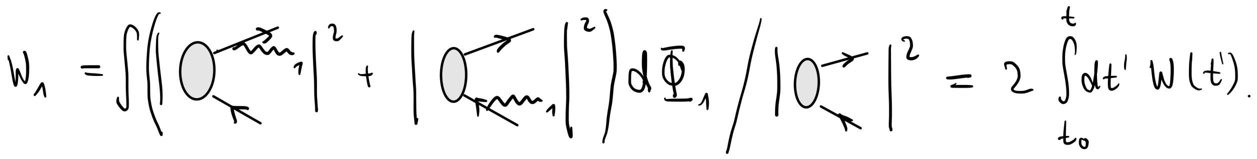

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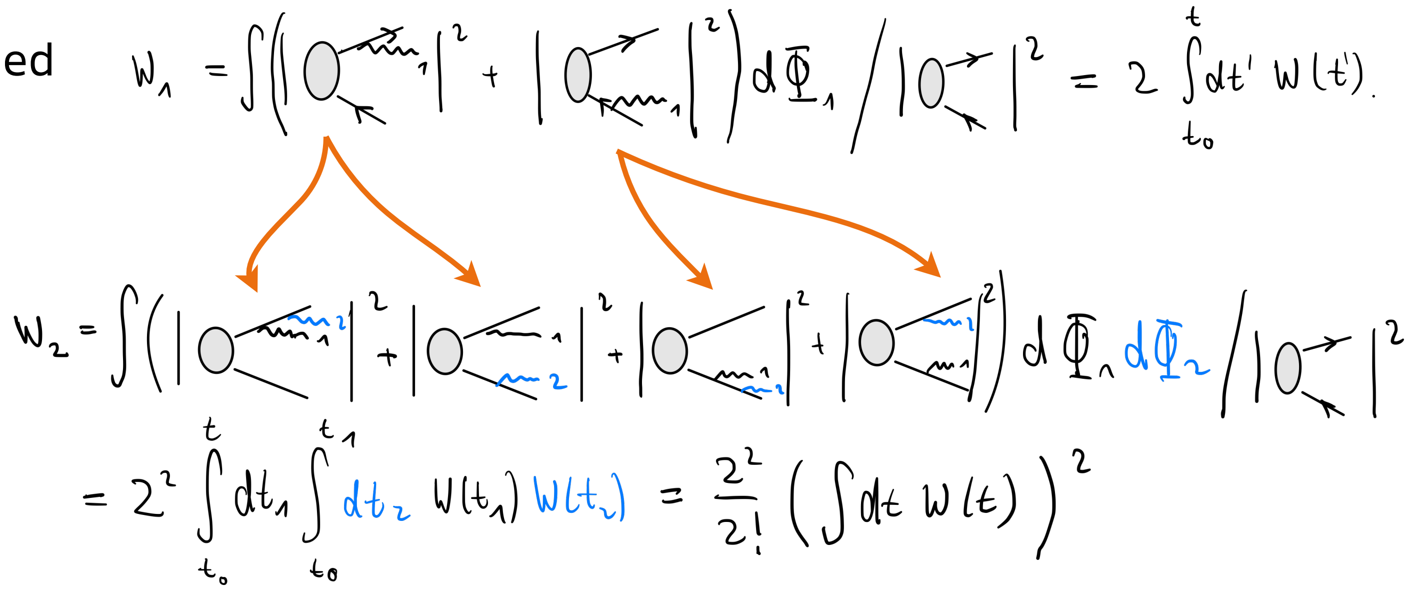

where we use $\int_{t_0}^{t} dt_1 \ldots \int_{t_0}^{t_{n-1}} dt_n W(t_1) \ldots W(t_n) = \frac{1}{n!} \left(\int_{t_0}^{t} dt' W(t') \right)^n$

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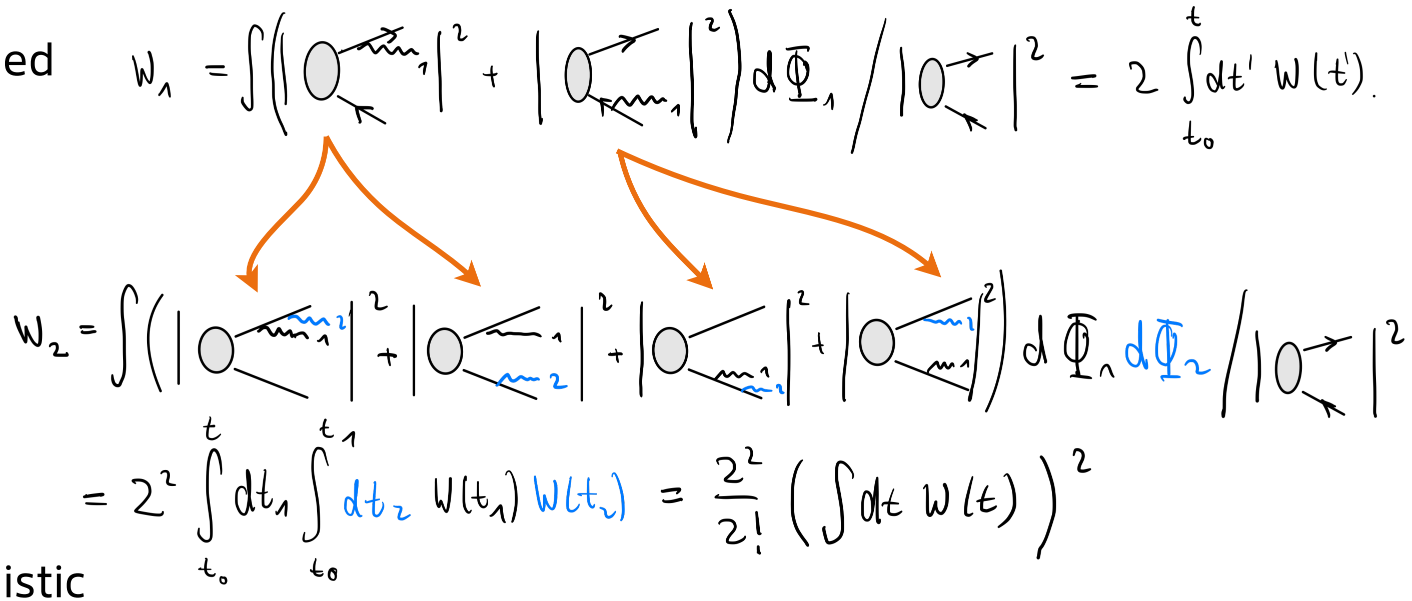

- Note: **no interferences**, iteration taken in probabilistic manner. Want to check this approximation explicitly

Sudakov Form Factor

[Lecture by S. Gieseke]

Iterating the single emission result

- Generalize to n emissions by induction:

Sudakov Form Factor

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- Where we defined the Sudakov Form Factor:

$$\Delta(t_0, t) = \exp\left[-\int_{t_0}^{t} \mathrm{d}t'\, W(t')\right]$$

Sudakov Form Factor

[Lecture by S. Gieseke]

... and how to interpret it

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Sudakov Form Factor

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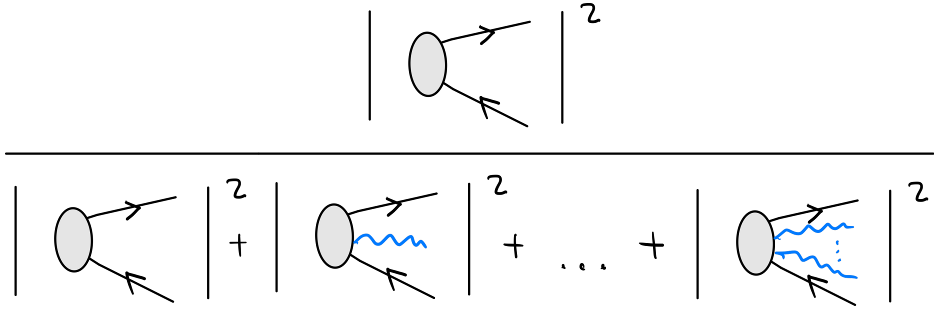

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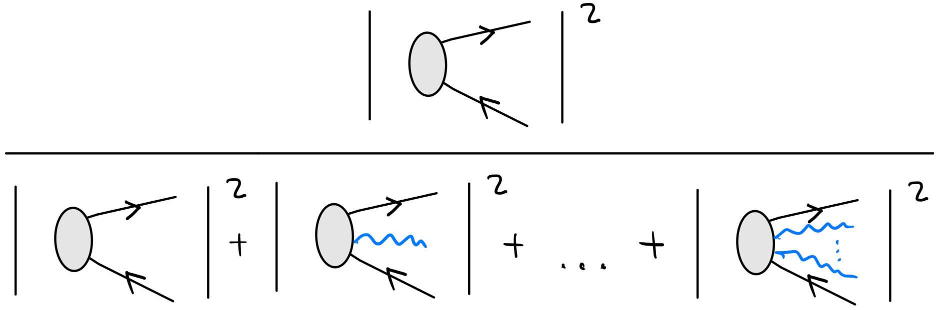

- So $\Delta(t_0, t)$ corresponds to the **probability of having no emissions** on **one leg** in the range $t \to t_0$ (hard scale to lower resolution scale)

  ➡ Note: we find $\Delta^2 = \text{Prob.}^2$ because we studied two independent charged legs

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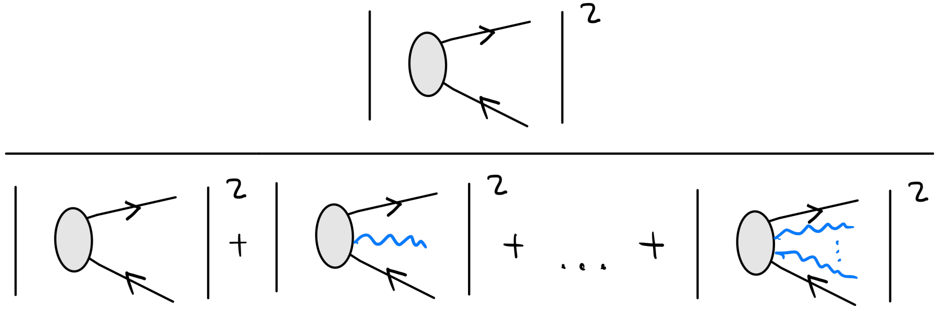
$$\sigma_{\text{all}} = \sigma_2 + \sigma_{>2} = \sigma_2 + \sigma_2 \left(\frac{1}{\Delta^2(t_0, t)} - 1 \right) \Longrightarrow \Delta^2(t_0, t) = \frac{\sigma_2}{\sigma_{\text{all}}} =$$



- So $\Delta(t_0, t)$ corresponds to the **probability of having no emissions** on **one leg** in the range $t \to t_0$ (hard scale to lower resolution scale)

  ➡ Note: we find $\Delta^2 =$ Prob.^{2} because we studied two independent charged legs

- Result: **probability density to sample over** (what e.g. HERWIG does to determine how to emit partons):

$$\mathrm{d}P(\text{1st emission at } t) = \frac{\mathrm{d}t}{t} \int_{z_-}^{z_+} \frac{\alpha(z,t)}{2\pi} \hat{P}(z,t)\mathrm{d}z \times \exp[-\Delta(t_0, t)]$$