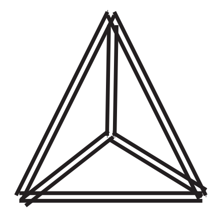
Color in orthogonal multiplet bases using Wigner 6j coefficients

- General treatment of color structure
- Orthogonal multiplet bases
- All you need is 
- Color structure treatment using group invariant Wigner 3j and 6j symbols

Thanks to my collaborators: Judith Alcock-Zeilinger, Stefan Keppeler, Simon Plätzer, Johan Thorén

Graz
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Malin Sjödahl

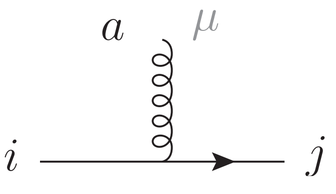
Dealing with color space

Due to confinement we never observe individual colors

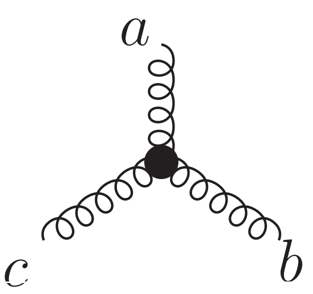
- We average over incoming colors
- We sum over outgoing colors
- $\rightarrow$ we sum over the colors of all external partons
- As always in quantum mechanics we also sum over all degrees of freedom that can interfere with each other $\rightarrow$ we sum over the colors of all internal particles
- $\rightarrow$ We sum over all colors of all particles

Notation:

I will use the graphical birdtrack notation

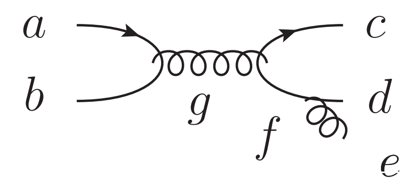


$$\equiv (t^a)^i{}_j$$



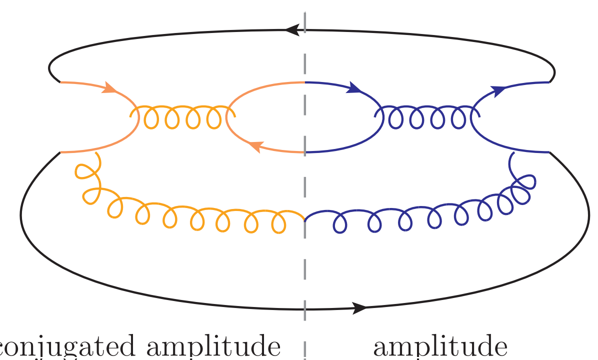
$$\equiv i f^{abc}$$

and implicitly sum over color indices of internal lines

Example: If $A = (t^g)^a{}_b (t^g)^f{}_c (t^e)^d{}_f =$  , then

$$\langle A|A\rangle = \sum_{a,b,c,d,e,f,g,h,i} \left[(t^h)^a{}_b (t^h)^i{}_c (t^e)^d{}_i\right]^* (t^g)^a{}_b (t^g)^f{}_c (t^e)^d{}_f$$

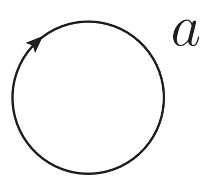
$$= \sum_{a,b,c,d,e,f,g,h,i} (t^h)^b{}_a (t^h)^c{}_i (t^e)^i{}_d (t^g)^a{}_b (t^g)^f{}_c (t^e)^d{}_f$$

$=$ 

The first equality holds since the generators are Hermitian, and the last holds since we always sum over the color of internal lines

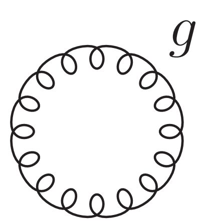
As seen above we can represent the squared amplitude with a picture. We can also calculate with graphs! To do so we need just a few rules

- There are N_c possible quark colors



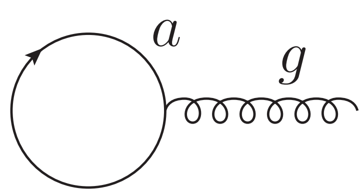
$$= N_c \qquad \sum_{a=1}^{N_c} \delta^a{}_a = N_c$$

- There are $N_g = N_c^2 - 1$ possible gluon colors



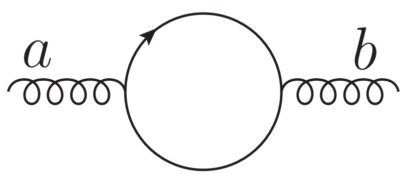
$$= N_c^2 - 1 \qquad \sum_{g=1}^{N_c^2-1} \delta^{gg} = N_c^2 - 1$$

- The generators are traceless

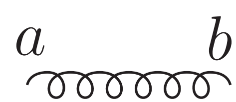


$= 0 \qquad \sum_{a=1}^{N_c} (t^g)^a{}_a = 0$

- Generator normalization

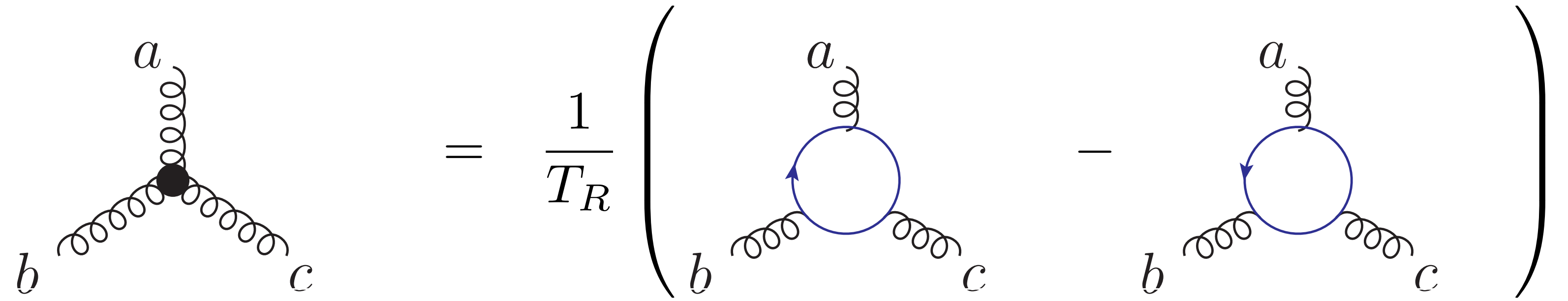


$= T_R$



$\mathrm{Tr}[t^a t^b] = T_R \delta^{ab}$

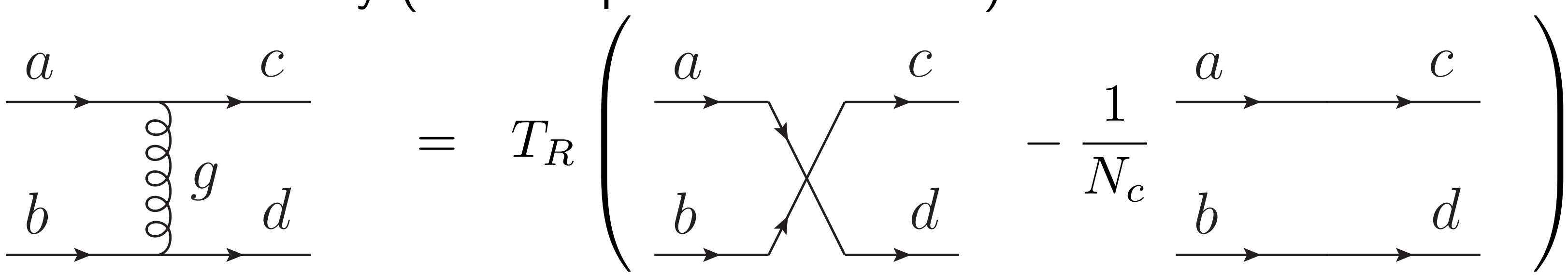
- The algebra $[t^a, t^b] = i f^{abc} t^c \Rightarrow$



$$i f^{abc} = \frac{1}{T_R}\left[\mathrm{Tr}[t^a t^b t^c] - \mathrm{Tr}[t^b t^a t^c]\right]$$

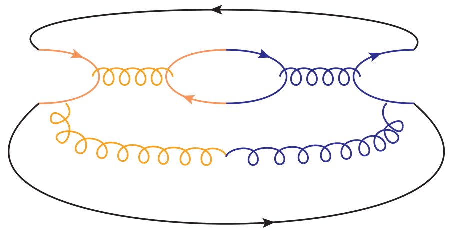
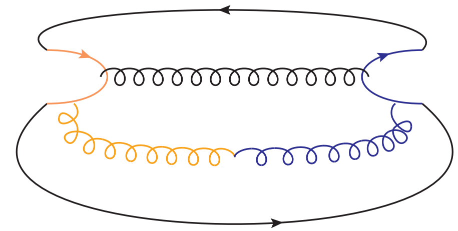
(Note: different arrow conventions in different sources)

- The Fierz identity (the completeness relation)

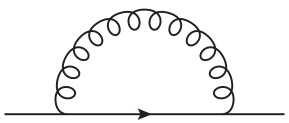
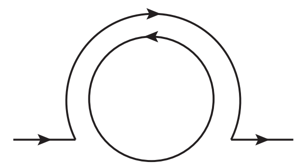


$$(t^g)^a{}_c (t^g)^b{}_d = T_R\left[\delta^a{}_d \delta^b{}_c - \frac{1}{N_c}\delta^a{}_c \delta^b{}_d\right]$$

Let's apply the rules to our example

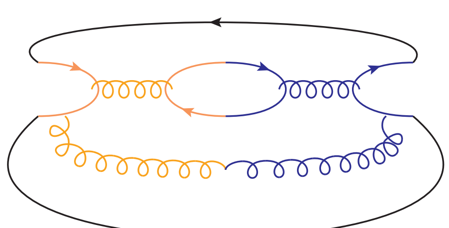
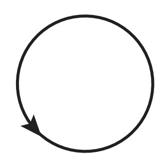
 $= T_R$ 

To further simplify the color structure we note using Fierz

 $= T_R \Big($  $- \frac{1}{N_c}$  $\Big) = T_R \left(N_c - \frac{1}{N_c} \right)$ 

$= T_R \frac{N_c^2 - 1}{N_c}$  $\equiv C_F$ 

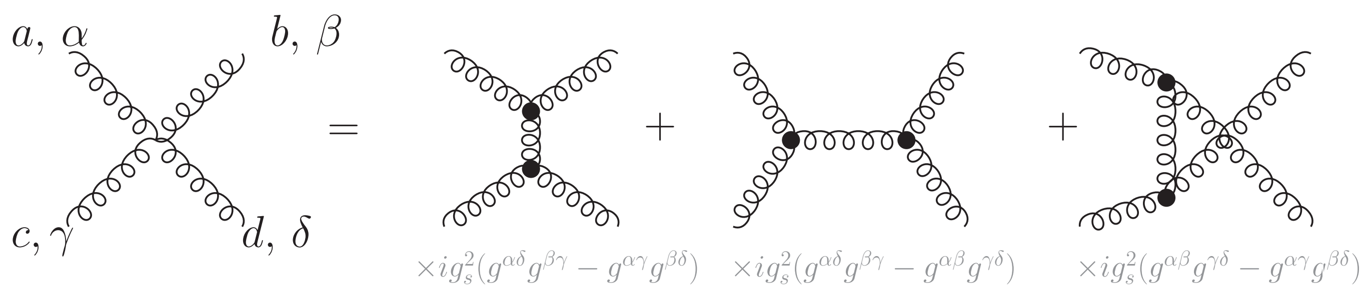
Giving, for the squared amplitude

 $= T_R C_F^2$  $= T_R C_F^2 N_c$

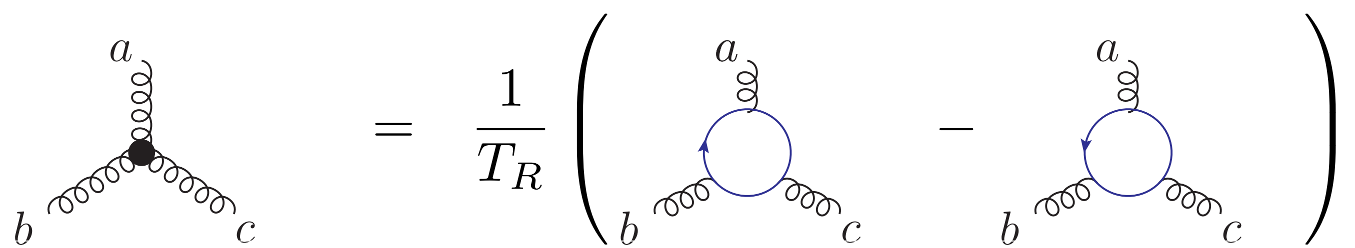
- In this way we can square any color amplitude and calculate any interference term.
- One way of dealing with color space is to just square the amplitudes one by one as one encounters them
- Alternatively, we may use any basis (spanning set)

The most popular bases: Trace bases

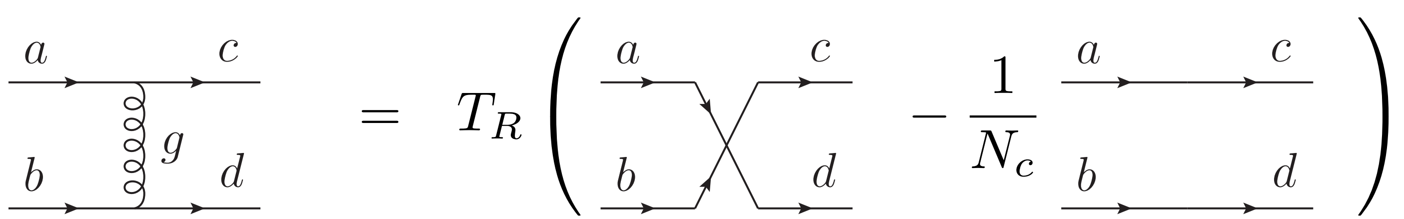
- Every 4g vertex can be replaced by 3g vertices:



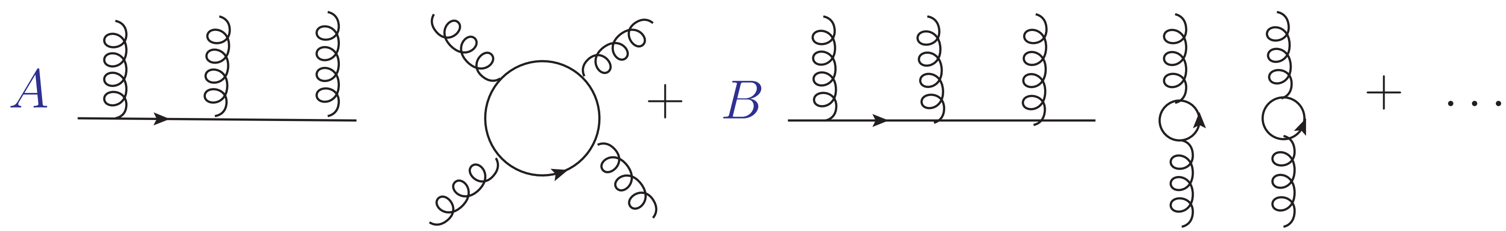
- Every 3g vertex can be replaced using:



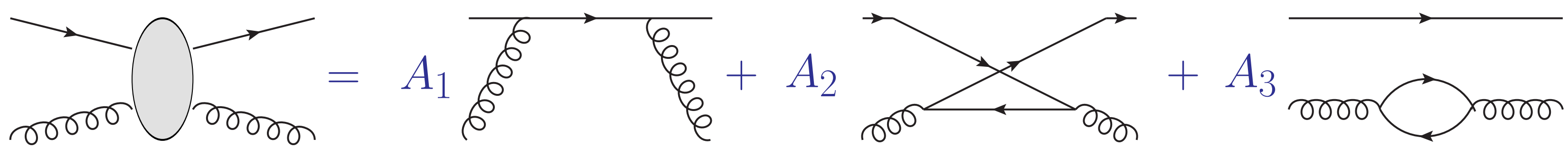
- After this every internal gluon can be removed using Fierz:



- This can be applied to any QCD amplitude, tree level or beyond
- In general an amplitude can be written as linear combination of different color structures, like



- For example for 2 (incoming + outgoing) gluons and one $q\bar{q}$ pair



(an incoming quark is the same as an outgoing anti-quark)

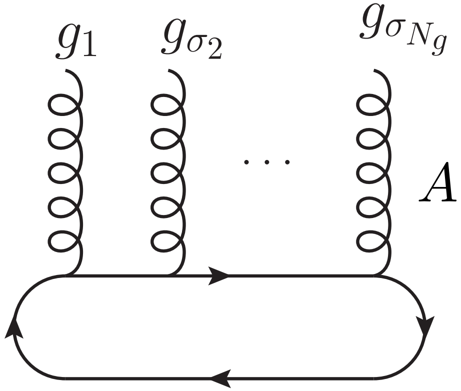
- The above type of color structure can be used as a spanning set, a "trace basis"

These bases have some nice properties

- Conceptual simplicity
- Can be reduced for a given *order* in perturbation theory, for example, for tree-level N_g-gluon amplitudes we have $(N_g-1)!$ color structures of form

$$\mathcal{M}(g_1, g_2, \ldots, N_g) = \sum_{\sigma \in S_{N_g-1}} \mathrm{Tr}(t^{g_1} t^{g_{\sigma_2}} \ldots t^{g_{\sigma_{N_g}}}) A(\sigma)$$

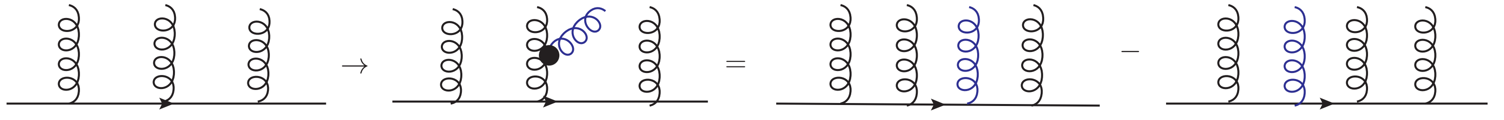
$$= \sum_{\sigma \in S_{N_g-1}}$$



$$A(\sigma),$$

whereas for higher orders we also have products of traces.

- Taking the leading N_c limit is trivial and results in a flow of colors
- The basis vectors are orthogonal when $N_c \to \infty$
- The effect of gluon emission is easily described:



We get just one new basis vector if the emitter is an (anti-)quark and two if the emitter is a gluon

- So is the effect of gluon exchange (MS 0906.1121 (JHEP), implementation in ColorFull 1412.3967 (EPJC))

There are also drawbacks with trace bases

- Not orthogonal
  $\rightarrow$ When squaring amplitudes almost all cross terms have to be taken into account $\rightarrow N^2_{\text{basis}}$ terms
- Overcomplete
  For $N_g + N_{q\bar{q}} > N_c$ the bases are also overcomplete
- The size of the vector space asymptotically grows as an exponential in the number of gluons/$q\bar{q}$-pairs for finite N_c

- For general N_c the basis size grows as a factorial

$$N_{\text{vec}}[N_q, N_g] \quad = \quad N_{\text{vec}}[N_q, N_g - 1](N_g - 1 + N_q) + N_{\text{vec}}[N_q, N_g - 2](N_g - 1)$$

where

$$\begin{aligned} N_{\text{vec}}[N_q, 0] &= N_q! \\ N_{\text{vec}}[N_q, 1] &= N_q N_q! \end{aligned}$$

(S. Keppeler & M.S. 1207.0609 (JHEP))

- For general N_c and gluon only amplitudes (to all order) the size is given by Subfactorial(N_g)$\approx N_g!/e$
- For tree-level gluon amplitudes traces may be used as spanning vectors giving $(N_g - 1)!$ spanning vectors

Example: Number of spanning vectors for N_g gluons (without imposing charge conjugation invariance). These numbers are representative also for N_g gluons plus $q\bar{q}$-pairs.

N_g	Vectors $N_c = 3$	Vectors $N_c \to \infty$	LO Vectors $N_c \to \infty$
4	8	9	3!=6
5	32	44	4!=24
6	145	265	120
7	702	1 854	720
8	3 598	14 833	5 040
9	19 280	133 496	40 320
10	107 160	1 334 961	362 880
11	614 000	14 684 570	3 628 800
12	3 609 760	176 214 841	39 916 800

(Y. Du, M.S. & J. Thorén, JHEP 1505 (2015) 119, 1503.00530)

The dimension of the full vector space (all orders) for $N_c = 3$

N_g	$N_{q\bar{q}} = 0$	N_g	$N_{q\bar{q}} = 1$	N_g	$N_{q\bar{q}} = 2$
4	8	3	10	2	13
5	32	4	40	3	50
6	145	5	177	4	217
7	702	6	847	5	1 024
8	3 598	7	4 300	6	5 147
9	19 280	8	22 878	7	27 178
10	107 160	9	126 440	8	149 318
11	614 000	10	721 160	9	847 600
12	3 609 760	11	4 223 760	10	4 944 920

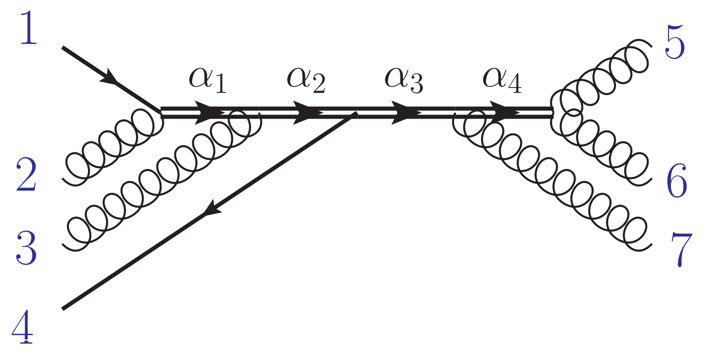
(M.S. & J. Thorén, 1507.03814, JHEP)

- Color flow bases fast for evaluation of color delta functions, and good for sampling over color but has a similar scaling (Color flow rules: Maltoni, Stelzer, Paul, Willenbrock, hep-ph/0209271, example of sampling De Angelis, Forshaw, Plätzer, 2007.09648, Forshaw Holguin, Plätzer, 2112.13124)

Multiplet bases

- QCD is based on SU(3) $\rightarrow$ the color space may be decomposed into irreducible representations
- Orthogonal basis vectors corresponding to irreducible representations may be constructed
- The construction of the corresponding basis vectors is non-trivial, and a general strategy was presented relatively recently (S. Keppeler & M.S. JHEP09(2012)124, 1207.0609, MS & J. Thoren JHEP 11 (2018) 198 , 1809.05002)
- With general, I mean general: general number of quarks and gluons, general order in α_s and general N_c

- In multiplet bases partons are grouped into representations



and can thus be characterized by a chain of representations $\alpha_1, \alpha_2, \ldots$ (In principle we have to differentiate between different vertices as well)

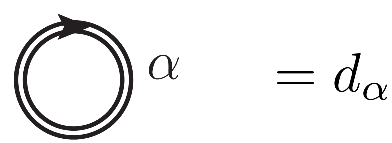
- These vectors are orthogonal ($\rightarrow$ minimal) by construction

- Multiplet bases can potentially speed up exact calculations in color space very significantly, as squaring amplitudes becomes much quicker
- But before squaring, amplitudes must be decomposed in multiplet bases
- How can amplitudes be expressed in multiplet bases?

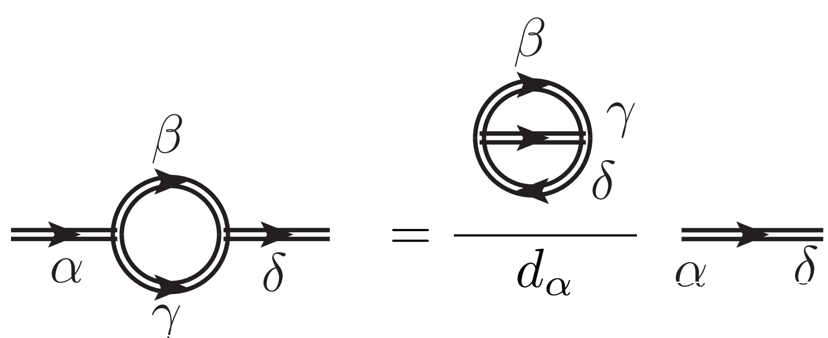
Decomposing color structure in multiplet bases

To simplify the color structure we need a few rules:

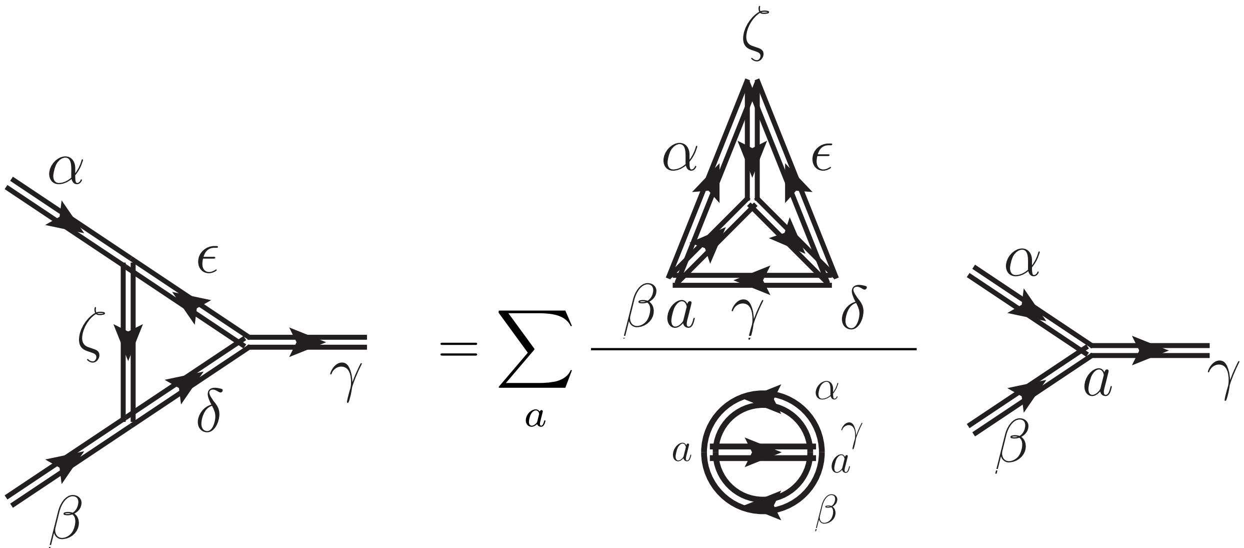
- Dimension relation



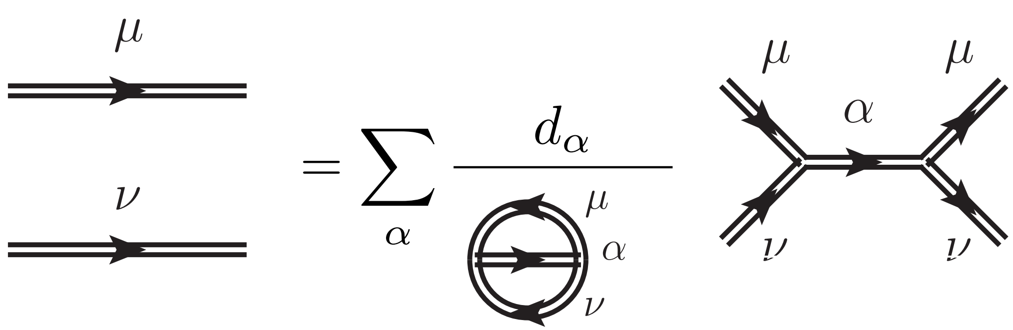
- Two-vertex loops give just a constant

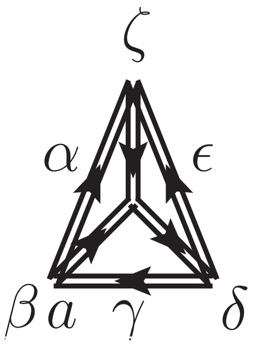


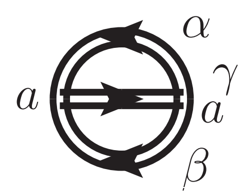
- Vertex correction relation



- For longer loops we need the completeness relation (which the Fierz identity is a special case of)



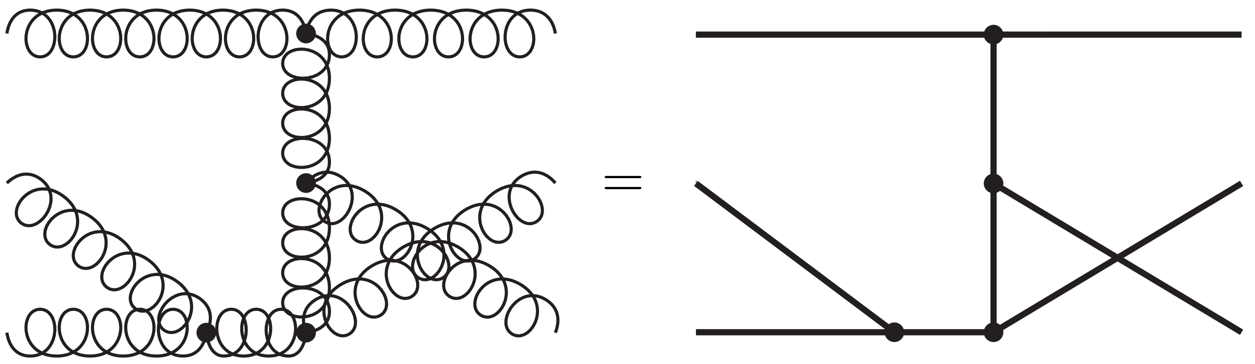
- The symbols 

and 

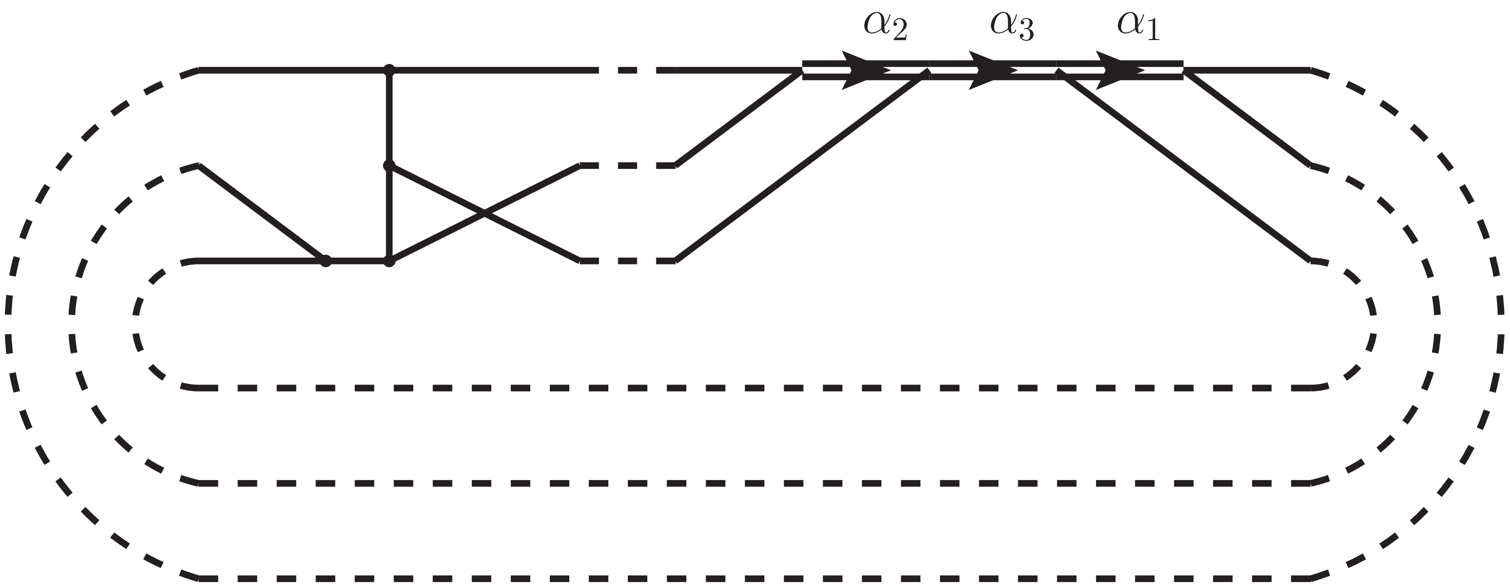
are Wigner 6j and 3j coefficients and their values can be calculated once and for all
- Knowing the 3j and 6j Wigner coefficients we can calculate with color without explicitly writing down color bases

Decomposing color with 6j and 3j coefficients

As an example consider the color structure of the Feynman diagram:

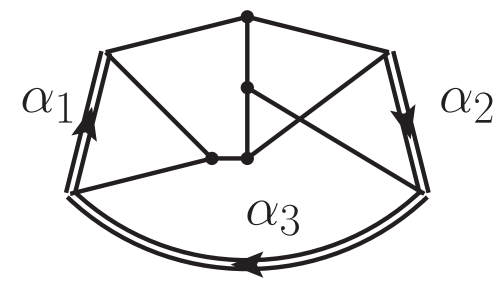


The scalar product between the color structure and a basis vector is given by:

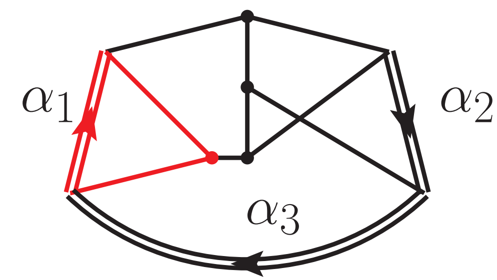


In a more compact form:

$$A(\alpha_1, \alpha_3, \alpha_2) \quad =$$

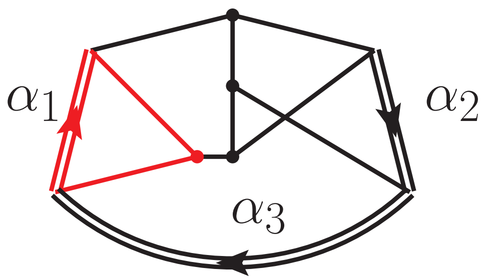


Here we note that we have a vertex correction:

$A(\alpha_1, \alpha_3, \alpha_2) \quad =$ 

Using the vertex correction results in:

$A(\alpha_1, \alpha_3, \alpha_2) \quad =$



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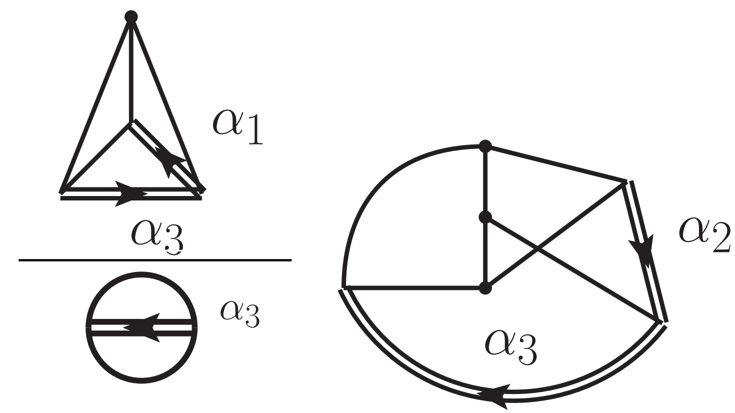
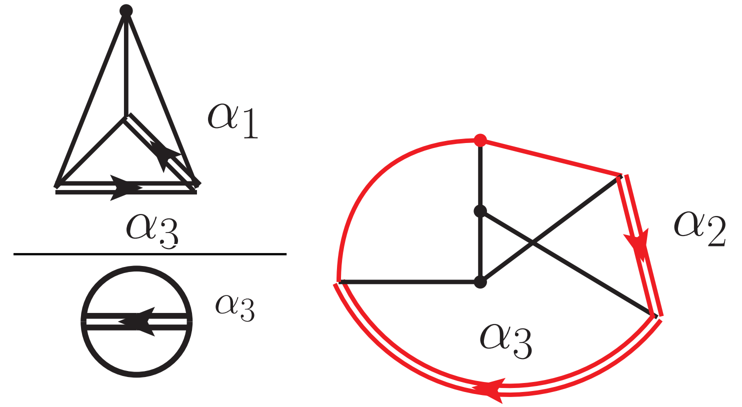
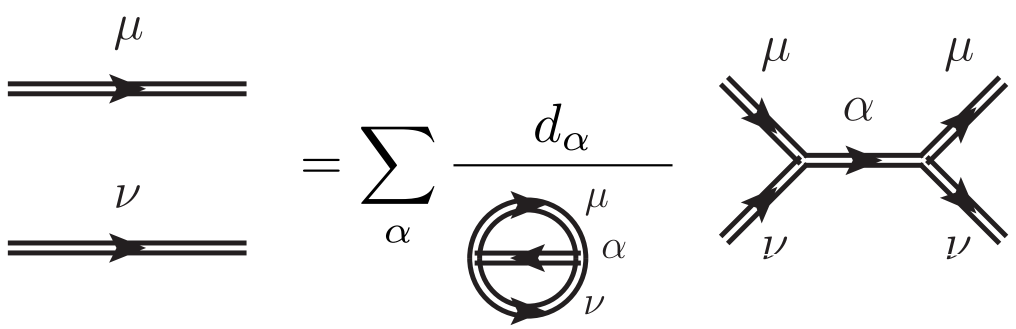


Now there is no trivial color structure, but we can pick any loop...

$A(\alpha_1, \alpha_3, \alpha_2) =$

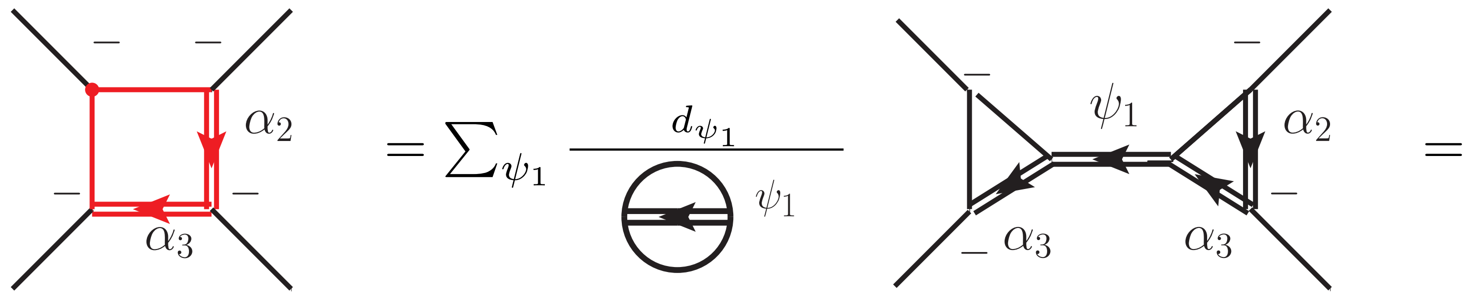


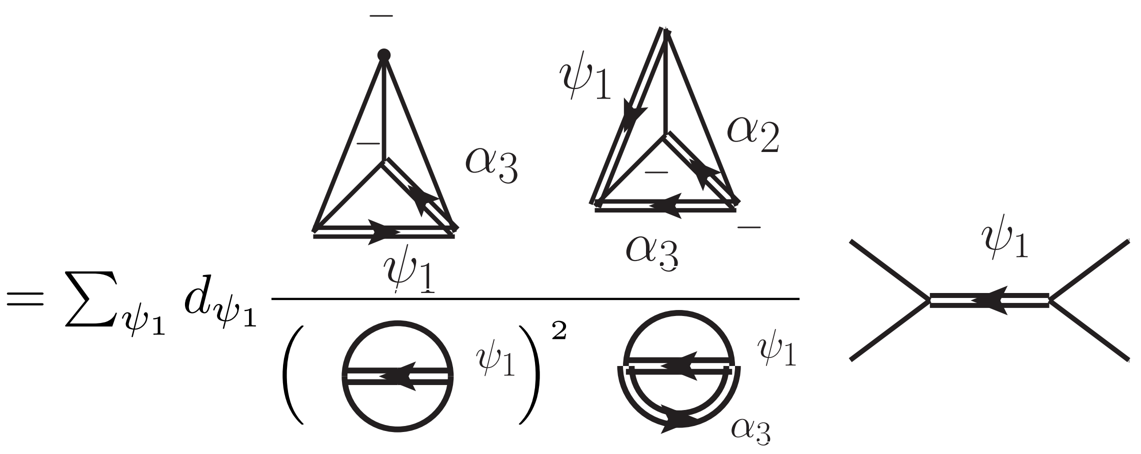
and use the completeness relation



to remove it

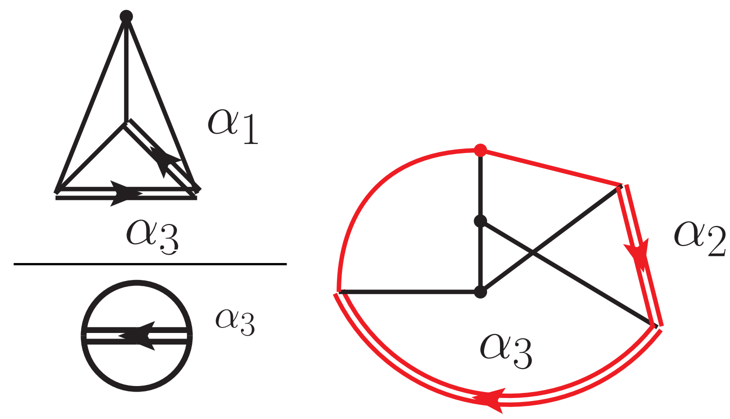
Applying the completeness relation and removing vertex corrections:



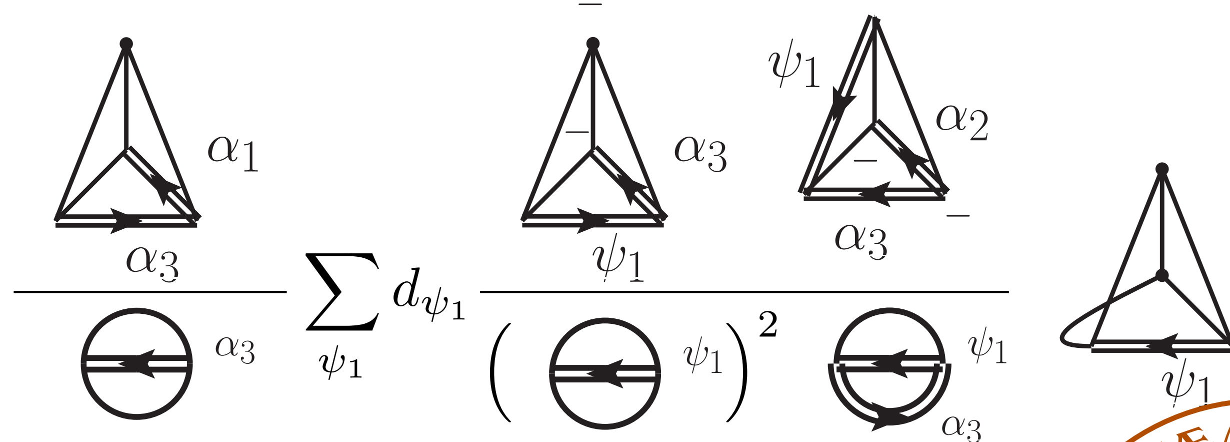


Removing the 4-vertex loop we get:

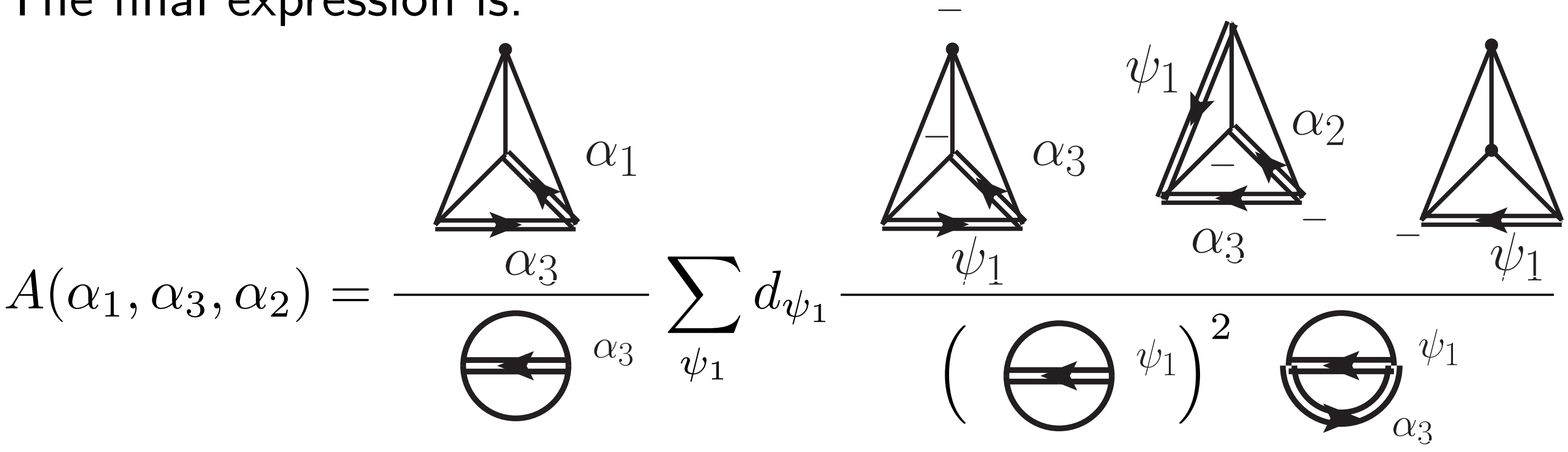
$A(\alpha_1, \alpha_3, \alpha_2) =$



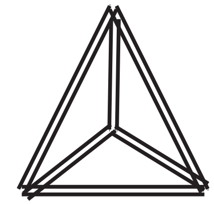
$=$

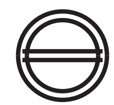
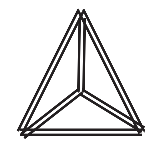
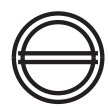
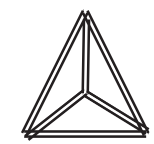


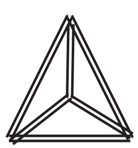
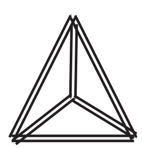
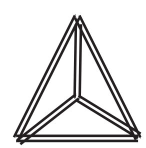
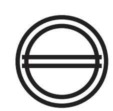
The final expression is:



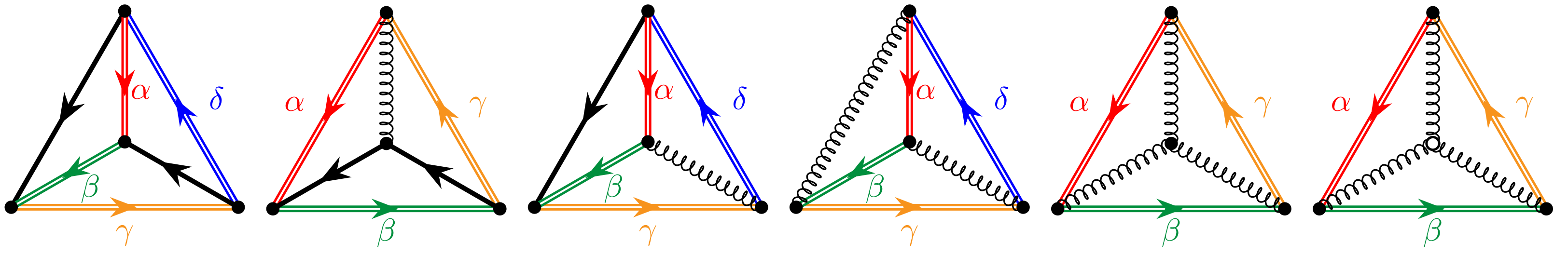
- Knowing the 3j and 6j Wigner coefficients we can immediately write down the scalar product with any basis vector!
- This only has to be done once for each Feynman diagram, not once for each Feynman diagram *and* each basis vector
- We only need to care about non-zero projections, we could list the non-zero 6j-coefficients
- Each sum contains at most 8 terms for SU(3), at most $N_c^2 - 1$ for SU(N_c)

All you need is 

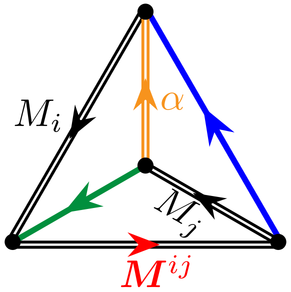
- In the above example we saw that we could decompose the color structure fully using only d_α,  , 
- We can normalize  =1, so we really only need 
- **Question:** If we can get all the color structure as a function of 6js can we then also get the 6js as a function of 6js?
- Can we calculate 6js (recursively)?

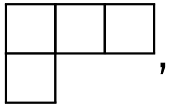
 $=$  $\Big($other  ,  $= 1, d_\alpha\Big)$?

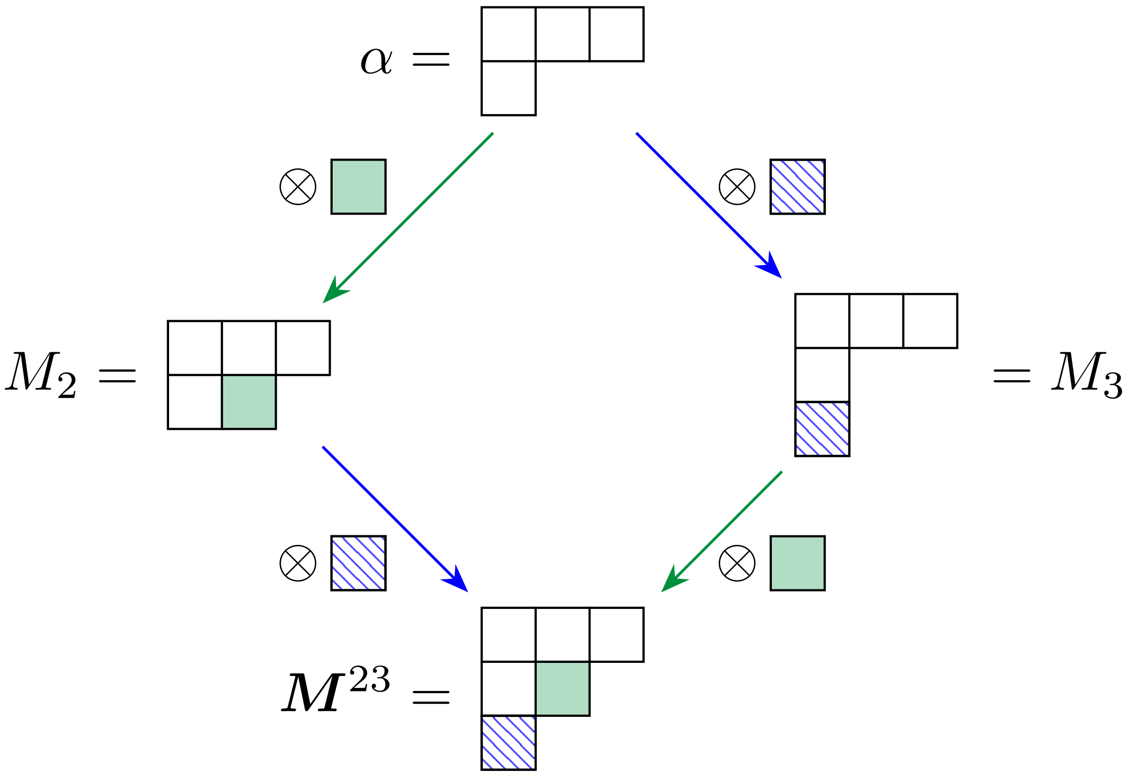
- For QCD, where every representation is 8, 3 or $\overline{3}$, it turns out that we only need 6js of form



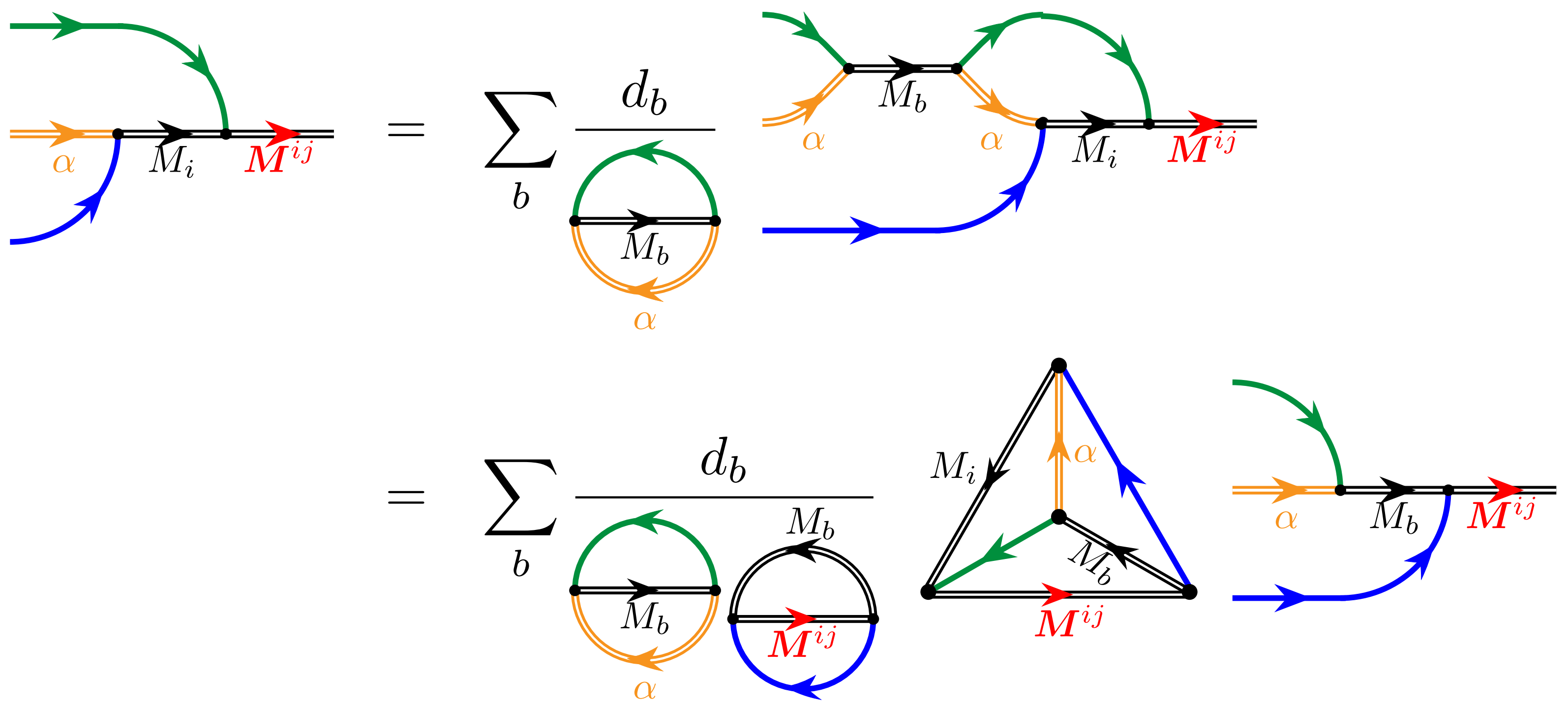
- Wigner 6j and 3j coefficients and their values can be calculated once and for all (Some in M.S. & J. Thorén, 1507.03814 (JHEP), 1809.05002 (JHEP)) ... but this still builds on constructing bases which builds on symmetrizers and anti-symmetrizers

Consider first 

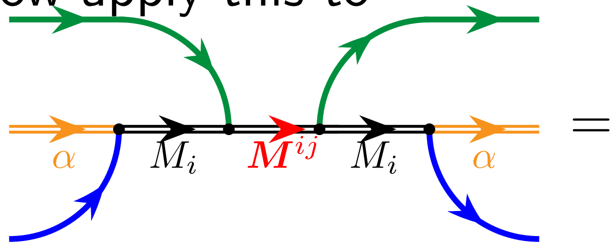
, say $\alpha =$  ,

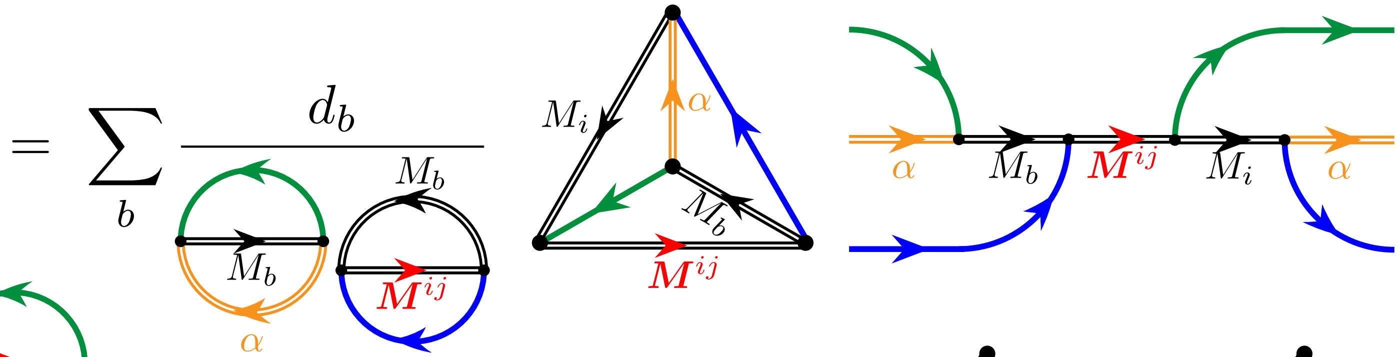


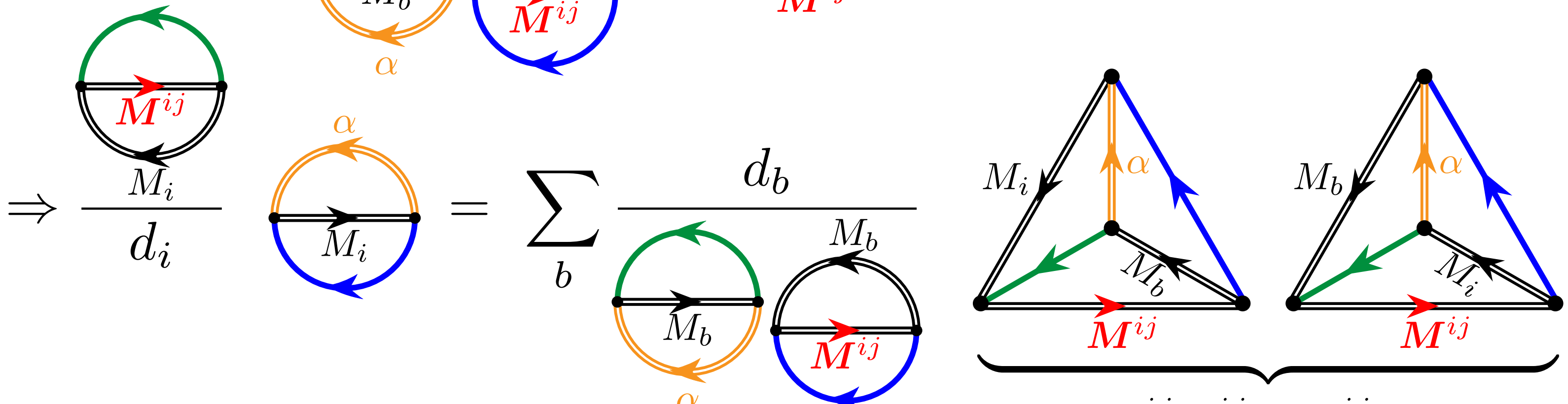
By repeated use of the completeness relation and the vertex correction relation (giving 6js), we can constrain the 6js. Consider for example



Now apply this to







$S^{ij}_{i,b} S^{ij}_{b,i} = (S^{ij}_{i,b})^2$

$$\Rightarrow 1 = d_i \sum_b d_b (S^{ij}_{i,b})^2$$ (as 3js are 1)

By similar methods we find a set of equations, for $N_c = 3$

1. For a given representation M^{ij}, we obtain

$$1 = (d_i)^2 (S^{ij}_{i,i})^2 + d_i d_j (S^{ij}_{i,j})^2 \quad 0 = d_i S^{ij}_{i,i} S^{ij}_{i,j} + d_j S^{ij}_{i,j} S^{ij}_{j,j}$$

2. For two given representations M_i and M_j, we obtain

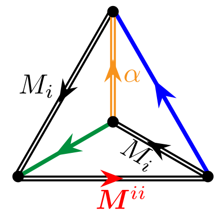
$$\frac{1}{d_\alpha} = \sum_{M^{ab}} d_{ab} (S^{ab}_{i,j})^2 \ ,$$

   where d_{ab} is the dimension of the representation M^{ab}.

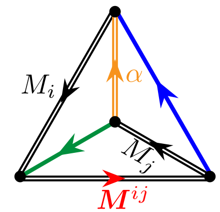
3. For a given representation M_i, we have

$$1 = \sum_b d_{ib} S^{ib}_{i,i} \ .$$

- This equation system can be solved giving

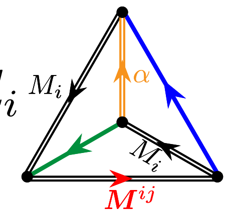


$= \frac{1}{d_i}$,

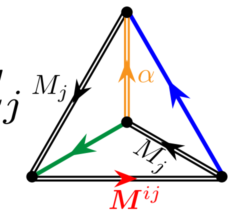


$= \pm \frac{1}{\sqrt{d_\alpha d_{ij}}}$

d_i

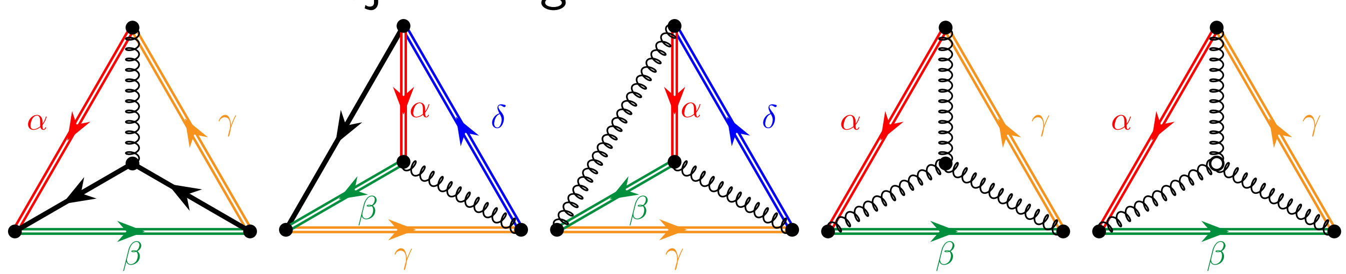


$= \pm \sqrt{1 - \frac{d_i d_j}{d_\alpha d_{ij}}} = d_j$

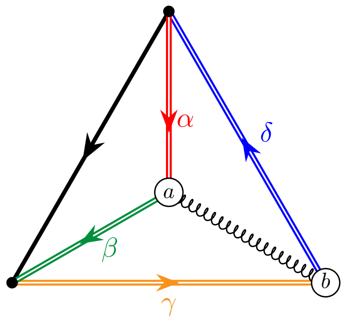


(Judith Alcock-Zeilinger, Stefan Keppeler, Simon Plätzer and MS, 2209.15013 (J. Math. Phy.))

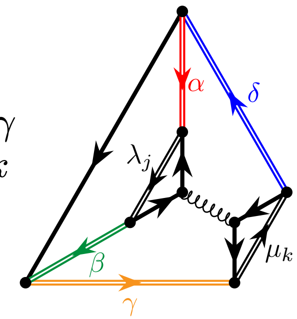
- Also need the 6js with gluons



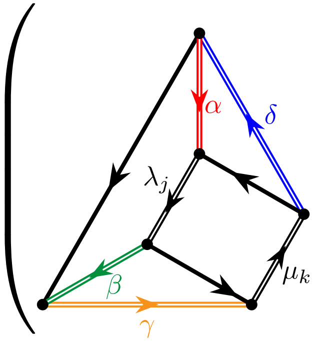
- Idea: split gluon into $q\overline{q}$-pair, for example we have



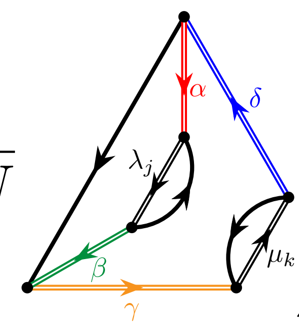
$$= \sum_{j=1}^{a} \sum_{k=1}^{b} C_{aj}^{\beta\alpha} C_{bk}^{\delta\gamma}$$



$$= \sum_{j=1}^{a} \sum_{k=1}^{b} \frac{C_{aj}^{\beta\alpha} C_{bk}^{\delta\gamma}}{N^2-1} \Bigg($$

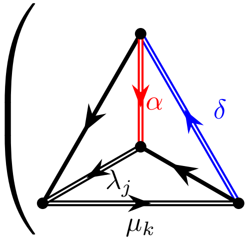


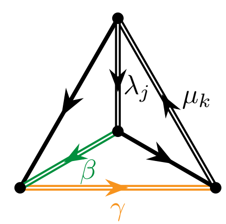
$$- \frac{1}{N}$$



$$\Bigg)$$

$$= \sum_{j=1}^{a} \sum_{k=1}^{b} \frac{C_{aj}^{\beta\alpha} C_{bk}^{\delta\gamma}}{N^2-1} \Bigg($$





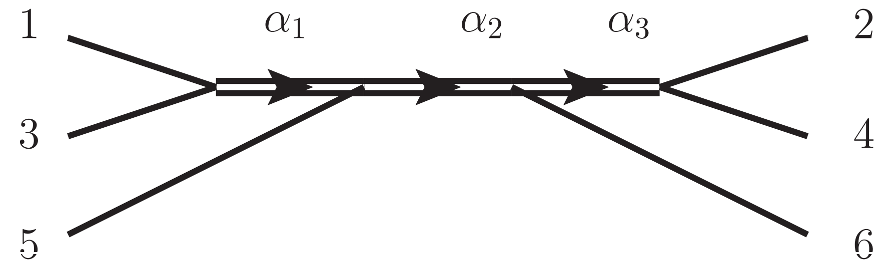
$$- \frac{\delta_{\alpha\beta}\, \delta_{\gamma\delta}}{N d_\alpha\, d_\gamma} \Bigg)$$

- By similar methods the other 6js with gluons are derived
  2312.16688 (JHEP), Stefan Keppeler, Simon Plätzer and MS
- Not more complicated to calculate 6js for high representations
- Multiplicity (> 1 instance of a vertex) is an issue... but can be addressed
- Number of required 6js scale only as N_q^2
- $\rightarrow$ We have all the ingredients for using representation based orthogonal bases for QCD also for very high multiplicities

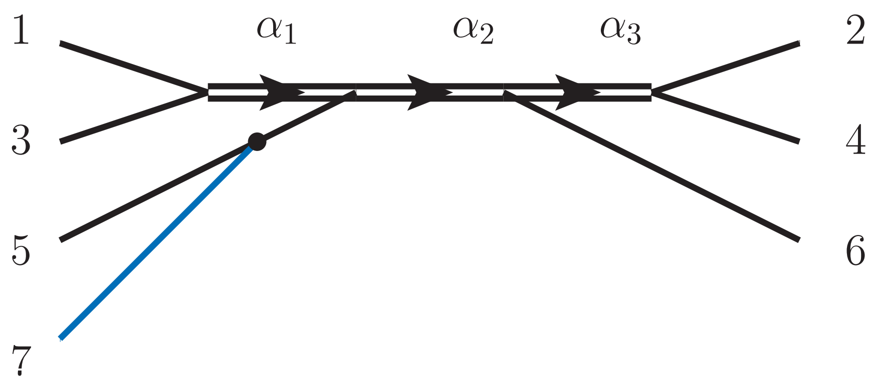
A parton shower perspective

- In a parton shower we start with some amplitude which we can assume that we have decomposed in the multiplet basis

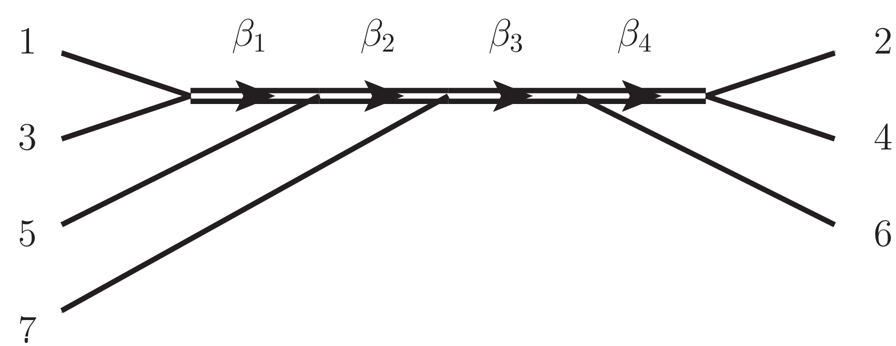
$$\text{Amp} = \sum_{\alpha_1,\alpha_2,\alpha_3} c_{\alpha_1,\alpha_2,\alpha_3}$$



- Knowing the decomposition for $N_g - 1$ gluons, how can we decompose the N_g gluon amplitude?

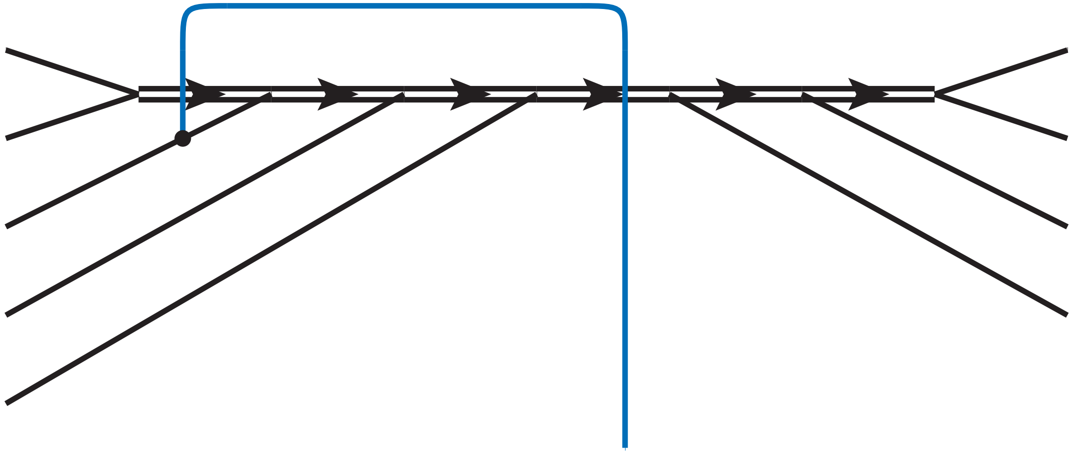


$$= \sum_{\beta_1,\beta_2,\ldots} \tilde{c}_{\beta_1,\beta_2,\ldots}$$

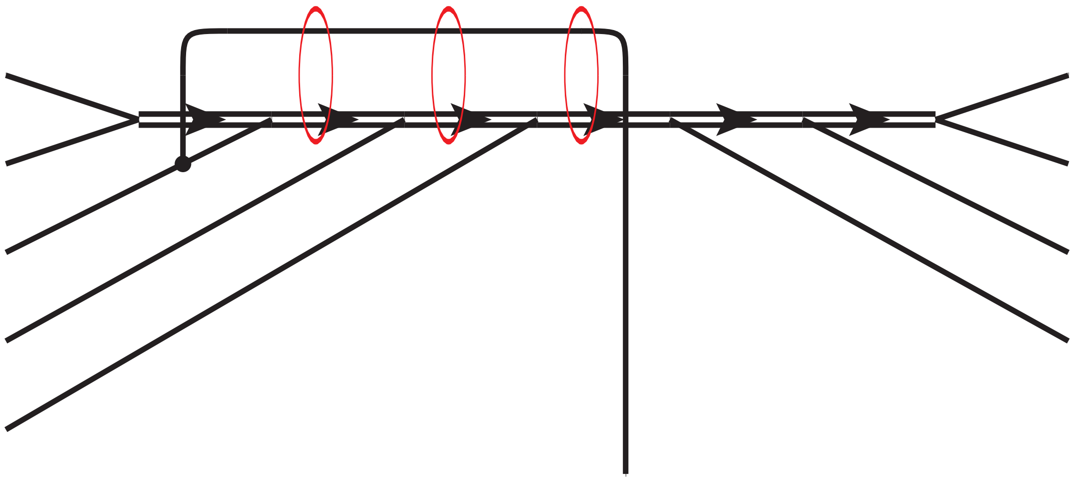


- Scalar products? Too slow!

Let one of the gluons emit a new gluon:

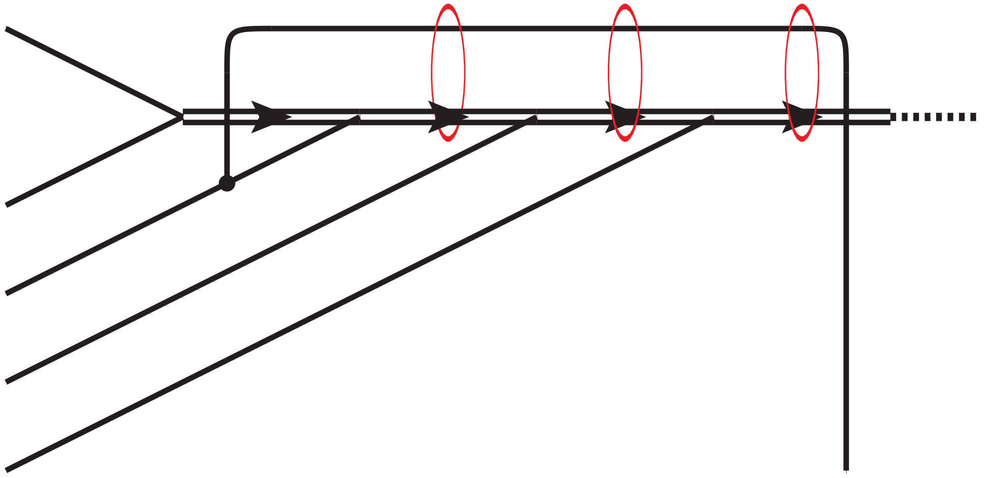


To decompose the affected side, we may insert the completeness relation repeatedly:

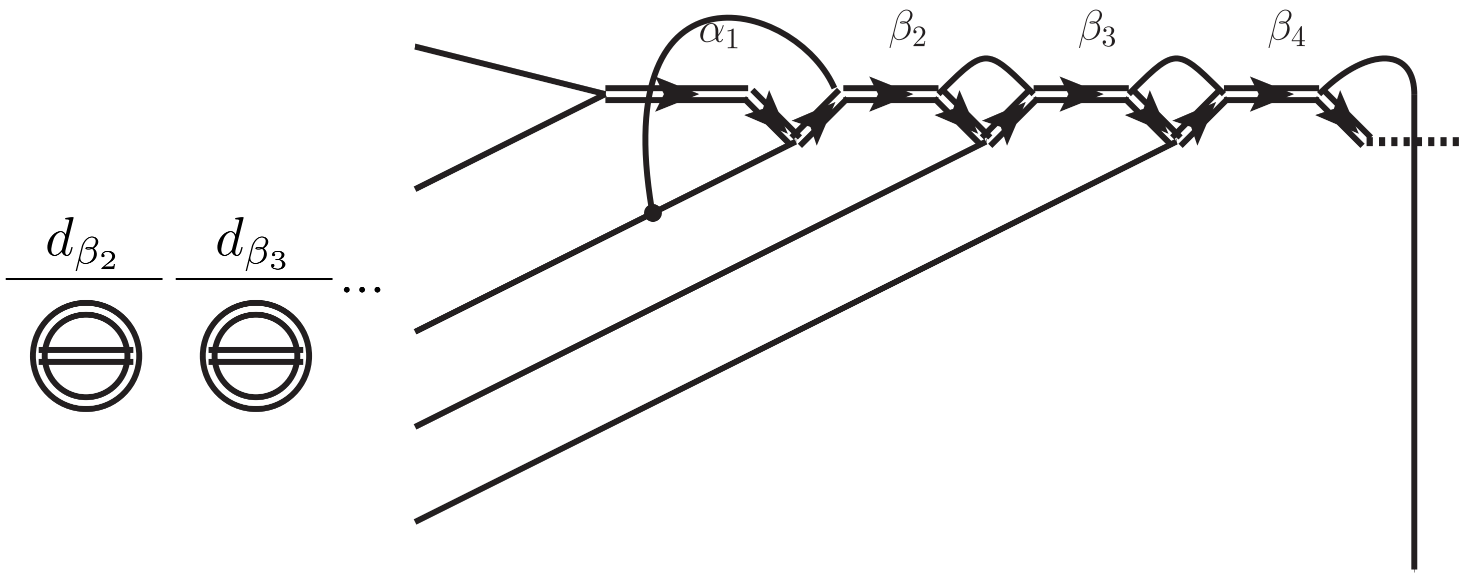


The representations on the other side (here right) don't change

Consider the affected side:

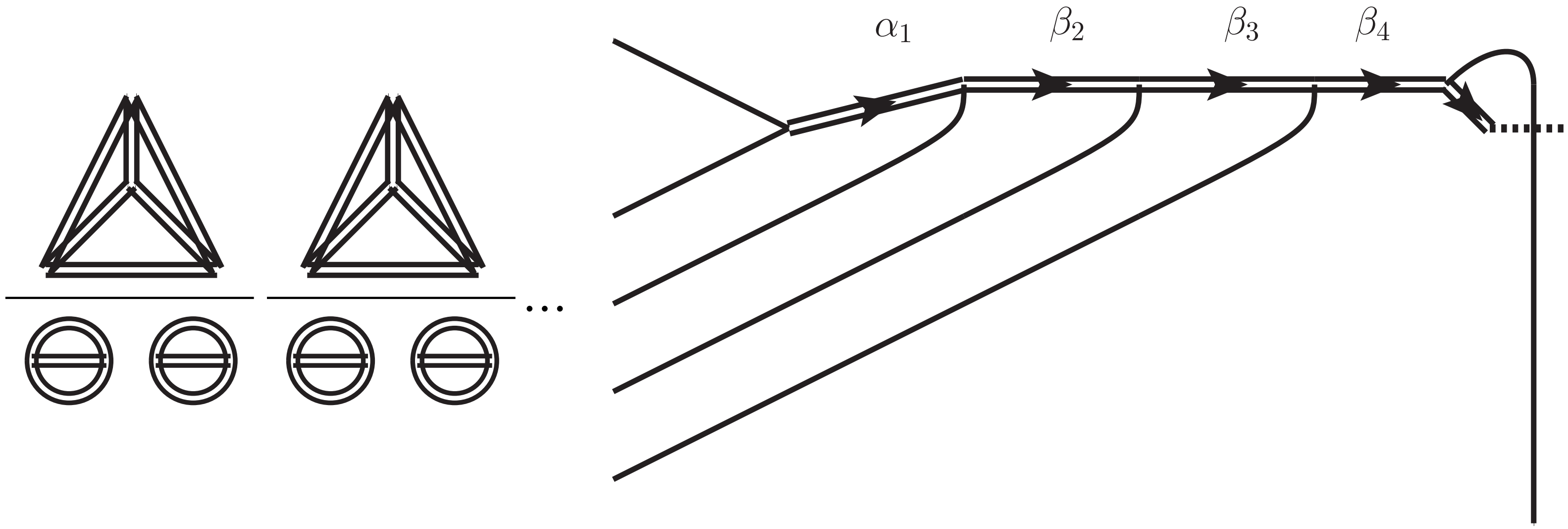


Inserting completeness relations we get a sum of terms of form:



What we have here are just vertex corrections which can be rewritten in terms of 3j and 6j coefficients

Giving us a sum of terms of form:



i.e., knowing the 3j and 6j symbols we can write down the resulting vectors

- By inserting the new gluon "in the middle" in the basis we guarantee that the emitted gluon need never "be transported" across more than $\sim$ half of the reps
- Typically we get only a small fraction of all basis vectors in the larger basis:

N_g	5→6	6→7	7→8	8→9	9→10
$N_c = 3$	0.094	0.027	0.012	0.0032	0.0014
$N_c \geq N_g$	0.071	0.014	0.0054	0.00092	0.00032

(Y. Du, M.S. & J. Thorén, JHEP 1505 (2015) 119, 1503.00530)

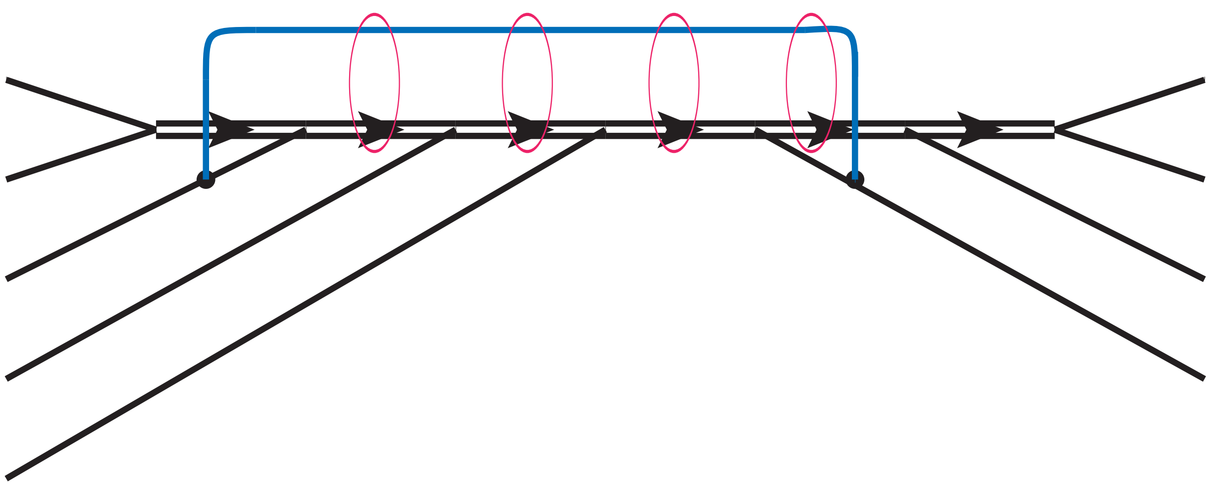
Consider the sum of all terms from all emissions (all emitters and all vectors) and compare to the number encountered when squaring a tree-level amplitude

N_g	Fraction ($N_c = 3$)	All terms ($N_c = 3$)	(# tree vectors)^2 (any N_c)
5→6	0.094	2 184	$(120)^2$
6→7	0.027	16 372	$(720)^2$
7→8	0.012	212 914	$(5\ 040)^2$
8→9	0.0032	1 758 620	$(40\ 320)^2 \sim 10^9$
9→10	0.0014	25 407 328	$(362\ 880)^2 \sim 10^{11}$

Numbers will be somewhat reduced by clever vertex choices, and non-general linear combinations

Gluon exchange

- For higher order calculations or for resummation we need to describe the effect of gluon exchange on the color structure
- Gluon exchange may be treated similar to emission



- Here we get a linear combination of basis vectors where only the intermediate representations can have changed

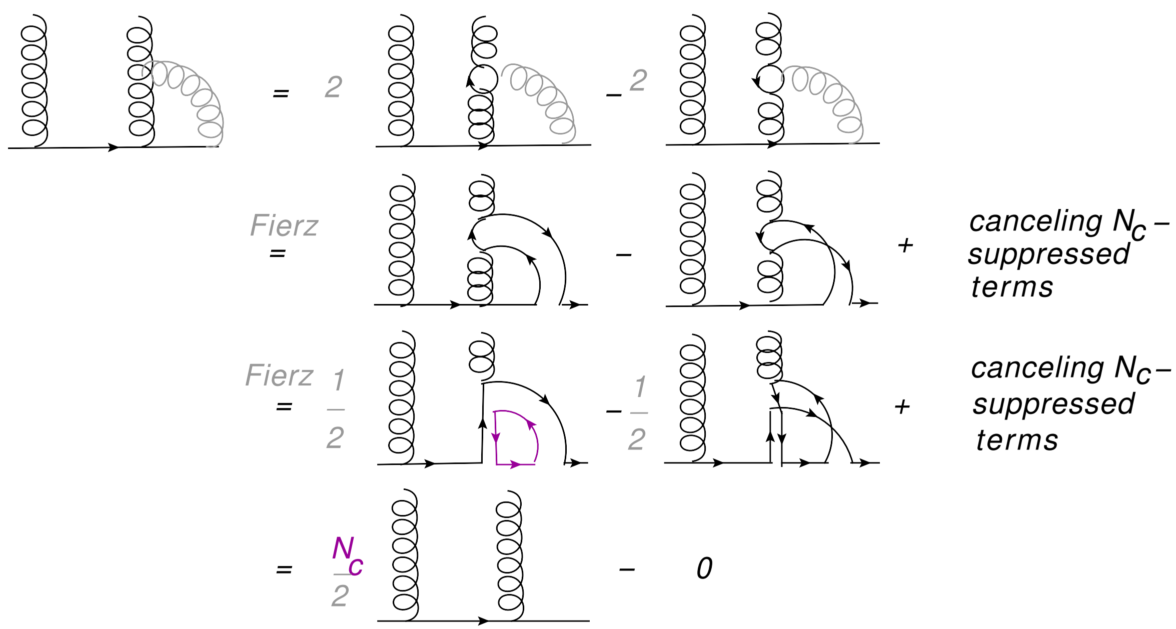
Summary

- We can calculate in orthogonal multiplet bases *without* explicitly constructing the corresponding bases
- Instead only the Wigner 6j coefficients are needed
- We can calculate them in a way which scales only as the square of the number of quarks
- An implementation on its way

Thank you for your attention!

Backup: Gluon exchange

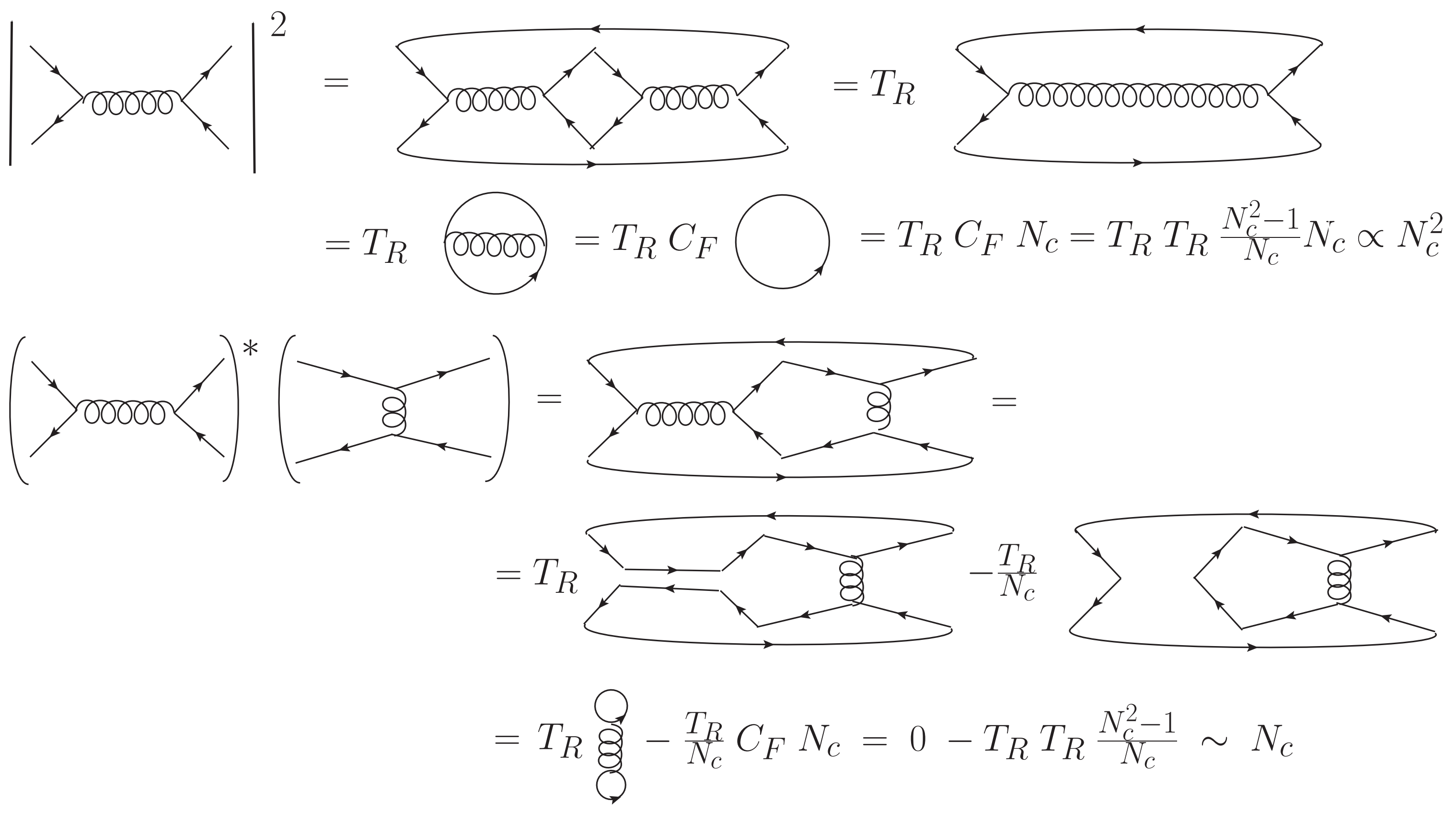
A gluon exchange in this basis "directly" i.e. without using scalar products gives back a linear combination of (at most 4) basis tensors



- N_c-enhancement possible only for near by partons
  $\rightarrow$ only "color neighbors" radiate in the $N_c \rightarrow \infty$ limit

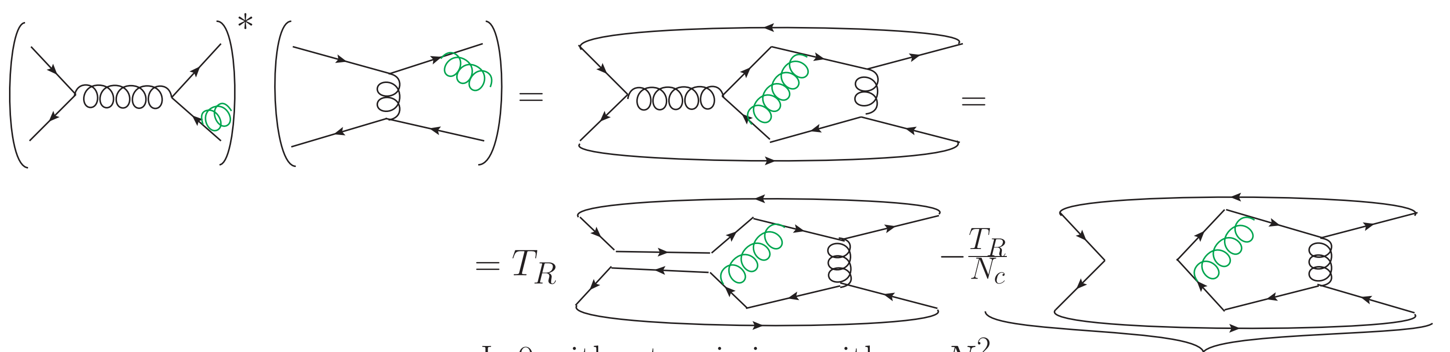
Backup: N_c-suppressed terms

That non-leading color terms are suppressed by $1/N_c^2$, is guaranteed only for same order α_s diagrams with only gluons ('t Hooft 1973)



Backup: N_c-suppressed terms

For a parton shower there may also be terms which only are suppressed by one power of N_c



Is 0 without emission, with $\sim N_c^2$ did not enter in any form, genuine "shower" contribution

Is $\sim N_c$ without emission, with $\sim N_c^2$ "included" in shower, contribution from hard process

The leading N_c contribution scales as N_c^2 before emission and N_c^3 after

For many partons the size of the vector space is much smaller for $N_c = 3$ (exponential), than for $N_c \to \infty$ (factorial)

N_g	Vectors $N_c = 3$	Vectors $N_c \to \infty$ trace bases	LO Vectors $N_c \to \infty$ LO trace bases
4	8	9	3!=6
5	32	44	4!=24
6	145	265	120
7	702	1 854	720
8	3 598	14 833	5 040
9	19 280	133 496	40 320
10	107 160	1 334 961	362 880

Number of basis vectors for N_g gluons *without* imposing vectors to appear in charge conjugation invariant combinations

... but the real advantage comes when squaring as the multiplet bases are orthogonal and the trace bases are not

N_g	Vectors $N_c = 3$	Vectors $N_c \to \infty$ trace bases	LO Vectors $N_c \to \infty$ LO trace bases
4	8	$(9)^2$	$(6)^2$
5	32	$(44)^2$	$(24)^2$
6	145	$(265)^2$	$(120)^2$
7	702	$(1\ 854)^2$	$(720)^2$
8	3 598	$(14\ 833)^2$	$(5\ 040)^2$
9	19 280	$(133\ 496)^2 \sim 10^{10}$	$(40\ 320)^2 \sim 10^9$
10	107 160	$(1\ 334\ 961)^2 \sim 10^{12}$	$(362\ 880)^2 \sim 10^{11}$

Number of terms from color when squaring for N_g gluons *without* imposing charge conjugation invariant combinations