UNIVERSITÄT GRAZ

NAWI Graz
Natural Sciences

# PDFs and TMDs from Functional Methods

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*with Gernot Eichmann & Alfred Stadler*

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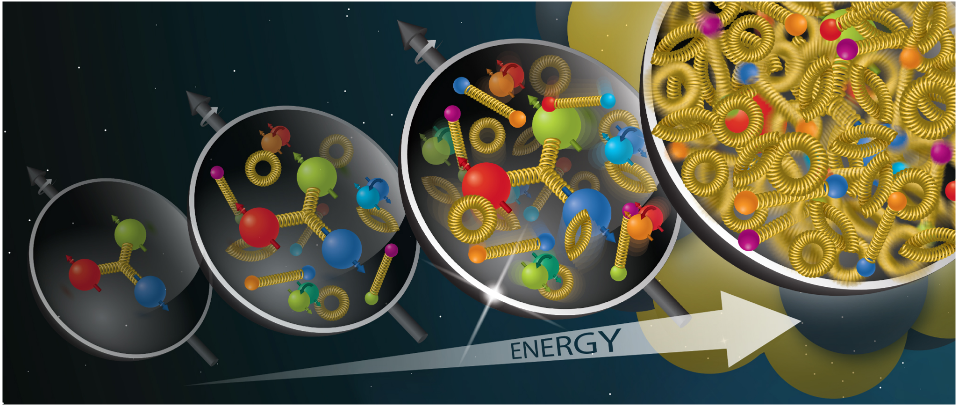
# Partonic Structure Functions

- PDFs, TMDs, etc... describe this picture:




(Picture source: EIC website)

- Different energy scales probe different aspects of the hadron.
- Contribution from **valence** and **sea** partons.

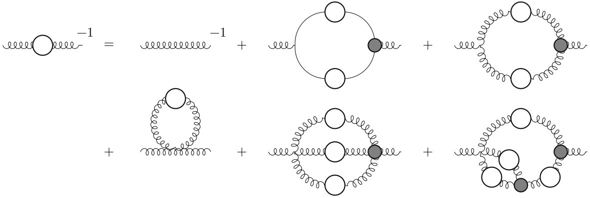
# 1 Non-perturbative methods

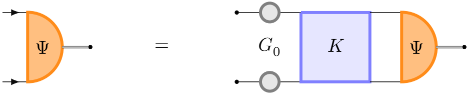
2 PDFs/TMDs

3 Four-point function

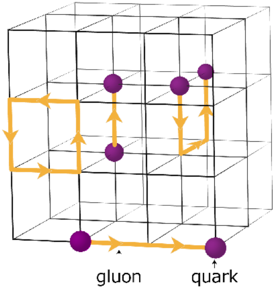
# Theoretical tools

- FUNctional Methods




- Lattice QCD



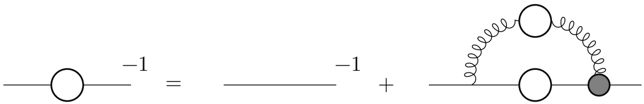

(Bi, Xiao, Guo, Ming; 2020)

Figure: *"Minecraft QCD"*

- Amplitude analysis
- Effective theories – $\chi$EFT
- Pheno models

# Dyson-Schwinger Equations

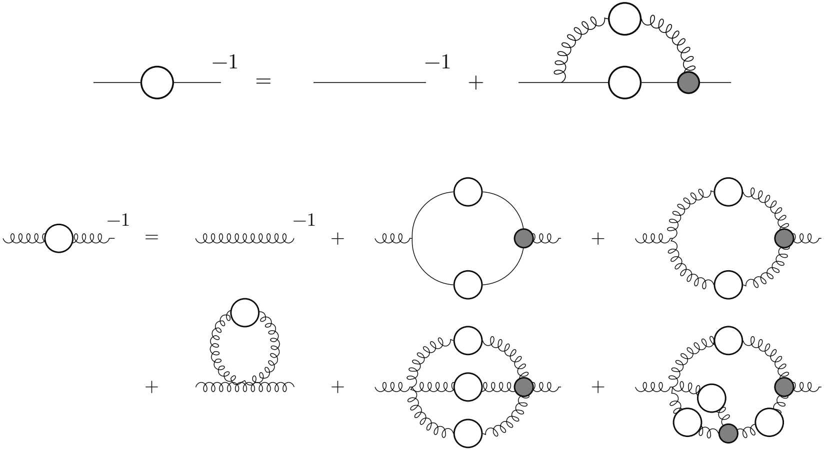
- Quantum equations of motion
- We can relate $n$-point functions with each other:




- The **full inverse quark propagator** is the inverse bare propagator plus self-energy $\Sigma$
- $\Sigma$ depends on gluon and $qqg$ vertex

# Dyson-Schwinger Equations

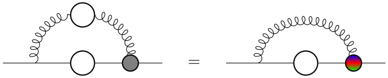
- Quantum equations of motion
- We can relate $n$-point functions with each other:



- Coupled equations: $n$-point function depends on higher order correlations.

# Truncation *Is the name of the game*

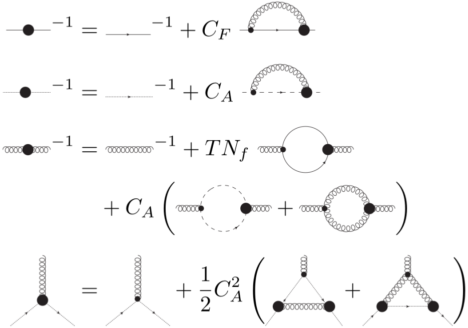
- Calculations only possible with truncations



$$\frac{\alpha(k^2)}{k^2} = \pi\eta^7\left(\frac{k^2}{\Lambda^2}\right)^2 e^{-\eta^2\frac{k^2}{\Lambda^2}} + \frac{2\pi\gamma_m\left(1-e^{-\frac{k^2}{\Lambda_t^2}}\right)}{\ln\left[e^2-1+\left(1+\frac{k^2}{\Lambda_{QCD}^2}\right)^2\right]}$$

**Rainbow-ladder truncation**

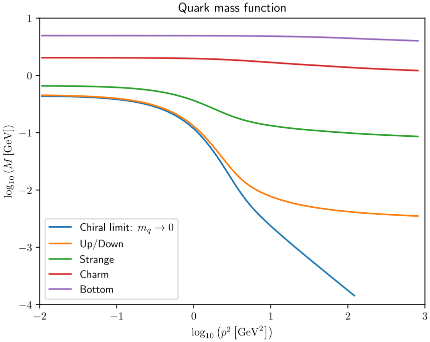
(Maris, Roberts; 1997) (Maris, Tandy; 1999) (Maris, Tandy; 2000)

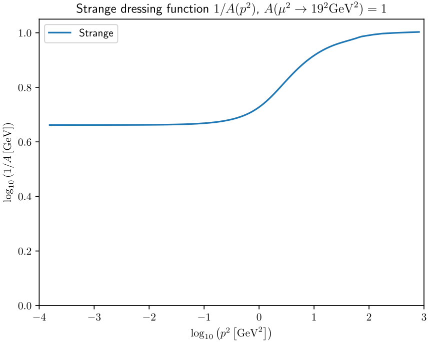



(Alkofer, Zierler; 2023)

# Truncation *Is the name of the game*

- Calculations only possible with truncations



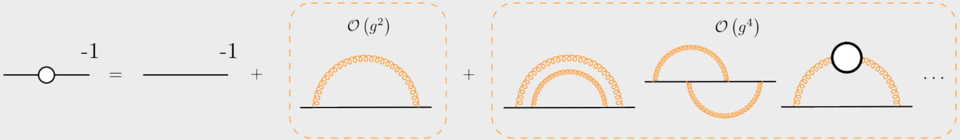

$$S(p) = \sigma_s(p^2)\mathbb{1} - i\sigma_v(p^2)\not{p} = \frac{1}{A(p^2)}\frac{-i\not{p} + M(p^2)}{p^2 + M(p^2)^2}$$

(Numerical results by: Raúl Torres)

## **BUT!** *It's not all-orders perturbation theory*

- Non-perturbative physics $\ggg$ all-orders perturbation theory

### D$\chi$SB is purely non-perturbative!



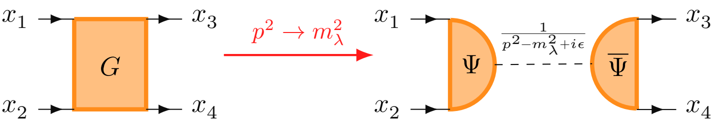

Series expansion: $\frac{1}{1-x} = f(x) \approx 1 + x + x^2 + x^3 + \ldots,\qquad |x| < 1$

Exact result: $\frac{1}{1-x} = f(x) = 1 + x f(x) = 1 + x + x^2 f(x) \ldots,\qquad \text{for all } x$

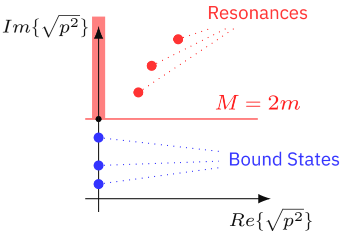
# Bethe-Salpeter Equation *How singularities relate to bound-states*

- The Bethe-Salpeter Wavefunction (BSWF) appears as the residue of a correlation function $G(p)$:

$$\Psi(x, P) = \langle 0 | \, \mathrm{T}\phi(0)\phi(x) \, | P \rangle \qquad \Psi(k, P) = \int d^4x e^{-ik\cdot x} \Psi(x, P)$$



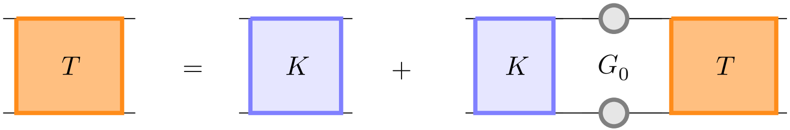

- Poles in correlation functions encode the theory's bound-state spectrum
- Position on the $p^2$ complex plane indicates their nature: bound-state, resonance, ...
- Euclidean metric!




(Eichmann, Sanchis-Alepuz, Williams, Alkofer, Fischer; 2016)

# Bethe-Salpeter Equation *Self-consistent bound-state equation*

- Take the scattering equation: $G = G_0 + G_0 T G_0 \Leftrightarrow T = K + K G_0 T$

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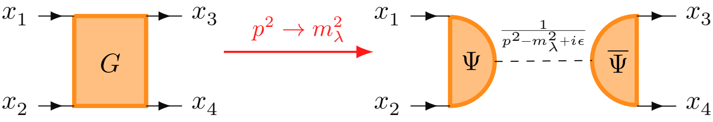
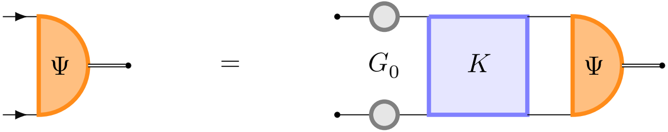



yields




(Eichmann, Sanchis-Alepuz, Williams, Alkofer, Fischer; 2016)

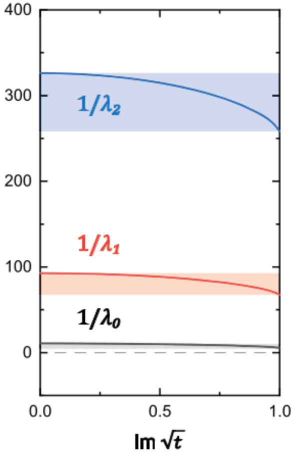
# Bethe-Salpeter Equation *Eigenvalue/vector equation*

- The BSE is usually solved as an eigenvalue equation:

$$\lambda_i \left(P^2\right) \psi_i = \mathbf{K G_0} \psi_i$$

- States found when $\lambda_i(P^2) \equiv 1$.
- Mass is given by $P^2 = -M^2$
- Eigenvalue/vector spectrum gives ground state and excited states.
- Like DSE, analytic structure is important, and **contour deformations** may be needed!

**QFT "version" of the Schrödinger Equation**



1 Non-perturbative methods

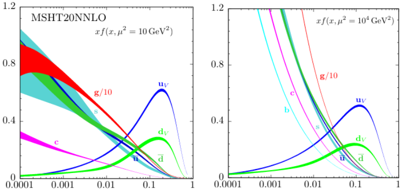
# 2 PDFs/TMDs

3 Four-point function

# Hadrons on the Light Front

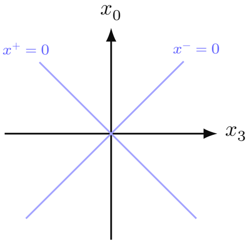
**Goal:** Use DSE/BSE to study hadrons on the light front, $x^+ = 0$.

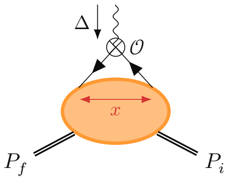
- Natural frame for defining parton distribution functions: PDFs, TMDs, ...




- Future: COMPASS/AMBER @ CERN
  EIC @ Brookhaven National Laboratory.

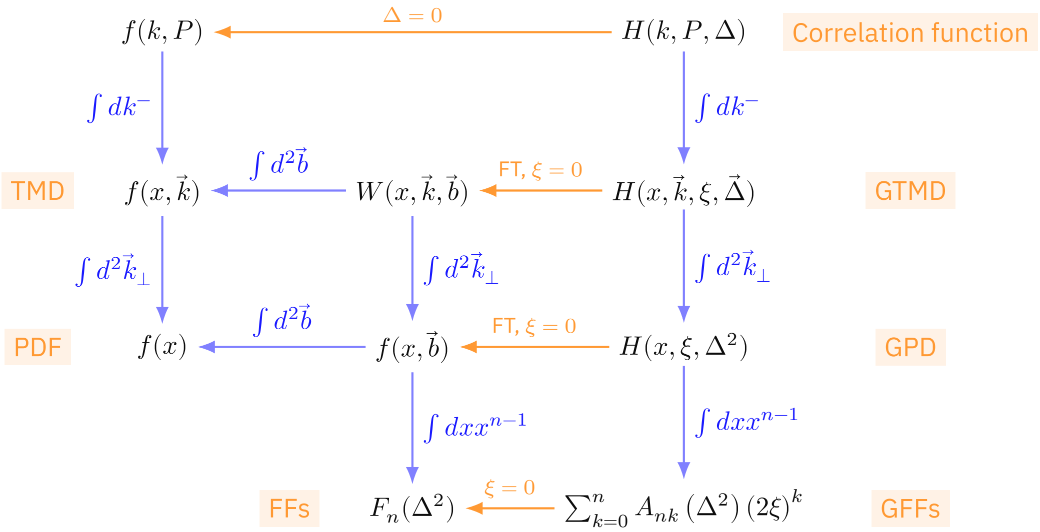
(AMBER: arXiv:1808.00848)
(EIC: arXiv:2305.14572)




$$\langle P_f | \, \mathrm{T}\Phi(x)\mathcal{O}\Phi(0) \, | P_i \rangle$$

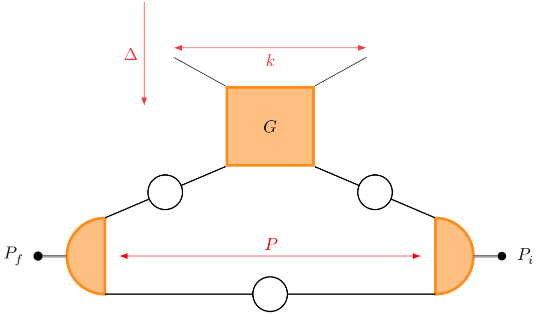
# Hadronic quantities




(Lorce, Pasquini, Vanderhaeghen; 2011), (Picture adapted from: Diehl, 2016), (Diehl, 2003), (Meißner, Goeke, Metz, Schlegel; 2008), (Meißner, Metz, Schlegel; 2009)

# Our Goal

- **Main Goal:** Get partonic distribution functions from hadron-hadron correlations via **FUN**ctional Methods




- $G$ is the four-point quark correlation function, calculated with scattering equation.
- The quark propagator is calculated via quark DSE.
- The BSWF is calculated via the meson BSE.

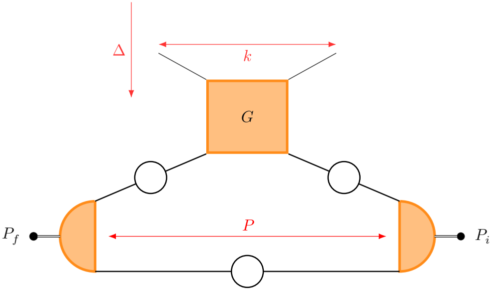
(Mezrag; 2015), (Diehl, Gousset; 1998), (Tiburzi, Miller; 2003), (Mezrag, Chang, Moutarde, Roberts, Rodríguez-Quintero, Sabatié, Schmidt; 2015), many others, ...

$$\mathcal{G}^{[\Gamma]}(P,k,\Delta) = \frac{1}{2}\mathrm{Tr}\left[\int dk^- \int \frac{d^4z}{2\pi^4} e^{ik\cdot z} \langle P_f | \overline{\psi}(z)\mathcal{W}\Gamma\psi(0)\, |P_i\rangle\right]$$

- Partonic distributions are calculated by integrating the correlator in $k^-$ and taking appropriate traces.

# Triangle Diagram

- We start by solving a simple model, and gradually build up the complexity of the calculation.
  - Hadrons are on-shell: $P^2 = -M^2$

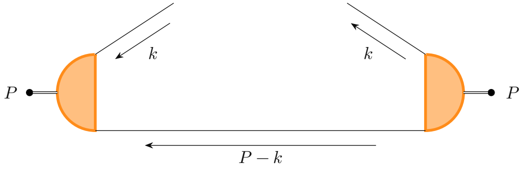



$$\Delta = \sqrt{\Delta^2}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad P = 2m\sqrt{t}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$k = xP + M\sqrt{R}\begin{pmatrix}0\\0\\\sqrt{1-Z^2}\\Z\end{pmatrix}$$

# Triangle Diagram

- We start by solving a simple model, and gradually build up the complexity of the calculation.
  - Hadrons are on-shell: $P^2 = -M^2$
  - **Forward limit:** $\Delta \to 0$ – We get the PDFs and TMDs



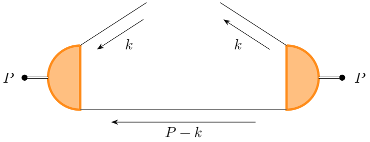

$$P = 2m\sqrt{t}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \quad k = xP + M\sqrt{R}\begin{pmatrix}0\\0\\\sqrt{1-Z^2}\\Z\end{pmatrix}$$

- Steps:
  1. Calculate BSWF;
  2. Calculate the correlator;
  3. Project to the light-front;

# Light-front Projection

- The TMD is defined as:

$$\int dk^-$$




- In our kinematic variables:

$$k^- \propto -2i\sqrt{R}Z$$

- Need the BSWF in $Z \in (-\infty, \infty)$.
- Use Padé approximants:

$$f(Z) = \frac{a_0 + a_1 Z + a_2 Z^2 + \cdots + a_N Z^N}{1 + b_1 Z + b_1 Z^2 + \cdots + b_M Z^M}$$

(L. Schlessinger, 1968) (Tripolt et al., 2019) (D. Binosi, R-A. Tripolt; 2019)

- $N - M$ is fixed, can control behaviour at very large $Z$.

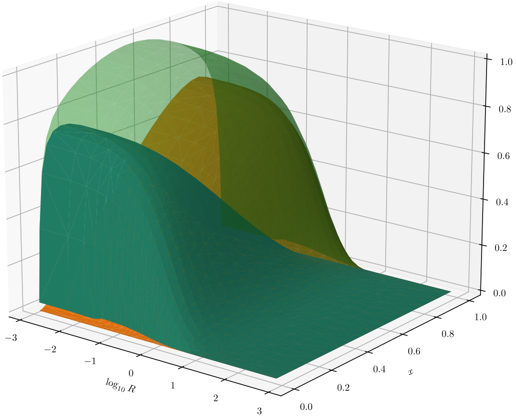
**Definition of the TMD**

$$\mathrm{TMD}(R, x) \propto -2i\sqrt{R} \int_{-\infty}^{\infty} dZ\, \mathcal{G}(R, Z, x)$$

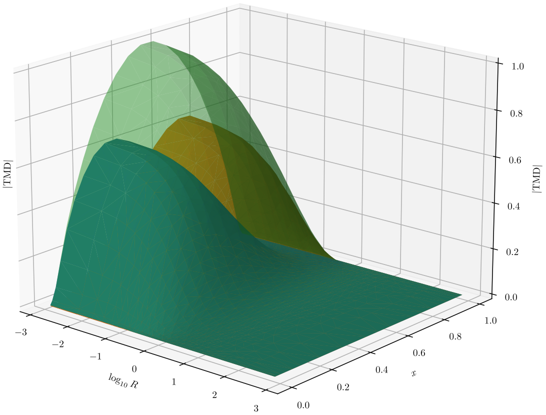
(Eichmann, EF, Stadler; 2022)

# TMD: Some results

- $\gamma = 1.5, \beta = 4, c = 1$



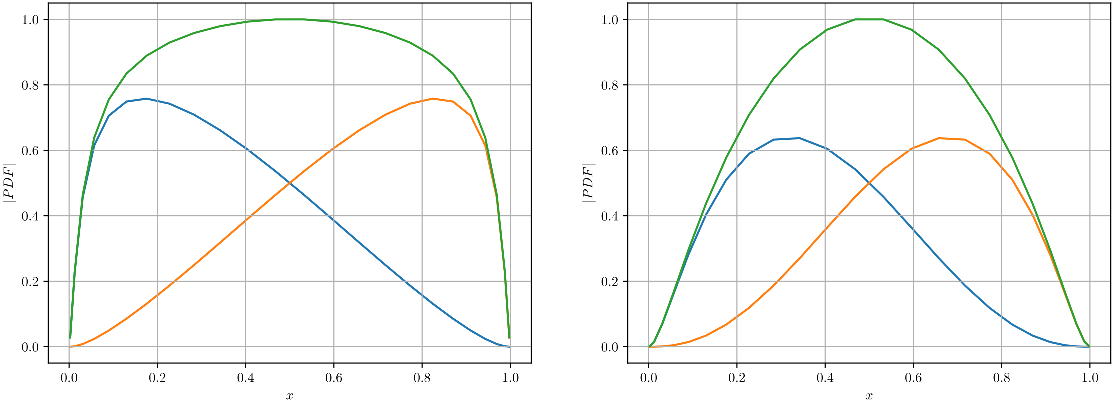

- $\gamma = 1, \beta = 1, c = 1$




■ Upper line open ■ Lower line open ■ Sum of both

# PDFs

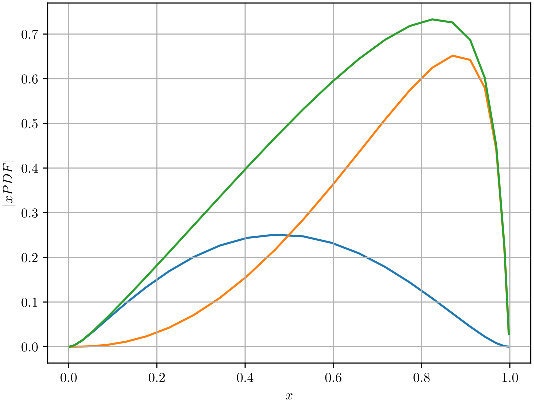
- $\sqrt{t} = 0.33i, \beta = 4, c = 1$
- $\sqrt{t} = 0.5i, \beta = 1, c = 1$



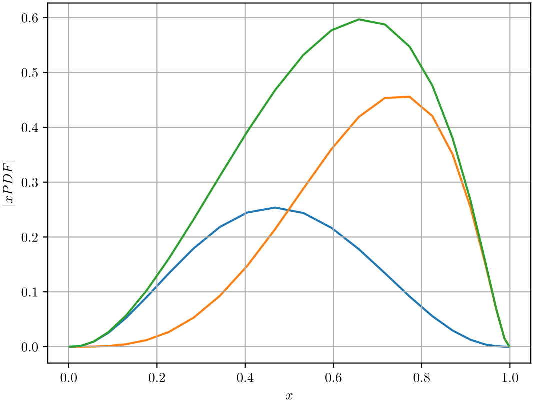

Upper line open
Lower line open
Sum of both

# PDFs

- $\sqrt{t} = 0.33i, \beta = 4, c = 1$




- $\sqrt{t} = 0.5i, \beta = 1, c = 1$

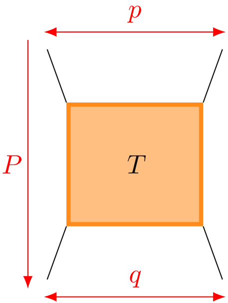



■ Upper line open ■ Lower line open ■ Sum of both

# 3 Four-point function

# 4-point function

- 4- point function determined from scattering equation:
  $\mathbf{G} = \mathbf{G_0} + \mathbf{G_0 T G_0} \implies \mathbf{T} = \mathbf{K} + \mathbf{K G_0 T} \implies \mathbf{T} = (\mathbb{1} - \mathbf{K G_0})^{-1}\,\mathbf{K}.$




- Fully off-shell: 6 Lorentz invariants
  - 3 radial: $X, t, R$;
  - 3 angular: $Y, Z, Q$;

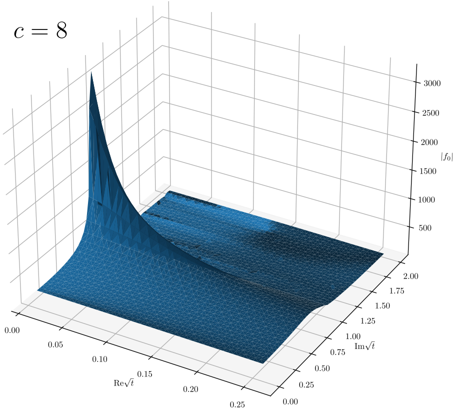
$$T(t,X,R,Z,Y,Q) = K(X,R,p\cdot q) + \frac{1}{2}\frac{m^4}{2\pi^4}\int_0^\infty dx\, x \int_{-1}^{1} dz\, \sqrt{1-z^2}\, G_0\,(x,z,t) \times \int_{-1}^{1} dy \int_0^{2\pi} d\Psi\, K(X,x,k\cdot q)\, T(t,x,y,z,R,Q)$$

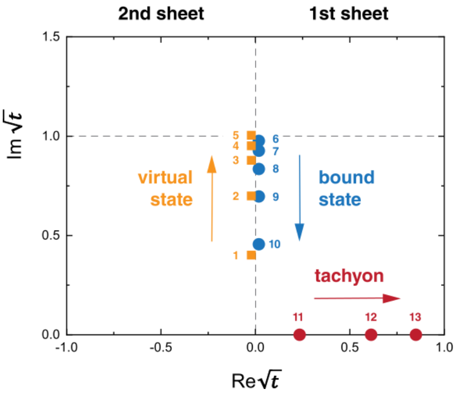
- Same $G_0$ and $K$ as in the BSE
- Solved numerically as linear system
- Contains *all* dynamics of 2 $\phi$ particles
  - **Must produce bound state poles dynamically!**

(Eichmann, Duarte, Peña, Stadler; 2019)

# 4-point function

- $T$ is made fully on-shell
- Partial-wave expansion shows bound-state pole in the first Riemann sheet



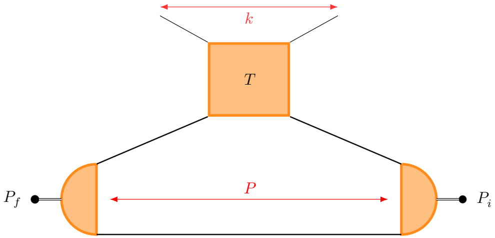




(Eichmann, Duarte, Peña, Stadler; 2019)

- Describes both long-range and short-range $qq$ dynamics.

# Full triangle

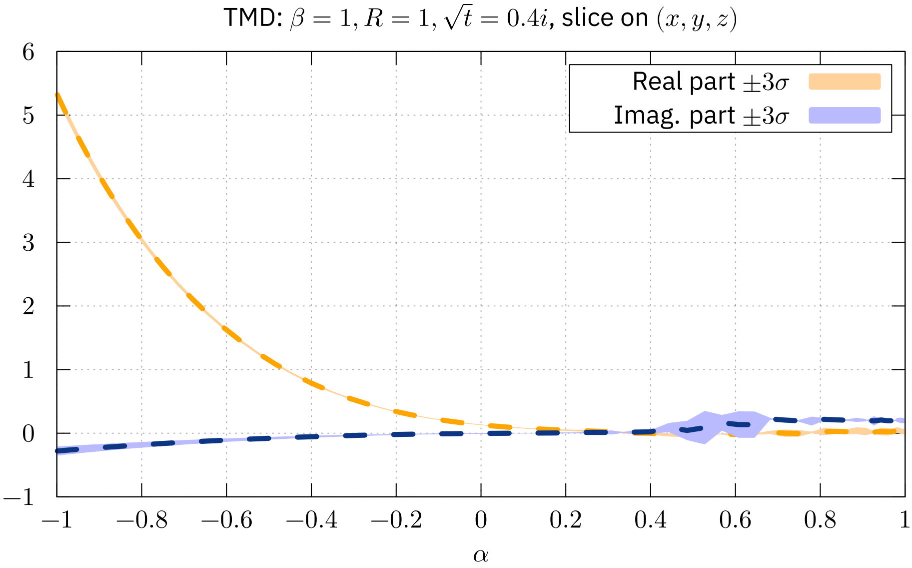
- Can we use the full 4-point function? – **YES!**




1. Calculate 4-point function
2. Calculate BSE
3. Do the loop integration
4. Project to LF
5. Do statistics on extrapolation

- We solve one triangle for each point in the $R, x$ grid – **HPC Needed!**
  - $N_x \times N_R \times N_Z$ 4-point functions!

# Results *Preliminary*



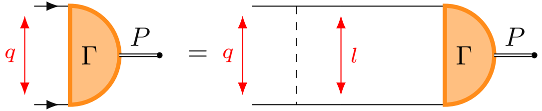
# Conclusions

- We have a method for TMDs/PDFs from first-principles self-consistent functional methods
- We have incorporated BSE solutions
- We have implemented contour deformations – resonances!
- We have made considerable progress in the inclusion of the 4-point function

Backup

# Scalar toy model *Bethe-Salpeter Equation*

- Scalar model:
  - $\phi$ of mass $m$
  - $\chi$ of mass $\mu$
- BS amplitude for bound-state of two $\phi$:




$$q = \frac{\alpha}{2}P + m\sqrt{\rho}\begin{pmatrix}0\\ \times\\ \times\\ \times\end{pmatrix} \qquad l = \frac{\alpha}{2}P + m\sqrt{l}\begin{pmatrix}\times\\ \times\\ \times\\ \times\end{pmatrix}$$

(Wick; 1954) (Cutkosky; 1954)

- The BSWF is a function of the kinematic invariants:

$$-M^2 = P^2 \qquad \frac{q^2}{m^2} \propto \rho \qquad z \propto \hat{k}\cdot\hat{P}$$

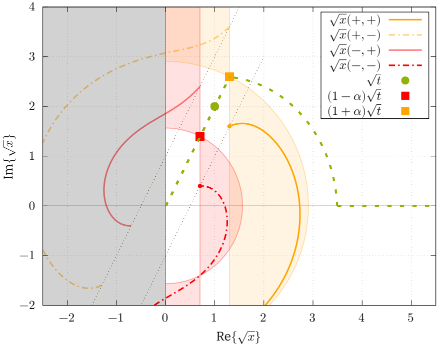
$$\psi(\rho, z, x) = \frac{m^4}{(2\pi)^3}\frac{1}{2}\int_0^\infty dl\, l$$
$$\times \int_{-1}^{1} dz_l \sqrt{1-{z_l}^2}\,\mathbf{G_0}(l, z_l, \alpha)$$
$$\times \int_{-1}^{1} dy_l\,\mathbf{K}(\rho, z, l, z_l, y_l)\,\psi(l, z_l, \alpha)$$

# Analytic Structure of BSE $\mathrm{Im}\sqrt{t} > \min\left(\frac{1}{1\pm\alpha}\right)$

$$\mathbf{G_0}^{-1} = (l_+^2 + m^2)(l_-^2 + m^2)$$

- Branch cuts in complex $\sqrt{l}$ plane

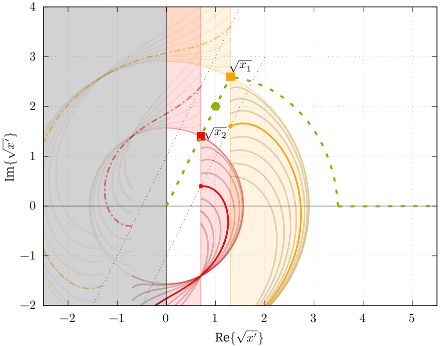
$$\sqrt{l}_\pm^\lambda = \mp(1\pm\alpha)\sqrt{t}\left[z_l + i\lambda\sqrt{1 - z_l^2 + \frac{1}{(1\pm\alpha)^2 t}}\right]$$




$$\mathbf{K}^{-1} = (q-l)^2 + \mu^2$$

- Branch cuts in complex $\sqrt{l}$ plane depend on path taken:

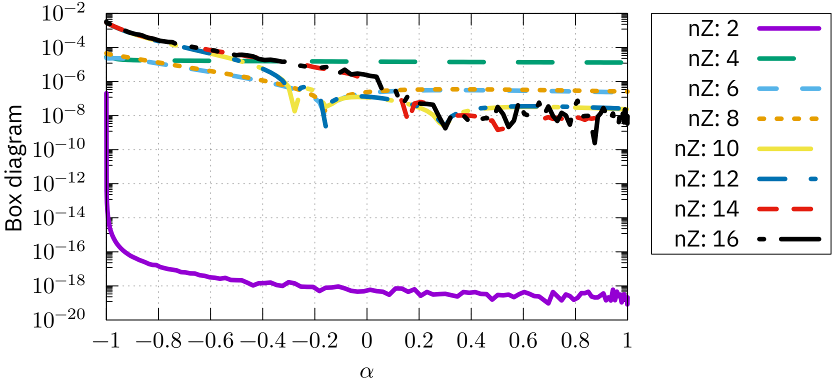
$$\sqrt{l} = \sqrt{\rho}\left(\Omega \pm i\sqrt{1 - \Omega^2 + \frac{\beta^2}{\rho}}\right)$$



# Extrapolation (can be) unreliable

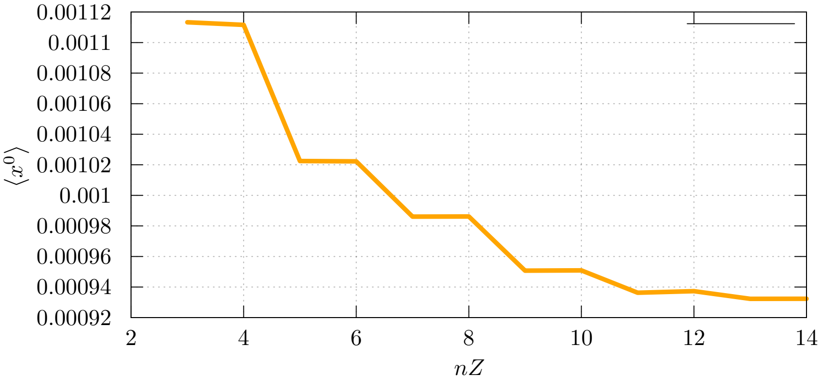
$$R(Z) = \cfrac{f_0}{1 + \cfrac{a_0(Z - Z_0)}{1 + \cfrac{a_1(Z - Z_1)}{1 + \dots}}}$$

- Schlessinger method is fast but limited – at most $\propto \frac{1}{Z}$
  - Coefficients must be very accurately calculated
- Different input points $\implies$ different output!



# Use Padé Approximants

- In this case we know the analytic dependence – can improve extrapolation
- Uncertainty **massively** reduced – automate search for best parameters
  - Use statistics to “approximate” uncertainty




- Figure of merit: **First Mellin moment** $\langle x^0 \rangle = \int d\alpha f(\alpha)$.

# Use Padé Approximants

- In this case we know the analytic dependence – can improve extrapolation
- Uncertainty **massively** reduced – automate search for best parameters
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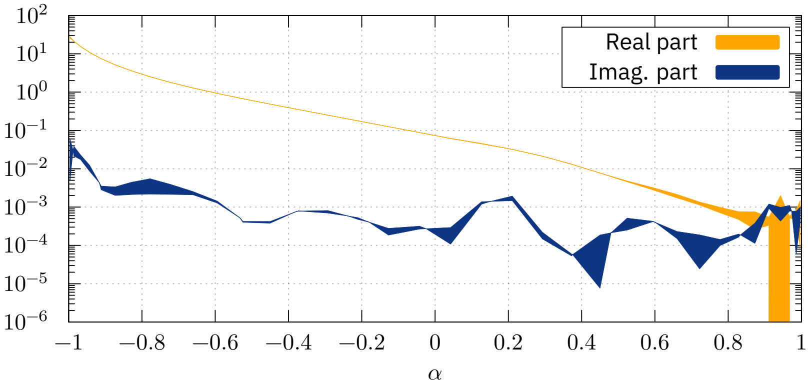
```
[main::main()]                  Points selected: (-0.971882,0) (-0.964138,0) (-0.637737,0) (-0.559991,0) (-0.367686,0) (-0.108877,0) (-0.00673456,0) (0.333391,0) (0.670008,
0) (0.775018,0) (0.929139,0) (0.895408,0)
[main::main()]                  Points selected: (-0.906222,0) (-0.991732,0) (-0.816613,0) (-0.536235,0) (-0.373951,0) (-0.0363416,0) (0.119572,0) (0.346285,0) (0.568612,0)
 (0.756554,0) (0.884481,0) (0.988616,0)
[main::main()]                  Points selected: (-0.994135,0) (-0.788311,0) (-0.861096,0) (-0.537644,0) (-0.301774,0) (-0.168043,0) (0.092449,0) (0.345567,0) (0.642733,0)
(0.72315,0) (0.856472,0) (0.96919,0)
[main::main()]                  Points selected: (-0.956588,0) (-0.919778,0) (-0.756584,0) (-0.503692,0) (-0.366042,0) (-0.105799,0) (0.110391,0) (0.349409,0) (0.601376,0)
(0.731116,0) (0.850736,0) (0.940282,0)
[main::main()]                  Points selected: (-0.971467,0) (-0.919462,0) (-0.801734,0) (-0.548751,0) (-0.307002,0) (-0.135181,0) (0.167636,0) (0.316577,0) (0.560352,0)
(0.823707,0) (0.945959,0) (0.824111,0)
[main::main()]                  Points selected: (-0.996096,0) (-0.947657,0) (-0.756661,0) (-0.517286,0) (-0.396726,0) (-0.202665,0) (0.17211,0) (0.420647,0) (0.61719,0) (0
.769154,0) (0.797286,0) (0.947448,0)
[main::main()]                  Points selected: (-0.957069,0) (-0.923619,0) (-0.744116,0) (-0.604607,0) (-0.285585,0) (-0.247233,0) (0.159705,0) (0.428975,0) (0.581043,0)
(0.670109,0) (0.839225,0) (0.954249,0)
[main::main()]                  Points selected: (-0.99626,0) (-0.830503,0) (-0.763215,0) (-0.582946,0) (-0.318577,0) (-0.126642,0) (0.104791,0) (0.361273,0) (0.627808,0) (
0.725125,0) (0.844745,0) (0.970048,0)
[main::main()]                  Points selected: (-0.894138,0) (-0.916913,0) (-0.744792,0) (-0.498678,0) (-0.387726,0) (-0.085339,0) (0.039761,0) (0.328953,0) (0.54071,0) (
0.617861,0) (0.909084,0) (0.841881,0)
[main::main()]                  Points selected: (-0.985678,0) (-0.859571,0) (-0.733059,0) (-0.611597,0) (-0.379311,0) (-0.18972,0) (0.184463,0) (0.279375,0) (0.64143,0) (0
.809775,0) (0.867791,0) (0.932604,0)
[main::main()]                  Points selected: (-0.918332,0) (-0.933731,0) (-0.672537,0) (-0.616266,0) (-0.328011,0) (-0.132243,0) (0.132344,0) (0.427782,0) (0.692186,0)
(0.718246,0) (0.849771,0) (0.965346,0)
[main::main()]                  Points selected: (-0.886897,0) (-0.924523,0) (-0.722975,0) (-0.424517,0) (-0.363058,0) (-0.124189,0) (0.119086,0) (0.419112,0) (0.556684,0)
(0.678858,0) (0.822687,0) (0.962312,0)
[main::main()]                  Points selected: (-0.875028,0) (-0.890325,0) (-0.781825,0) (-0.562061,0) (-0.362633,0) (-0.18577,0) (0.206556,0) (0.298088,0) (0.638054,0) (
0.758299,0) (0.86561,0) (0.905794,0)
[main::main()]                  Points selected: (-0.957465,0) (-0.796093,0) (-0.772831,0) (-0.590603,0) (-0.389053,0) (-0.111661,0) (0.119117,0) (0.400803,0) (0.550732,0)
(0.688639,0) (0.889395,0) (0.977207,0)
[main::main()]                  Points selected: (-0.90401,0) (-0.865011,0) (-0.842518,0) (-0.537244,0) (-0.398679,0) (-0.113835,0) (0.0799175,0) (0.380676,0) (0.591231,0)
(0.91713,0) (0.779767,0) (0.912919,0)
[main::main()]                  Points selected: (-0.947619,0) (-0.928872,0) (-0.65895,0) (-0.584036,0) (-0.380337,0) (-0.133252,0) (0.148025,0) (0.468123,0) (0.50868,0) (0
.724906,0) (0.935442,0) (0.971511,0)
[main::main()]                  Points selected: (-0.981875,0) (-0.859104,0) (-0.780963,0) (-0.587309,0) (-0.37501,0) (-0.228738,0) (0.170687,0) (0.320618,0) (0.554563,0) (
0.855494,0) (0.952401,0) (0.899179,0)
[main::main()]                  Points selected: (-0.994747,0) (-0.936452,0) (-0.799706,0) (-0.502598,0) (-0.293855,0) (-0.238063,0) (0.0710795,0) (0.418727,0) (0.528205,0)
 (0.791017,0) (0.841999,0) (0.922319,0)
[main::main()]                  Points selected: (-0.981901,0) (-0.942334,0) (-0.817244,0) (-0.502693,0) (-0.362869,0) (0.0531426,0) (0.188461,0) (0.378999,0) (0.517289,0)
(0.76008,0) (0.958332,0) (0.829141,0)
[main::main()]                  Points selected: (-0.830801,0) (-0.895413,0) (-0.753161,0) (-0.533404,0) (-0.292988,0) (-0.116918,0) (0.126602,0) (0.398628,0) (0.602596,0)
(0.730918,0) (0.883785,0) (0.984913,0)
[main::main()]                  Points selected: (-0.920836,0) (-0.91827,0) (-0.686716,0) (-0.599125,0) (-0.331364,0) (-0.143183,0) (0.14999,0) (0.304703,0) (0.628909,0) (0
.78723,0) (0.821294,0) (0.927566,0)
[main::main()]                  Points selected: (-0.939713,0) (-0.896893,0) (-0.773103,0) (-0.538324,0) (-0.32484,0) (-0.0845858,0) (0.0429227,0) (0.300719,0) (0.535192,0)
 (0.759677,0) (0.940532,0) (0.958998,0)
[main::main()]                  Points selected: (-0.912195,0) (-0.928642,0) (-0.786102,0) (-0.560542,0) (-0.356796,0) (-0.123163,0) (0.118005,0) (0.335118,0) (0.571327,0)
```

- Generate large collection of $Z$ points ($\approx 50$).

# Use Padé Approximants

- In this case we know the analytic dependence – can improve extrapolation
- Uncertainty **massively** reduced – automate search for best parameters
  - Use statistics to “approximate” uncertainty




- Print the results as $\overline{f}(\alpha) \pm \sigma_f(\alpha)$.